Fluctuations of the free energy of spherical spin glass

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Joint work with Ji Oon Lee (KAIST, Korea)

- arXiv:1505.07349. Fluctuations of the free energy of the spherical Sherrington–Kirkpatrick model
- Free energy of the spherical Sherrington–Kirkpatrick model with non-zero mean interactions, in preparation

Random Hermitian/Symmetric matrix

Largest eigenvalue

$$\lambda_1 = \max_{\|x\|=1} \langle x, Mx \rangle$$

Finite temperature version:

$$F_N(\beta) = \frac{1}{N} \log \left[\int_{\|x\|=1} e^{N\beta \langle x, Mx \rangle} d\Omega(x) \right]$$

Spherical Sherrington-Kirkpatrick (SSK) model of spin glass.

Plan

1. Spin glass
2. Results
3. Proof
4. Spin glass + Ferromagnetic Hamiltonian

Spin glass

- Spin glass: Edwards–Anderson
- Mean-field spin glass: Sherrington–Kirkpatrick (SK)
- Spherical mean-field spin glass: Spherical Sherrington–Kirkpatrick (SSK)

Edwards–Anderson model:

Given random J_{xy} for $x \sim y \in \mathcal{L}_N \subset \mathbb{Z}^d$, set $H_N(\sigma; J) := -\sum_{x\sim y} J_{xy}\sigma_x\sigma_y$

Gibbs measure $p_N(\sigma; J) = \frac{1}{Z_N(J)} e^{-\beta H_N(\sigma;J)}$

Free energy $F_N = \frac{1}{N} \log Z_N(J) = \frac{1}{N} \log \left[\sum_\sigma e^{-\beta H_N(\sigma;J)}\right]$

Sherrington–Kirkpatrick model (SK model)

$$\sigma = (\sigma_1, \cdots, \sigma_N) \in \{-1, 1\}^N$$

$$H_N(\sigma) = -\sum_{i \neq j} J_{ij}\sigma_i\sigma_j = -\langle \sigma, J\sigma \rangle$$

J is a symmetric random matrix with zero diagonal ($J_{ij} = J_{ji}$ and $J_{ii} = 0$)

J_{ij} are iid with mean 0 and variance $\frac{1}{N}$ for $i < j$.

Some extensions: we don't consider them here

external magnetic field $H_N(\sigma) = -\sum_{i,j} J_{ij}\sigma_i\sigma_j + h\sum_i \sigma_i$

p-spin $H_N(\sigma) = -\sum_{i_1,\cdots,i_p} J_{i_1,\cdots,i_p}\sigma_{i_1}\cdots\sigma_{i_p}$

dynamics; aging

Free energy of SK

$F_N = \frac{1}{N} \log \left[\sum_\sigma e^{\beta \sum_{i \neq j} J_{ij} \sigma_i \sigma_j} \right], \qquad \mathbb{E}[J_{ij}] = 0, \quad \mathbb{E}[J_{ij}^2] = \frac{1}{N}$

Parisi's formula (1980) $F = \lim_{N\to\infty} F_N$

Rigorous proof by Talagrand (2006) for Gaussian J_{ij}.

Universality: Guerra and Toninelli (2002), Carmona and Hu (2006)

Transition at $\beta_c = \frac{1}{2}$

high temperature $\frac{1}{N}\mathbb{E}[\log Z_N] - \frac{1}{N}\log \mathbb{E}[Z_N] \to 0$

low temperature $\frac{1}{N}\mathbb{E}[\log Z_N] - \frac{1}{N}\log \mathbb{E}[Z_N] \not\to 0$

Free energy of SK: fluctuations

For high temperature ($\beta < 1/2$): Aizenmann, Lebowitz, Ruelle (1987)

$$\left(F_N - \left(\log 2 + \beta^2\right)\right) N \Rightarrow \mathcal{N}(f, c), \quad c = -\frac{1}{2}\log(1-4\beta^2) - 2\beta^2$$

