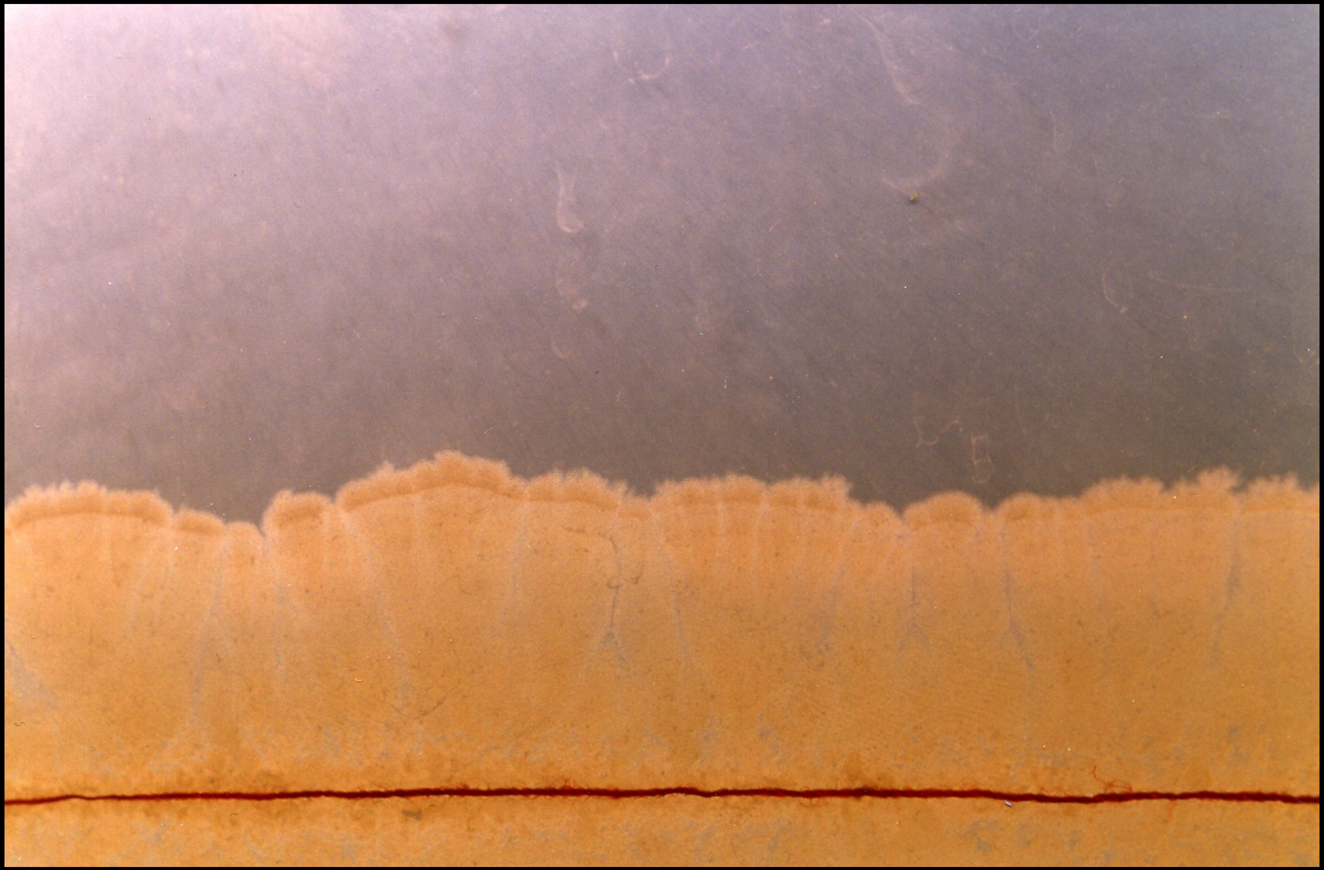
KPZ Playbook

-George Palasantzas (Groningen
-Kazumasa Takeuchi (Tokyo)
-Yuexia Lin (Barnard)

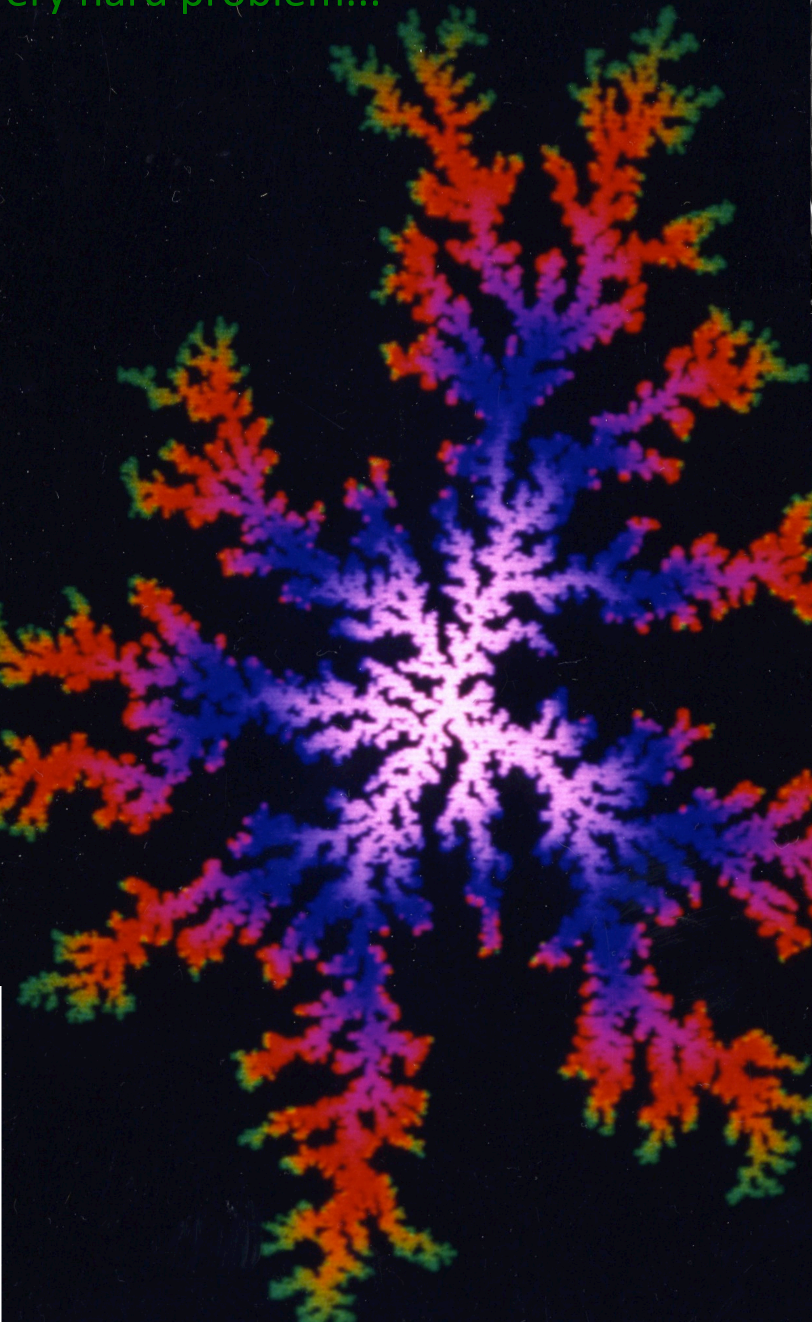
Universal Exponents, Distributions & Correlators

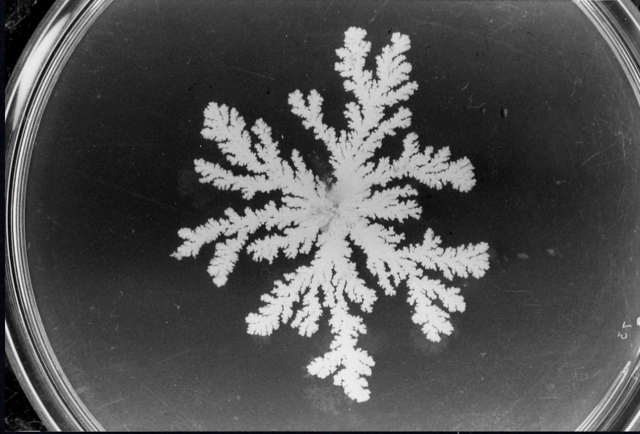


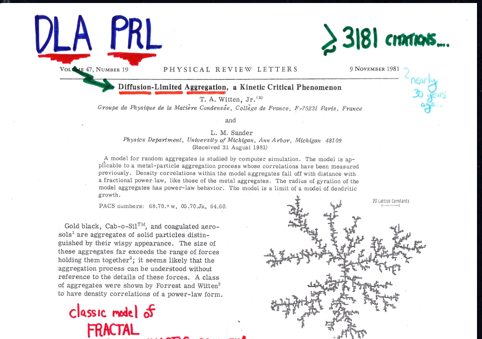
Kinetic Roughening, Nonequilibrium Stochastic Growth, Directed Polymers,...

Parallel Track: DLA

Rich, seductive, but very hard problem…







DLA PRL

≳ 3181 CITATIONS….

VOLUME 47, NUMBER 19 PHYSICAL REVIEW LETTERS 9 NOVEMBER 1981

≳ nearly 30 years ago…

Diffusion-Limited Aggregation, a Kinetic Critical Phenomenon

T. A. Witten, Jr.(a)

Groupe de Physique de la Matière Condensée, Collège de France, F-75231 Paris, France

and

L. M. Sander

Physics Department, University of Michigan, Ann Arbor, Michigan 48109

(Received 31 August 1981)

A model for random aggregates is studied by computer simulation. The model is applicable to a metal-particle aggregation process whose correlations have been measured previously. Density correlations within the model aggregates fall off with distance with a fractional power law, like those of the metal aggregates. The radius of gyration of the model aggregates has power-law behavior. The model is a limit of a model of dendritic growth.

