The asymmetric KMP model, and its duality.

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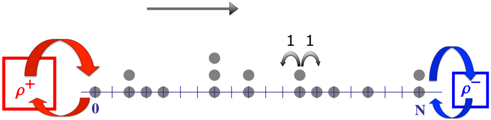
Outline

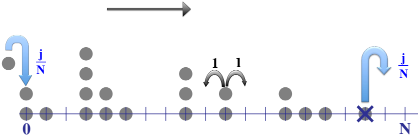
- The symmetric KMP model
- Lie algebraic approach to duality theory
- $\mathfrak{su}_q(1,1)$ algebra : ASIP, ABEP, AKMP
- Applications

Non-equilibrium in 1d: particle transport

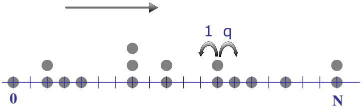
- density reservoirs



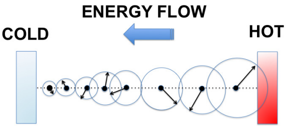
- current reservoirs



- asymmetry



Non-equilibrium in 1d: energy transport



Fourier law $\quad J = \kappa \nabla T$

KMP model (1982)

Energies at every site: $\quad z = (z_1, \ldots, z_N) \in \mathbb{R}_+^N$

$$L^{KMP} f(z) =$$

$$\sum_{i=1}^{N} \int_0^1 dp \left[f(z_1, \ldots, p(z_i + z_{i+1}), (1-p)(z_i + z_{i+1}), \ldots, z_N) - f(z) \right]$$

$\rightarrow$ conductivity $0 < \kappa < \infty$; model solved by duality.

(Stochastic) Duality

Definition

$(\eta_t)_{t\geq 0}$ Markov process on Ω with generator L

$(\xi_t)_{t\geq 0}$ Markov process on Ω_{dual} with generator L_{dual}

ξ_t is dual to η_t with duality function $D : \Omega \times \Omega_{dual} \to \mathbb{R}$ if $\forall t \geq 0$

$$\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \qquad \forall (\eta, \xi) \in \Omega \times \Omega_{dual}$$

η_t is self-dual if $L_{dual} = L$.

In terms of generators:

$$LD(\cdot, \xi)(\eta) = L_{dual} D(\eta, \cdot)(\xi)$$

Duality

- Why is it usefull
  - the dual process is simpler: “from many to few”
  - duality is a signature of integrability
- Questions
  - how to find a dual process & duality function?
  - how to construct processes with duality?

Lie algebraic approach

to duality theory

Algebraic approach

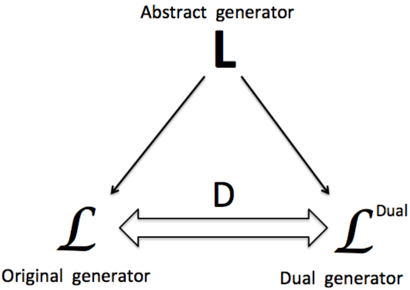
1. The Markov generator, in abstract form, is an element of a (quantum) Lie algebra.

2. Duality is related to a change of representation.
   Duality functions are the intertwiners.

3. Self-duality is associated to symmetries.

Conversely, the approach can be turned into a constructive method.

Duality



Abstract generator
L
D
$\mathcal{L}$
$\mathcal{L}^{\text{Dual}}$
Original generator
Dual generator

Trivial self-duality

Consider Markov chains with countable state space Ω and with a reversible measure μ.

A trivial (i.e. diagonal) self-duality function is

$$\mathbf{d}(\eta, \xi) = \frac{1}{\mu(\eta)} \delta_{\eta,\xi}$$

Indeed

$$\sum_{\eta' \in \Omega} \mathbf{L}(\eta, \eta') \mathbf{d}(\eta', \xi) = \sum_{\xi' \in \Omega} \mathbf{L}(\xi, \xi') \mathbf{d}(\eta, \xi')$$

