q-Q.H.E. and Topology

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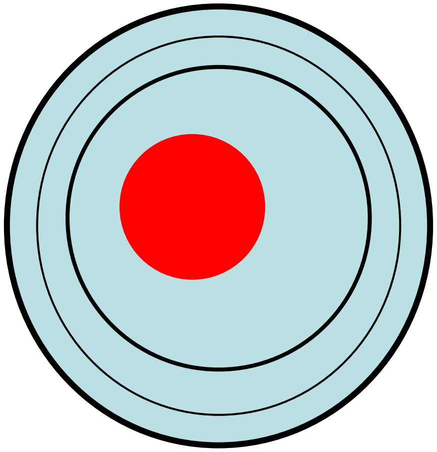
From the Hall Effect to integrability

1. Hall effect.
2. Transfer Matrix.
3. Annular algebras.
4. deformed Hall effect wave function as a toy model for TQFT.
5. Conclusion.

Hall effect

- Lowest Landau Level wave functions

$$\psi_n(z) = \frac{z^n}{n!} e^{-\frac{z\bar{z}}{l^2}}$$



$r = l\sqrt{n}$

n=1,2, …, $\frac{A}{2\pi l^2}$

$$\frac{A}{2\pi l^2} = n_0$$

Number of available cells also
the **maximal degree** in each variable

$$z_1^{\lambda_1} \ldots z_n^{\lambda_n}$$

Is a basis of **states** for the system

Interactions

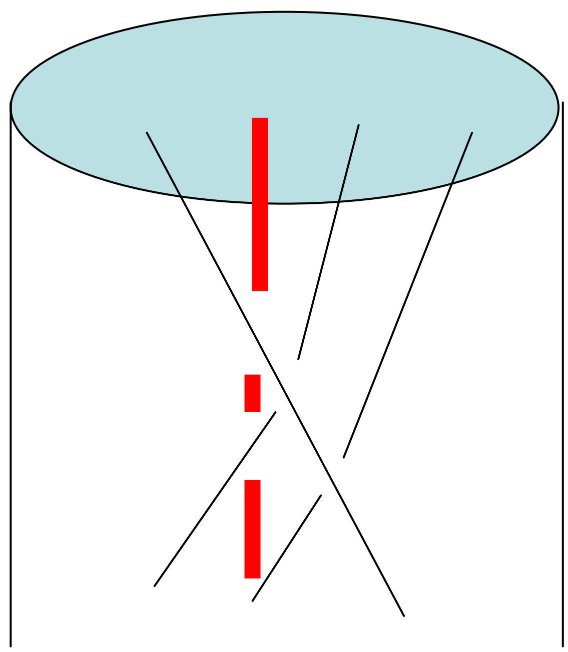
$$\left(z_i - z_j\right)^m$$

m measures the strength of the interactions.

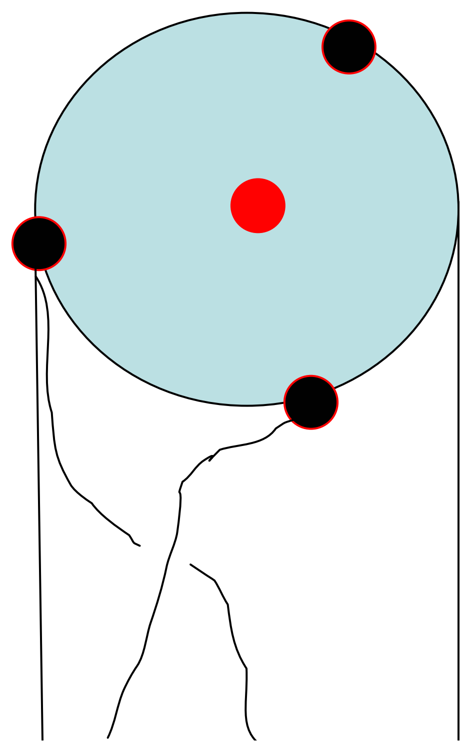
Competition between interactions which spread electrons apart and high Compression which minimizes the degree n.

With adiabatic time QHE=TQFT

- Bulk and edge.



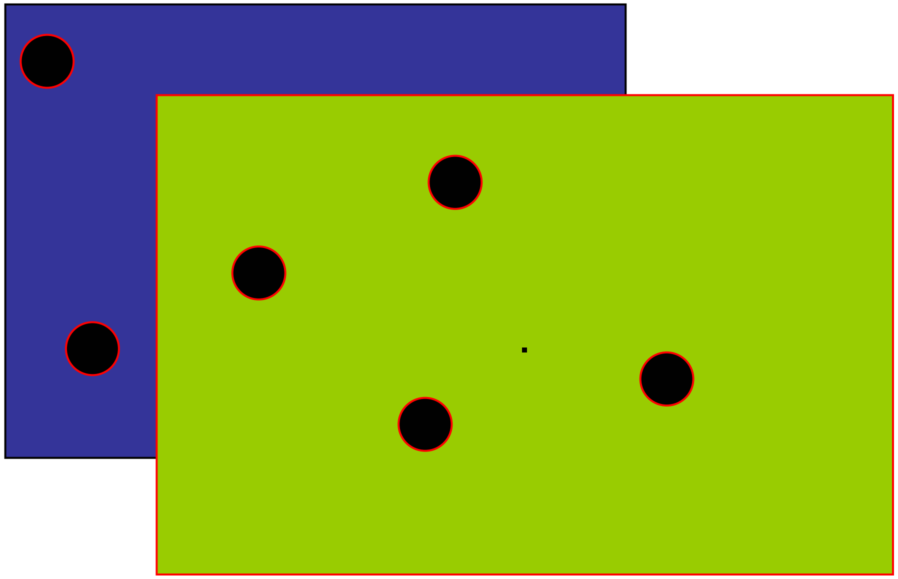
Compute Feynman path integrals



Two layer system.

- Spin singlet projected system of 2 layers

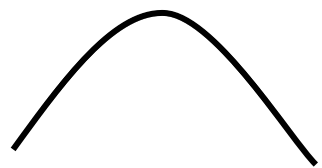
$$\prod (x_i - x_j)^m (y_i - y_j)^m$$



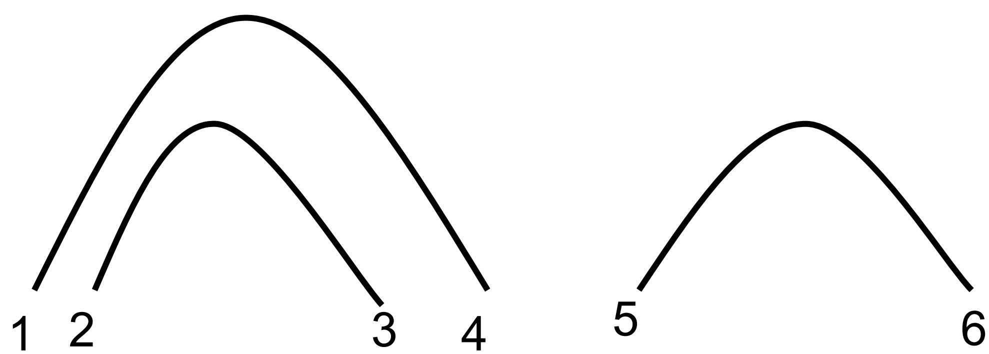
When **3 electrons** are put together, the wave function vanishes as: ε^m

