Kavli Insitute for Theoretical Physics , March 7, 2016

New approaches to non-equilibrium and random systems:

KPZ integrability, universality, applications and experiments

Coord.: Ivan Corwin, Pierre Le Doussal, Tomohiro Sasamoto

$$\frac{\partial h(x,t)}{\partial t} = \frac{\lambda}{2}\left(\frac{\partial h(x,t)}{\partial x}\right)^2 + \nu\frac{\partial^2 h(x,t)}{\partial x^2} + \sqrt{D}\eta(x,t)$$

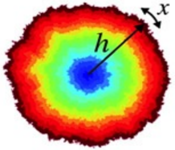

Image from K. A. Takeuchi et al. Sci. Rep. 1, 34 (2011)

Localization in one-dimensional log-gamma polymers

Francis Comets

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Purpose

Directed polymer model : random walk in a random potential.
Likelihood of path (energy) = Cumulated potential it meets on its way.

- Constant potential $\implies$ path spreads over distances O ($\sqrt{\text{length}}$).
- For potential with large variability (Strong Disorder), one expects
  - ☞ path localizes on a few corridors with small width
  - ☞ spreads at "abnormally" large distance $\text{length}^\xi, \xi > 1/2$ (superdiffusivity)

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• Recently, explicitely slovable models of planar polymers have been discovered and used to adress the second effect - typical of the KPZ universality class.

Our final Aim: use Log-Gamma polymer to get sharp results for localization.

Outline

Generalities: Directed Polymer Model

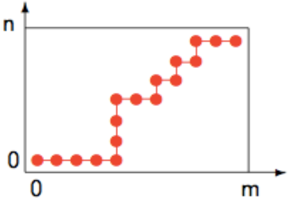
Environment: $(\omega(x), x \in \mathbb{Z}^2)$ real, independent identically distributed r.v.'s

Polymer path: over up-right paths $\mathbf{x} = (x_t; 0 \leq t \leq n)$, $x_t - x_{t-1} \in \{\mathbf{e_1}, \mathbf{e_2}\}$.

Point-to-line polymer measure of a path of length n is

$$Q_n^{\omega}(\mathbf{x}) = \frac{1}{Z_n^{\omega}} \exp\left\{\sum_{t=1}^{n} \omega(x_t)\right\}$$

with $Z_n = Z_n^{\omega}$ the sum over all up-right paths $\mathbf{x}$ starting at $x_0 = (0,0)$.



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with $Z_n = Z_n^\omega$ the sum over all up-right paths $\mathbf{x}$ starting at $x_0 = (0,0)$.
• Quenched asymptotics: For a typical realization ω of the environment, what is the behavior of the polymer of large size $n \to \infty$?

Assumption: $\mathbb{E}e^{r\omega(x)} < \infty$ for r in a neighborhood of $[0,1]$.

Then $\exists$ *Quenched free energy* $\lim_{n\to\infty} n^{-1} \ln Z_n$.

• The model is well defined and is of interest in arbitrary dimension:
$\mathbf{x}_t = (t, S_t), t \leq n$ and S n.n. path in $\mathbb{Z}^d$, $\quad Q_n^\omega(\mathbf{x}) = \frac{1}{Z_n^\omega} \exp\left\{ \sum_{t=1}^n \omega(t, S_t) \right\}$

The KPZ universality class

Polymer models belongs to Kardar-Parisi-Zhang universality class:

Non-gaussian scaling limits and statistics, characterized by a few exponents.

Logarithm of P2P partition fn $h = \log Z_t(x)$ solves (a version of) KPZ eq'n:

$$\frac{\partial}{\partial t} h(t,x) = \frac{\partial^2}{\partial x^2} h(t,x) + \left| \frac{\partial}{\partial x} h(t,x) \right|^2 + \dot{W}(t,x)$$

Many efforts on planar polymer models to understand physics predictions:
• Last passage percolation: Johansson 1998, Prahofer-Spohn 2004,. . . ;
• Alberts-Khanin-Quastel 2014: P2P part'n f'n with diffusive scaling at intermediate disorder converges to SHE (d=1);
Different for $d = 2$: Gaussian limit by Caravenna-Sun-Zygouras 2015$^+$
• See Quastel 2012 and Corwin 2012 for recent surveys.