PACS numbers: 68.70.+w, 05.70.Jk, 64.60.

Gold black, Cab-o-Sil^{TM}, and coagulated aerosols[1] are aggregates of solid particles distinguished by their wispy appearance. The size of these aggregates far exceeds the range of forces holding them together[2]; it seems likely that the aggregation process can be understood without reference to the details of these forces. A class of aggregates were shown by Forrest and Witten[3] to have density correlations of a power-law form.

classic model of FRACTAL

Outline:

i) 1+1 KPZ/ASEP*

exps, amplitudes, LD

TW-GOE, TW-GUE, Baik-Rains F_o

ii) 2+1 KPZ Class

-Simple Height Distributions (HD)

-SLRD & EVS (local)

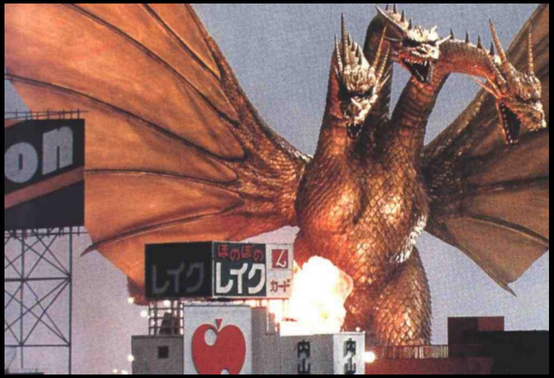
-Universal Limit Distribution

(2+1 analogs: TW & BR)

-Universal Spatial (Airy_1)

& Temporal Covariance

iii) Conclude: Open Problems***



*JSP 160(4) August 2015 special KPZ issue....

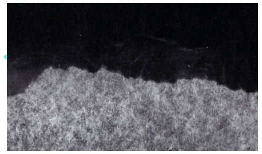
In the beginning...

van Beijeren, Kutner & Spohn- PRL 54, 2026 (1985)
Forster, Stephen & Nelson- PRA 16, 732 (1977)

KPZ PRL

≳ 2200 CITATIONS

9 PHYSICAL REVIEW LETTERS 3 MARCH 1986



Dynamic Scaling of Growing Interfaces

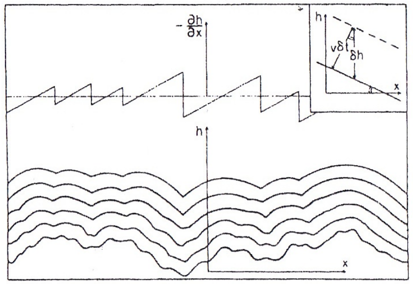
Mehran Kardar
Physics Department, Harvard University, Cambridge, Massachusetts 02138

Giorgio Parisi
Physics Department, University of Rome, I-00173 Rome, Italy

and

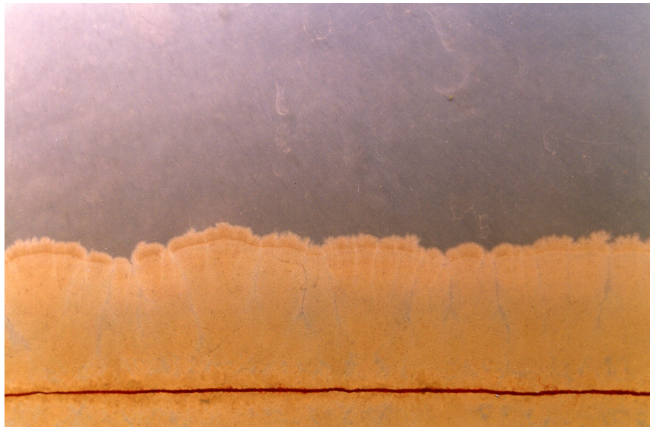
Yi-Cheng Zhang
hysics Department, Brookhaven National Laboratory, Upton, New York 11973

(Received 12 November 1985)



A model is proposed for the evolution of the profile of a growing interface. The deterministic growth is solved exactly, and exhibits nontrivial relaxation patterns. The stochastic version is studied by dynamic renormalization-group techniques and by mappings to Burgers's equation and to a random directed-polymer problem. The exact dynamic scaling form obtained for a one-dimensional interface is in excellent agreement with previous numerical simulations. Predictions are made for more dimensions.

$$\frac{\partial h}{\partial t} = \nu \underbrace{\nabla^2 h}_{\text{RELAXATION}} + \underbrace{\frac{1}{2}\lambda(\nabla h)^2}_{\text{NONLINEARITY}} + \underbrace{\eta}_{\text{STOCHASTIC NOISE}}$$



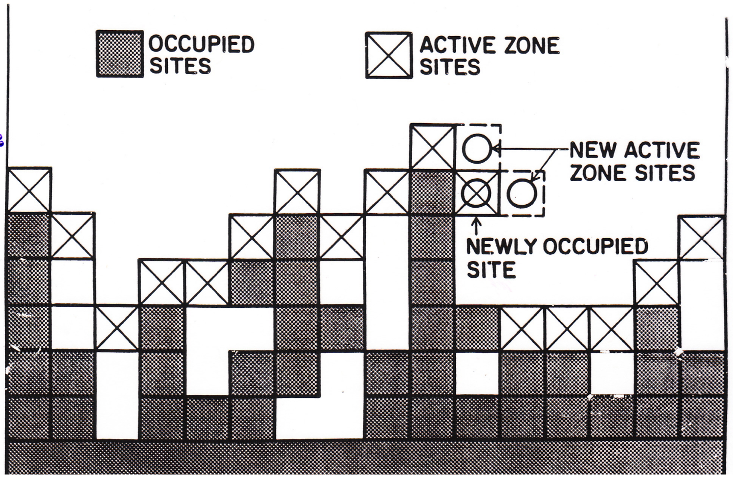
$$\langle \eta(x,t)\eta(x',t')\rangle = D\,\delta(x-x')\,\delta(t-t')$$

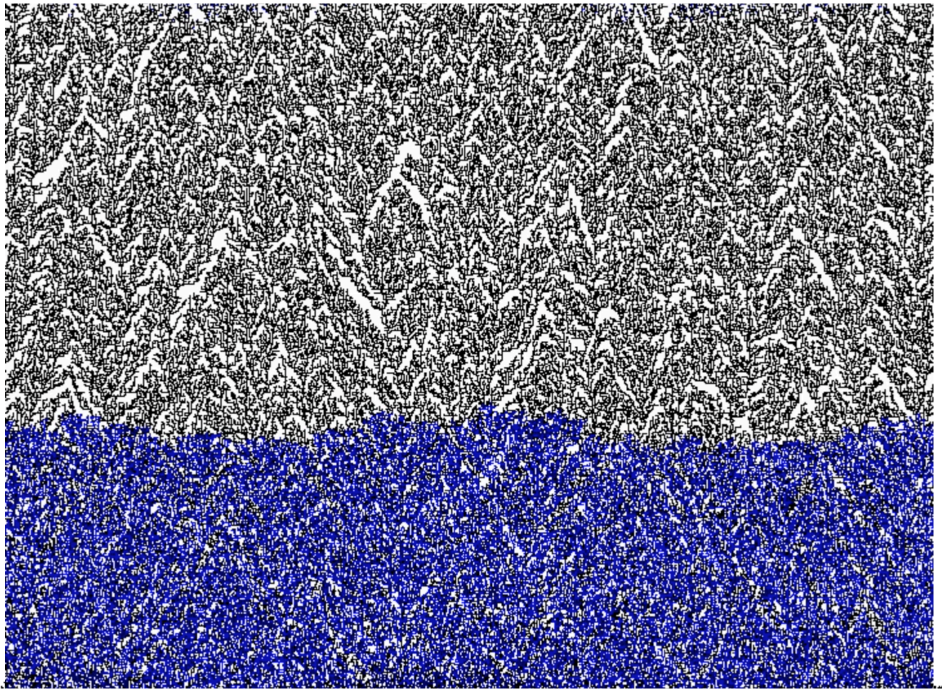
BALLISTIC DEPOSITION:

(THIN FILM GROWTH, MBE,) ~ NSF ##

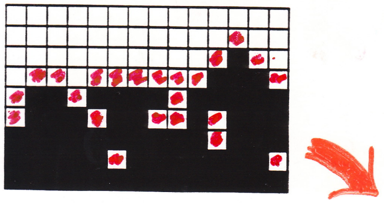
TETRIS...

STOCHASTIC GROWTH RULE = VERTICAL DROP + STICK UPON FIRST CONTACT





EDEN CLUSTER:

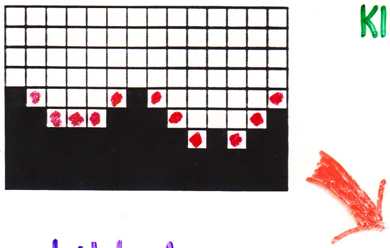


(BACTERIAL COLONY, FOREST FIRE PROPAGATION)

RULE?