RVB-BASIS

Projection onto the **singlet state**

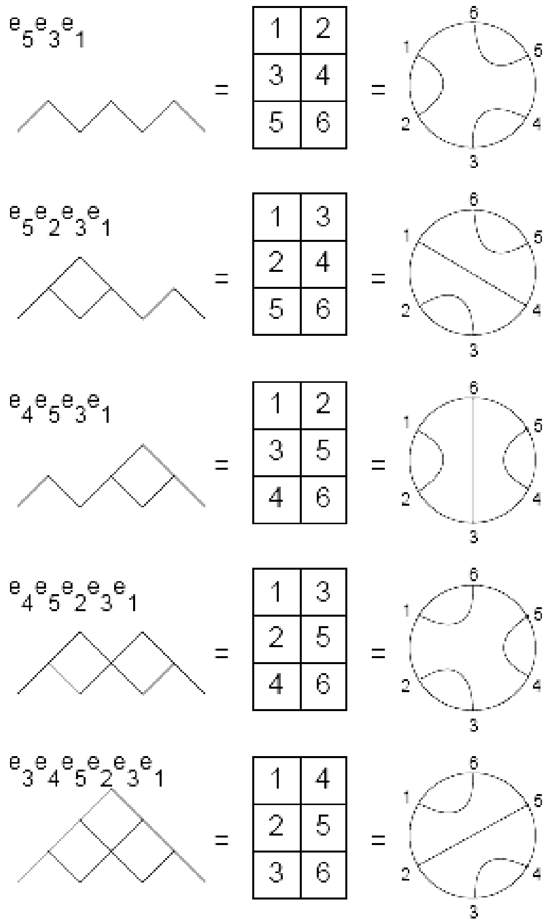
$\uparrow \; \downarrow \; - \; \downarrow \; \uparrow \; =$ 

Crossings forbidden to avoid double counting



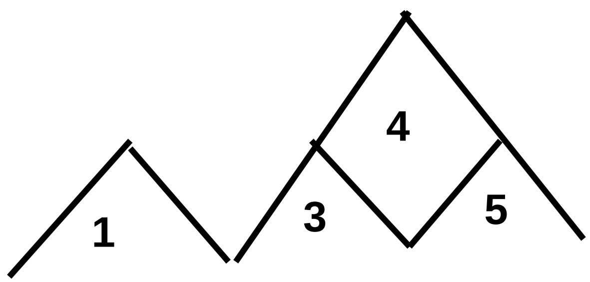
RVB basis:

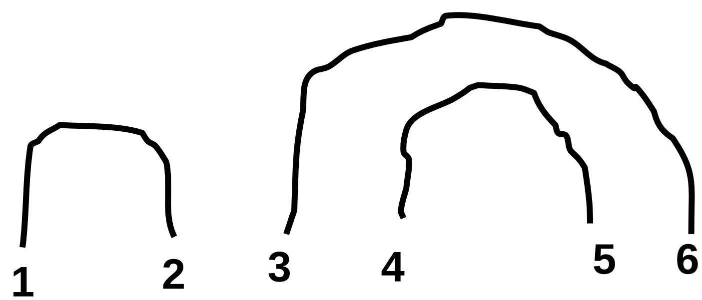
Projection onto the **singlet state**



RVB Basis

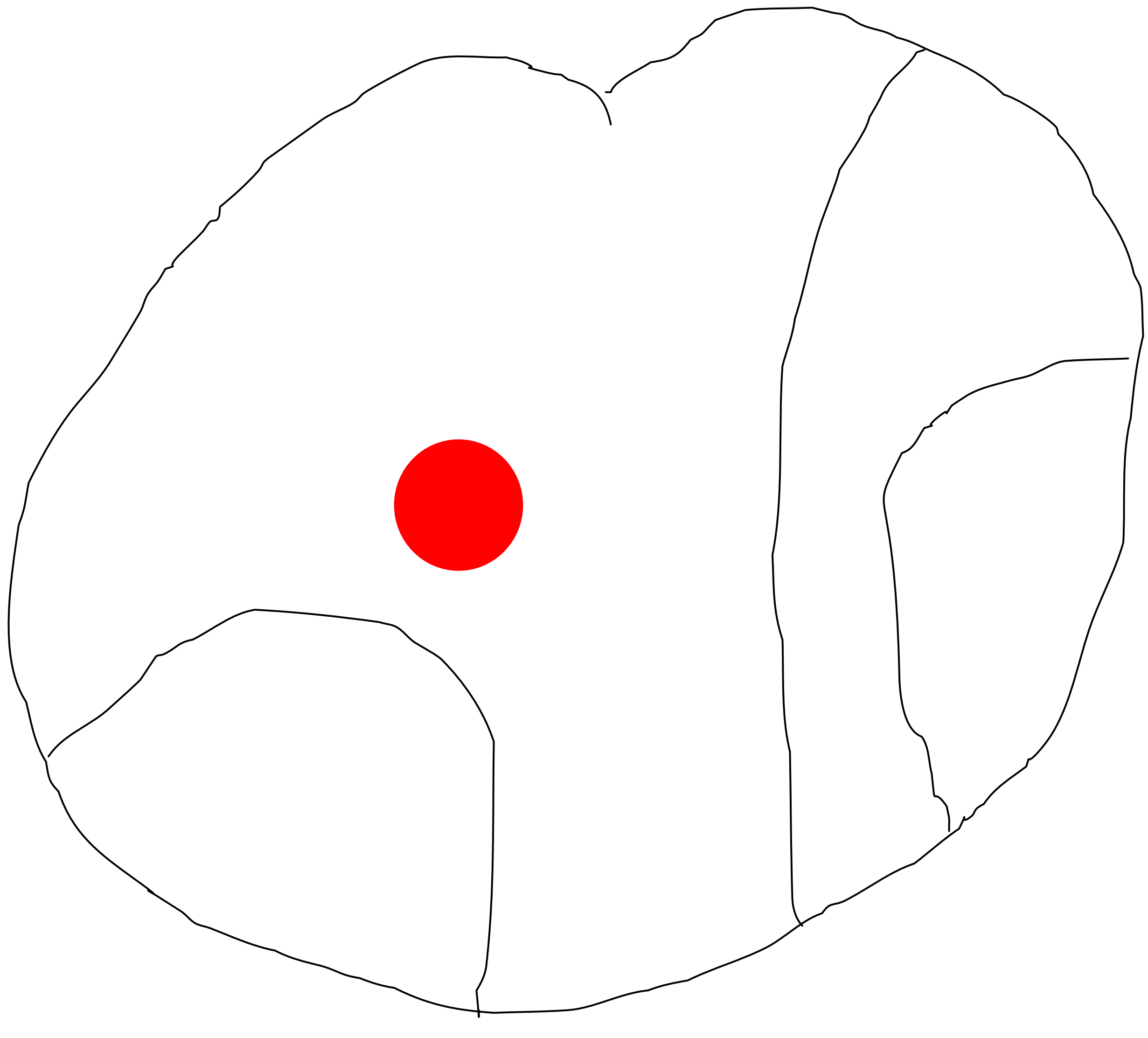
$$e_4 e_5 e_3 e_1$$





1 2
3 4
5 6

Also with fluxes

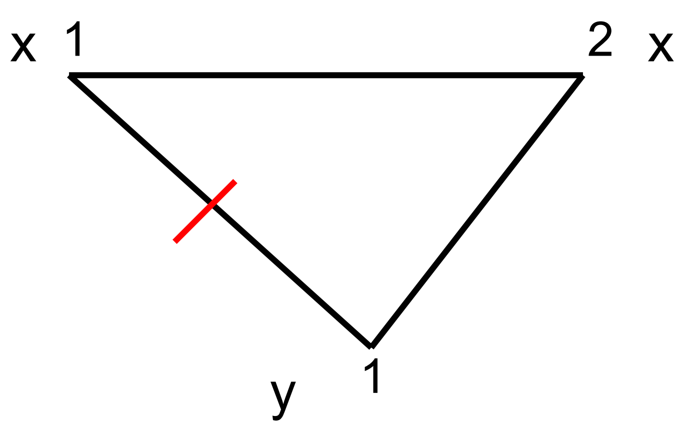


Exemples of wave functions

- Haldane-Rezayi: singlet state for 2 layer system.

