The Atlas model, in and out of equilibrium

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Jointly with Tsai (in equilibrium)
and
with Cabezas, Sarantsev, Sidoravicius (out of equilibrium)

Markov processes & Brownian motions

$\underline{X}(t) = (X_i(t), i \geq 0)$ is $\mathbb{R}^{\mathbb{N}}$-valued stochastic process.

Markov process: for $t \geq s \geq 0$ and suitable $f(\cdot) \in \mathcal{S}$:

$\mathbb{E}[f(\underline{X}(t))|\mathcal{F}_s^{\mathbf{X}}] = \int p_{t-s}(\underline{X}(s), \mathrm{d}\underline{y}) f(\underline{y}) =: (p_{t-s} f)(\underline{X}(s))$.

$\{p_u(\cdot,\cdot)\}$ transition probabilities semi-group:

(a) $p_u(\underline{x}, \cdot)$ probability measure on $\mathbb{R}^{\mathbb{N}}$, per $u, \underline{x}$.
(b) $p_u(\cdot, A)$ Borel function, per u, $A \subset \mathbb{R}^{\mathbb{N}}$ Borel.
(c) Semi-group: $p_{u+v}(\underline{x}, A) = \int p_u(\underline{x}, \mathrm{d}\underline{y}) p_v(\underline{y}, A)$, for $u, v \geq 0$.

Brownian Motion: $t \mapsto W_i(t)$ continuous, Markov process

$p_t(x, A) = \int_A p_t(x-y)\mathrm{d}y$, heat kernel $p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ (here $\mathbb{N} = 1$).
$u(t,x) = (p_t f)(x)$ solves HE: $u_t = \frac{1}{2} u_{xx}$.

Brownian scaling: $W_i^b(t) = b W_i(t/b^2) \overset{(d)}{=} W_i(t)$ for any $b > 0$.

$(W_i(t), t \geq 0), i \geq 0$ independent BM-s $\Leftrightarrow$ product measures.

If $f(\underline{x}) = \prod_i g_i(x_i)$ then $(p_t f)(\underline{x}) = \prod_i (p_t g_i)(x_i)$.

Poisson process, Martingales & Ito's lemma

$$Z_k \sim \mathrm{Exp}(\lambda) \qquad \iff \qquad \mathbf{P}(Z_k \geq z) = e^{-\lambda z},\ z \geq 0,\ \lambda > 0.$$

Independent Exponentials $\underline{Z}^{(\lambda)} := (Z_k, k \geq 1) \sim \rho_\lambda = \bigotimes_{k=1}^{\infty} \mathrm{Exp}(\lambda)$.

Poisson process has points at $\underline{Y} = (Y_k, k \geq 1)$:
$(Y_k, k \geq 1) \sim \mathrm{PPP}_+(\lambda) \ \Leftrightarrow\ Y_1 = 0,\ Y_{k+1} = Y_k + Z_k\,,\ \ k \geq 1$

Continuous $\mathbb{R}$-valued $t \mapsto M(t)$ is L^2-MG

$\iff \qquad \mathbb{E}[M_t | \mathcal{F}_s^{\mathbf{M}}] = M_s \quad \& \quad \mathbb{E}[M_t^2] < \infty \quad \forall t \geq s \geq 0.$

$\implies \quad M_t^2 - [M]_t \quad$ is MG (for quadratic variation $[M]_t$), by Doob-Meyer.

For $f \in C_b^{1,2}(\mathbb{R})$ let $\mathcal{L}f = f_t + \frac{1}{2} f_{xx}$.

Ito's lemma: $M_t^f := f(t, W(t)) - f(0, W(0)) - \int_0^t (\mathcal{L}f)(s, W(s))\mathrm{d}s$ is L^2-MG

$[M^f]_t = \int_0^t f_x^2(s, W(s))\mathrm{d}s.$

Interacting particles: SSEP; Hydrodynamics

Interacting particles: Markov process $\underline{R}(t)$ with interaction.

SSEP: $\underline{R}(t) \in \{0,1\}^{\mathbb{Z}}$.

Jumps: $\Delta_k(i) \in \{-1,+1\}$ i.i.d. $\mathbf{P}(\Delta_k(i) = +1) = \frac{1}{2}$
independent of i.i.d. $\text{PPP}_+(1)$ 'clock' processes $\{\tau_k(i)\}$ for $i \in \mathbb{Z}$.