ALL PERIMETER SITES EQUALLY LIKELY

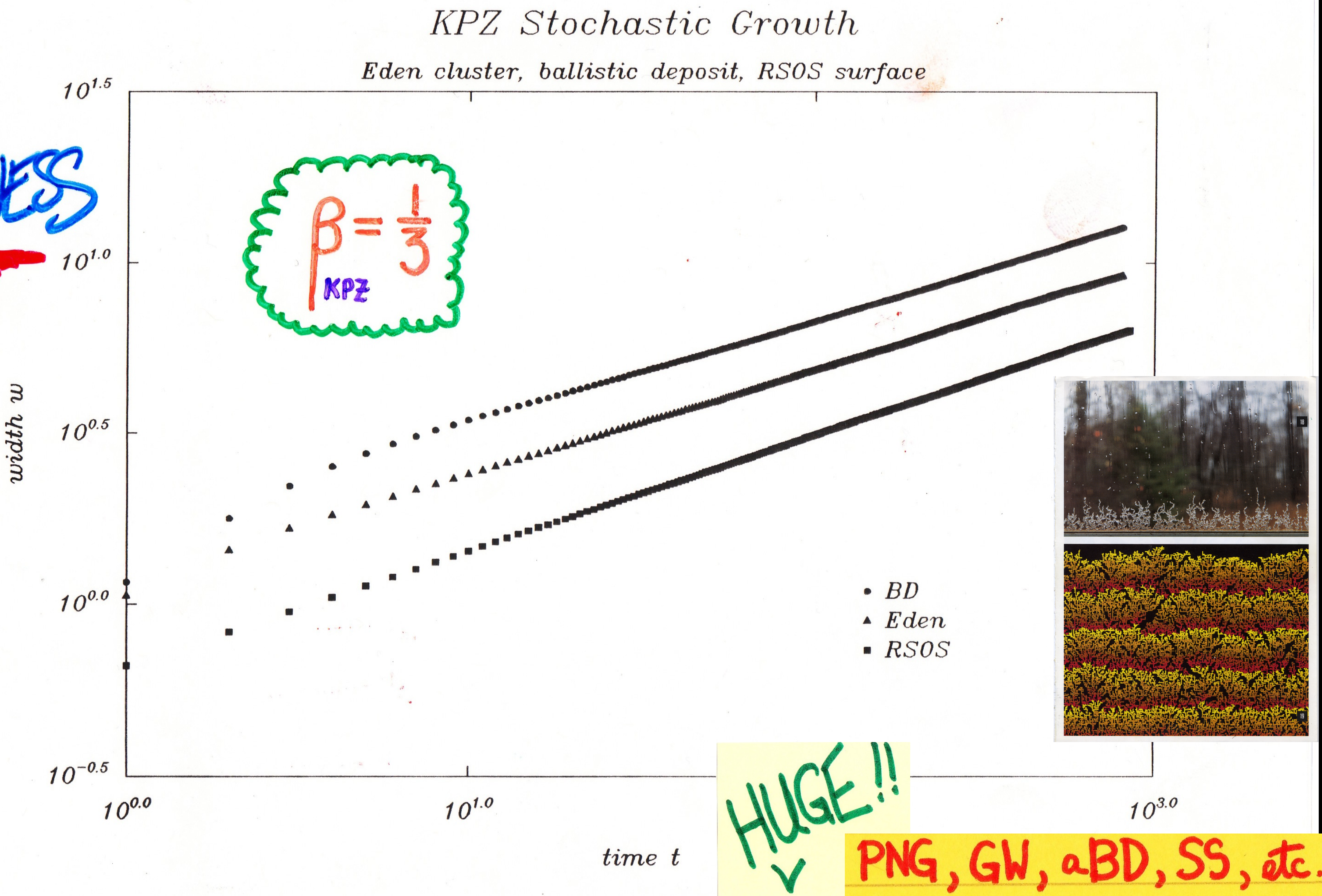
RSOS MODEL:



KIM + KOSTERLITZ PHYS. REV. LETT. (1989)

$|\Delta h| \leq 1$

EARLY TIME ROUGHNESS



$\beta_{KPZ} = \frac{1}{3}$

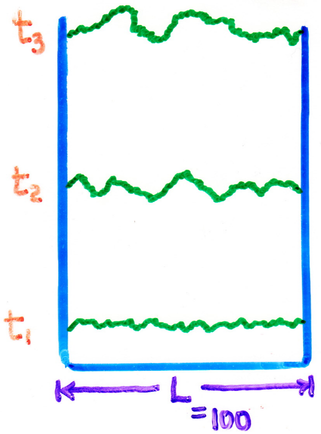
$w \sim t^{\beta}$

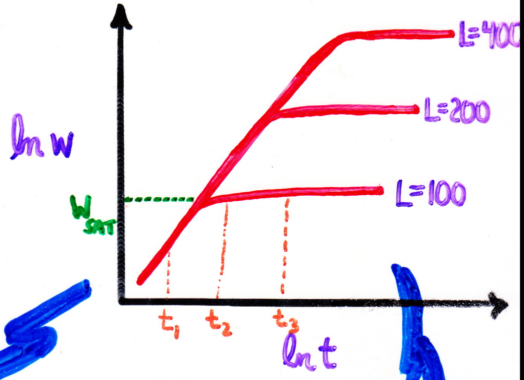
HUGE!!

PNG, GW, aBD, SS, etc...

→ A SINGLE UNIVERSALITY CLASS...

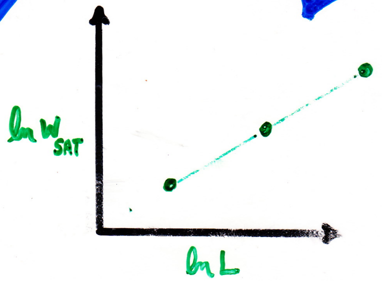
GENERIC BEHAVIOR:





"EARLY-TIME ROUGHNESS"

$W \sim t^{\beta}$

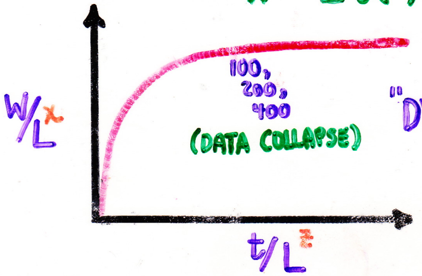


"SATURATION WIDTH"

$W_{SAT} \sim L^{\chi}$

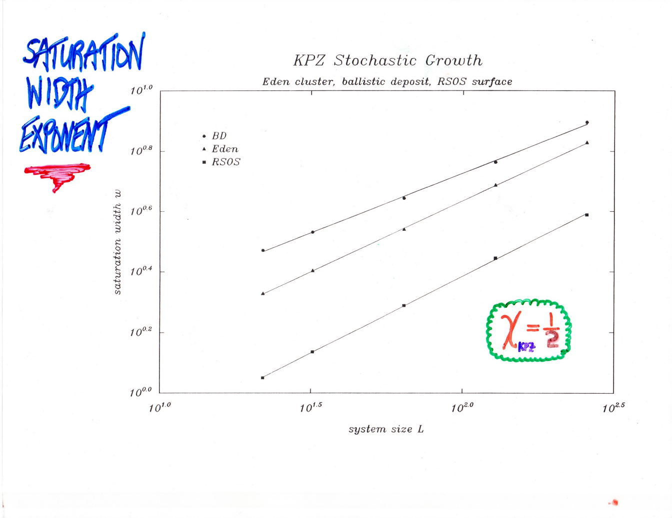
"FAMILY-VICSEK SCALING ANSATZ"

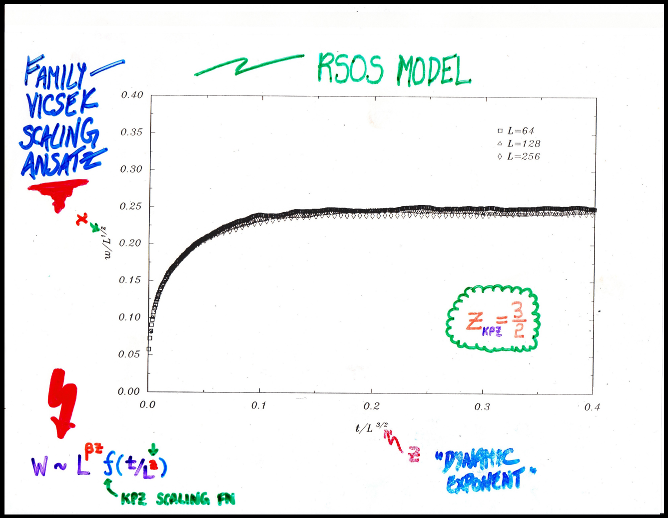
$W \sim L^{\chi} f(t/L^{z})$



"DYNAMIC EXPONENT"

$$\boxed{z = \frac{\chi}{\beta}}$$





¿ UNIVERSALITY ~ DYNAMIC SCALING NONEQUILIBRIUM KINETIC ROUGHENING ?

ANS: YES !!

KPZ EQUATION-

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(\vec{x},t)$$

SURFACE RELAXATION

NONLINEARITY

STOCHASTIC NOISE

(UNCORRELATED, GAUSSIAN)

⇙

$$\langle \eta(\vec{x}',t')\eta(\vec{x},t)\rangle = D\,\delta(\vec{x}'-\vec{x})\,\delta(t-t')$$

KARDAR, PARISI, & ZHANG
(PRL 1986)

DYNAMIC RG :

1-LOOP : MEDINA et al (1990)
2-LOOP : FREY & TAUBER (1994-6)
K. WIESE (DEC. 1997 Phys. Rev. E)

(PERTURBATIVE IN λ...)

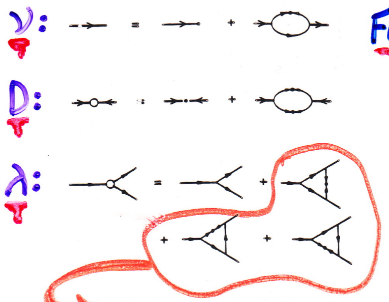
$$h(\vec{k},\omega) = h_0(\vec{k},\omega) - \frac{1}{2}\lambda G_0(\vec{k},\omega)\int_{\vec{q},\Omega} \vec{q}\cdot(\vec{k}-\vec{q})\,h(\vec{q},\Omega)\,h(\vec{k}-\vec{q},\omega-\Omega)$$

$G_0(\vec{k},\omega)\,\eta(\vec{k},\omega)$

$\dfrac{1}{-i\omega+\nu k^2}$

AND

$$\langle\eta\eta\rangle = 2D\,\delta(\vec{k}+\vec{q})\,\delta(\omega+\Omega)$$



CANCELLATION !!