$$\mathrm{Perm}\left[\frac{1}{x_i - y_j}\right]\prod(x_i - x_j)(y_i - y_j)(x_i - y_j)$$

When **3 electrons** are put together, the wave function vanishes as: ε^2



Moore Read

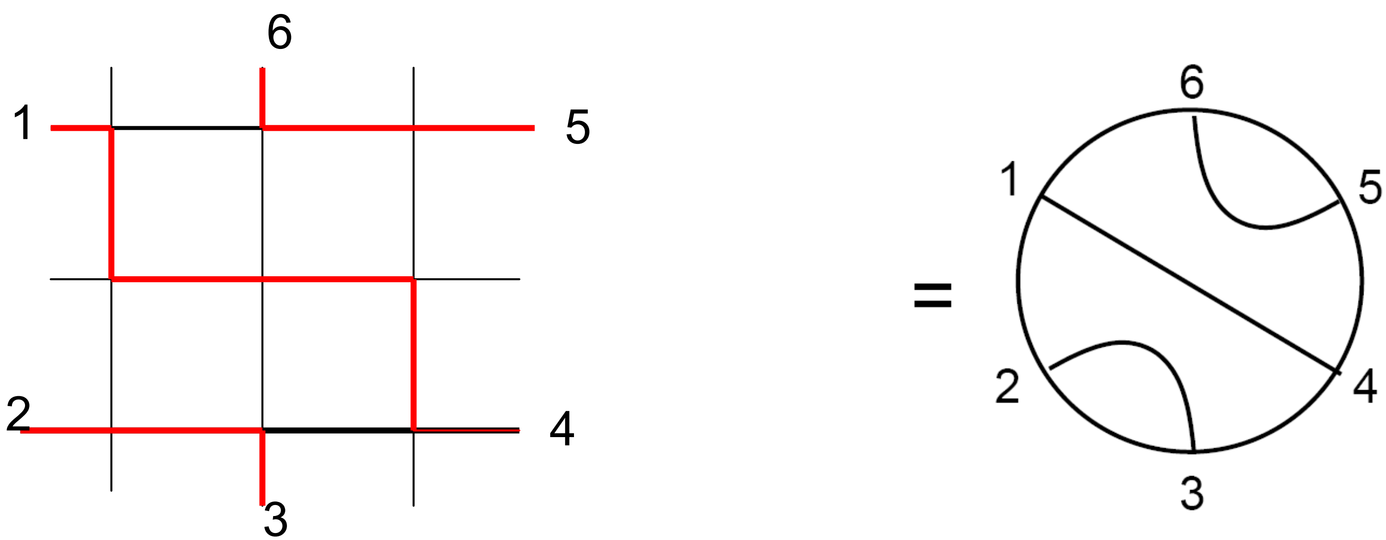
No spin

When **3 electrons** are put together, the wave function vanishes as: ε^2

$$\text{Pfaff}\left[\frac{1}{z_i - z_j}\right]\prod(z_i - z_j)$$

Razumov Stroganov Conjectures

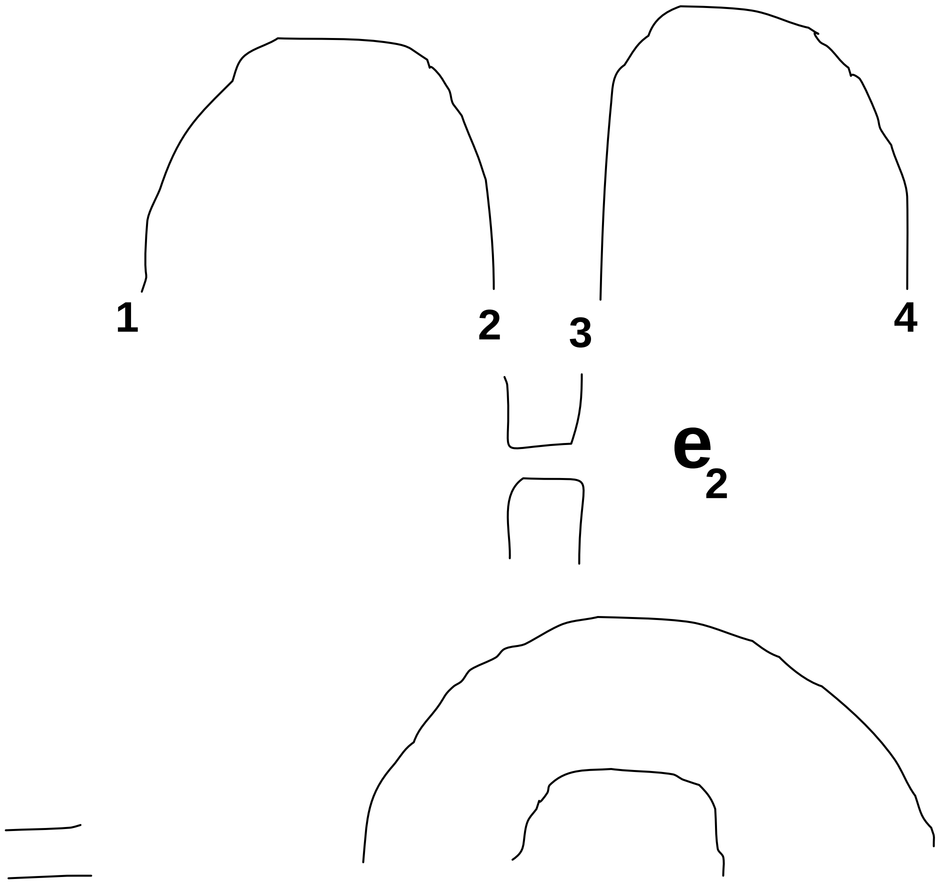
I.K. Partition function:

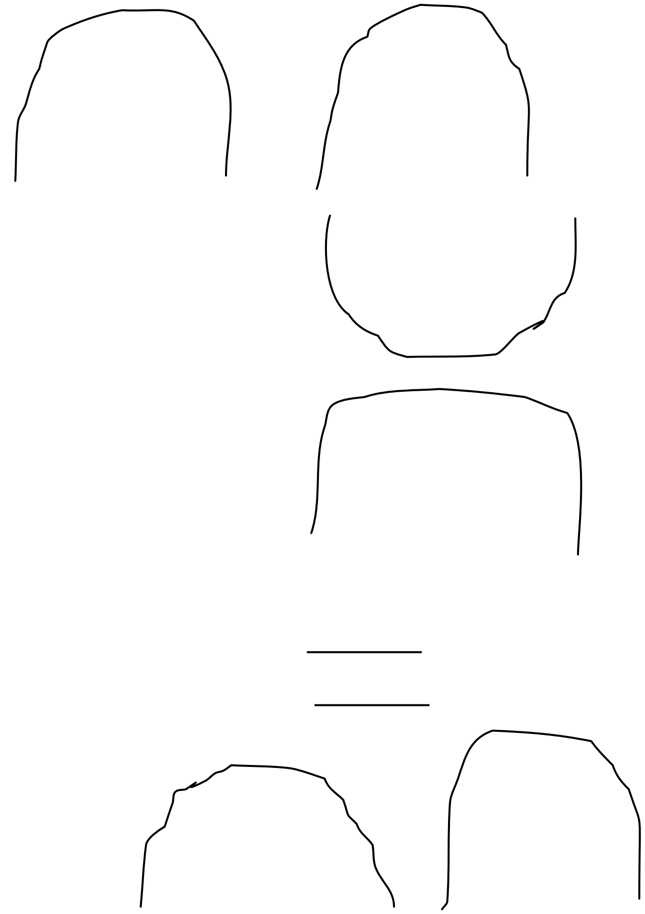


Also eigenvector of:

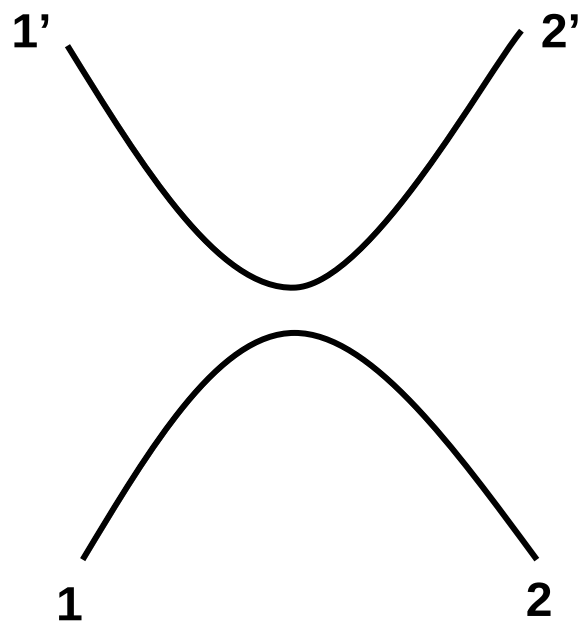
$$H = \sum_{i=1}^{6} e_i$$

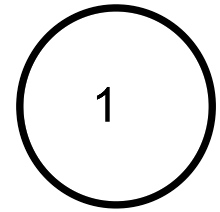
Stochastic matrix


1
2
3
4
e_2



$e_1 =$





$$H = \begin{pmatrix} \underline{3} & 2 & 2 & 0 & 2 \\ 1 & \underline{2} & 0 & 1 & 0 \\ 1 & 0 & \underline{2} & 1 & 0 \\ 0 & 2 & 2 & \underline{3} & 2 \\ 1 & 0 & 0 & 1 & \underline{2} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

Stochastic matrix

If d=1

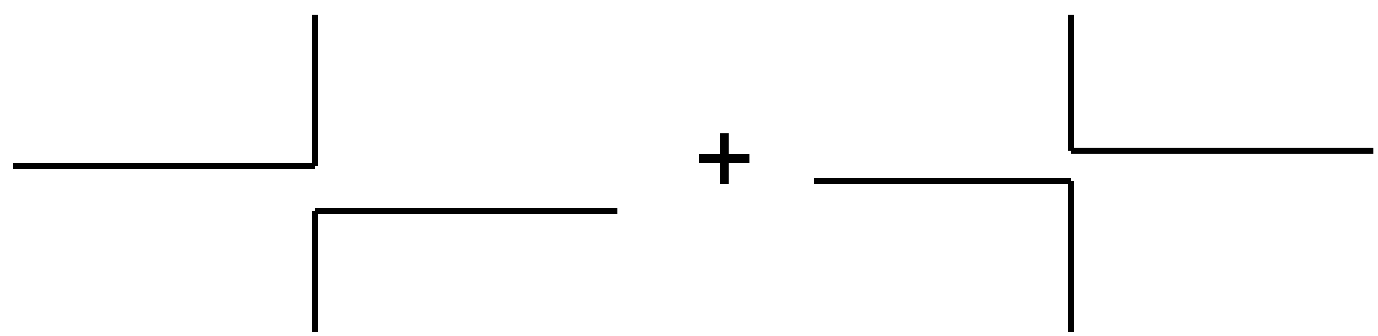
Not hermitian

Transfer Matrix

Consider inhomogeneous transfer matrix:

$$T(z, z_i) = tr\left(L\left(\frac{z_1}{z}\right) \ldots\ldots L\left(\frac{z_n}{z}\right)\right)$$

L=



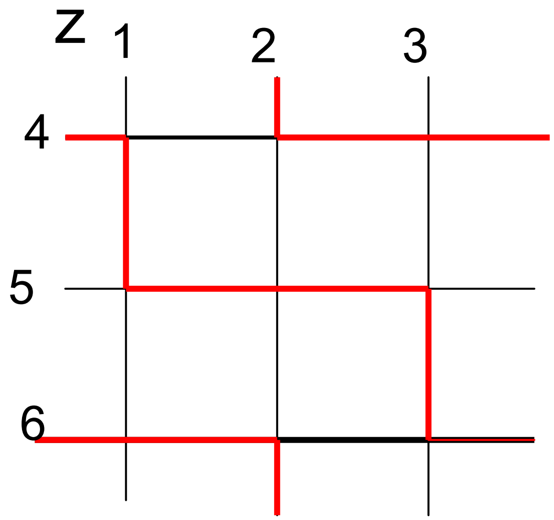
$$q z - q^{-1} z^{-1} \qquad\qquad z - z^{-1}$$

Transfer Matrix

$$T(z,z_i)\Psi(z_1\ldots z_n)=\Lambda(z_1\ldots z_n)\Psi$$

$$\left[T(z),T(w)\right]=0$$

I.K. Partition function



$$Z(z_1 \cdots z_n)$$

Total partition function can be expressed as a Gaudin Determinant

When d=q+1/q=1 symmetrical and given by a Schur function.

Stroganov

Hecke and Yang generators

- Consider Hecke algebra generators:

$$t_1 t_2 t_1 = t_2 t_1 t_2$$

Braid group relations

$$(t_1 - q)(t_1 + q^{-1}) = 0$$

- And Yang operators:

$$Y_1 = \frac{z_1 t_1 - z_2 t_1^{-1}}{q z_2 - q^{-1} z_1}$$

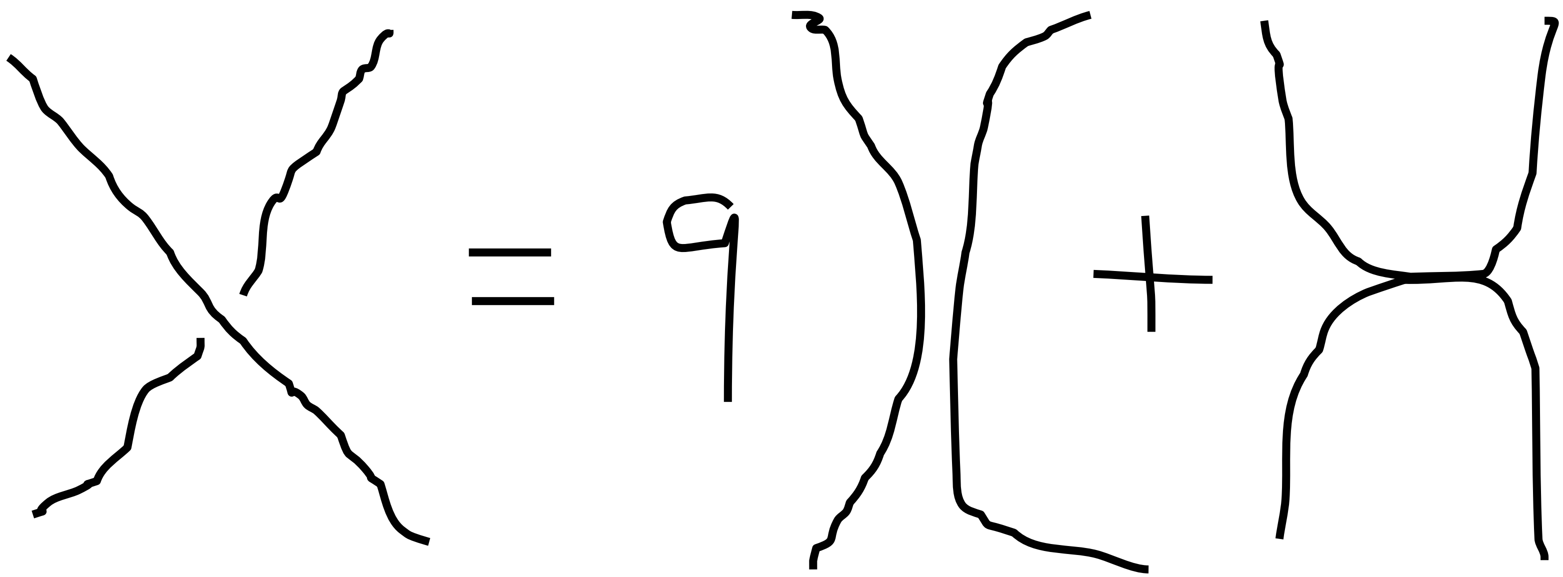
$$Y_1 Y_2 Y_1 = Y_2 Y_1 Y_2$$

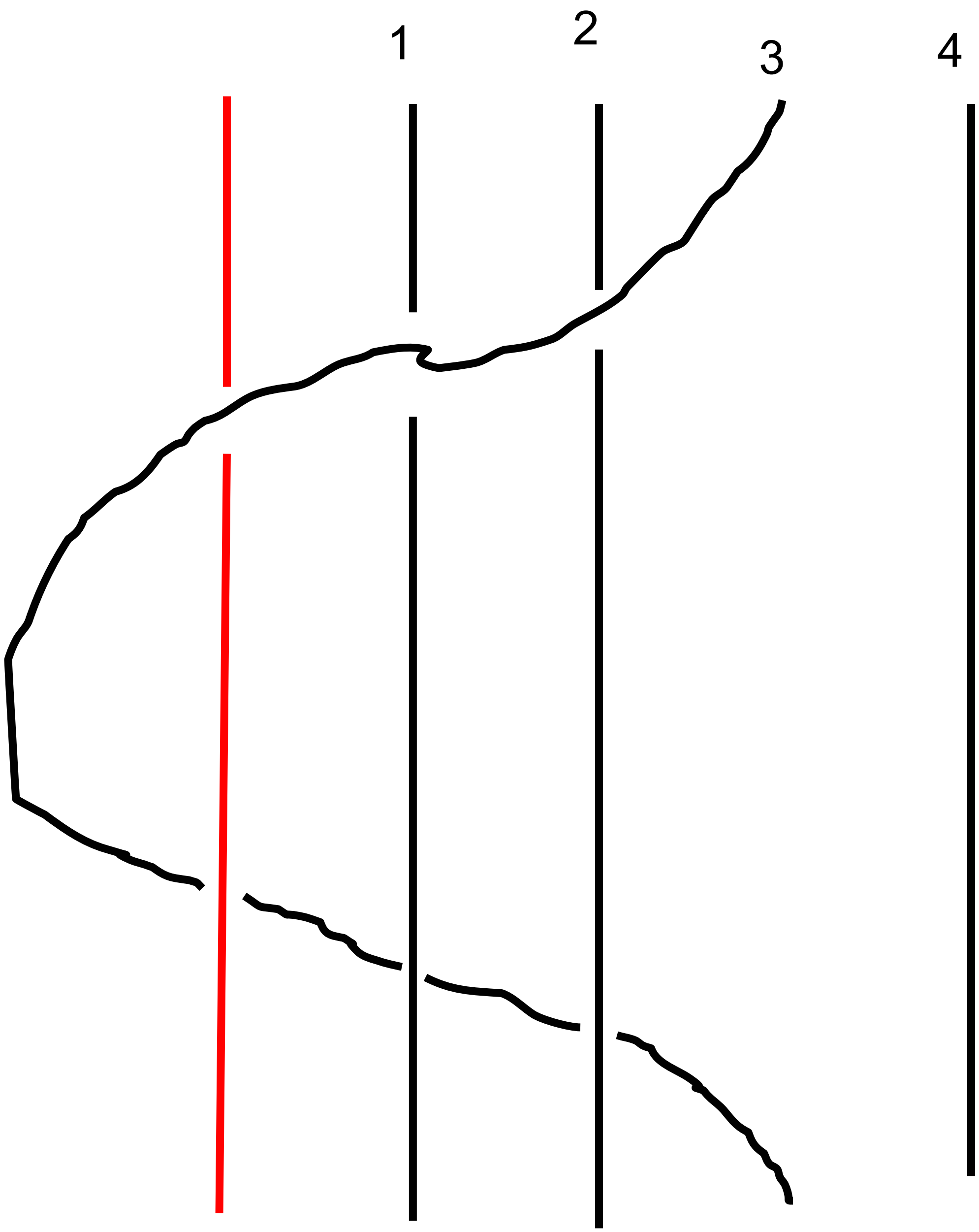
$$Y_1 Y_1 = 1$$

Permutation relations

Or Yang-Baxter algebra

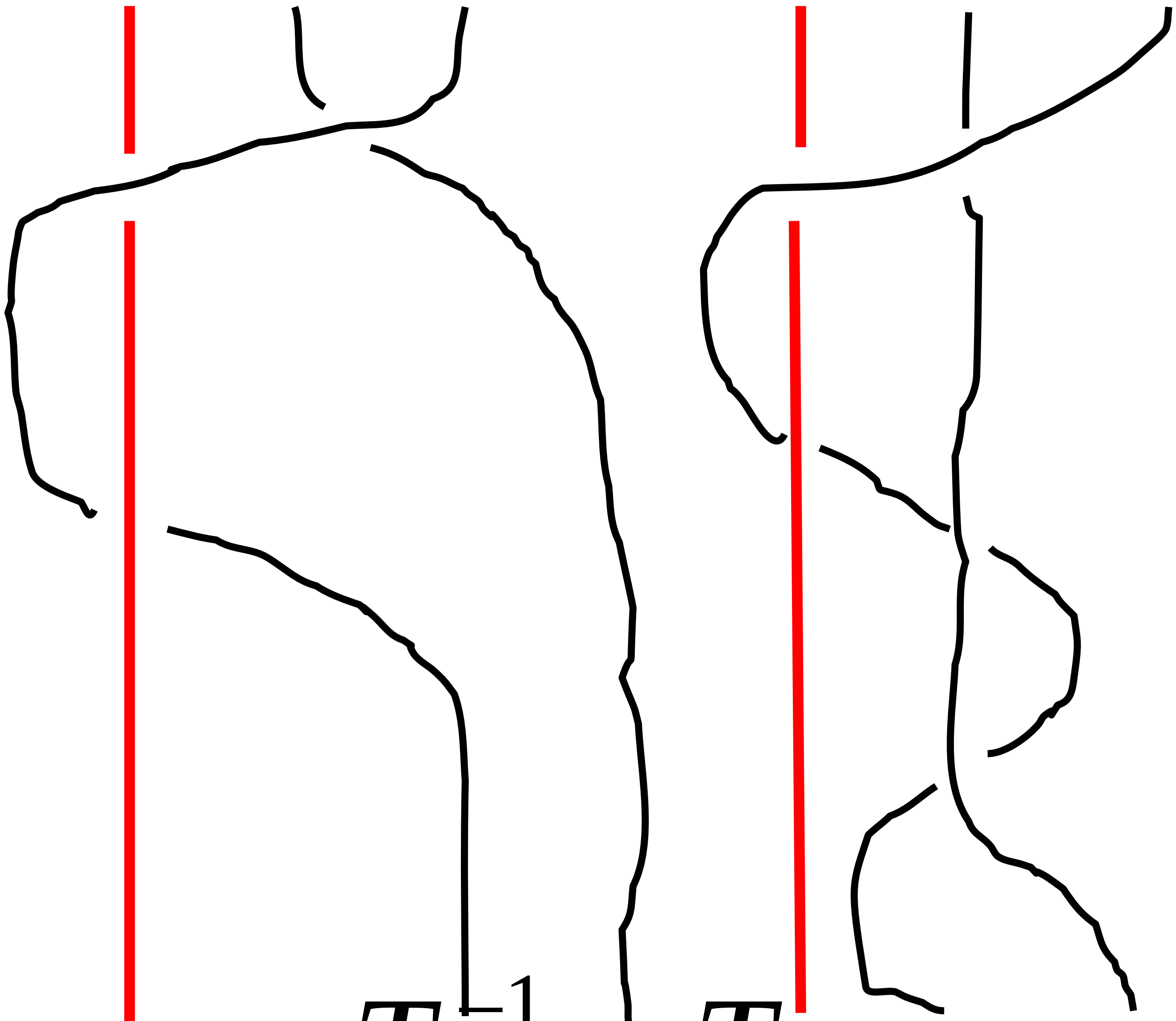
T.L. (Jones)