Order $\{\tau_k(i)\}$, $i \in \mathbb{Z}$ and $k \geq 2$.
Sequentially, if $R_i(\tau_k(i)) = 1$ and $R_{i+\Delta_k(i)}(\tau_k(i)) = 0$ exchange these values.
Otherwise, do nothing (exclusion).

Hydrodynamics: $b \sum_{i=0}^{x/b} \; R_i(t/b^2) \to Q_\star(t,x)$ as $b \to 0$
$Q_\star$ non-random solves some PDE (for suitable $\underline{R}(0)$).

$\mathrm{ATLAS}_\infty(\lambda)$ model

$$X_i(t) = X_i(0) + W_i(t) + \int_0^t \mathbf{1}_{\{X_i(s)=X_{(0)}(s)\}}\,\mathrm{d}s\,, \quad i \geq 0\,.$$

$(W_i(t), t \geq 0)$, $i \geq 0$ independent BM-s.

$$\underline{X}(0) = (X_i(0), i \geq 0) \sim \mathrm{PPP}_+(\lambda), \quad \lambda \in (0,\infty)\,,$$

$$\iff \quad \underline{Z}(0) = \underline{Z}^{(\lambda)} \sim \rho_\lambda = \bigotimes_{k=1}^{\infty} \mathrm{Exp}(\lambda)\,.$$

$X_{(0)}(t) = \min_i\{X_i(t)\}$ left-most particle.

Ranked process $\underline{Y}$ and *spacings process* $\underline{Z}$:

$$Z_k(t) := Y_{k+1}(t) - Y_k(t) := X_{(k)}(t) - X_{(k-1)}(t)\,, \qquad k \geq 1$$

($Y_k(\cdot)$ and $Z_k(\cdot)$ are k-th ranked particle and k-th spacing, resp.).
[Ichiba-Karatzas-Shkolnikov 13, Pal-Pitman 08] $\exists$ unique, rankable weak sol. $\underline{X}$.

Reflected Brownian Motion

RBM representation for $\underline{Z}(t)$ based on

$$Y_k(t) - Y_k(0) = t\mathbf{1}_{\{k=1\}} + B_k(t) + L_{k-1}(t) - L_k(t)$$

$(\underline{B}(t))$ independent BM-s

$L_0(t) = 0$, $L_k(t)$ local time at $\{Z_k(s) = 0\}$, $k \geq 1$ (collisions).

$\text{ATLAS}_\infty(2)$ an equilibrium case

[Pal-Pitman 08] $\lambda = 2 \Rightarrow$ Spacings equilibrium ($\underline{Z}(t) \overset{(d)}{=} \underline{Z}(0)$).
(utilizing [Williams 87] work on RBM-s on polyhedra).

[Conj. 2]: Unique invariant measure (Open).

[Conj. 3]: (resolved in [D-Tsai 15]).

$$t^{-1/4} X_{(0)}(t) \overset{(d)}{\to} N(0, c)\,, \qquad t \to \infty\,, \quad \text{some } c \in (0, \infty)\,.$$

(tagged particle of Harris system [Harris 65, Dürr-Goldstein-Lebowitz 85], and of SSEP [Arratia 83, Rost-Vares 85, Landim-Volchan 00, De Masi-Ferrari 02]).

By spacing equilibrium, [D-Tsai 15] resolve [Conj. 3, PP08] by showing that asymptotic fluctuation at scale $b^{-1/2}$ follows ASHE with Neumann BC at 0.

Question: Out of equilibrium? Expects

$$X_{(0)}(s) \to \pm\infty\,, \quad \text{according to} \quad \text{sgn}(2 - \lambda)\,.$$

Hydrodynamics for $\text{ATLAS}_\infty(\lambda)$: Setting

Asymptotics $b \downarrow 0$ of point processes on $\mathbb{R}_+ \times \mathbb{R}$

$$Q^b(t,\cdot) := b \sum_{i=0}^{\infty} \delta_{t, X_i^b(t)}\,, \quad X_i^b(t) = bX_i(t/b^2), \quad i \geq 0\,.$$

$Q^b(t,\cdot) \in M_*(\mathbb{R}) = \{\text{all Borel } \mu \geq 0 \text{ with } \mu((-\infty, r]) \text{ finite } \forall r\}$,

$\mathcal{C}_* := \{f \in \mathcal{C}_b(\mathbb{R}) \text{ eventually zero}\}$-topology, metrizable by d_*.

