Applications of the Bethe ansatz for the finite ASEP with open boundaries

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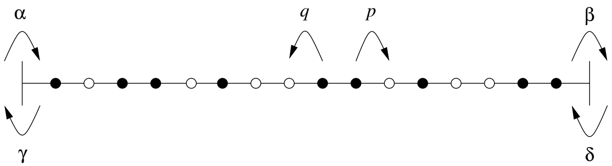
ACEMS

Outline

Partially Asymmetric Exclusion Process (PASEP)



It is convenient to make use of parameters $\kappa^{\pm}_{\alpha,\gamma}$ which are solutions of

$$\alpha\kappa^2 + (p - q - \alpha + \gamma)\kappa + \gamma = 0.$$

We will use the notation

$$a = \kappa^{+}_{\alpha,\gamma}, \quad b = \kappa^{+}_{\beta,\delta}, \quad c = \kappa^{-}_{\alpha,\gamma}, \quad d = \kappa^{-}_{\beta,\delta}.$$

Features

Denote the probability to find a configuration at time t by $P_\tau(t)$ and the transition matrix by M.

The time evolution is given by the Master Equation:

$$\frac{\mathrm{d}}{\mathrm{d}\,t} P_\tau(t) = -\sum_\sigma M_{\tau\sigma} P_\sigma(t)$$

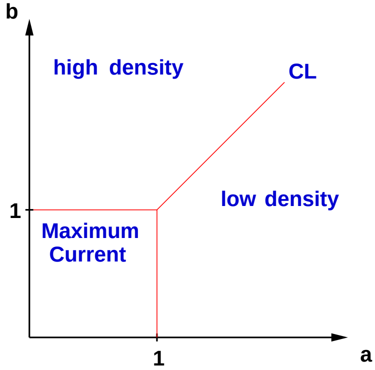
Stationary state:

$$M \cdot P(\infty) = 0.$$

Matrix product formalism:

$$P_{\tau_1,\ldots,\tau_L}(\infty) = \frac{1}{Z_L} \langle W | \prod_{i=1}^{L} [\tau_i D + (1-\tau_i) E] \, | V \rangle.$$

Revision: Stationary phase diagram



CL: Coexistence line

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- What about the spectral gap (second eigenvalue of M)?
- Transition matrix M is equivalent to the Hamiltonian of the XXZ quantum spin chain with open boundary conditions
- Use Yang-Baxter integrability (Bethe's ansatz) to diagonalise M
- Works well for periodic boundary conditions (Gwa & Spohn '92 and Kim '95 in context of growth model)
- Difficulty for open boundaries: no particle conservation

Bethe's ansatz

Example: Single diffusive particle on a ring

$$\begin{aligned}(H\psi)(x) &= p\psi(x-1) + q\psi(x+1) - (p+q)\psi(x) \\ &= \Lambda\psi(x)\end{aligned}$$

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Plane wave ansatz:

$$\psi(x) = z^x$$

$$\Lambda = pz^{-1} + qz - p - q$$

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Results of Bethe's Ansatz for m particles:
Each eigenvalue can be expressed in roots of a system of high degree polynomials.

$$\Lambda = \sum_{j=1}^{m} \lambda(z_j), \qquad P_L(z_j; z_1, \ldots, z_m) = 0 \quad (j = 1, \ldots, m).$$

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Constraint in ASEP parameters:

$$(Q^{L+2k} - 1)(\alpha\beta - \gamma\delta Q^{L-2k-2}) = 0, \qquad Q = \sqrt{\frac{q}{p}}.$$

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Need two sets of Bethe equations, all eigenvalues are given by

$$E_1=E_0+\sum_{j=1}^{L/2-k-1}\lambda(\{z_j\}), \qquad E_2=\sum_{j=1}^{L/2+k}\lambda(\{w_j\}).$$

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- Take $k = -L/2$
- One energy level $E_2 = 0$ (stationary state).
- All excited states described by E_1.