Y_4

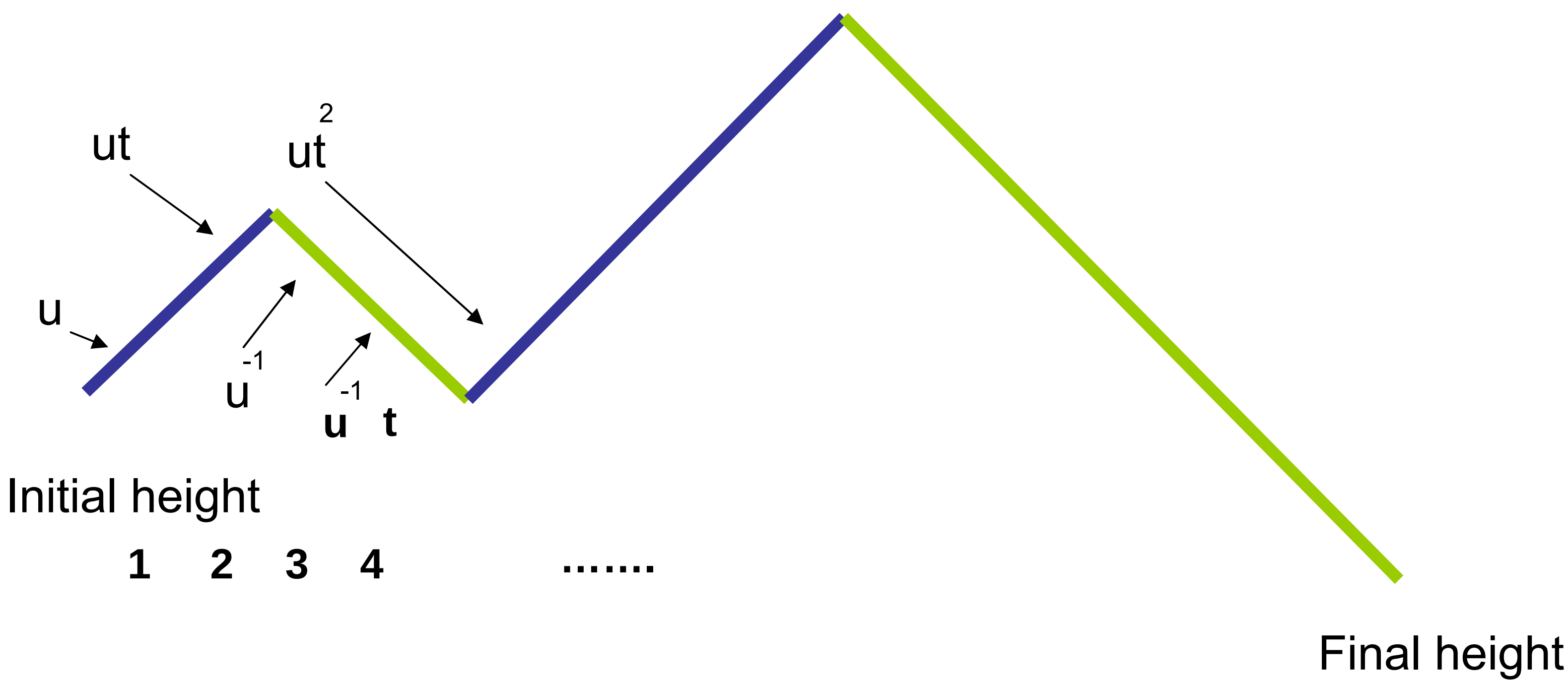
$$[Y_i, Y_j] = 0$$



$$y_1 T^{-1} = T y_2$$

Content of Representation

Read eigenvalues of y_i



Bosonic Ground State

$$\Psi = 1 = \sum \pi \otimes \hat{\pi}$$

$$t_i \Psi = \Psi \bar{t}_i,$$

$$y_i \Psi = \Psi \overline{y}_i,$$

Look for dual representation of AHA on polynomials

AHA

- Spin representation

$$y_1$$

$$y_2 = t_1 y_1 t_1$$

$$y_3 = t_2 y_2 t_2 \cdots$$

- Polynomial representation

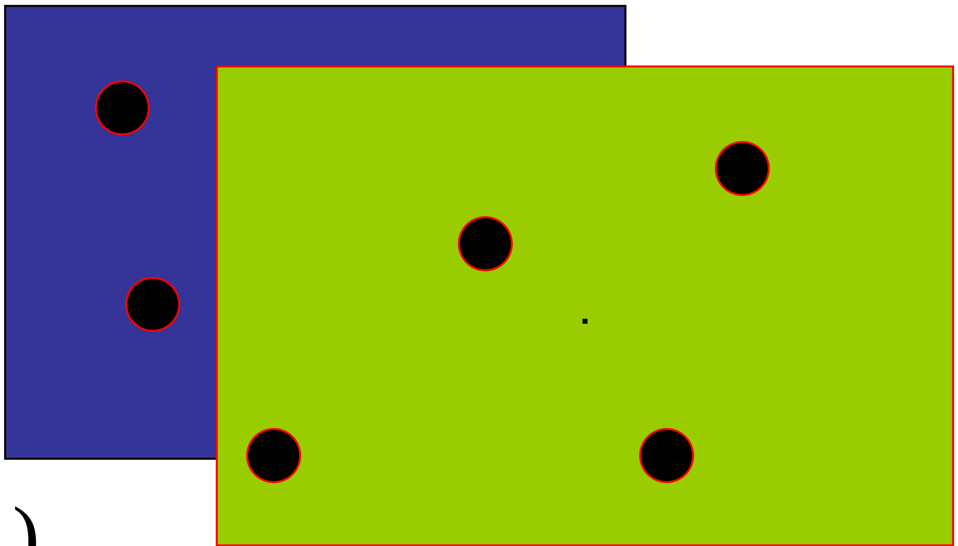
$$\overline{y}_1 = \overline{t}_1 \cdots \overline{t}_n \overline{\sigma}$$

$$with\ :\ z_i \overline{\sigma} = z_{i+1}$$

$$and\ :\ z_{i+n} = s z_i$$

Two q-layer system.