$Q^b(\cdot,\cdot) \in \mathfrak{C} = \{\text{all continuous } t \mapsto \mu(t,\cdot) : \mathbb{R}_+ \to (M_*(\mathbb{R}), d_*)\}$,

with topology of uniform convergence on compacts in $\mathbb{R}_+$.

Hydrodynamics for $\textsc{Atlas}_\infty(\lambda)$: Result

Theorem (CDSS 15)

For $\textsc{Atlas}_\infty(\lambda)$ *as* $b \to 0$ *we have* $Q^b(\cdot,\cdot) \to Q_*(\cdot,\cdot)$ *in* $\mathfrak{C}$.

The Q_**-density with respect to Lebesgue*

$$u_*(t,x) := \left[c_1 + c_2 \Phi(x/\sqrt{t})\right] \mathbf{1}_{\{x > y_*(t)\}}, \quad y_*(t) := \kappa\sqrt{t}, \quad \forall t > 0$$

$\Phi(\cdot)$ CDF *of N(0,1) and*

$$c_1 := \frac{2 - \lambda\Phi(\kappa)}{1 - \Phi(\kappa)}, \quad c_2 := \frac{\lambda - 2}{1 - \Phi(\kappa)}.$$

$\operatorname{sgn}(\kappa) = \operatorname{sgn}(2 - \lambda)$ *for* κ *unique such that*

$$\frac{\kappa(1 - \Phi(\kappa))}{\Phi'(\kappa)} = 1 - \frac{\lambda}{2}.$$

Left-most particle $X^b_{(0)}(t) \to y_*(t)$ *as* $b \to 0$ *(uniformly on compacts).*

Stefan problem for $\mathrm{ATLAS}_\infty(\lambda)$

$y_*(t) = \inf\{x : u_*(t,x) > 0\}$ differentiable and $u_*(t,x)$ unique, uniformly bounded and uniformly positive on $x > y(t)$, solution of 1-sided Stefan problem:

$$u_t(t,x) = \frac{1}{2} u_{xx}(t,x)\,, \qquad \forall x > y(t)\,. \quad \text{HE}$$

$$\lim_{t \downarrow 0} u(t,x) = \lambda \mathbf{1}_{x>0}\,, \qquad \forall x \neq 0\,. \quad \text{IC}$$

$$u(t, y(t)^+) := \lim_{x \downarrow y(t)} u(t,x) = 2\,, \qquad \forall t > 0\,. \quad \text{EQ-LBV}$$

$$u(t, y(t)^+) \frac{dy}{dt}(t) + \frac{1}{2} u_x(t, y(t)^+) = 0\,,\ \forall t > 0\,. \quad \text{FLX-BD}$$

The flux condition: consequences

$$\frac{dy}{dt} = -\frac{1}{4} u_x(t, y(t)^+)\,,\ \forall t > 0\,. \quad \text{FLX-BD}$$

$$\begin{aligned} \lambda - 2 > 0 \quad &\Longrightarrow \quad \kappa < 0 \quad \text{(expanding)},\\ \lambda - 2 < 0 \quad &\Longrightarrow \quad \kappa > 0 \quad \text{(contracting)}. \end{aligned}$$

Non-random rate of expansion/contraction

$$\lim_{s\to\infty} \frac{Y_1(s)}{\sqrt{s}} = \kappa\,.$$

$u_*(1,\cdot)$ as limiting particle density profile:

$$\lim_{s\to\infty} Q^{1/\sqrt{s}}(1, x + [-\epsilon, \epsilon]) = \int_{-\epsilon}^{\epsilon} u_*(1, x + r)dr\,, \quad \epsilon > 0\,.$$

# of particles at time $s \gg 1$ near $\sqrt{s}x$ has density $u_*(1, x)$.

Stochastic monotonicity and spacing at the edge

Definition

$$\underline{\xi} \preceq \underline{\xi}' \quad \Leftrightarrow \quad \mathbf{P}(\underline{\xi} \geq \underline{y}) \leq \mathbf{P}(\underline{\xi}' \geq \underline{y}), \qquad \forall \underline{y} \in \mathbb{R}^{\mathbb{N}}.$$

Theorem (CDSS 15)

$\underline{Z}(0) = \underline{Z}^{(\lambda)} \sim \rho_\lambda$.