$F_N \approx (\log 2 + \beta^2) + \frac{1}{N}\mathcal{N}(f, c)$

Low temperature? open question

Spherical SK model (SSK model)

spin variables are on the sphere, $\|\sigma\| = \sqrt{\sigma_1^2 + \cdots + \sigma_N^2} = \sqrt{N}$

$$F_N(\beta) = \frac{1}{N} \log \left[\int_{\|\sigma\|=\sqrt{N}} e^{\beta\langle\sigma, J\sigma\rangle} d\Omega(\sigma) \right]$$

Kosterlitz, Thouless, Jones (1976) Crisanti and Sommers (1992)

Random matrix theory: for real Wigner matrices,

$$F_N(\infty) := \max_{\|x\|=1} \langle x, Jx\rangle = \lambda_{\max}, \quad (F_N(\infty) - 2)N^{2/3} \Rightarrow TW_{GOE}$$

SSK: Known results

$F = \lim_{N\to\infty} F_N$ Crisanti and Sommers (1992), Talagrand (2006)

F is explicit (2-spin): Panchenko and Talagrand (2007), Guionnet and Maïda (2005), Kosterlitz, Thouless, Jones (1976)

$F(\beta)$ is C^2 but not C^3 at $\beta = \beta_c$.

$$F(\beta) = \begin{cases} \beta^2, & \beta < 1/2, \\ 2\beta - \frac{1}{2}\log(2\beta) - \frac{3}{4}, & \beta > 1/2 \end{cases}$$

Theorem 1 for SSK: high temperature regime $\beta < 1/2$

Let $J = \frac{M}{\sqrt{N}}$. For $i < j$, assume $\mathbb{E}[M_{ij}] = 0$, $\mathbb{E}[M_{ij}^2] = 1$, $(\mathbb{E}[M_{ij}^4] = 3,)$ and all moments of M_{ij} are finite.

$$(F_N - \beta^2)N \Rightarrow \mathcal{N}(f, c)$$

The variance $c = -\frac{1}{2}\log(1 - 4\beta^2) - 2\beta^2$ is same as SK (Aizenman, Lebowitz, Ruelle)

The mean is $f = \frac{1}{4}\log(1 - 4\beta^2) - 2\beta^2$

Theorem 2 for SSK: low temperature $\beta > 1/2$

Let $J = \frac{M}{\sqrt{N}}$. For $i < j$, assume $\mathbb{E}[M_{ij}] = 0$, $\mathbb{E}[M_{ij}^2] = 1$ and all moments of M_{ij} are finite.

$$\frac{1}{\beta - 1/2} N^{2/3} \left(F_N - (2\beta - \frac{1}{2} \log(2\beta) - \frac{3}{4}) \right) \Rightarrow TW_{GOE}$$

Order of fluctuations

1. $\beta < 1/2$: $F_N \approx F(\beta) + \frac{1}{N}\mathcal{N}(f, c)$
2. $\beta > 1/2$: $F_N \approx F(\beta) + \frac{1}{N^{2/3}}(\beta - 1/2)\, TW_{GOE}$

Here $c = -\frac{1}{2}\log(1-4\beta^2) - 2\beta^2$

(Hermitian J and complex spin σ: TW_{GUE})

Intuitively,

1. High temperature: all eigenvalues contribute (linear statistic)
2. Low temperature: λ_1 dominates.

This is true even on the level of fluctuations; TW vs Gaussian

Integral representation of Z_N for SSK

Using $J = O^T \Lambda O$, we have $\int_{S^{N-1}} e^{\langle x, Jx \rangle} d\Omega(x) = \int_{S^{N-1}} e^{\sum_i \lambda_i x_i^2} d\Omega(x)$

$$Z_N = \frac{-i}{|S_{N-1}|} \left(\frac{\pi}{N\beta} \right)^{N/2-1} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{N\beta z}}{\prod_{k=1}^N \sqrt{z - \lambda_k}} dz, \quad \text{where } \gamma > \lambda_1$$

Kosterlitz, Thouless, Jones (1976), Mo (2012), Wang (2013)

Integral representation of Z_N for SSK

Thus,

$$Z_N = C_N \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\frac{N}{2} G(z)} dz, \quad \text{where } \gamma > \lambda_1$$

and

$$G(z) = 2\beta z - \frac{1}{N} \sum_{k=1}^{N} \log(z - \lambda_k)$$

1. Method of steepest-descent
2. Critical point $G'(z) = 0$ in $Re(z) > \lambda_1$.
3. Key point: $G(z)$ is random, but almost deterministic since the eigenvalues λ_i are rigid!