- Spin singlet projected system of 2 layers



$$\prod (q\, x_i - q^{-1} x_j)(q\, y_i - q^{-1} y_j)$$

(P) If i<j<k cyclically ordered, then $\boxed{\Psi(z_i = z, z_j = q^2 z, z_k = q^4 z) = 0}$

Imposes $\boxed{s = q^6}$ for no new condition to occur

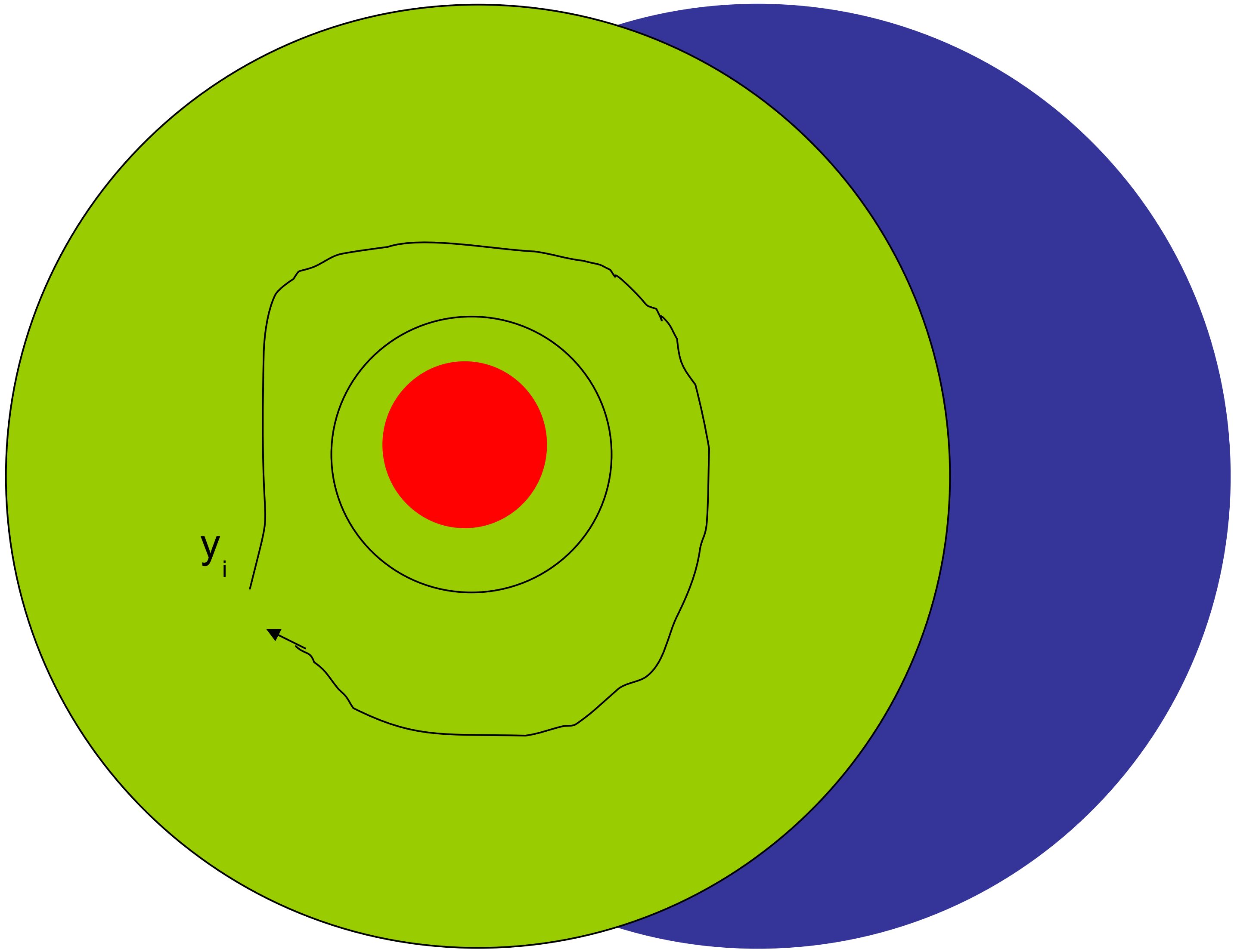
T.L. and Measure

$$(e+\tau)\Psi = (\Psi(z_1, z_2) - \Psi(z_2, z_1))\frac{z_1 q - z_2 q^{-1}}{z_1 - z_2}$$

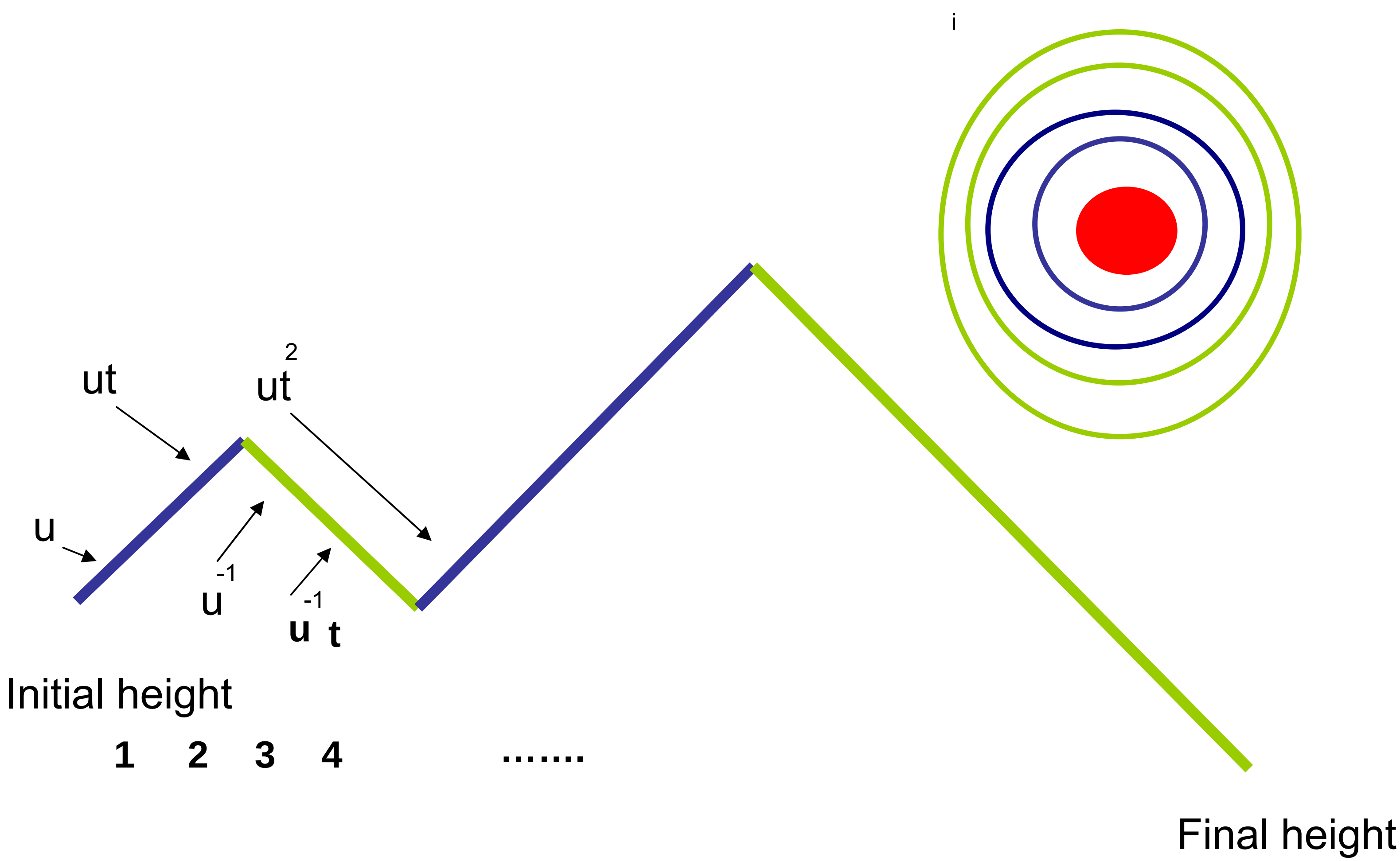
e+τ projects onto polynomials divisible by:

$$z_1 q - z_2 q^{-1}$$

e Measures the Amplitude for 2 electrons to be In the same layer


y_i

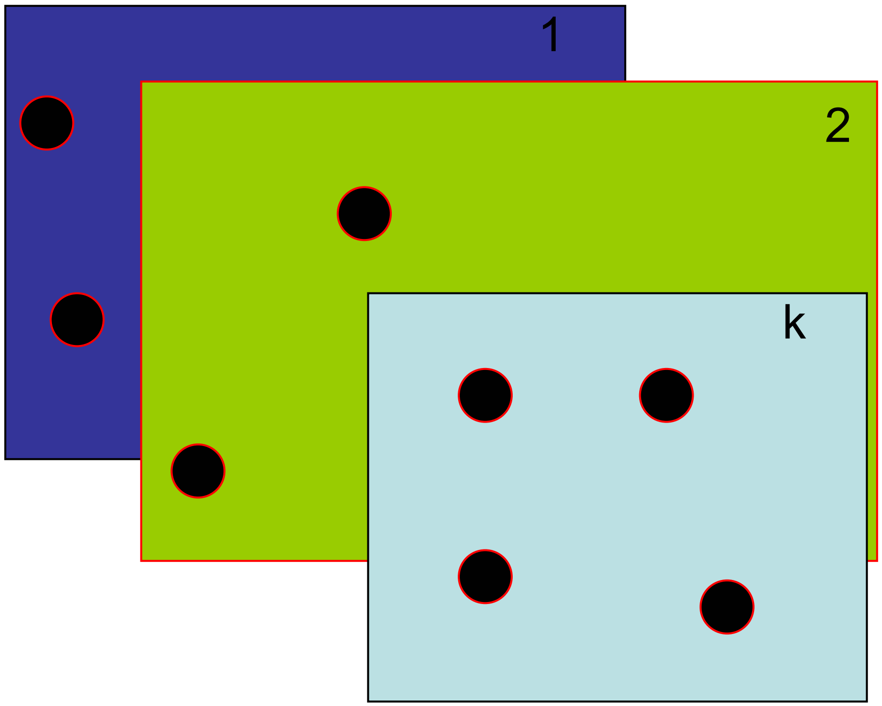
Flux seen by particles



k q-layer system

- Spin singlet projected system of k layers

$$\prod_{a=1}(q\ x_i^a - q^{-1} x_j^a)$$



If i<j<k cyclically ordered, then

(P) $$\Psi(z_i = z, z_j = q^2 z, z_k = q^{2k} z) = 0$$

Imposes $s = q^{2(k+1)}$ for no new condition to occur

Other generalizations

- q-Haldane-Rezayi

$$\mathrm{Det}\left[\frac{1}{(x_i - y_j)(qx_i - q^{-1}y_j)}\right]$$

Generalized Wheel condition, Gaudin Determinant

Related in some way to the **Izergin-Korepin partition function?**

Fractional hall effect Flux ½ electron

k and r

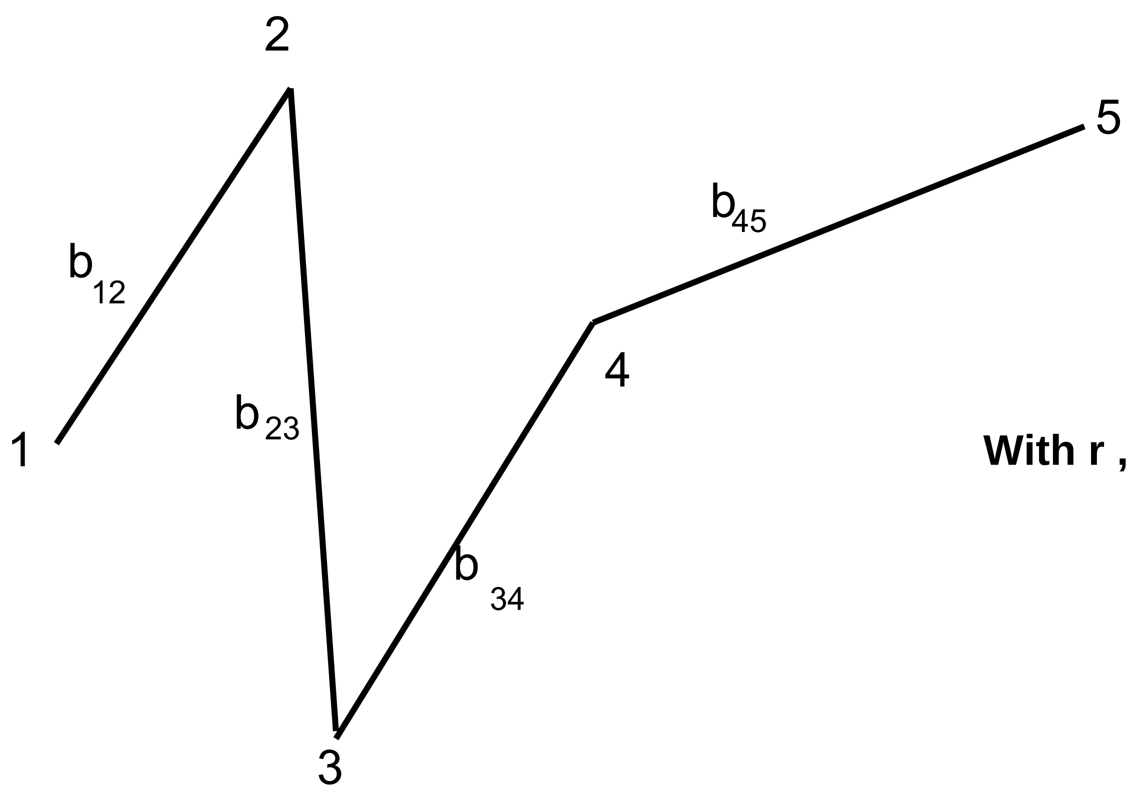
Kasatani wheel conditions

$$t^{k+1}q^{r-1}=1$$

$$\frac{z_{i_{a+1}}}{z_{i_a}}=tq^{b_{aa+1}}$$

$$b_{aa+1}=0\Rightarrow i_{a+1}>i_a$$

$$\sum b_{aa+1}\le r-2$$



With r , Flux=1/r particle.

Moore-Read

- Property (P) with s arbitrary.
- Affine Hecke replaced by Birman-Wenzl-Murakami,.
- R.S replaced by Nienhuis De Gier in the symmetric case. $sq^{2(k+1)} = 1$

$$\mathrm{Pfaff}\left[\frac{1}{qx_i - q^{-1}x_j}\right]$$

The case q root of unity

- When q+1/q=1, Hecke representation is no more semisimple and degenerates into a trivial representation.
- Stroganov Partition function (Schur function) can be recovered as the unique symmetrical polynomial of the minimal degree obeying (P).

- Other roots of unity?

Conclusions

- T.Q.F.T. realized on q-deformed wave functions of the Hall effect .
- All connected to Razumov-Stroganov type conjectures. Proof of conjecture still missing.
- Relations with works of Feigin, Jimbo,Miwa, Mukhin and Kasatani on polynomials obeying wheel condition.
- Excited states of the Hall effect.

cond-mat/0506075

math.QA/0507364