Integrable systems. Main ones for polymers: O'Connell-Yor 2003, Seppäläinen 2012. . . .

Log-gamma polymer

Change notations:

$$Y_{i,j} = e^{\omega(i,j)}$$

so that $Z_n = \sum_{\mathbf{x}} \prod_{t=1}^{n} Y_{x_t}$.

Def. (Seppäläinen 2012): Log-gamma polymer with parameter $\mu > 0$ is when $Y_x^{-1} \sim \mathrm{Gamma}(\mu)$ with density

$$Y_{i,j}^{-1} \sim \Gamma(\mu)^{-1} r^{\mu-1} e^{-r} \quad , \qquad r > 0.$$

Log-gamma polymer is in KPZ class

Seppäläinen 2012 discovered the stationarity property of this model, which makes it explicitely solvable:

- Seppäläinen obtains the value of the free energy

$$n^{-1} \ln Z_n \xrightarrow{n\to\infty} -\Psi_0(\mu/2), \qquad \Psi_0 = \Gamma'/\Gamma$$

  and proves that the volume and wandering exponents for fluctuations are

$$\chi = 1/3, \qquad \xi = 2/3.$$

- Large deviations of the partition function Georgiou-Seppäläinen 2013
- GUE Tracy-Widom fluctuations for Z_n at scale $n^{1/3}$
  Borodin-Corwin-Remenik 2013: for small μ,

$$n^{-1/3}\left(\ln Z_n + n\Psi_0(\mu/2)\right) \xrightarrow{\text{law}} F_{GUE}$$

  the GUE Tracy-Widom distribution.

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  the GUE Tracy-Widom distribution.
- Explicit formula for the Laplace transform of the partition function at finite size Corwin-O'Connell-Seppäläinen-Zygouras 2014; integral formula which can be turned into a Fredholm determinant.

Weak and Strong disorder

Back to general envir. and dim. $1+d$: Normalized partition function

$$W_n = Z_n/\mathbb{E}Z_n$$

Weak Disorder	Strong Disorder	Very Strong Disorder
$W_n \to W_\infty > 0$	$W_n \to 0$	$W_n \simeq e^{-n\psi(\beta)}$
$\beta < \beta_c, d \geq 3$		$\beta > \beta_c'$
polymer diffusive		polymer localized

Bolthausen'89, Sinai'92, C-Yoshida'06 See below .

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- Speed of convergence:

$$n^{\frac{d-2}{4}} \frac{W_n - W_\infty}{W_n} \to \mathcal{N}(0, \sigma^2)$$

in a subregion of (WD). [C-Liu'16$^+$].

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- Different from polymer on trees [Derrida-Spohn'88] (Branching Random Walk): $e^{-cn} \times \frac{W_n - W_\infty}{W_n^{1/2}}$ is asymptotically normal.

Outline

Localization in general framework

Can be measured by probability of the "favourite endpoint"
(= largest probability among endpoints)

$$\mathcal{I}_n = \max_x Q^{\omega}_{n-1}\{x_n = x\} \in (0, 1).$$

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General fact: Localization $\iff$ Very Strong Disorder:

Theorem (Carmona-Hu'02, C-Shiga-Yoshida'03)

$$\lim_n n^{-1} \ln Z_n < \lim_n n^{-1} \ln \mathbb{E} Z_n \quad \Longrightarrow \quad \exists c_0 > 0 : \liminf_n n^{-1} \sum_{k=1}^{n} \mathcal{I}_k \geq c_0$$

$$\lim_n n^{-1} \ln Z_n = \lim_n n^{-1} \ln \mathbb{E} Z_n \quad \Longrightarrow \quad \lim_n n^{-1} \sum_{k=1}^{n} \mathcal{I}_k = 0.$$

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Strong disorder holds for planar models (C-Vargas'06, Lacoin'10). Then,

$$\limsup_{n\to\infty} \mathcal{I}_n \geq c_0$$

A sharp contrast with decay $O(n^{-1/2})$ when $\omega =$Cst.