Formally, approximate by semi-circle $\frac{d\sigma_{sc}}{dx} = \frac{1}{2\pi}\sqrt{4-x^2}$

$$G(z) = 2\beta z - \frac{1}{N}\sum_{k=1}^{N} \log(z-\lambda_k) \to G_\infty(z) = 2\beta z - \int_{-2}^{2} \log(z-x)\sigma_{sc}(x)dx$$

Steepest-descent analysis of non-random integral

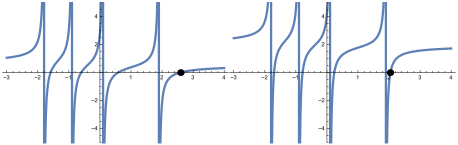
$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{N}{2}G_\infty(z)}dz \qquad \text{Restriction: } \gamma > 2$$

If $\beta < 1/2$, $z_c^\infty = 2\beta + \frac{1}{2\beta} > 2$.

If $\beta > 1/2$, no critical point. Main contribution from $z = 2$ (branch point). Kosterlitz, Thouless, Jones (1976)

Critical point of $G(z)$ itself

$$G'(z) = 2\beta - \frac{1}{N}\sum_{k=1}^{N}\frac{1}{z-\lambda_k}$$



If $\beta < 1/2$, $z_c \approx z_c^\infty$, $O(1)$ away from the branch point $z = 2$.

If $\beta > 1/2$, $z_c = \lambda_1 + O(N^{-1+\epsilon})$ with high probability

To make it rigorous and to compute the fluctuations,

we use:

1. Rigidity of the eigenvalues, Erdös, Yau and Yin (2012)

$$|\lambda_k - \gamma_k| \le \hat{k}^{-1/3} N^{-2/3+\epsilon}, \qquad \hat{k} = \min\{k, N+1-k\}$$

2. Tracy-Widom limit of the largest eigenvalue, Soshinkov (1999), Tao and Vu (2010), Erdös, Yau and Yin (2012)

$$N^{2/3}(\lambda_1 - 2) \Rightarrow TW_{GOE}$$

3. Gaussian fluctuations of linear statistics $L_f = \sum_{i=1}^N f(\lambda_i)$, Johansson (1998), Bai and Yao (2005) (for $f(z) = \log(c - z)$ where $c > 2$) For f analytic in a neighborhood of $[-2, 2]$,

$$L_f - \mathbb{E}[L_f] \Rightarrow \mathcal{N}(0, \sigma^2)$$

Lemma 1: $z_c = \lambda_1 + O(N^{-1+\epsilon})$ from rigidity

Lemma 2: It is still true that

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{N}{2}G(z)}dz \approx K e^{\frac{N}{2}G(z_c)}, \qquad N^{-c} \le K \le C$$

Lemma 3: Using $z_c = \lambda_1 + O(N^{-1+\epsilon})$ and $\lambda_1 = 2 + O(N^{-2/3+\epsilon})$,

$$\begin{aligned}
G(z_c) &= 2\beta z_c - \frac{1}{N}\sum_{i=1}^{N}\log(z_c - \lambda_i) \\
&\approx 2\beta\lambda_1 - \frac{1}{N}\sum_{i=i_0}^{N}\log(\lambda_1 - \lambda_i) \qquad (\lambda_1 - \lambda_i = O(N^{-2/3+\epsilon})) \\
&\approx 2\beta\lambda_1 - \frac{1}{N}\sum_{i=i_0}^{N}\left[\log(2-\lambda_i) + \frac{1}{2-\lambda_i}(\lambda_1 - 2)\right] \\
&\approx 2\beta\lambda_1 - \int_{-2}^{2}\log(2-s)\sigma(s)ds - \int_{-2}^{2}\frac{\sigma(s)ds}{2-s}(\lambda_1 - 2) \\
&= (2\beta - 1)\lambda_1 + \frac{3}{2}.
\end{aligned}$$