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$$\frac{\mathbf{L}(\eta,\xi)}{\mu(\xi)} = \sum_{\eta'\in\Omega}\mathbf{L}(\eta,\eta')\mathbf{d}(\eta',\xi) = \sum_{\xi'\in\Omega}\mathbf{L}(\xi,\xi')\mathbf{d}(\eta,\xi') = \frac{\mathbf{L}(\xi,\eta)}{\mu(\eta)}$$

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Indeed

$$\frac{\mathbf{L}(\eta,\xi)}{\mu(\xi)} = \sum_{\eta'\in\Omega}\mathbf{L}(\eta,\eta')\mathbf{d}(\eta',\xi) = \sum_{\xi'\in\Omega}\mathbf{L}(\xi,\xi')\mathbf{d}(\eta,\xi') = \frac{\mathbf{L}(\xi,\eta)}{\mu(\eta)}$$

Thus

$$\mathbf{L}\mathbf{d} = \mathbf{d}\mathbf{L}^{T}$$

Symmetries and self-duality

S: symmetry of the Markov generator, i.e. $[\mathbf{L}, \mathbf{S}] = 0$

$\mathbf{d}$: trivial self-duality function

$\longrightarrow \quad \mathbf{D} = \mathbf{Sd}$ is a self-duality function

Indeed

$$\mathbf{LD} = \mathbf{LSd} = \mathbf{SLd} = \mathbf{SdL}^T = \mathbf{DL}^T$$

Construction of Markov generators with algebraic structure

Ingredients:

- (*Algebra*): Start from a Lie algebra $\mathfrak{g}$.
- (*Casimir*): Pick an element C in the center of $\mathfrak{g}$, e.g. the Casimir.
- (*Co-product*): Consider a co-product $\Delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ conserving the commutation relations.

Steps:

(i) (*Quantum Hamiltonian*): Compute $H = \Delta(C)$.

(ii) (*Symmetries*): $S = \Delta(X)$ with $X \in \mathfrak{g}$

$$[H, S] = [\Delta(C), \Delta(X)] = \Delta([C, X]) = \Delta(0) = 0.$$

(iii) (*Markov generator*): Apply a "positive ground state transformation" to turn H into a Markov generator L.

Quantum $\mathfrak{su}_q(1,1)$ algebra

q-numbers

For $q \in (0,1)$ and $n \in \mathbb{N}_0$ introduce the q-number

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Remark: $\lim_{q \to 1} [n]_q = n$.

The first q-number's are:

$$[0]_q = 0, \qquad [1]_q = 1, \qquad [2]_q = q + q^{-1}, \qquad [3]_q = q^2 + 1 + q^{-2}, \quad \dots$$

Quantum Lie algebra $\mathfrak{su}_q(1,1)$

For $q \in (0,1)$ consider the generators K^+, K^-, K^0 with

$$[K^0, K^\pm] = \pm K^\pm, \qquad [K^+, K^-] = -[2K^0]_q$$

where

$$[2K^0]_q := \frac{q^{2K^0} - q^{-2K^0}}{q - q^{-1}}$$

Irreducible representations are infinite dimensional. E.g., for $n \in \mathbb{N}$

$$\begin{cases} K^+ e^{(n)} &= \sqrt{[n+2k]_q [n+1]_q}\; e^{(n+1)} \\ K^- e^{(n)} &= \sqrt{[n]_q [n+2k-1]_q}\; e^{(n-1)} \\ K^o e^{(n)} &= (n+k)\; e^{(n)} \end{cases}$$

Casimir element

$$C = [K^o]_q [K^o - 1]_q - K^+ K^-$$

In this representation

$$C\, e^{(n)} = [k]_q [k-1]_q \, e^{(n)} \qquad\qquad k \in \mathbb{R}_+$$

Co-product

A co-product $\Delta : U_q(\mathfrak{su}(1,1)) \to U_q(\mathfrak{su}(1,1))^{\otimes 2}$

$$\begin{aligned} \Delta(K^{\pm}) &= K^{\pm} \otimes q^{-K^o} + q^{K^o} \otimes K^{\pm} \\ \Delta(K^o) &= K^o \otimes 1 + 1 \otimes K^o \end{aligned}$$

and it extends via

$$\Delta[A \cdot B] = \Delta[A] \cdot \Delta[B] \qquad \Delta[A + B] = \Delta[A] + \Delta[B]$$

