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Outline

- Bogoliubov-de-Gennes geometry
- $p_x + ip_y$ order-parameter
- Edge states
- Vortex-core states

References

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- D. A. Ivanov, Phys. Rev. Lett, **86**, 268-7 (2001).
- A. Stern, F. von Oppen, E. Mariani, Phys. Rev. B **70**, 205338 (2004).
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$$\hat{H}_{\text{Bogoliubov}} = a_i^\dagger H_{ij} a_j + \frac{1}{2}\Delta_{ij} a_i^\dagger a_j^\dagger + \frac{1}{2}\Delta_{ij}^\dagger a_i a_j$$

$$\hat{H}_{\text{Bogoliubov}} = \frac{1}{2} \begin{pmatrix} a_i^\dagger & a_i \end{pmatrix} \begin{pmatrix} H_{ij} & \Delta_{ij} \\ \Delta_{ij}^\dagger & -H_{ij}^T \end{pmatrix} \begin{pmatrix} a_j \\ a_j^\dagger \end{pmatrix} + \frac{1}{2} \operatorname{tr} H.$$

Connection with $\mathrm{SO}(2N)$

$$\gamma_i = (a_i + a_i^\dagger), \quad \gamma_{i+N} = i(a_i^\dagger - a_i).$$

$$\{\gamma_i, \gamma_j\} \equiv \gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}.$$

$$\Rightarrow \hat{H}_{\mathrm{Bogoliubov}} = \frac{1}{2}\sum_{i,j=1}^{2N} h_{ij}\Gamma_{ij} + \frac{1}{2}\mathrm{tr}\, H$$

$$\Gamma_{ij} = \frac{1}{4i}[\gamma_i, \gamma_j] \in \mathrm{Lie}\,[\mathrm{SO}(2N)]$$

$$h_{ij} = \begin{pmatrix} -\mathrm{Im}\,(H+\Delta), & -\mathrm{Re}(H-\Delta) \\ \mathrm{Re}(H+\Delta), & -\mathrm{Im}(H-\Delta) \end{pmatrix}_{ij}.$$

Bogoliubov transformation

$$\begin{aligned} a_i &= u_{i\alpha} b_\alpha + v^*_{i\alpha} b^\dagger_\alpha \\ a^\dagger_i &= v_{i\alpha} b_\alpha + u^*_{i\alpha} b^\dagger_\alpha . \end{aligned}$$

$$\begin{pmatrix} H & \Delta \\ \Delta^\dagger & -H^T \end{pmatrix} \begin{pmatrix} u_\alpha \\ v_\alpha \end{pmatrix} = E_\alpha \begin{pmatrix} u_\alpha \\ v_\alpha \end{pmatrix}, \quad E_\alpha \geq 0$$

$$\Rightarrow \begin{pmatrix} H & \Delta \\ \Delta^\dagger & -H^T \end{pmatrix} \begin{pmatrix} v^*_\alpha \\ u^*_\alpha \end{pmatrix} = -E_\alpha \begin{pmatrix} v^*_\alpha \\ u^*_\alpha \end{pmatrix} .$$

$\Rightarrow E \leftrightarrow -E$ symmetry

$$
\begin{aligned}
\hat{H}_{\text{Bogoliubov}} &= \sum_{\alpha=1}^{N} \frac{1}{2} E_\alpha (b_\alpha^\dagger b_\alpha - b_\alpha b_\alpha^\dagger) + \frac{1}{2} \operatorname{tr} H \\
&= \sum_{\alpha=1}^{N} E_\alpha \frac{1}{4i} [\gamma_{\alpha+N}, \gamma_\alpha] + \frac{1}{2} \operatorname{tr} H \\
&= \sum_{\alpha=1}^{N} E_\alpha b_\alpha^\dagger b_\alpha - \frac{1}{2} \sum_{\alpha=1}^{N} E_\alpha + \frac{1}{2} \sum_{i=1}^{N} E_i^{(0)}.
\end{aligned}
$$

($\gamma_\alpha = O_{\alpha i} \gamma_i$, where $O \in \mathrm{SO}(2N)$.)