High temperature $\beta < 1/2$

$z_c \approx z_c^\infty = 2\beta + \frac{1}{2\beta} > 2$

$$\frac{2}{N} \log \left[\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\frac{N}{2} G(z)} dz \right] \approx G(z_c) \approx G(z_c^\infty)$$

$$G(z_c^\infty) = 2\beta z_c^\infty - \frac{1}{N} \sum_{i=1}^{N} \log(z_c^\infty - \lambda_i)$$

Linear statistics of $\varphi(x) = \log(z_c^\infty - x)$, Johansson (1998), Bai-Yao (2005)

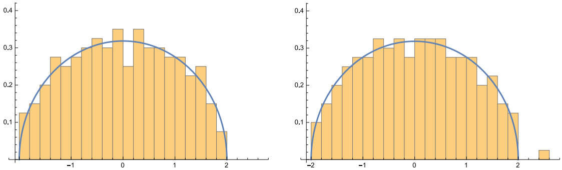
$H(\sigma) = \sum_{i \neq j} J_{ij}\sigma_i\sigma_j + \tilde{J}_0 \sum_{i \neq j} \sigma_i\sigma_j$ for non-random $\tilde{J}_0 = \frac{J_0}{N}$

$H(\sigma) = \langle \sigma, \hat{J}\sigma \rangle$ with $\hat{J}_{ij} = J_{ij} + \frac{J_0}{N}$

$$\hat{J} \simeq \text{GOE} + \frac{J_0}{N} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} (1, \cdots, 1)$$

Spiked random matrix: finite rank perturbed GOE

Spiked random matrix



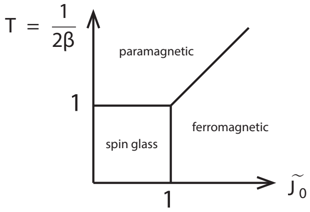
Baik, Ben Arous, Péché 2005, Péché 2006, Féral, Péché 2007, ...

$$\lambda_1 \approx 2 + \frac{c_1}{N^{2/3}} TW_{GOE} \quad \text{if } J_0 < 1$$

and

$$\lambda_1 \approx \frac{1}{2}\left(J_0 + \frac{1}{J_0}\right) + \frac{1}{N^{1/2}} \mathcal{N}(c_2, c_3) \quad \text{if } J_0 > 1$$

Results: Fluctuations of free energy



- spin glass: TW_{GOE} with $N^{-2/3}$ ($z_c \approx \lambda_1 \approx 2$)
- paramagnetic: Gaussian with N^{-1} ($z_c \approx z_c^{\infty} > 2$)
- ferromagnetic: Gaussian with $N^{-1/2}$ ($z_c \approx \lambda_1 > 2$)

SG-FM transition $J_0 = 1 + J' N^{-2/3}$
Hermitian: $F_2 \to F_1^2 \to$ *Gaussian* (Baik, Ben Arous, Péché 2005)
Symmetric: $F_1 \to F \to$ *Gaussian* (Bloemendel, Virag, Mo, Wang)

Extensions: work in progress

(1) Bi-partite SSK $H_N(\sigma,\tau) = \sum_{i=1}^{N}\sum_{j=1}^{M} J_{ij}\sigma_i\tau_j$

Sample covariance matrix/ Wishart ensemble

(2) Near critical temperature $\beta = \beta_N \to \beta_c$ as $N \to \infty$.

Same fluctuations as before when $\beta = \beta_c + N^{-\delta}$ with $\delta < 1/3$.

Expect when $\beta = \beta_c - N^{-\delta}$ with $\delta < 1/3$.

Perhaps the critical window is $\beta = \frac{1}{2} + O(\frac{\sqrt{\log N}}{N^{1/3}})$? Standard deviations are of order $\frac{\sqrt{-\log(1-2\beta)}}{N}$ and $\frac{(\beta-\frac{1}{2})}{N^{2/3}}$.

(Directed polymer: Intermediate disorder regime $\beta = \frac{B}{N^{1/4}}$ (Alberts, Khanin, Quastel 2014))