Bethe ansatz for ASEP with open boundaries

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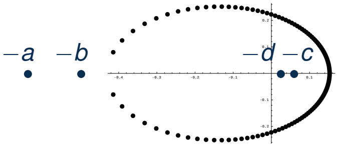
$$K(z) = \frac{(z+a)(z+b)(z+c)(z+d)}{(qaz+1)(qbz+1)(qcz+1)(qdz+1)}$$

Root distribution

For the first excited state Λ_1, Bethe roots lie on a simple curve



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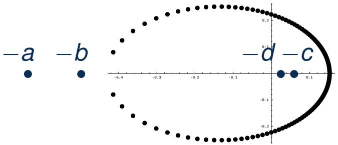

- Solution depends on locus of $a, b, c, d \Rightarrow$ Phase transitions are non-analytic points
- Integral equation

$$K(z) = \frac{(z+a)(z+b)(z+c)(z+d)}{(qaz+1)(qbz+1)(qcz+1)(qdz+1)}$$

Logarithmic form

$$\left[\frac{qz_j - 1}{1 - z_j}\right]^{2L} K(z_j) = \prod_{l \neq j}^{L-1} \left[\frac{qz_j - z_l}{z_j - qz_l}\right] \left[\frac{q^2 z_j z_l - 1}{z_j z_l - 1}\right]$$

Taking log gives:

$$Y_L(z) := g(z) + \frac{1}{L} k(a, b, c, d; z) + \frac{1}{L} \sum_{l=1}^{L-1} \log S(z_l, z), \qquad Y(z_j) = 2\pi I_j$$

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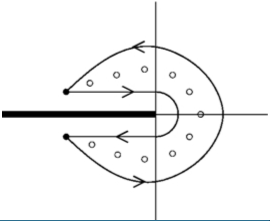
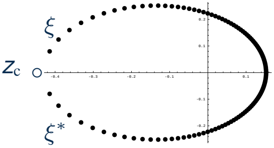
Now use Cauchy:

$$\frac{1}{L} \sum_{j=1}^{L-1} \lambda(z_j) = \oint_{C_1 + C_2} \frac{\mathrm{d}\, z}{2\pi i} \lambda(z) \cot\left(\frac{1}{2} L Y_L(z)\right)$$



Integral equation



Euler-Maclaurin:

$$iY_L(z) = g(z) + \frac{1}{L}k(a,b,c,d;z) + \frac{1}{2\pi}\int_{\xi^*}^{\xi} S(w,z)\,Y_L'(w)\,\mathrm{d}\,w + \mathcal{O}(L^{-2}).$$

Solve by expanding

$$Y_L(w) = Y_L(\xi) + Y_L'(\xi)(w-\xi) + \dots$$

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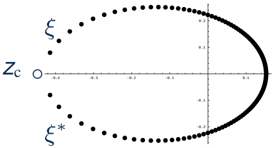

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Solve by expanding

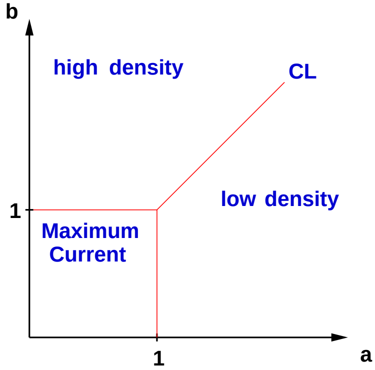
$$Y_L(w) = Y_L(\xi) + Y_L'(\xi)(w-\xi) + \dots$$

and

$$Y_L(w) = \sum_{n\geq 0} L^{-n} y_n(w), \qquad \xi = z_c + \sum_{n\geq 1} L^{-n}(\delta_n + i\eta_n)$$

(Saddle point when $Y_L'(z_c) = 0$).

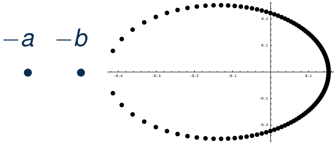
Reminder: Stationary phase diagram



CL: Coexistence line

Low density I

$$-a, -b < z_c$$



$$\lambda_1 = c_1(a,b) + \frac{c_2(a,b)}{L^2} + \mathcal{O}(L^{-3})$$

Finite gap $\Rightarrow$ exponential relaxation.