⇓

SYMMETRY

$(h \to h + \vec{\epsilon}\cdot\vec{x},\ \vec{x} \to \vec{x} + \lambda\vec{\epsilon}t)$

⇓

$$\boxed{\chi + z = 2}$$

KPZ EXPONENT IDENTITY

FLOW EQNS -

$$\frac{d\nu}{d\ell} = \left[z - 2 + \frac{K_d\bar{\lambda}^2(2-d)}{4d}\right]\nu$$

$$\frac{dD}{d\ell} = \left[z - d - 2\chi + \frac{K_d\bar{\lambda}^2}{4}\right]D$$

$$\frac{d\lambda}{d\ell} = \left[\chi + z - 2\right]\lambda$$

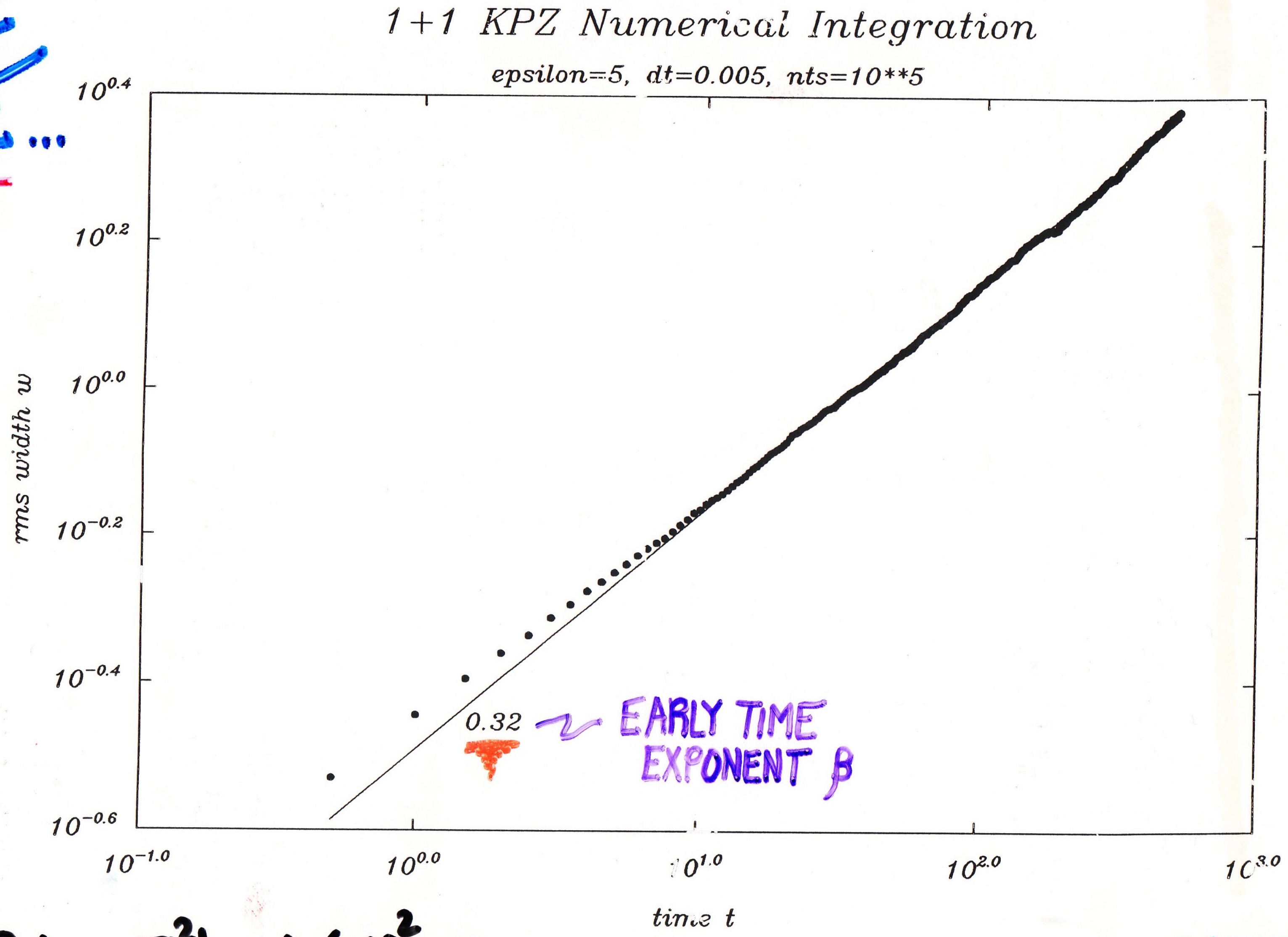
⇓

FIXED POINT
1+1 DIMENSIONS

$$\chi = \frac{1}{2},\ z = \frac{3}{2},\ \beta = \frac{1}{3}$$

FOKKER-PLANCK

BRUTE FORCE...



EARLY TIME EXPONENT β

KPZ: $\partial_t h = \nabla^2 h + \lambda (\nabla h)^2 + \eta$

RELAX

NONLINEARITY

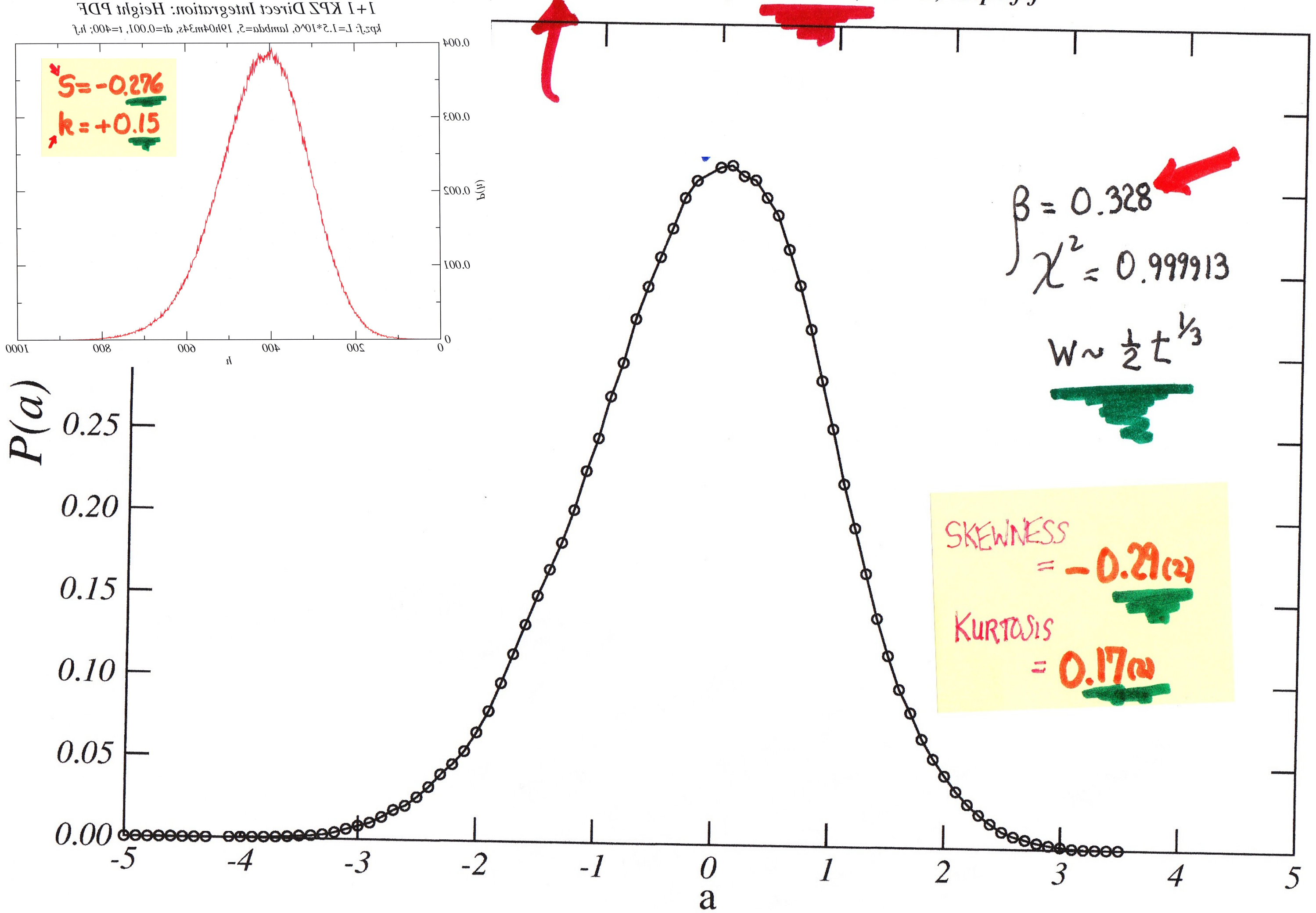
STOCHASTIC NOISE

BONNIE TAMMINGA & THH (1993)

METHOD #1

1+1 RSOS Kinetic Roughening

L=10^6; nr=1; nt=7500; 16m18s; -345; kkpdf.f



S = -0.276

k = +0.15

$\beta = 0.328$

$\chi^2 = 0.999913$

$W \sim \frac{1}{2} t^{1/3}$

SKEWNESS = -0.29(2)

KURTOSIS = 0.17(2)