BCS Ground State

Ground state $b_i|0\rangle_b = 0$,

$$\Rightarrow \quad (a_i + a_k^\dagger v_{k\alpha}^* (u^{*-1})_{\alpha i})|0\rangle_b = 0, \quad i = 1, \dots N.$$

$$S_{ij} = v_{i\alpha}^* (u^{*-1})_{\alpha j}, \quad S_{ij} = -S_{ji}$$

$$\exp\left\{\frac{1}{2} a_i^\dagger a_j^\dagger S_{ij}\right\} a_k \exp\left\{-\frac{1}{2} a_i^\dagger a_j^\dagger S_{ij}\right\} = a_k + a_i^\dagger S_{ik}.$$

$$\Rightarrow |0\rangle_b = \mathcal{N} \exp\left\{\frac{1}{2} a_i^\dagger a_j^\dagger S_{ij}\right\} |0\rangle_a$$

$2n$-particle Pfaffian wavefunction

Let

$$|S\rangle \stackrel{\mathrm{def}}{=} \exp\left\{\frac{1}{2} a_i^\dagger a_j^\dagger S_{ij}\right\} |0\rangle_a,$$

and

$$|i_1, i_2, \ldots, i_{2n}\rangle = a_{i_{2n}}^\dagger a_{i_{2n-1}}^\dagger \ldots a_{i_1}^\dagger |0\rangle_a.$$

Then

$$\begin{aligned} \langle i_1, i_2, \ldots, i_{2n}|S\rangle &= \frac{1}{2^n n!} \epsilon_{i_1 i_2 \ldots i_{2n}}^{j_1 j_2 \ldots j_{2n}} S_{j_1 j_2} \cdots S_{j_{2n-1} j_{2n}} \\ &= \epsilon_{i_1 i_2 \ldots i_{2n}} \mathrm{Pf}(S). \end{aligned}$$

(Freeman Dyson?)

BCS Normalization

$$|S\rangle \stackrel{\text{def}}{=} \exp\left\{\frac{1}{2} a_i^\dagger a_j^\dagger S_{ij}\right\} |0\rangle_a,$$

$$\Rightarrow \langle S_1|S_2\rangle = \det{}^{1/2}(I + S_1^\dagger S_2),$$

$$\Rightarrow \mathcal{N} = \det{}^{-1/4}(1 + S^\dagger S).$$

- This $\det{}^{1/2}$ is sometimes called a “Pfaffian”. Although it is not the usual object of that name, it has the Pfaffian property of being a polynomial in its entries.
- The collection of ground-state wavefunctions is therefore a *Pfaffian line bundle* over the symmetric space $\mathrm{SO}(2N)/\mathrm{U}(N)$.
- The states $|S\rangle$ are holomorphic in the parameters S_{ij}.

Berry Phase

$$\ln \mathcal{N} = -\frac{1}{4} \ln \det (I + S^\dagger S), \quad \text{(Kähler Potential)}$$

$$\begin{aligned} iA_{\text{Berry}} &= \sum_{i<j} \left(\frac{\partial \ln \mathcal{N}}{\partial S^*_{ij}} dS^*_{ij} - \frac{\partial \ln \mathcal{N}}{\partial S_{ij}} dS_{ij} \right) \\ &= \frac{1}{2} \sum_{\alpha=1}^{N} \left(\, v_\alpha \quad u_\alpha \, \right) d \begin{pmatrix} v^*_\alpha \\ u^*_\alpha \end{pmatrix} + \frac{i}{2} d \left\{ \text{Arg} \left(\det u \right) \right\} \end{aligned}$$

(Remember that it is the (v^*, u^*) that have negative energy, and so are "occupied" in ground state.)

$$p_x + ip_y$$

Bogoliubov-de Gennes equation

$$\begin{bmatrix} -\frac{1}{2m}\nabla^2 - \mu & \hat{\Delta} \\ \hat{\Delta}^\dagger & +\frac{1}{2m}\nabla^2 + \mu \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = E \begin{pmatrix} u \\ v \end{pmatrix}$$