General techniques to prove localization

- $W_n = Z_n / \mathbb{E} Z_n$ positive martingale, its logarithm is supermartingale with Doob decomposition

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- Estimate

$$A_n \asymp \sum_{k=}^{n} \mathcal{J}_k\,, \quad \langle M\rangle_n \asymp \sum_{k=}^{n} \mathcal{J}_k$$

with $\mathcal{J}_k = (Q^{\omega}_{k-1})^{\otimes 2}\{x_k = \tilde{x}_k\}$ and $a_n \asymp b_n \iff c \leq \frac{a_n}{b_n} \leq C$.

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- By soft martingale arguments:
$$\lim_n n^{-1} \ln Z_n < \lim_n n^{-1} \ln \mathbb{E}Z_n \quad \iff \liminf_n n^{-1} \sum_{k=1}^{n} \mathcal{J}_k \geq c_0 > 0.$$

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- From $\mathcal{I}_k^2 \le \mathcal{J}_k \le \mathcal{I}_k$, conclude that
$$\lim_n n^{-1} \ln Z_n < \lim_n n^{-1} \ln \mathbb{E} Z_n \quad \iff \liminf_n n^{-1} \sum_{k=1}^{n} \mathcal{I}_k \ge c_0 > 0.$$

General techniques

✖ In Gaussian environment, integration by parts makes replica overlap appear in the estimates:

$$\mathcal{R}_n = \frac{1}{n}\sum_{t=1}^{n}(Q^{\omega}_{n-1})^{\otimes 2}\{x_t = \tilde{x}_t\}.$$

Overlap can be compared to $\mathcal{J}_n$.

✖ In some models (with IBP), path localization at SSD (C-Cranston'14, C-Yoshida'14):
$\exists x^* : [0,n] \to \mathbb{Z}^d$ such that $\lim_{n\to\infty} E\frac{1}{n}\sum_{t=1}^{n} \mathbf{1}_{x_t = x_t^*} = c(\beta) > 0$.

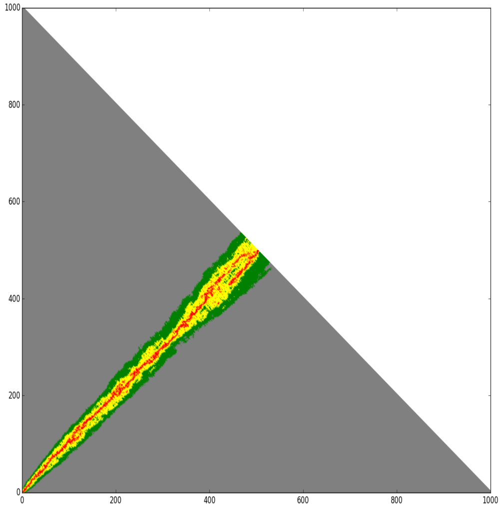
✖ In absence of moments, an alternative approach was developped by Vargas'07 leading directly to $\mathcal{I}_n$.

✍ All these methods are indirect.
In the log-gamma model we have explicit computations.

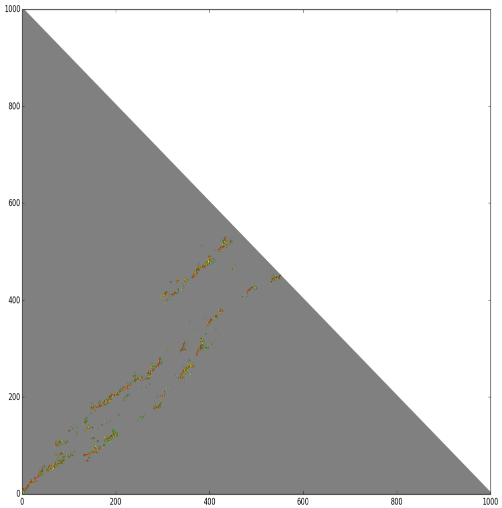
Our aim here

Our aim is to sharpen localization results for the log-gamma model.
Natural questions for the end-point distribution are:

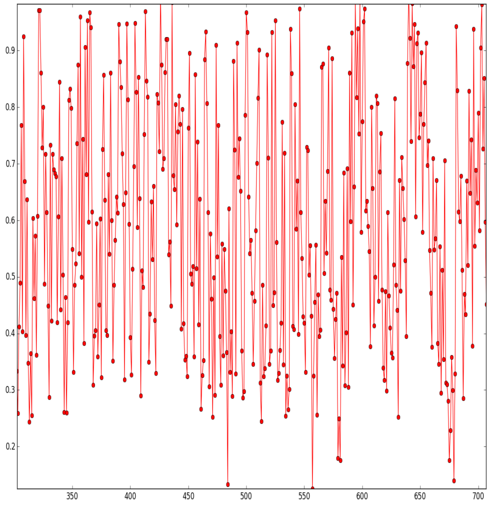
- ✘ What are the regions which contribute the most to the full measure ?
- ✘ Are they many ? How far are they ? how wide ?



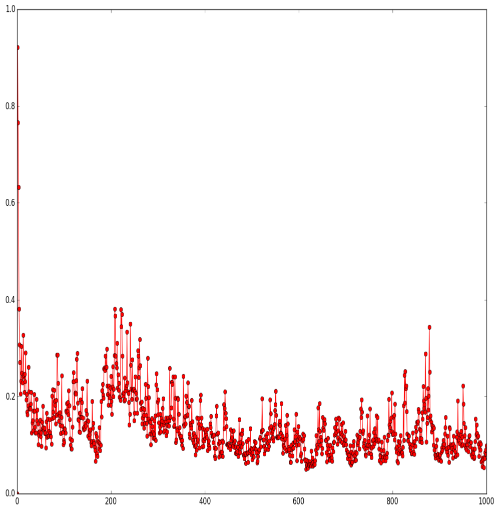
Simulation by Vu-Lan Nguyen:
Endpoint distribution for log-gamma parameter 100: red > yellow > green.



Endpoint distribution for log-gamma parameter 1. (Vu-Lan Nguyen)



Maximal endpoint mass for log-gamma parameter 1. (Vu-Lan Nguyen)



Maximal endpoint mass for log-gamma parameter 10. (Vu-Lan Nguyen)

. . . What to do next ???

It [Localization] has yet to receive adequate mathematical treatment, and one has to resort to the indignity of numerical simulations to settle even the simplest questions about it.

Philip Anderson, from his Nobel Lecture, 8 December 1977

(Fortunately, much has been done since that time !)

Outline

Boundary Conditions

Assign distinct weight distributions on the boundaries and in the bulk:

$$U_{i,0} = Y_{i,0} \text{ and } V_{0,j} = Y_{0,j} \text{ for } i,j \in \mathbb{N} := \{1,2,\ldots\}.$$

Model b.c.(θ): For $\theta \in (0,\mu)$, denote by b.c.(θ) the model with

$$\begin{cases} \{U_{i,0}, V_{0,j}, Y_{i,j} : i,j \in \mathbb{N}\} \text{ are independent with distributions} \\ U_{i,0}^{-1} \sim \text{Gamma}(\theta), \quad V_{0,j}^{-1} \sim \text{Gamma}(\mu-\theta), \quad Y_{i,j}^{-1} \sim \text{Gamma}(\mu). \end{cases}$$

Recall the point-to-point partition function

$$Z_{m,n} = \sum_{\mathbf{x}:\mathbf{0}\mapsto(m,n)} \prod_{t=1}^{m+n} Y_{x_t},$$

and define new weights on horizontal or vertical edges

$$U_{m,n} = \frac{Z_{m,n}}{Z_{m-1,n}} \text{ and } V_{m,n} = \frac{Z_{m,n}}{Z_{m,n-1}}.$$

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Facts (Seppäläinen 2012): Along any down-right path the variables U, V's are mutually independent with marginal distributions

$$U^{-1} \sim \text{Gamma}(\theta), \; V^{-1} \sim \text{Gamma}(\mu-\theta) \quad .$$