$$\lambda < 2 \quad \Longrightarrow \quad \underline{Z}^{(2)} \preceq \underline{Z}(t) \preceq \underline{Z}(s) \preceq \underline{Z}^{(\lambda)}, \qquad \forall t \geq s \geq 0,$$

and $\underline{Z}(t) \to \underline{Z}^{(2)}$ *(convergence of f.d.d.).*

$$\lambda > 2 \quad \Longrightarrow \quad \underline{Z}^{(\lambda)} \preceq \underline{Z}(s) \preceq \underline{Z}(t) \preceq \underline{Z}^{(2)}, \qquad \forall t \geq s \geq 0.$$

SSEP versus Ito, PDE and the proof

[Landim-Olla-Volchan 98] get same Stefan problem for effect of tagged asymmetric particle on (truely) doubly-infinite SSEP, by [Arratia 85] transform of spacings in -SEP to constant rate zero-range process.

Here purely one-sided system. Stochastic monotonicity (RBM theory) plus LD for i.i.d. BM-s and for $\text{PPP}_+(\lambda)$ give pre-compactness/regularity of $\{Q^b, b > 0\}$ ($\mathfrak{C}$-limit-points Q^0 as $b \to 0$, with bounded Q^0-density and $X^b_{(0)}(t) \to y_{Q^0}(t)$).

By Ito's lemma (diminishing martingale term as $b \to 0$), all limit points satisfy same weak (distributional) form of our Stefan problem. A-priori regularity and standard PDE tools [Ishii 81] give uniqueness of solution.

Space-time particle fluctuations at $\lambda = 2$: Setting

Asymptotics $b \downarrow 0$ of re-scaled point processes on $\mathbb{R}_+ \times \mathbb{R}$

$$\widehat{Q}^b(t,\cdot) := \sqrt{b/2}\Big[\sum_{i=0}^{\infty} \delta_{X_i^b(t)} - (2/b)\mathrm{Leb}(\mathbb{R}_+)\Big]\,, \quad X_i^b(t) = bX_i(t/b^2), \quad i \ge 0\,.$$

Heat kernel $\quad p_t(x) = \Phi_t'(x) \quad$ for $\quad \Phi_t(x) = \Phi(x/\sqrt{t}) - 1.$

Neumann kernel $p_t^N(y,x) = \partial_y \Psi_t(y,x)$ for

$$\Psi_t(y,x) := \Phi_t(y-x) + \Phi_t(y+x)\,.$$

$B(\cdot)$ Brownian motion, $\mathcal{W}(t,x)$ standard white noise are independent.

$$\widehat{W}_t(x) := \int_0^{\infty} \Psi_t(y,x)\mathrm{d}B(y)\,,$$

$$\widehat{M}_t(x) := \int_0^t \int_0^{\infty} p_{t-s}^N(y,x)\mathrm{d}\mathcal{W}(y,s)\,.$$

$C(\mathbb{R}_+^2;\mathbb{R})$-valued Gaussian process $\widehat{U}^0(t,x) = \widehat{W}_t(x) + \widehat{M}_t(x)$, solves the ASHE

$$(\partial_t - \frac{1}{2}\partial_{xx})\widehat{U}^0(t,x) = \dot{\mathcal{W}}(t,x)\,, \qquad \widehat{U}^0(0,x) = B(x)\,.$$

Space-time particle fluctuations at $\lambda = 2$: Result

Equip $D(\mathbb{R}_+^2)$ with uniform convergence on compacts and let

$$\widehat{U}^b(t,x) := \sqrt{b/2}\Big(2X_{(\lfloor x/(2b)\rfloor)}(t/b^2) - \lfloor x/(2b)\rfloor\Big)\,.$$

Theorem (D-Tsai 15)

For $\text{ATLAS}_\infty(2)$ *as* $b \to 0$,

$$\widehat{U}^b(\cdot,\cdot) \Rightarrow \widehat{U}^0(\cdot,\cdot)\,.$$

In particular, $\quad b^{-1/2}X_{(0)}(t/b^2) \Rightarrow (2/\pi)^{1/4}V(t)$ *a* $1/4$-FBM.

$\widehat{U}^b(t,x) \approx F^{b,r}(t,x) := \langle \widehat{Q}^b(t,\cdot), \Psi_{b^{1+r}}(\cdot, x+b^r)\rangle$ (some $0 < r < 1/2$).

Ito's lemma for $F^{b,r}(t,x)$:

martingale contribution goes to $\widehat{M}_t(x)$,

IC contribution goes in law to $\widehat{W}_t(x)$,

HE and choice of Ψ eliminate $\mathcal{L}F$ part.

Thank you!