KPZ → HEIGHT FLUCTUATION PDF

VOLUME 79, NUMBER 8 PHYSICAL REVIEW LETTERS 25 AUGUST 1997

FLAMELESS FIRE FRONTS

Kinetic Roughening in Slow Combustion of Paper

J. Maunuksela,[1] M. Myllys,[1] O.-P. Kähkönen,[1] J. Timonen,[1] N. Provatas,[2,3] M. J. Alava,[4,5] and T. Ala-Nissila[2,6,*]
[1]*Department of Physics, University of Jyväskylä, P.O. Box 35, FIN-40351 Jyväskylä, Finland*
[2]*Helsinki Institute of Physics, University of Helsinki, P.O. Box 9, FIN-00014 Helsinki, Finland*
[3]*Department of Physics and Mechanical Engineering, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois 61801-3080*
[4]*Laboratory of Physics, Helsinki University of Technology, P.O. Box 1000, FIN-02150 HUT, Espoo, Finland*
[5]*NORDITA, Blegdamsvej 17, DK-2100 Copenhagen, Denmark*
[6]*Department of Physics, Brown University, Providence, Rhode Island 02912*
(Received 18 March 1997)

PRL 83, 2054 (1999), $\chi = 0.46$

We present results from an experimental study on the kinetic roughening of slow combustion fronts in paper sheets. The sheets were positioned inside a combustion chamber and ignited from the top to minimize convection effects. The emerging fronts were videotaped and digitized to obtain their time-dependent heights. The data were analyzed by calculating two-point correlation functions in the saturated regime. Both the growth and roughening exponents were determined and found consistent with the Kardar-Parisi-Zhang equation, in agreement with recent theoretical work. [S0031-9007(97)03836-2]

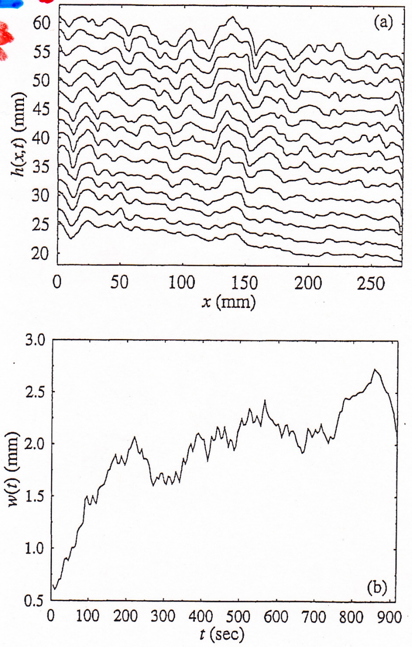


FIG. 2. (a) A series of successive digitized flame fronts taken every 5 s following the ignition of copier paper. (b) Evolution of the time-dependent surface width $w(t)$.

also, PHYS. REV. E 64, 036101 (2001)
issue of scaling & noise
PRL 84, 1946 (2000)

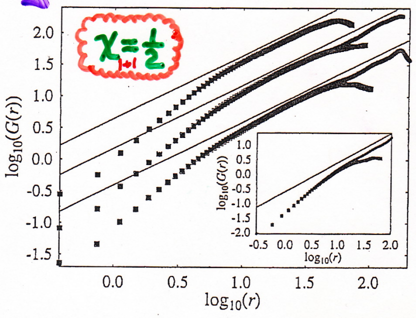


FIG. 3. The spatial correlation function $G(r)$ for three different burns of the copier paper (data have been shifted for clarity and the units are in mm). Filled circles denote the case where the average global tilt of the interface has been subtracted out. The solid lines denote $2\chi = 1$. Inset shows corresponding data for the cigarette paper.

KR - PENETRATING FLUX FRONTS
HIGH T_c THIN FILM SUPERCONDUCTORS
R. WIJNGAARDEN - AMSTERDAM

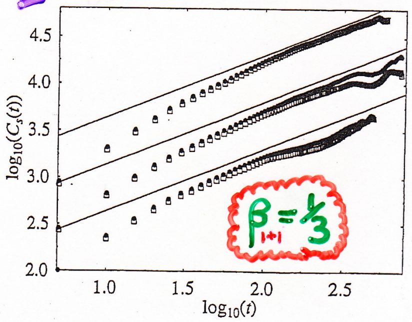
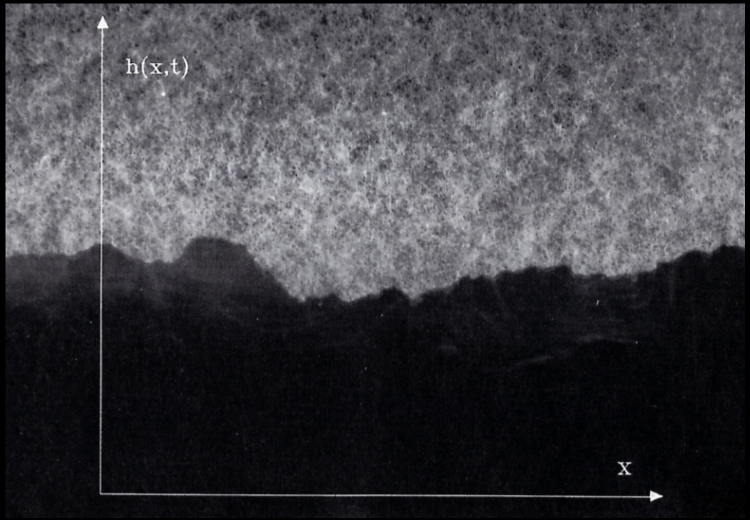


FIG. 4. Time-dependent correlation functions $C_s(t)$ for the data used in Fig. 3. The solid lines denote $2\beta = 2/3$.

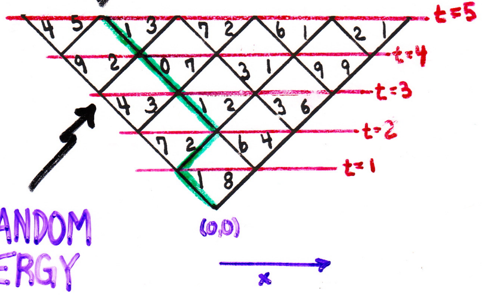


JUN ZHANG, et al.
Physica A 189, 383 (1992)
"MODELING FOREST-FIRE BY PAPER BURNING EXPT"

DIRECTED POLYMERS IN RANDOM MEDIA

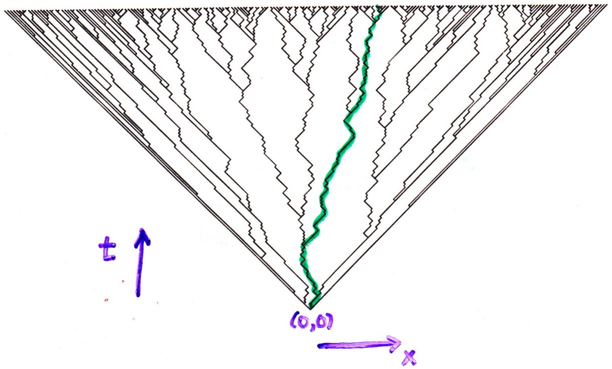
HUSE & HENLEY, PRL (1985)
KARDAR, PRL (1985)
KARDAR + ZHANG, PRL (1987)

GLOBALLY OPTIMAL PATH (5 STEPS)



"RANDOM ENERGY LANDSCAPE"

ENSEMBLE OF LOCALLY OPTIMAL PATHS:



POSITION, ENERGY ~ FLUCTUATING STATISTICAL VARIABLES!

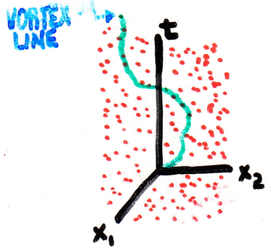
DPRM ~ CONTINUUM FORMULATION:

$$\underbrace{\mathcal{Z}(\vec{x},t)}_{\text{DPRM}} = \int_{\text{ALL PATHS}} \mathcal{D}\vec{x}\; e^{-\int dt\left\{\frac{1}{2}\left(\frac{d\vec{x}}{dt}\right)^2 + \mathcal{V}_p(\vec{x},t)\right\}}$$

ELASTICITY

RANDOM PINNING POTENTIAL
(UNCORRELATED, GAUSSIAN

$$\langle \mathcal{V}_p(\vec{x}',t')\mathcal{V}_p(\vec{x},t)\rangle = \delta(\vec{x}'+\vec{x})\,\delta(t'-t)$$



NOTE -

① ELASTIC ENERGY $\sim \frac{x^2}{t} \Rightarrow \boxed{\omega = 2\zeta - 1}$

② FEYNMAN PATH INTEGRAL ⟶ SCHRODINGER EQN

$$\partial_t \mathcal{Z} = \left[\tfrac{1}{2}\nabla^2 + \mathcal{V}_p\right]\mathcal{Z}$$

$$\Big\downarrow \quad \mathcal{Z} = e^{h}$$

$$\partial_t h = \tfrac{1}{2}\nabla^2 h + \tfrac{1}{2}(\nabla h)^2 + \mathcal{V}_p$$

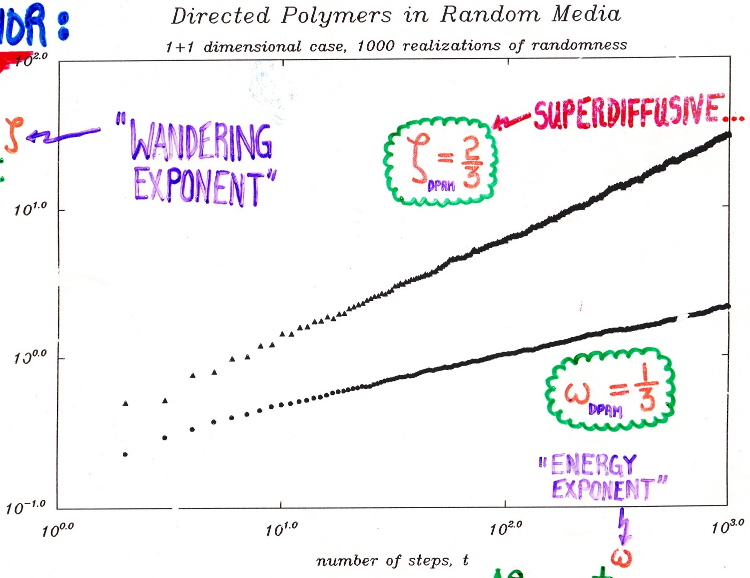
FREE ENERGY ~ KPZ!