Random walk representation for ratios of partition functions

Considering the down-right path along the vertices $x : x \cdot (\mathbf{e_1} + \mathbf{e_2}) = n$, we deduce the representation

$$\frac{Z_{k,n-k}}{Z_{0,n}} = \exp(-\sum_{i=0}^{k} X_i^n).$$

with a collection $X_k^n = -\log(\frac{U_{k,n-k}}{V_{k-1,n-k+1}})$ of i.i.d.r.v.'s. The endpoint distribution is

$$Q_n^\omega\{x_n = (k, n-k)\} = \frac{Z_{k,n-k}}{\sum_{i=0}^n Z_{i,n-i}} = \frac{\exp(-S_k^n)}{\sum_{i=0}^n \exp(-S_i^n)}$$

with $\boxed{S_k^n = \sum_{i=1}^{k} X_i^n}$ a random walk; It is centered iff $\theta = \mu/2$.

The favorite endpoint is

$$\boxed{I_n^n = \arg\min\{S_k^n; 0 \leq k \leq n\}}$$

Outline

main result for P2L

For every n, consider the end-point distribution centered around favorite endpoint,

$$\hat{\xi}_k^{(n)} = Q_n^\omega \{x_n = (l_n^n + k, n - l_n^n - k)\}, \quad k \in \mathbb{Z}.$$

Thus, $\hat{\xi}^{(n)} = (\hat{\xi}_k^{(n)}; k \in \mathbb{Z})$ is a random element of $\mathcal{M}_1(\mathbb{Z})$.

Theorem (C-Nguyen 2015$^+$)

For the model b.c.(θ) with $\theta \in (0, \mu)$, we have convergence in law

$$\hat{\xi}^{(n)} \xrightarrow{\mathcal{L}} \xi \qquad \text{in the space } (\mathcal{M}_1, \|\cdot\|_{TV}),$$

where $\|\mu - \nu\|_{TV} = \sum_k |\mu(k) - \nu(k)|$ is the total variation distance.

Consequences of main result: answers to our questions

A few consequences:

☞ Mass of favourite endpoint converges

$$\mathcal{I}_n \xrightarrow{\mathcal{L}} \max\{(\xi(k)+\xi(k+1))/2; k \in \mathbb{Z}\} > 0.$$

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☞ Tightness of the endpoint: Letting $\overrightarrow{l_n} = (l_n, n - l_n)$,

$$\lim_{K \to \infty} \limsup_{n \to \infty} Q_n^\omega[|x_n - \overrightarrow{l_n}| \geq K] = 0 \quad \text{in probability}$$

Cf. uniqueness of geodesics in First/Last Passage percolation (Newman'95, ...) and related models (Damron-Hanson'14, Bakhtin-Cator-Khanin'14, Georgiou-Rassoul-Agha-Seppalainen'15)

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- Tightness of the endpoint: Letting $\overrightarrow{I_n} = (I_n, n-I_n)$,

$$\lim_{K\to\infty} \limsup_{n\to\infty} Q_n^\omega[|x_n - \overrightarrow{I_n}| \geq K] = 0 \quad \text{in probability}$$

- Scaling limit of endpoint: For $\theta = \mu/2$, Donsker's invariance principle: RW $S_k \simeq W_t$ Brownian Motion, and then

$$\frac{I_n}{n} \xrightarrow{\mathcal{L}} \arg\min_{t\in[0,1]} W_t \ ,$$

the arcsine law. (And so does $\frac{x_n}{n}$ by previous point.)

- When $\theta > \mu/2$, the limit is 0. (In fact, I_n converges in law to a finite limit.)

Consequences of main result, continued

Large deviations of the polymer endpoint:

- For $\theta = \mu/2$,

$$Q_n^\omega\{x_n = ([ns], n-[ns])\} \simeq e^{-\sqrt{n}[W(s)-\min_{[0,1]} W]},$$

- ...whereas for $\theta > \mu/2$,

$$Q_n^\omega\{x_n = ([ns], n-[ns])\} \simeq e^{-ns|\Psi_0(\theta)-\Psi_0(\mu-\theta)|}.$$

Observe:

- Change of speed in the LDP from equilibrium to non-equilibrium.
- Rate function is random in the first case (it depends on the environment), and deterministic linear in the second one.

b.c. are crucial

Observe that these results disagree with KPZ scaling, e.g. on where the polymer localizes (not at distance $\mathcal{O}(n^{2/3})$ from diagonal) !