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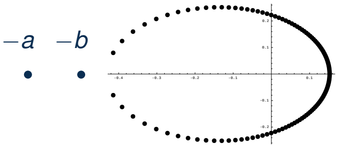

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Finite gap $\Rightarrow$ exponential relaxation.

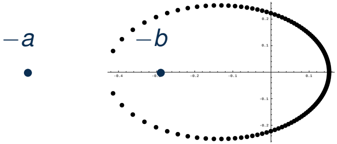
Coexistence line:

$$a = b: \qquad \lambda_1 = \frac{c_2(a,a)}{L^2}.$$

Diffusive relaxation

Low density II

$$-a < z_c < -b$$



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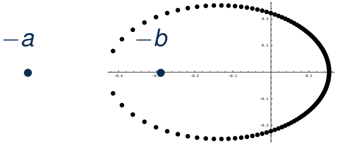

$$\lambda_1 = c_1(a, b_c) + \frac{c_2(a, b_c)}{L^2} + \mathcal{O}(L^{-3})$$

$$b_c = \frac{1}{\sqrt{ab_c}}$$

So λ_1 only depends on a.

Maximum current

Saddle point $Y_L'(z_c) = 0$ and need to expand to second order:

$$Y_L(w) = Y_L(z_c) + \frac{1}{2} Y_L''(z_c)(w - z_c)^2 + \dots$$

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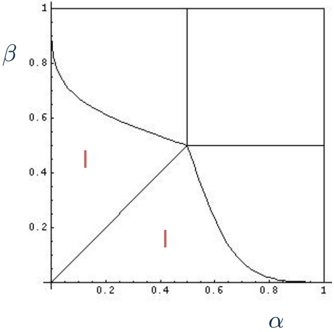
But technical difficulty specific to open boundaries prohibits further analysis

Extremely convincing numerics finds $\Lambda_1 \propto L^{-3/2}$.

$(\gamma = \delta = 0)$ Phase diagram



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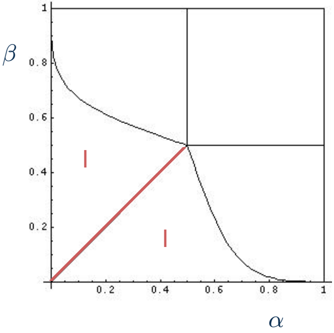

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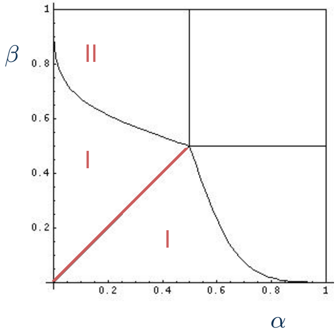

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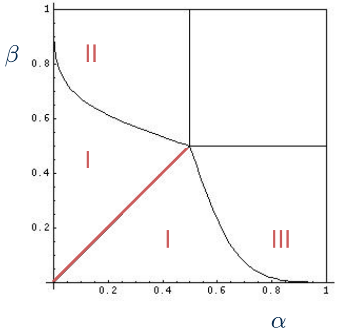

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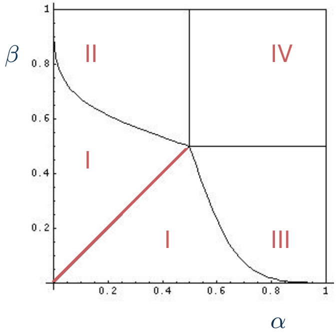

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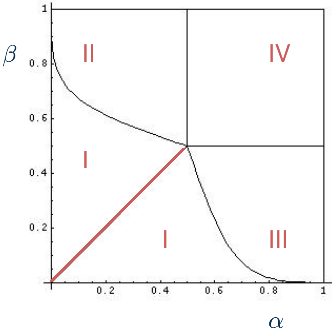

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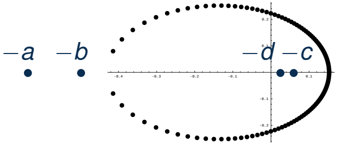
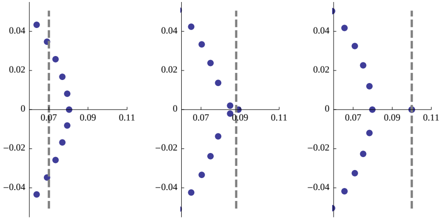
$a = (1 - \alpha)/\alpha$, $b = (1 - \beta)/\beta$,