The co-product conserves the commutation relations:

$$[\Delta(K^o), \Delta(K^{\pm})] = \pm\Delta(K^{\pm}) \qquad [\Delta(K^+), \Delta(K^-)] = [2\Delta(K^o)]_q$$

Iteratively $\Delta^n : U_q(\mathfrak{su}(1,1)) \to U_q(\mathfrak{su}(1,1))^{\otimes(n+1)}$, i.e. for $n \geq 2$

$$\begin{aligned} \Delta^n(K^{\pm}) &= \Delta^{n-1}(K^{\pm}) \otimes q^{-K^0_{n+1}} + q^{\Delta^{n-1}(K^0_i)} \otimes K^{\pm}_{n+1} \\ \Delta^n(K^0) &= \Delta^{n-1}(K^0) \otimes 1 + 1^{\otimes n} \otimes K^0_{n+1} \end{aligned}$$

Quantum Hamiltonian

$$\Delta(C_i) = q^{K_i^0} \left\{ K_i^+ \otimes K_{i+1}^- + K_i^- \otimes K_{i+1}^+ - B_i \otimes B_{i+1} \right\} q^{-K_{i+1}^0}$$

Quantum Hamiltonian

$$\Delta(C_i) = q^{K_i^0}\left\{K_i^+ \otimes K_{i+1}^- + K_i^- \otimes K_{i+1}^+ - B_i \otimes B_{i+1}\right\}q^{-K_{i+1}^0}$$

out-of-diagonal: ≥ 0

$$\begin{aligned} B_i \otimes B_{i+1} &= \frac{(q^k + q^{-k})(q^{k-1} + q^{-(k-1)})}{2(q-q^{-1})^2}\left(q^{K_i^0} - q^{-K_i^0}\right) \otimes \left(q^{K_{i+1}^0} - q^{-K_{i+1}^0}\right) \\ &+ \frac{(q^k - q^{-k})(q^{k-1} - q^{-(k-1)})}{2(q-q^{-1})^2}\left(q^{K_i^0} + q^{-K_i^0}\right) \otimes \left(q^{K_{i+1}^0} + q^{-K_{i+1}^0}\right) \end{aligned}$$

Quantum Hamiltonian

$$\Delta(C_i) = q^{K_i^0}\left\{K_i^+ \otimes K_{i+1}^- + K_i^- \otimes K_{i+1}^+ - B_i \otimes B_{i+1}\right\}q^{-K_{i+1}^0}$$

$$H^{(L)} := \sum_{i=1}^{L-1}\left(\mathbf{1}^{\otimes(i-1)} \otimes \Delta(C_i) \otimes \mathbf{1}^{\otimes(L-i-1)} + c_{q,k}\mathbf{1}^{\otimes L}\right)$$

$$c_{q,k} = \frac{(q^{2k} - q^{-2k})(q^{2k-1} - q^{-(2k-1)})}{(q-q^{-1})^2} \qquad s.t. \qquad H \cdot \left(\otimes_{i=1}^{L} e_i^{(0)}\right) = 0$$

Symmetries

Lemma

Let $a \in \{+,-,0\}$, then $K^a = \Delta^{L-1}(K_1^a)$ are symmetries:

$$[H^{(L)}, K^a] = 0$$

Explicitly

$$K^{\pm} := \sum_{i=1}^{L} q^{K_1^0} \otimes \cdots \otimes q^{K_{i-1}^0} \otimes K_i^{\pm} \otimes q^{-K_{i+1}^0} \otimes \ldots \otimes q^{-K_L^0}$$

$$K^0 := \sum_{i=1}^{L} \underbrace{1 \otimes \cdots \otimes 1}_{(i-1)\text{ times}} \otimes K_i^0 \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(L-i)\text{ times}}$$

Proof:

For $n=2$: $[H^{(2)}, K^a] = [\Delta(C_1), \Delta(K_1^a)] = \Delta([C_1, K_1^a]) = \Delta(0) = 0.$

For $n>2$: induction.