Bogoliubov-de Gennes equation

$$\begin{bmatrix} -\frac{1}{2m}\nabla^2 - \mu & \hat{\Delta} \\ \hat{\Delta}^\dagger & +\frac{1}{2m}\nabla^2 + \mu \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = E \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\hat{\Delta} = \frac{1}{2}\left(\frac{|\Delta|}{k_f}\right) e^{i\Phi/2}\left\{\hat{\Sigma}, \hat{P}\right\} e^{i\Phi/2}.$$

$$\Sigma_{\alpha\beta} = (i(\sigma \cdot \mathbf{d})\sigma_2)_{\alpha\beta}.$$

$\Phi =$ local phase

Bogoliubov-de Gennes equation

$$\begin{bmatrix} -\frac{1}{2m}\nabla^2 - \mu & \hat{\Delta} \\ \hat{\Delta}^\dagger & +\frac{1}{2m}\nabla^2 + \mu \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = E \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{aligned} \hat{P} &= -i(\hat{p}_x + i\hat{p}_y) \\ &= -(\partial_x + i\partial_y), \\ &= -e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right), \end{aligned}$$

Bogoliubov-de Gennes equation

$$\begin{bmatrix} -\frac{1}{2m}\nabla^2 - \mu & \hat{\Delta} \\ \hat{\Delta}^\dagger & +\frac{1}{2m}\nabla^2 + \mu \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = E \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{aligned} \hat{P} e^{il\theta} J_l(kr) &= k e^{i(l+1)\theta} J_{l+1}(kr) \\ \hat{P}^\dagger e^{il\theta} J_l(kr) &= k e^{i(l-1)\theta} J_{l-1}(kr) \end{aligned}$$

Bulk Modes

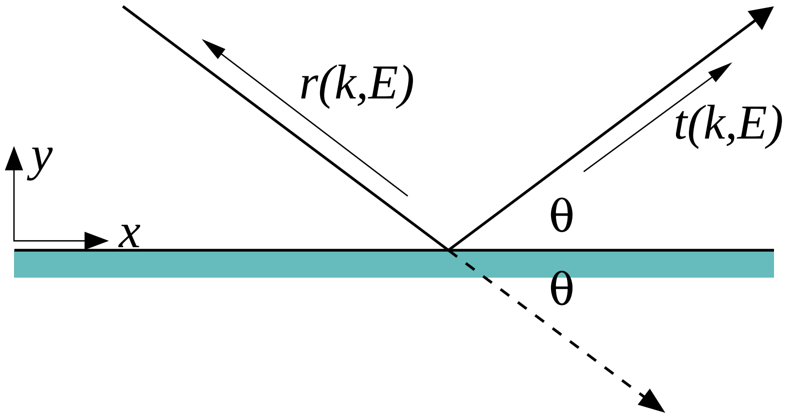
$$\Psi_{E,l}(r,\theta) = \frac{1}{2\sqrt{E(E+\Delta)}} \begin{bmatrix} (E+\epsilon+\Delta)e^{i(l+1)\theta} J_{l+1}(kr) \\ (E-\epsilon+\Delta)e^{il\theta} J_l(kr) \end{bmatrix},$$

$$E = +\sqrt{\epsilon^2(k) + (k/k_f)^2\Delta^2}, \quad \epsilon = k^2/2m - \mu.$$

$$\Psi_{-|E|,l}(r,\theta) = \frac{1}{2\sqrt{|E|(|E|-\Delta)}} \begin{bmatrix} (|E|-\Delta-\epsilon)e^{i(l+1)\theta} J_{l+1}(kr) \\ (|E|-\Delta+\epsilon)e^{il\theta} J_l(kr) \end{bmatrix},$$

$$E = -\sqrt{\epsilon^2(k) + (k/k_f)^2\Delta^2}.$$

Edge Mode

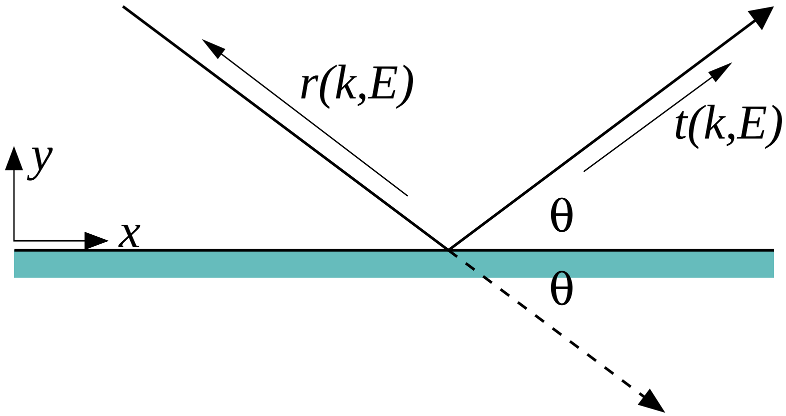


- $r(k, E)$: Andreev reflection.
- $t(k, E)$: Specular reflection $=$ "transmission".