$\beta_c = (1 + a^{-1/3})^{-1}$ and $\alpha_c = (1 + b^{-1/3})^{-1}$

Reversed bias



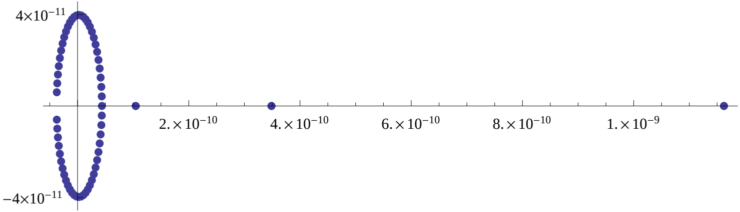
What happens if $-c$ or $-d$ lies outside?



(a) $c = -0.07$ (b) $c = -0.088$ (c) $c = -0.1$

Reversed bias

Isolated roots outside curve:



This is reversed bias boundaries: $-q^m c < a^{-1} < -q^{m-1} c$ and leads to different asymptotics.

E.g. coexistence line:

$$\lambda_1 \propto \frac{1}{(L-2m)^2}.$$

Completely different expression if isolated roots dominate

Asymptotic current generating function

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$$E(\lambda) = \lim_{L\to\infty} \lim_{t\to\infty} \frac{1}{t} \log\langle \mathrm{e}^{\lambda Q_1(t)} \rangle.$$

This implies that the probability $P(j_1, t)$ to observe a current $j_1 = Q_1(t)/t$ at the first site obeys

$$P(j_1, t) \sim \mathrm{e}^{-t\widehat{E}(j_1)}$$

where

$$\widehat{E}(j_1) = \max_{\lambda} \{\lambda j_1 - E(\lambda)\}$$

is the Legendre transform of $E(\lambda)$.

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- Sequence 1: $\lambda_n = n \log q, \qquad n = 0, 1, 2 \ldots.$
- Sequence 2: $\lambda_n = (L-1-n)\log q + \log(abcd), \quad n = L-1, L-2, \ldots.$

Recall that ASEP Bethe equations for lowest eigenvalue are different than for excited states!

Bethe equations

Eigenvalue is expressed in auxiliary variables:

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$$\left[\frac{1-z_j}{1-qz_j}\right]^{2L} K(z_j) = \prod_{l \neq j}^{n} \left[\frac{z_j - qz_l}{z_l - qz_j}\right] \left[\frac{1-qz_j z_l}{1-z_j z_l}\right]$$

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Bethe equations

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Sequence 1: $\lambda_n = n \log q, \quad n = 0, 1, 2 \ldots.$

For the low density phase the solution of the Bethe ansatz equations consists of only n roots ($n = 0, 1, 2, \ldots$)

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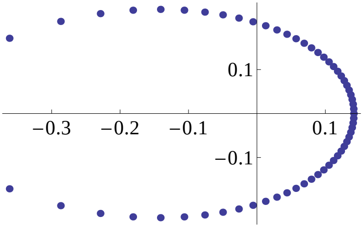
leading to

$$E(\lambda_n) = (1-q)\left(\frac{a}{a+1} - \frac{a}{a+q^n}\right).$$

Sequence 2

Sequence 2: $\lambda_n = (L-1-n)\log q + \log(abcd), \quad n = L-1, L-2, \ldots.$

Now the solution of the Bethe ansatz equations consists of $L-1-n$ roots ($n = 0, 1, \ldots$)



leading (eventually) to

$$E(\lambda_n) = (1-q)\left(\frac{a}{a+1} - \frac{1}{1+bcdq^{L-1-n}}\right).$$

Summarising our results:

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So we conjecture that

$$E(\lambda) = (1-q)\left(\frac{a}{a+1} - \frac{a}{a+\lambda}\right)$$

for all values of λ.
(This has been confirmed by MPA calculations)

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- Extend infinite lattice methodology to finite size (duality, random matrix methods, $\dots$)?