Markov processes

with $\mathfrak{su}_q(1,1)$ symmetry

Ground state transformation

Lemma

Let H be a matrix with $H(\eta, \eta') \geq 0$ if $\eta \neq \eta'$.
Suppose g is a positive ground state, i.e. $H\,g = 0$ and $g(\eta) > 0$.
Let G be the matrix $G(\eta, \eta') = g(\eta)\delta(\eta, \eta')$. Then

$$L = G^{-1} H\, G$$

is a Markov generator.

Indeed

$$L(\eta, \eta') = \frac{H(\eta, \eta') g(\eta')}{g(\eta)}$$

Therefore

$$L(\eta, \eta') \geq 0 \quad \text{if} \quad \eta \neq \eta' \qquad\qquad \sum_{\eta'} L(\eta, \eta') = 0$$

Exponential symmetries

- $g = \otimes_{i=1}^{L} e_i^{(0)}$ is a ground state, i.e. $Hg = 0$.
- For every symmetry $[H, S] = 0$ another ground state is $g_S := Sg$.
- The exponential symmetry

$$S^+ = \exp_{q^2}(E) = \sum_{n \geq 0} \frac{(E)^n}{[n]_q!} q^{-n(n-1)/2}$$

with

$$E = \Delta^{(L-1)}(q^{K_1^0}) \cdot \Delta^{(L-1)}(K_1^+)$$

gives a positive ground state

$$g_{S^+} := S^+ g = \sum_{\ell_1, \ldots, \ell_L \geq 0} \otimes_{i=1}^{L} \left(\sqrt{\binom{\ell_i + 2k - 1}{\ell_i}_q} \cdot q^{\ell_i(1-k+2ki)} \right) e^{(\ell_i)}$$

(1): Asymmetric Inclusion Process: ASIP(q,k)

For $k \in \mathbb{R}_+$ the interacting particle system ASIP(q, k) on $[1, L] \cap \mathbb{Z}$ with state space $\{0, 1, \ldots\}^L$ is defined by

$$(L^{ASIP(q,k)} f)(\eta) = \sum_{i=1}^{L-1} (L_{i,i+1} f)(\eta)$$

with

$$\begin{aligned}(L_{i,i+1} f)(\eta) &= q^{\eta_i - \eta_{i+1} + (2k-1)} [\eta_i]_q [2k + \eta_{i+1}]_q (f(\eta^{i,i+1}) - f(\eta)) \\ &+ q^{\eta_i - \eta_{i+1} - (2k-1)} [2k + \eta_i]_q [\eta_{i+1}]_q (f(\eta^{i+1,i}) - f(\eta))\end{aligned}$$

- $q = 1 \rightarrow SIP(k)$: symmetric inclusion
  jump right at rate $\eta_i(2k + \eta_{i+1})$, jump left at rate $(2k + \eta_i)\eta_{i+1}$

Properties of ASIP(q,k)

- The $ASIP(q,k)$ on $[1,L] \cap \mathbb{Z}$ has a family (labeled by $\alpha > 0$) of inhomogeneous reversible product measures with marginals

$$\mathbb{P}_\alpha(\eta_i = x) = \frac{\alpha^x}{Z_{i,\alpha}} \binom{x+2k-1}{x}_q \cdot q^{4kix}$$

- $q = 1$: the reversible measure is homogeneous and product of Negative Binomials $(2k, \alpha)$

(2) Asymmetric Brownian Energy Process: ABEP(σ, k)

For $\sigma > 0$, let $(\eta^{(\epsilon)}(t))_{t\geq 0}$ be the $ASIP(1-\epsilon\sigma, k)$ process initialized with ϵ^{-1} particles. The scaling limit (weak asymmetry)

$$z_i(t) := \lim_{\epsilon\to 0} \epsilon\, \eta_i^{(\epsilon)}(t)$$

is the diffusion ABEP(σ, k) with generator $L^{ABEP(\sigma,k)} = \sum_{i=1}^{L-1} \mathcal{L}_{i,i+1}$

$$\mathcal{L}_{i,i+1} =$$
$$-\frac{1}{2\sigma}\left\{(1-e^{-2\sigma z_i})(e^{2\sigma z_{i+1}}-1) + 2k\left(2-e^{-2\sigma z_i} - e^{2\sigma z_{i+1}}\right)\right\}\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}}\right)$$
$$+\frac{1}{4\sigma^2}(1-e^{-2\sigma z_i})(e^{2\sigma z_{i+1}}-1)\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}}\right)^2$$

Properties of ABEP(σ, k)

- $\sigma \to 0^+$

  $$\mathcal{L}_{i,i+1} = -2k\,(z_i - z_{i+1})\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}}\right) + z_i z_{i+1}\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}}\right)^2$$

  The reversible measures are given by product of i.i.d. $Gamma(2k;\gamma)$.