d=1+1

$$\omega_{DPRM} = \beta_{KPZ} = \frac{1}{3}$$

$$\zeta_{DPRM} = z^{-1}_{KPZ} = \frac{2}{3}$$

DPRM SCALING BEHAVIOR:



$X_{RMS} \sim t^{\zeta}$ ← "WANDERING EXPONENT"

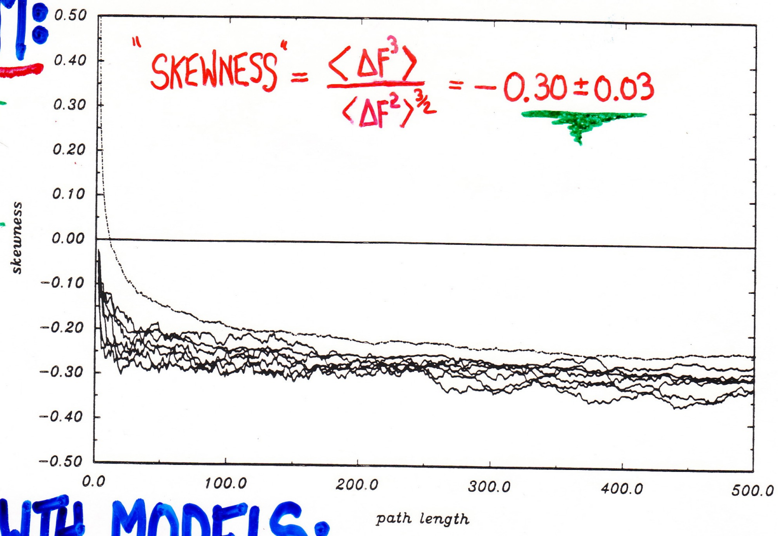
$\zeta_{DPRM} = \frac{2}{3}$ ← SUPERDIFFUSIVE...

$\omega_{DPRM} = \frac{1}{3}$

"ENERGY EXPONENT"

$\Delta e_{RMS} \sim t^{\omega}$

DPRM:



DIFFERENT TEMPS, DISORDER STRENGTH...

$$\text{"SKEWNESS"} = \frac{\langle \Delta F^3 \rangle}{\langle \Delta F^2 \rangle^{3/2}} = -0.30 \pm 0.03$$

GROWTH MODELS:

→ -0.28 ± 0.04

"..... UNIVERSALITY FOR DRIVEN INTERFACES + DPRM"

STATISTICS OF TORN PAPER EDGES:

KERTÉSZ, HORVATH, + WEBER – Les Houches '92

KINETIC ROUGHENING + DIRECTED POLYMERS..

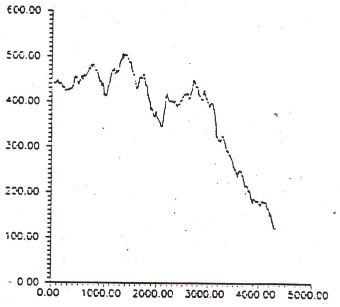


Figure 1. A digitized tear line. Note the different scales on the axes

- FRACTAL TEAR LINES
- EQUILATERALLY APPLIED TENSILE STRESS
- SHEETS 30 × 45 cm
- 5 DIFFERENT KINDS (INSTITUTE FOR PAPER RESEARCH, HUNGARY)
- STRENGTH, DENSITY, THICKNESS (FACTORS OF 2-3)

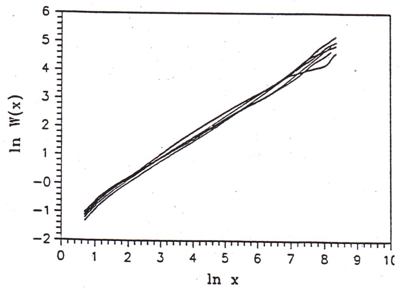


Figure 2. Scaling plots of tear lines in paper

$\zeta_{PAPER} = 0.67 \pm 0.05$

"ROUGHNESS OF CRACK INTERFACES" ~ $\zeta_{\text{BRITTLE FRACTURE}} \approx 0.7$

HANSEN, HINRICHSEN + ROUX
PHYS. REV. LETT.
(1991)

KPZ UNIVERSALITY: DPRM ~ RSOS ?

$$P(a) \approx \begin{cases} e^{-C_-|a|^{\eta_-}} & \text{LEFT TAIL} \\ e^{-C_+|a|^{\eta_+}} & \text{RIGHT TAIL} \end{cases}$$

NUMERICS

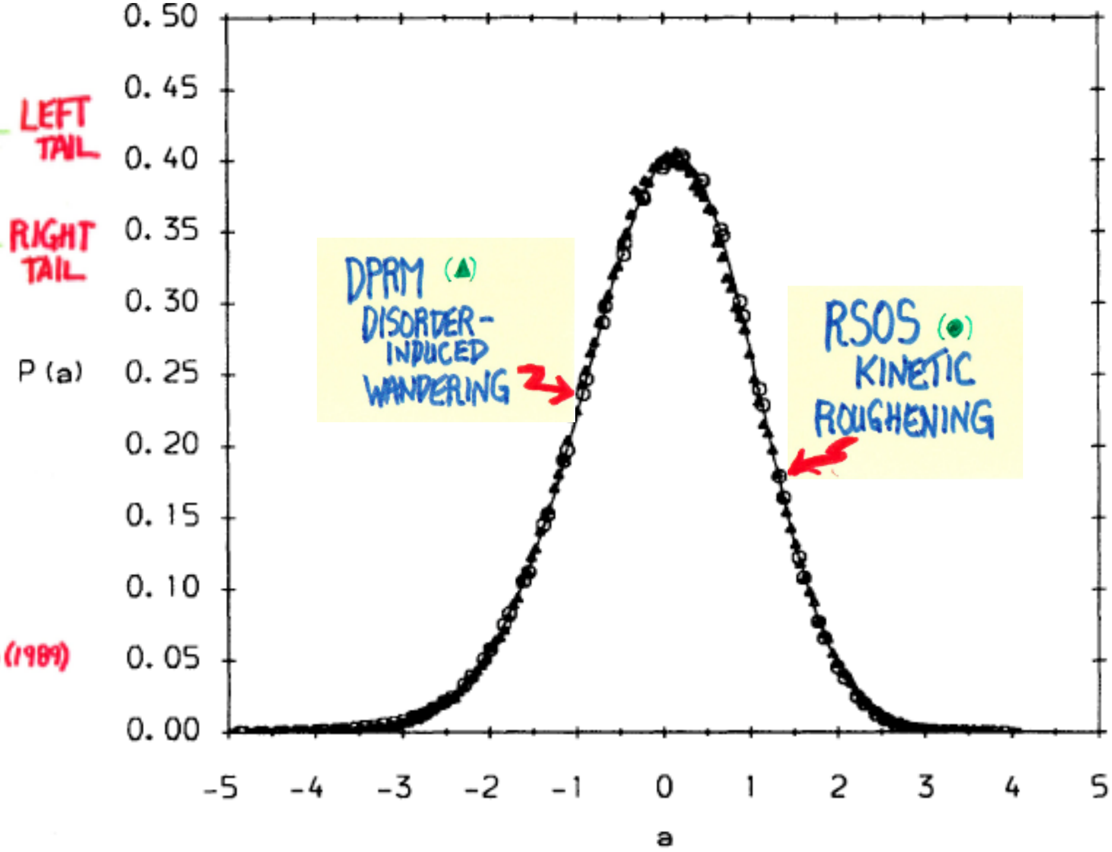
⇒ $\eta_- = 1.6 \pm 0.2$

$\eta_+ = 2.4 \pm 0.2$

THEORY ~ ZHANG, EPL 9, 113 (1989)

⇒ $\boxed{\eta_-^{(1+1)} = \frac{1}{1-\omega} = \frac{3}{2}}$ ✓

later, $\eta_+ = 3$ ✓ DOTSENKO, et al. PRL (2008)



IG. 10. The probability distribution $P(a,t)$ of height h as a function of $a = (h - \langle h \rangle)/\Delta h$ in a restricted-solid-on-solid growth model for $d=1+1$ ($t=300$, $L=40\,000$ and 20 runs; circle) and the probability distribution $P_1(a,t)$ of E_1 as a function of $a = (E_1 - \langle E_1 \rangle)/\Delta E_1$ in our model ($t=200$ and 400 000 runs; triangle). The continuous line is for the $P(a)$ of Fig. 7 for $d=1+1$.

Kim, Bray & Moore
Phys. Rev. A44, 2345 (1991)

KPZ UNIVERSALITY → RANDOM MATRICES & TRACY-WIDOM DIST.

OLUME 84, NUMBER 21 PHYSICAL REVIEW LETTERS 22 MAY 2000

Universal Distributions for Growth Processes in 1 + 1 Dimensions and Random Matrices

Michael Prähofer* and Herbert Spohn†
Zentrum Mathematik and Physik Department, TU München, D-80290 München, Germany

(Received 14 December 1999)

We develop a scaling theory for Kardar-Parisi-Zhang growth in one dimension by a detailed study of the polynuclear growth model. In particular, we identify three universal distributions for shape fluctuations and their dependence on the macroscopic shape. These distribution functions are computed using the partition function of Gaussian random matrices in a cosine potential.