$$\Psi_\theta(x, y) = \begin{bmatrix} a_L(y) \\ b_L(y) \end{bmatrix} e^{ik_f x \cos\theta + ik_f y \sin\theta} - \begin{bmatrix} a_R(y) \\ b_R(y) \end{bmatrix} e^{ik_f x \cos\theta - ik_f y \sin\theta},$$

a, b vary slowly w.r.t. k_f and $\Psi = 0$ on boundary.

Edge Mode



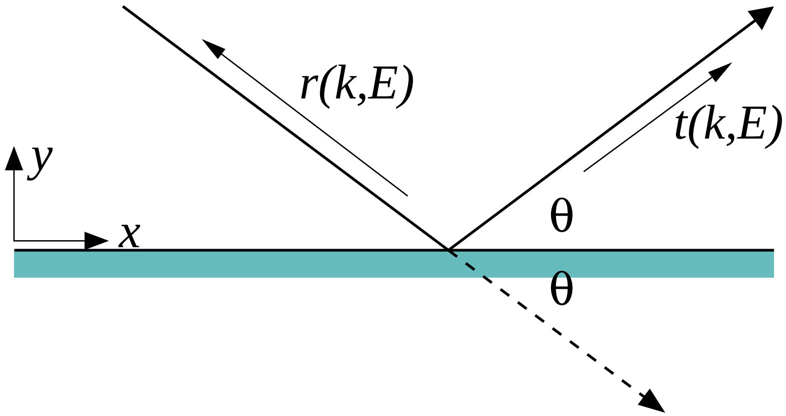
- $r(k,E)$: Andreev reflection.
- $t(k,E)$: Specular reflection $=$ "transmission".

$$H = -i\tau_3\partial_\xi + \Delta\tau_1 e^{-i\tau_3\phi(\xi)}, \quad y = \xi\sin\theta$$

$$\begin{bmatrix} -i\partial_\xi & \Delta e^{i\phi(\xi)} \\ \Delta e^{-i\phi(\xi)} & i\partial_\xi \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = E \begin{pmatrix} a \\ b \end{pmatrix},$$

On reflection ϕ jumps from $\phi_L = -\theta$ to $\phi_R = \theta$

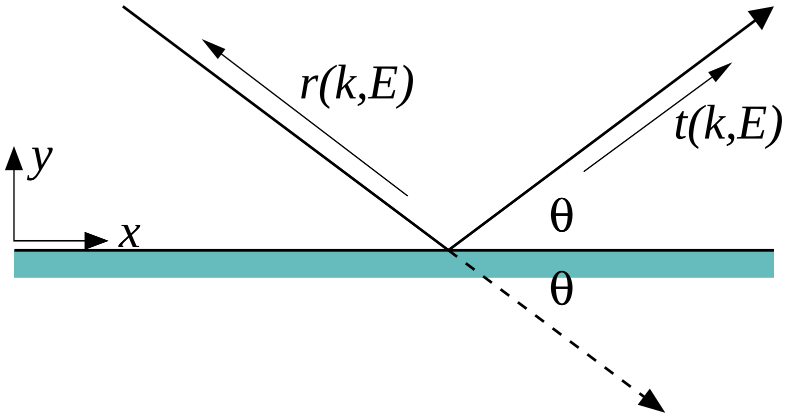
Edge Mode



- $r(k,E)$: Andreev reflection.
- $t(k,E)$: Specular reflection $=$ "transmission".

$$\begin{aligned} \psi &= \psi^{\phi_L}_{k,E} + r(k,E)\psi^{\phi_L}_{-k,E}, \qquad \xi < 0, \\ &= t(k,E)\psi^{\phi_R}_{k,E}, \qquad \xi > 0. \end{aligned}$$