- $\sigma \neq 0$

  the process is truly asymmetric, i.e. on the 1-d torus it has a non-zero current.

  On $\mathbb{Z}_+$it has inhomogeneous reversible product measures (labeld by $\gamma > -4\sigma k$) with marginal density

  $$\mu(z_i) = \frac{1}{\mathcal{Z}_{i,\gamma}}(1 - e^{-2\sigma z_i})^{(2k-1)} e^{-(4\sigma k i + \gamma) z_i}$$

(3) $KMP(k)$ process

Instantaneous thermalization limit:

$$L_{i,j}^{KMP(k)} f(z_i, z_j) := \lim_{t\to\infty} \left(e^{tL_{i,j}^{BEP(k)}} - 1 \right) f(z_i, z_j)$$
$$= \int_0^1 dp\, \nu^{(k)}(p)\, [f(p(z_i + z_j), (1-p)(z_i + z_j)) - f(z_i, z_j)]$$

$$Z_i, Z_j \sim \mathrm{Gamma}\,(2k, \theta) \quad \text{i.i.d.} \quad \Longrightarrow \quad P = \frac{Z_i}{Z_i + Z_j} \sim \mathrm{Beta}\,(2k, 2k)$$

$$\nu^{(k)}(p) = \frac{p^{2k-1}(1-p)^{2k-1}}{B(2k, 2k)}$$

For $k = \frac{1}{2}$: uniform redistribution, original KMP

$AKMP(\sigma, k)$ process

- $AKMP(\sigma, k)$

$$L_{i,j}^{AKMP(\sigma,k)} f(z_i, z_j) := \lim_{t\to\infty} \left(e^{tL_{i,j}^{ABEP(\sigma,k)}} - 1 \right) f(z_i, z_j)$$

$$= \int_0^1 dp\, \nu_\sigma^{(k)}(p|z_i + z_j)\, [f(p(z_i + z_j), (1-p)(z_i + z_j)) - f(z_i, z_j)]$$

with

$$\nu_\sigma^{(k)}(p|E) = \frac{1}{\mathcal{Z}_{\sigma,k}(E)} e^{2\sigma pE} \left\{ \left(e^{2\sigma pE} - 1 \right) \left(1 - e^{-2\sigma(1-p)E} \right) \right\}^{2k-1}$$

- Th-ASIP(q, k)

$$(n, m) \to (R_q, n + m - R_q)$$

with R_q a q-deformed Beta-Binomial $(n + m, 2k, 2k)$

Duality relations

Self-duality of ASIP(q,k)

Theorem [Carinci,G., Redig, Sasamoto (2015)]

The ASIP(q,k) is self-dual on

$$D(\eta,\xi^{(\ell_1,\dots,\ell_n)}) = \frac{q^{-4k\sum_{m=1}^n \ell_m - n^2}}{(q^{2k}-q^{-2k})^n} \cdot \prod_{m=1}^{n} (q^{2N_{\ell_m}(\eta)} - q^{2N_{\ell_{m+1}}(\eta)})$$

where $\xi^{(\ell_1,\dots,\ell_n)}$ is the configuration with n particles at sites $\ell_1,\dots,\ell_n$ and

$$N_i(\eta) := \sum_{k=i}^{L} \eta_k$$

- It follows from the explicit knowledge of the reversible measure and from the exponential symmetry S_+

Duality between $ABEP(\sigma, k)$ and $SIP(k)$

Theorem [Carinci,G., Redig, Sasamoto (2015)]