PNG MODEL:

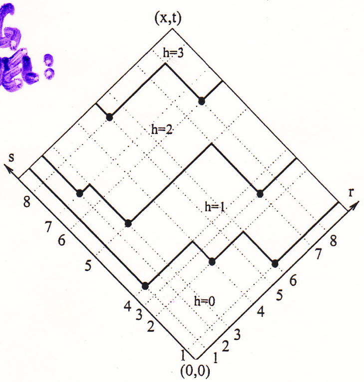


FIG. 1. The height h of a PNG droplet with nucleation events corresponding to the permutation $(4, 7, 5, 2, 8, 1, 3, 6)$.

2000 PS PRL:
PNG-Ulam Problem
LIS, Random Permutation
&…..
SS/DPRM/TASEP
Kurt Johansson

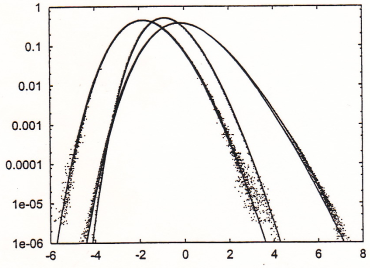


FIG. 2. From left to right: the probability densities of the universal distributions χ_2, χ_1, and χ_0 for curved, flat, and stationary self-similar growth, respectively.

RMT Tracy-Widom
Distributions, 1994:
TW-GUE (radial geometry),
TW-GOE (flat IC), …

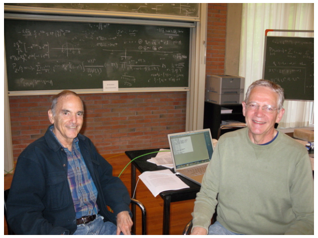


TABLE I. Mean, variance, skewness, and kurtosis for the distributions of χ_2, χ_1, and χ_0 as determined by numerically solving Painlevé II [19]. $\langle\chi^n\rangle_c$ denotes the nth cumulant.

	Curved (χ_2)	Flat (χ_1)	Stationary (χ_0)
$\langle\chi\rangle$	−1.771 09	−0.760 07	0
$\langle\chi^2\rangle_c$	0.813 20	0.638 05	1.150 39
$\langle\chi^3\rangle_c/\langle\chi^2\rangle_c^{3/2}$	0.2241	0.2935 ←	0.359 41
$\langle\chi^4\rangle_c/\langle\chi^2\rangle_c^2$	0.093 45	0.1652	0.289 16

scaled cumulants;
skewness s & kurtosis k

Experimental determination of KPZ height-fluctuation distributions

L. Miettinen[a], M. Myllys, J. Merikoski, and J. Timonen

Department of Physics, University of Jyväskylä, P.O. Box 35 (YFL), 40014 Jyväskylä, Finland

Received 16 December 2004 / Received in final form 30 March 2005
Published online 8 August 2005 – © EDP Sciences, Società Italiana di Fisica, Springer-Verlag 2005

Abstract. Height-fluctuation distributions of nonequilibrium interfaces were analyzed using slow-combustion fronts propagating in sheets of paper. All distributions measured were definitely non-Gaussian. The experimental distributions for transient and stationary regimes were well fitted by the theoretical distributions proposed by Prähofer and Spohn in reference [9]. Consistent with the Galilean invariance of the system, the same distributions were found for horizontal fronts and, when determined along the normal to the slope, for fronts with a non-zero average slope.

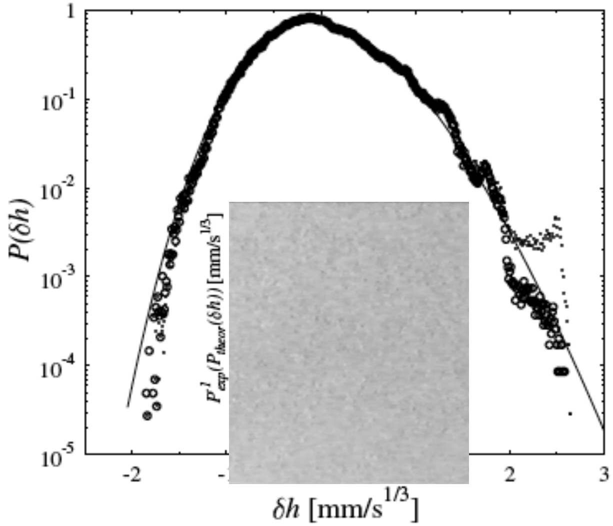
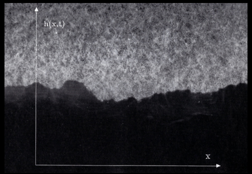
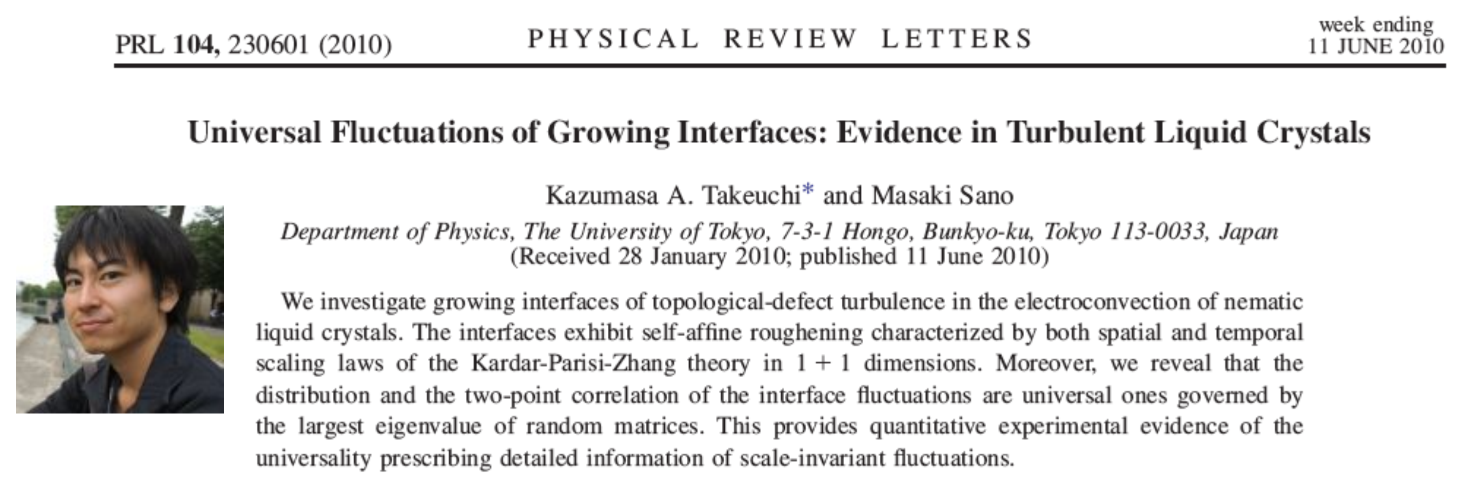


Fig. 3. Height-fluctuation distribution for horizontal fronts in the transient ($w \sim t^{1/3}$) regime, and a fit by a (scaled and shifted) theoretical distribution f_1. A theoretical inversion of the measured distribution is shown in the inset. The dots denote the measured data and the circles the data with an avalanche suppressed.

more Finnish flame front expts…
(2005)





PRL 104, 230601 (2010) PHYSICAL REVIEW LETTERS week ending 11 JUNE 2010

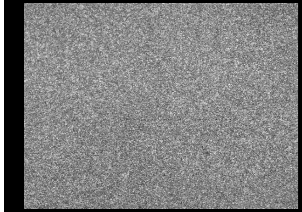
Universal Fluctuations of Growing Interfaces: Evidence in Turbulent Liquid Crystals

Kazumasa A. Takeuchi* and Masaki Sano

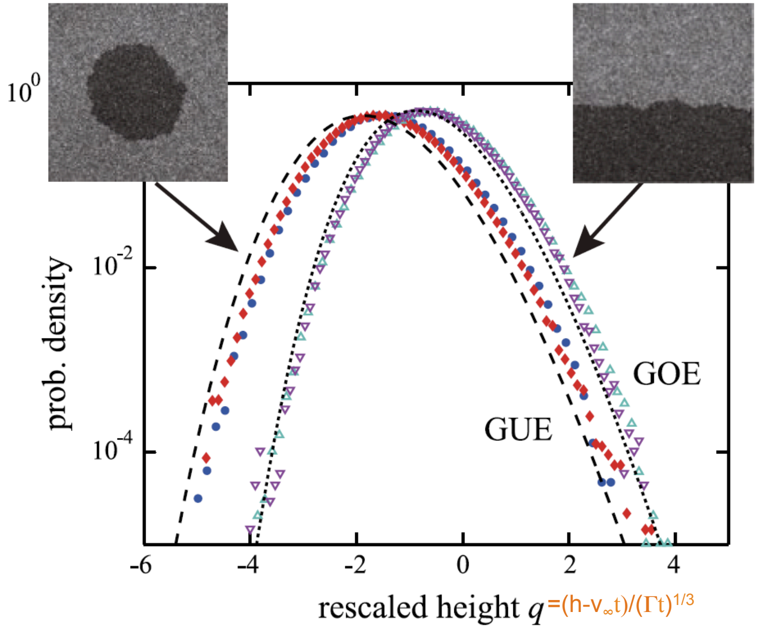
Department of Physics, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan

(Received 28 January 2010; published 11 June 2010)

We investigate growing interfaces of topological-defect turbulence in the electroconvection of nematic liquid crystals. The interfaces exhibit self-affine roughening characterized by both spatial and temporal scaling laws of the Kardar-Parisi-Zhang theory in 1 + 1 dimensions. Moreover, we reveal that the distribution and the two-point correlation of the interface fluctuations are universal ones governed by the largest eigenvalue of random matrices. This provides quantitative experimental evidence of the universality prescribing detailed information of scale-invariant fluctuations.