Edge Mode



- $r(k,E)$: Andreev reflection.
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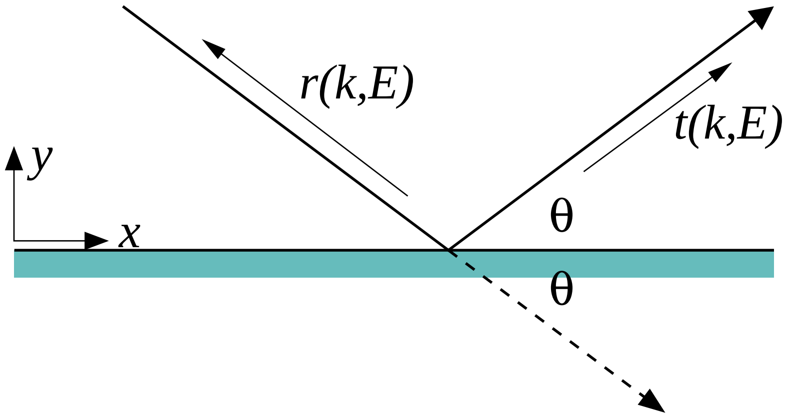
$$\chi_{\text{edge}}(k_x) = \sqrt{\frac{\Delta}{2v_f}} e^{ik_f x \cos\theta} \sin(k_f y \sin\theta) e^{-\Delta y/v_f} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$E_{\text{edge}}(k_x) = -\Delta\cos\theta = -\Delta(k_x/k_f). \quad \text{(Weyl)}$$

Edge Mode



- $r(k,E)$: Andreev reflection.
- $t(k,E)$: Specular reflection $=$ "transmission".

$$\begin{pmatrix} \hat{\psi} \\ \hat{\psi}^\dagger \end{pmatrix}_{\text{bound}} = \sum_{k_x} \hat{b}_{k_x} e^{ik_x x} \sin(k_f y \sin\theta) e^{-\Delta y \sin\theta / v_f} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \hat{b}_{-k_x} = \hat{b}^\dagger_{k_x} \text{ (Majorana)}$$

Dangerous argument

$$H = -i\tau_3\partial_x + m\tau_1 + m_5(x)\tau_2$$

$$\psi_{\text{topological}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp\left\{-\int_0^x m_5(\xi)\, d\xi\right\}$$

$$H\psi_{\text{topological}} = m\psi_{\text{topological}}.$$

Dangerous argument

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$$H\psi_{\rm topological} = m\psi_{\rm topological}.$$

$$\begin{aligned} Q &= \tau_1 H - m \\ QH &= -HQ \\ Q\psi_{\rm topological} &= 0. \end{aligned}$$

$\Rightarrow E \leftrightarrow -E$ symmetry?

Dangerous argument

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$$\psi_{\text{topological}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp\left\{-\int_0^x m_5(\xi)\, d\xi\right\}$$

$$H\psi_{\text{topological}} = m\psi_{\text{topological}}.$$

NO!

Dangerous argument

$$H = -i\tau_3\partial_x + m\tau_1 + m_5(x)\tau_2$$

$$\psi_{\rm topological} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp\left\{-\int_0^x m_5(\xi)\, d\xi\right\}$$

$$H\psi_{\rm topological} = m\psi_{\rm topological}.$$

Spectrum asymmetric, and edge current has contributions from scattering states.

Vortex-core states

For circulation n vortex with $e^{i\Phi} = e^{in\theta}$, seek solution of form:

$$\begin{pmatrix} u \\ v \end{pmatrix} = e^{il\theta} \begin{pmatrix} e^{i\theta(n+1)/2}[a(r)H_{l+1/2}(k_f r) + \text{c.c.}] \\ e^{-i\theta(n+1)/2}[b(r)H_{l-1/2}(k_f r) + \text{c.c}] \end{pmatrix}.$$

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Cosmetic redefinition:

$$\begin{pmatrix} a(r) \\ b(r) \end{pmatrix} = \begin{pmatrix} e^{-in\theta(x)/2}\tilde{a}(x) \\ e^{in\theta(x)/2}\tilde{b}(x) \end{pmatrix}$$