The disagrement comes from the boundary conditions.

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The disagrement comes from the boundary conditions.

So, what is a general message from this computation ???

We believe that "the trapping at the minimum of a RW" we prove here, *enters in an essential manner* the mechanism for localization in general models.

sketch of proof of main result for $\theta = \mu/2$

Recall that

$$Q_n^\omega\{x_n = (m, n-m)\} = \frac{\exp(-S_m^n)}{\sum_{i=0}^n \exp(-S_i^n)}$$

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Since we are only interested in the law of $Q_n^\omega\{x_n = (m, n-m)\}$, we drop the superscript n in S_n^n, X_i^n, I_n^n, etc...

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• Splitting a random walk S at its minimum is a well studied [Bertoin'91-94, Doney'89-94].

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Since we are only interested in the law of $Q_n^\omega\{x_n = (m, n-m)\}$, we drop the superscript n in S_n^n, X_i^n, l_n^n, etc. . .

• Splitting a random walk S at its minimum is a well studied [Bertoin'91-94, Doney'89-94].

• For $\theta = \mu/2$ (otherwise, quite different and simpler.): The process converges to 2 independent pieces on $\mathbb{Z}^+$ and $\mathbb{Z}^-$, glued at 0,

- (i) $S^\uparrow =$ random walk conditioned to stay non negative forever;
- (ii) $S^\downarrow =$ random walk with jumps $-X$ conditioned to stay strictly positive forever.

For fixed K,

$$(S_{l_n+k} - S_{l_n})_{1\le k\le K} \xrightarrow{\mathcal{L}} (S_k^\uparrow)_{1\le k\le K}\ ,$$

$$(S_{l_n+k} - S_{l_n})_{-1\ge k\ge -K} \xrightarrow{\mathcal{L}} (S_k^\downarrow)_{1\le k\le K}\ .$$

sketch of proof of main result for $\theta = \mu/2$

Since we condition by a null event, $(S_k^{\uparrow})$ and $(S_k^{\downarrow})$ are taboo processes and a correct definition is via Doob's *h*-transform.

sketch of proof of main result for $\theta = \mu/2$

Since we condition by a null event, $(S_k^{\uparrow})$ and $(S_k^{\downarrow})$ are taboo processes and a correct definition is via Doob's h-transform.

Ritter'81 allows to control the full (unbounded) sum:

$$S_{k+l_n} - S_{l_n} \simeq k^{1/2} \qquad \text{for large } k.$$

With the preceeding, we get for $m = \ell_n$:

$$\frac{1}{\sum_{k=0}^{n} \exp(-(S_k - S_{l_n}))} \xrightarrow{\mathcal{L}} \frac{1}{\sum_{k=0}^{\infty} \exp(-S_k^{\uparrow}) + \sum_{k=1}^{\infty} \exp(-S_k^{\downarrow})},$$

which writes also

$$Q_n^{\omega}\left\{x_n = (l_n^n, n - l_n^n)\right\} = \hat{\xi}_0^n \xrightarrow{\mathcal{L}} \xi_0.$$

sketch of proof of main result for $\theta = \mu/2$

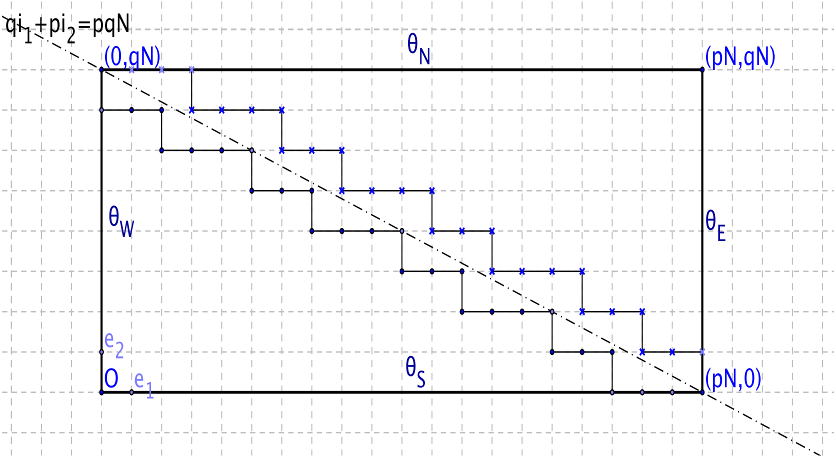
For general values of k, the same arguments lead to the expression of the limit of $\xi^n_{l_n+k} = \hat{\xi}^n_k$, explicitely:

$$\xi_k = \begin{cases} \dfrac{\exp(-S_k^{\uparrow})}{1 + \sum_{i=1}^{\infty} \exp(-S_i^{\uparrow}) + \sum_{i=1}^{\infty} \exp(-S_i^{\downarrow})}, & \text{if } k \geq 0 \\ \dfrac{\exp(-S_k^{\downarrow})}{1 + \sum_{i=1}^{\infty} \exp(-S_i^{\uparrow}) + \sum_{i=1}^{\infty} \exp(-S_i^{\downarrow})}, & \text{if } k < 0 \end{cases}$$

□

Outline

Point-to-Point polymer



Point to point polymer

Fix $\mu > 0$, $(p, q) \in (\mathbb{Z}_+^*)^2$ and for $N \in \mathbb{N}$, let R_N be the rectangle with diagonal $(0, 0), (pN, qN)$.

Model P2P-b.c.(θ): Assume

$$\begin{cases} Y_{i,j} : (i,j) \in R_N \setminus \{\mathbf{0}, (pN, qN)\} \text{ are independent with} \\ Y_{i,j}^{-1} \sim \text{Gamma}(\theta, 1) \text{ for } j \in \{0, qN\}, \\ Y_{i,j}^{-1} \sim \text{Gamma}(\mu - \theta, 1) \text{ for } i \in \{0, pN\}, \\ Y_{i,j}^{-1} \sim \text{Gamma}(\mu, 1) \text{ for } 0 < i < pN \text{ and } 0 < j < qN. \end{cases}$$

Point-to-point polymer measure is the probability measure

$$Q^{\omega}_{pN,qN}(\mathbf{x}) = \frac{1}{Z^{\omega}_{pN,qN}} \exp\left\{ \sum_{t=1}^{n} \omega(x_t) \right\}.$$

For a path $\mathbf{x}$ denote by t^- the "time it crosses the second diagonal". The transverse coordinate of the crossing point can be described by

$$F(\mathbf{x}) = (x_{t^-} + x_{t^-+1}) \cdot (q\mathbf{e_1} - p\mathbf{e_2}).$$

Middle-point distribution, P2P polymer

Theorem (C-Nguyen 2015$^+$)

For any $\theta \in (0, \mu)$, there exist a random integer m_N depending on ω and a random probability measure $\hat{\xi}$ on $\mathbb{Z}$ such that, as $N \to \infty$,

$$\Big(Q^{\omega}_{pN,qN}(F(\mathbf{x}) = m_N + k); k \in \mathbb{Z} \Big) \xrightarrow{\mathcal{L}} \hat{\xi},$$

in the space $(\mathcal{M}_1, \|\cdot\|_{TV})$.

- Recall that middle-point localization for the point-to-point measure is not covered by the usual semi-martingale approach to localization, and this result is totally new.

Conclusions:

✓ Polymer concentrates around the favourite location and spreads at distance $O(1)$ around it. No need of scaling.

✓ The second high peak does not contribute significantly.

✓ Localization comes as a distribution of the form $\boxed{\exp\{-\mathrm{RW}\}}$

Conclusions:

- ✓ Polymer concentrates around the favourite location and spreads at distance $O(1)$ around it. No need of scaling.
- ✓ The second high peak does not contribute significantly.
- ✓ Localization comes as a distribution of the form $\boxed{\exp\{-\mathrm{RW}\}}$

Could be a more general phenomenon ! Cf Trapping in Sinaï RWRE .