- For every σ (including 0^+), the process $\{z(t)\}_{t\geq 0}$ with generator $L^{ABEP(\sigma,k)}$ and the process $\{\eta(t)\}_{t\geq 0}$ with generator $L^{SIP(k)}$ are dual on

$$D(z,\xi) = \prod_{i=1}^{L} \frac{\Gamma(2k)}{\Gamma(2k+\xi_i)} \left(\frac{e^{-2\sigma E_{i+1}(z)} - e^{-2\sigma E_i(z)}}{2\sigma} \right)^{\xi_i}$$

with

$$E_i(z) = \sum_{l=i}^{L} z_l \qquad E_{L+1}(z) = 0$$

- Same duality holds between $AKMP(\sigma, k)$ and Th-$SIP(k)$

symmetric case $\sigma = 0^+$

$$L = \sum_{i=1}^{L-1} \Big(K_i^+ K_{i+1}^- + K_i^- K_{i+1}^+ - 2K_i^o K_{i+1}^o + 2k^2 \Big)$$

Two representations:

$$\begin{cases} K_i^+ e^{(\eta_i)} = (\eta_i + 2k)\, e^{(\eta_i+1)} \\ K_i^- e^{(\eta_i)} = \eta_i e^{(\eta_i-1)} \\ K_i^o e^{(\eta_i)} = (\eta_i + 4k)\, e^{(\eta_i)} \end{cases} \qquad \begin{cases} \mathcal{K}_i^+ = z_i \\ \mathcal{K}_i^- = z_i\, \partial_{z_i}^2 + 2k\partial_{z_i} \\ \mathcal{K}_i^o = z_i\, \partial_{z_i} + k \end{cases}$$

$L = L^{SIP(k)}$ $\qquad$ $L = L^{BEP(k)}$

$$\frac{\Gamma(2k)}{\Gamma(2k+\xi_i)} z_i^{\xi_i}$$

Duality fct $\equiv$ intertwiner

asymmetric case $\sigma \neq 0$

- The $ABEP(\sigma, k)$ can be mapped to $BEP(k)$ via the non-local transformation

$$z \mapsto g(z) \qquad\qquad g_i(z) := \frac{e^{-2\sigma E_{i+1}(z)} - e^{-2\sigma E_i(z)}}{2\sigma}$$

  Equivalently

$$L^{ABEP(\sigma,k)} = C_g \circ L^{BEP(k)} \circ C_{g^{-1}}$$

  with

$$(C_g f)(z) = (f \circ g)(z)$$

- Therefore, despite the asymmetry, the symmetry group of $ABEP(\sigma, k)$ is the same as for $BEP(k)$, namely $\mathfrak{su}(1,1)$. The representation is a non-local conjugation of the differential operator representation.

Applications

Example1: Current of ABEP(σ, k)

Definition

The current $J_i(t)$ during the time interval $[0, t]$ across the bond $(i-1, i)$ is defined as:

$$J_i(t) = E_i(z(t)) - E_i(z(0))$$

where

$$E_i(z) := \sum_{k \geq i} z_k$$

Remark: let $\xi^{(i)}$ be the configuration with 1 dual particle:

$$\xi^{(i)}_m = \begin{cases} 1 & \text{if } m = i \\ 0 & \text{otherwise} \end{cases}$$

then

$$D(z, \xi^{(i)}) = \frac{e^{-2\sigma E_{i+1}(z)} - e^{-2\sigma E_i(z)}}{4k\sigma}$$

Example1: Current of ABEP(σ, k)

Using duality between ABEP(σ, k) and SIP(k)

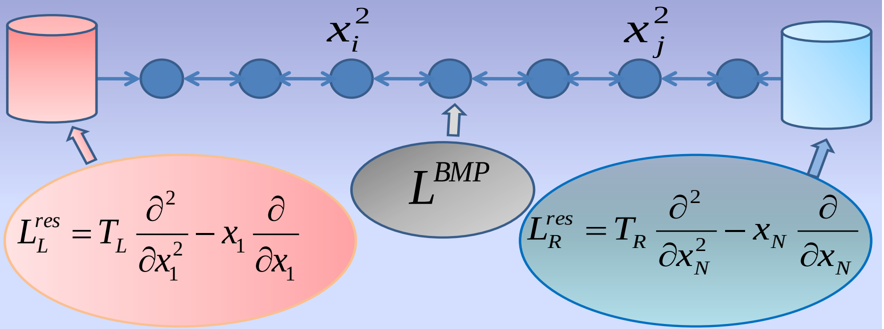
$$\mathbb{E}_z(e^{-2\sigma J_i(t)}) = e^{-4kt} \sum_{n \in \mathbb{Z}} e^{-2\sigma(E_n(z) - E_i(z))} I_{|n-i|}(4kt)$$

$I_n(t)$ modified Bessel function.