Vortex-core states

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Leads to Caroli–de-Dennes–Matricon equation:

$$\begin{pmatrix} -iv_f\partial_x & \Delta(x)e^{in\theta(x)} \\ \Delta(x)e^{-in\theta(x)} & +iv_f\partial_x \end{pmatrix} \begin{pmatrix} \tilde{a}(x) \\ \tilde{b}(x) \end{pmatrix} = E_l \begin{pmatrix} \tilde{a}(x) \\ \tilde{b}(x) \end{pmatrix}.$$

Vortex-core states

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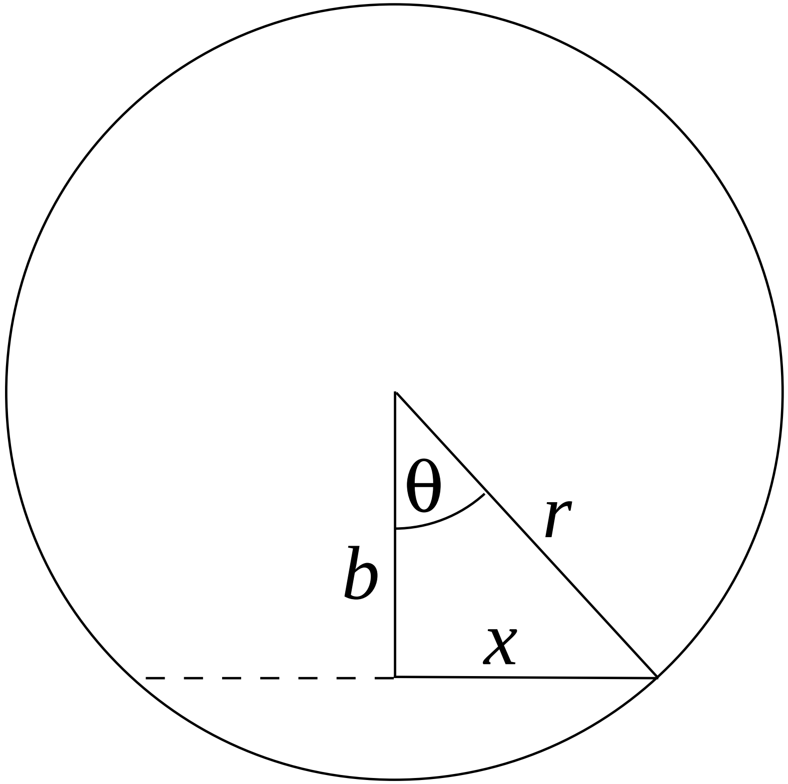
What is "x", "$\theta(x)$", and so on?

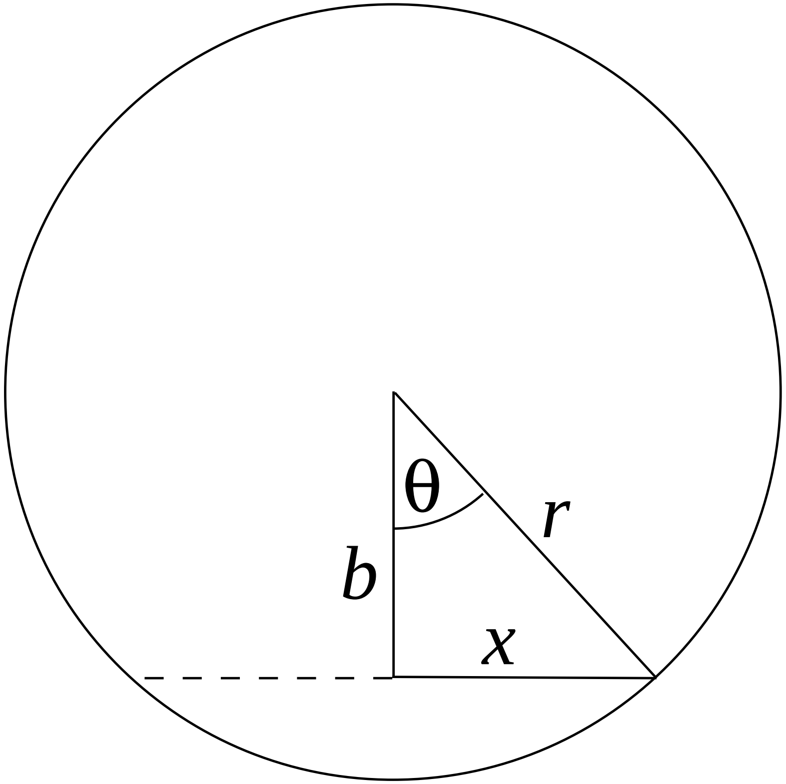
Bessel Asymptotics



- WKBJ approximation
- Impact parameter
  $bk = l$
- $x = \sqrt{b^2 - r^2}$
- $\theta(r) = \cos^{-1}(b/r)$

$$H_l(kr) \approx \sqrt{\frac{2}{\pi k x(r)}} \exp\left\{ i \left(k x(r) - l\theta(r) - \frac{\pi}{4} \right) \right\}, \qquad r \gg b.$$

Bessel Asymptotics



- WKBJ approximation
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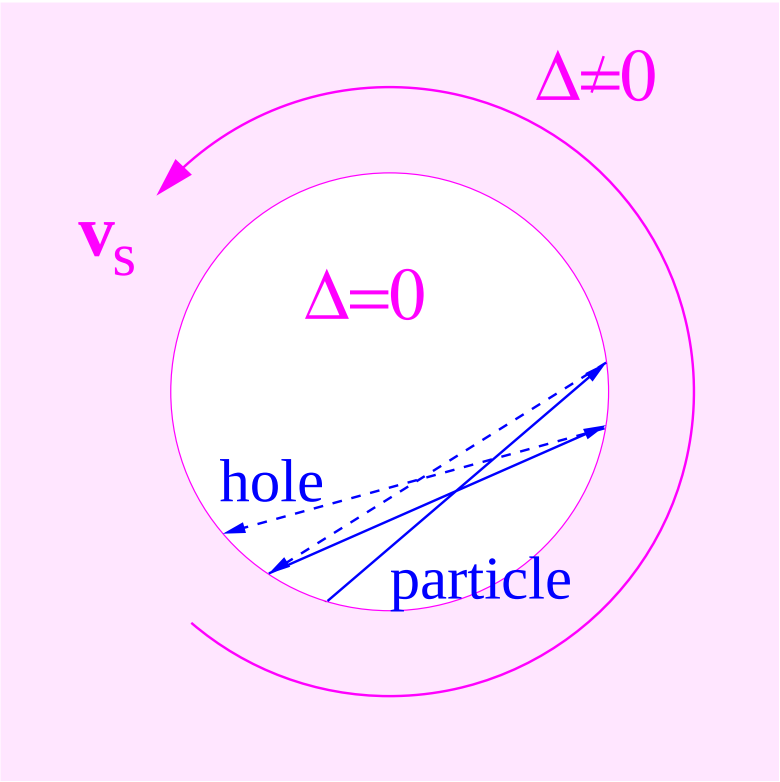
$$\begin{pmatrix} -iv_f\partial_x & \Delta(x)e^{in\theta(x)} \\ \Delta(x)e^{-in\theta(x)} & +iv_f\partial_x \end{pmatrix} \begin{pmatrix} \tilde{a}(x) \\ \tilde{b}(x) \end{pmatrix} = E_l \begin{pmatrix} \tilde{a}(x) \\ \tilde{b}(x) \end{pmatrix}.$$

vortex core II



Andreev bound state

- $\Rightarrow E_l = -\omega_0(l + \alpha)$
- $\frac{d\theta}{dt} = \frac{\partial E_l}{\partial l} = -\omega_0$
- Andreev reflection not *quite* retro-reflective $\Rightarrow$ backward creep

Zero Modes

$$n = +1: \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} e^{i\theta} f(r) \\ e^{-i\theta} f(r) \end{pmatrix}, \quad f(r) \text{ real}$$

$$n = -1: \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(r) \\ f(r) \end{pmatrix}.$$

Zero Modes

$$n=+1: \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} e^{i\theta} f(r) \\ e^{-i\theta} f(r) \end{pmatrix}, \quad f(r) \text{ real}$$

$$n=-1: \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(r) \\ f(r) \end{pmatrix}.$$

Global phase change $\Delta \to e^{i\chi}\Delta$ takes

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} e^{\chi/2} u \\ e^{-i\chi/2} v \end{pmatrix}$$

$\Rightarrow$ square-root monodromy.