The computation requires a single dual SIP particle, which is a simple symmetric random walk jumping at rate $2k$:

$$\mathbb{P}_i(X_t = n) = e^{-4kt} I_{|n-i|}(4kt).$$

Brownian Momentum Process with reservoirs

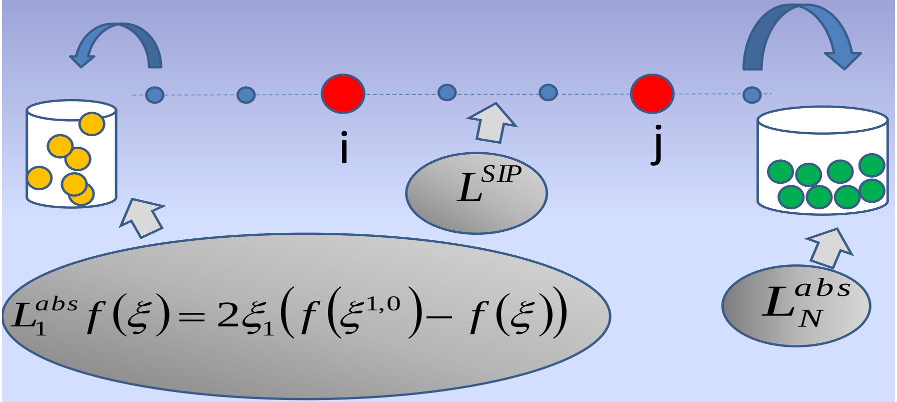


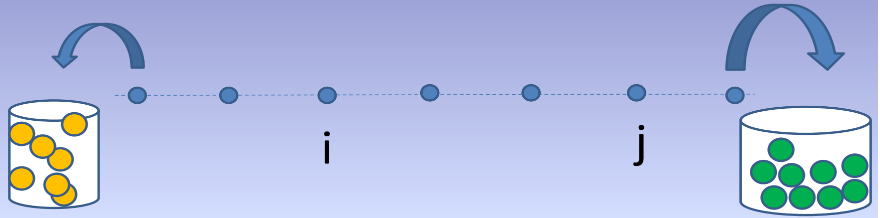
Example 2: Energy covariance in the boundary driven BEP

$$\text{If } \vec{\xi} = (0,\ldots,0,\underset{\text{site } i \nearrow}{1},0,\ldots,0,\underset{\text{site } j \nearrow}{1},0,\ldots,0) \quad \Rightarrow \quad D(x,\vec{\xi}) = x_i^2 x_j^2$$

In the dual process we initialize two SIP walkers $(X_t, Y_t)_{t\geq 0}$ with $(X_0, Y_0) = (i,j)$

Inclusion Process with absorbing reservoirs





$$\mathbf{E}\left(x_i^2 x_j^2\right) = T_L^2\,\mathrm{P}(\;) + T_R^2\,\mathrm{P}(\;) + T_L T_R\left(\mathrm{P}(\;) + \mathrm{P}(\;)\right)$$

Example 2: Energy covariance in the boundary driven BEP

$$\mathbb{E}\left(x_i^2 x_j^2\right) - \mathbb{E}\left(x_i^2\right)\mathbb{E}\left(x_j^2\right) = \frac{2i(N+1-j)}{(N+3)(N+1)^2}(T_R - T_L)^2$$

Remark: Long range correlations

$$\lim_{N\to\infty} N\, Cov(x^2_{s_1 N}, x^2_{s_2 N}) = 2s_1(1-s_2)(T_R - T_L)^2$$

Some references

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