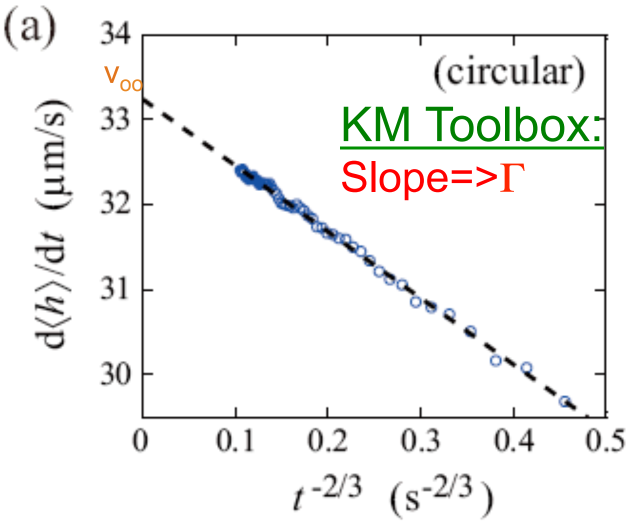
Random Matrix Theory: Tracy-Widom Limit Distributions



=(h-v∞t)/(Γt)$^{1/3}$

Fig. 8 Histogram of the rescaled local height $q \equiv (h - v_\infty t)/(\Gamma t)^{1/3}$ for the circular (*solid symbols*) and flat (*open symbols*) interfaces. The *blue circles* and *red diamonds* display the histograms for the circular interfaces at $t = 10$ s and 30 s, respectively, while the *turquoise up-triangles* and *purple down-triangles* are for the flat interfaces at $t = 20$ s and 60 s, respectively. The *dashed* and *dotted curves* show the GUE and GOE TW distributions, respectively, defined by the random variables χ_{GUE} and χ_{GOE}. (Color figure online)

Time-Dependent Growth Velocity:



v_∞

KM Toolbox:

Slope=>Γ

PRL **104,** 230602 (2010) PHYSICAL REVIEW LETTERS week ending 11 JUNE 2010

One-Dimensional Kardar-Parisi-Zhang Equation: An Exact Solution and its Universality

Tomohiro Sasamoto[*]
Department of Mathematics and Informatics, Chiba University, 1-33 Yayoi-cho, Inage, Chiba 263-8522, Japan

Herbert Spohn[†]
Zentrum Mathematik and Physik Department, TU München, D-85747 Garching, Germany

(Received 15 February 2010; revised manuscript received 10 May 2010; published 11 June 2010)

We report on the first exact solution of the Kardar-Parisi-Zhang (KPZ) equation in one dimension, with an initial condition which physically corresponds to the motion of a macroscopically curved height profile. The solution provides a determinantal formula for the probability distribution function of the height $h(x, t)$ for all $t > 0$. In particular, we show that for large t, on the scale $t^{1/3}$, the statistics is given by the Tracy-Widom distribution, known already from the Gaussian unitary ensemble of random matrix theory. Our solution confirms that the KPZ equation describes the interface motion in the regime of weak driving force. Within this regime the KPZ equation details how the long time asymptotics is approached.

RELATED PAPERS

SS model, TASEP, WASEP...

- SASAMOTO & SPOHN Nucl. Phys. B834, 523 (2010)
- SASAMOTO & SPOHN J. Stat. Phys. 137, 917 (2009) & 140, 209 (2010)
- TRACY & WIDOM Comm. Math. Phys. 290, 129 (2009)
- AMIR, CORWIN & QUASTEL - arXiv 1003.0443

$$(\lambda/2\nu)h(x,t) = -x^2/4\nu t - \tfrac{1}{12}\gamma_t^3 + 2\log\alpha + \gamma_t\xi_t,$$

where

$$\gamma_t = (\alpha^4\nu t)^{1/3}, \qquad \alpha = (2\nu)^{-3/2}\lambda D^{1/2}.$$

Full temporal evolution KPZ height fluctuation PDF:

$$\rho_t(s) = \int_{-\infty}^{\infty} du\,\gamma_t e^{\gamma_t(s-u)} \exp[-e^{\gamma_t(s-u)}] \times (\det[1 - P_u(B_t - P_{\mathrm{Ai}})P_u] - \det(1 - P_u B_t P_u)).$$

$$B_t(x,y) = \int_{-\infty}^{\infty} dw(1 - e^{-\gamma_t w})^{-1}\mathrm{Ai}(x+w)\mathrm{Ai}(y+w).$$

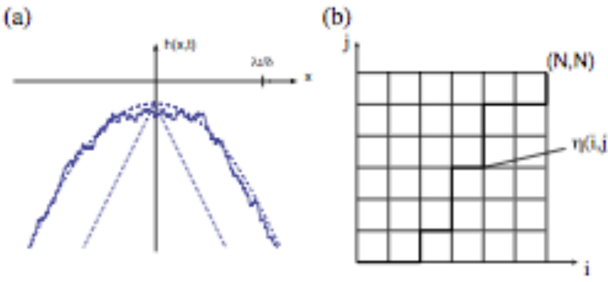
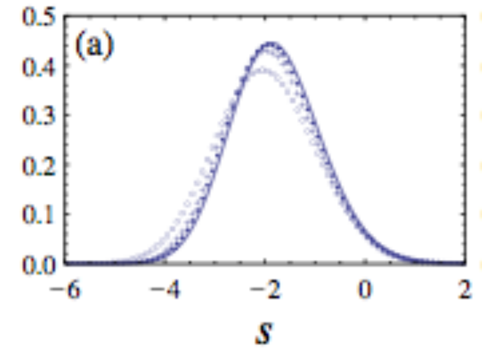


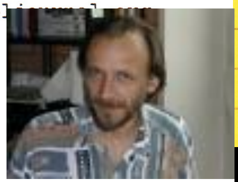
FIG. 1 (color online). (a) A typical realization of the droplet height function. (b) A directed polymer configuration.

Gaussian @early times, but…. asymptotically, Tracy-Widom



THE FRONTIERS OF PHYSICS
April 2010

EPL, **90** (2010) 20003
doi: 10.1209/0295-5075/90/20003
www.epl



also ↗
- DOTSENKO, arXiv 1004.4455
- CALABRESE, et al.
Eur. Phys. Lett 90, 2002
(2010)

Bethe ansatz derivation of the Tracy-Widom distribution for one-dimensional directed polymers

V. DOTSENKO[a]

LPTMC, Université Paris VI - 75252 Paris, France, EU and
L.D. Landau Institute for Theoretical Physics - 119334 Moscow, Russia

$$Z(N,L) = \prod_{a=1}^{N} \int_{\phi_a(0)=0}^{\phi_a(L)=0} \mathcal{D}\phi_a(\tau)\, \overline{\exp\left[-\beta \int_0^L d\tau \sum_{a=1}^{N} \{\frac{1}{2}[\partial_\tau \phi_a(\tau)]^2 + V[\phi_a(\tau),\tau]\}\right]}$$

$$Z(N,L) = \prod_{a=1}^{N} \int_{\phi_a(0)=0}^{\phi_a(L)=0} \mathcal{D}\phi_a(\tau)\, \exp\left[-\frac{1}{2}\beta \int_0^L d\tau \Big\{ \sum_{a=1}^{N} [\partial_\tau \phi_a(\tau)]^2 - \beta u \sum_{a,b=1}^{N} \delta[\phi_a(\tau) - \phi_b(\tau)] \Big\}\right]$$

Lieb-
Liniger
Model
(1d bosons)

$$Z(N,L) = \sum_{M=1}^{N} \frac{1}{M!} \left[\prod_{\alpha=1}^{M} \int_{-\infty}^{+\infty} \frac{dq_\alpha}{2\pi} \sum_{n_\alpha=1}^{\infty}\right] \delta\Big(\sum_{\alpha=1}^{M} n_\alpha,\ N\Big)\, |\Psi^{(M)}_{\mathbf{q},\mathbf{n}}(0)|^2\, \mathrm{e}^{-E_M(\mathbf{q},\mathbf{n})L}$$

$$E_M(\mathbf{q},\mathbf{n}) = \frac{1}{2\beta} \sum_{\alpha=1}^{M} n_\alpha q_\alpha^2 - \frac{\kappa^2}{24\beta} \sum_{\alpha=1}^{M} (n_\alpha^3 - n_\alpha)$$

$$\tilde{Z}(N.L) = N! \Big\{ \int_{-\infty}^{+\infty} \frac{dq}{2\pi\kappa N} \exp\Big[-\underline{\frac{L}{2\beta} N q^2 + \frac{\kappa^2 L}{24\beta} N^3}\Big] + \underset{GS}{}$$

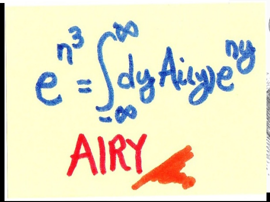
$$+ \sum_{M=2}^{N} \frac{1}{M!} \left[\prod_{\alpha=1}^{M} \sum_{n_\alpha=1}^{\infty} \int_{-\infty}^{+\infty} \frac{dq_\alpha}{2\pi\kappa n_\alpha}\right] \delta\Big(\sum_{\alpha=1}^{M} n_\alpha,\ N\Big) \prod_{\alpha<\beta} \frac{|q_\alpha - q_\beta - \frac{i\kappa}{2}(n_\alpha - n_\beta)|^2}{|q_\alpha - q_\beta - \frac{i\kappa}{2}(n_\alpha + n_\beta)|^2} \times$$

$$\times \exp\Big[-\frac{L}{2\beta} \sum_{\alpha=1}^{M} n_\alpha q_\alpha^2 + \frac{\kappa^2 L}{24\beta} \sum_{\alpha=1}^{M} n_\alpha^3\Big]\Big\}$$

GS

&
Excited
States!

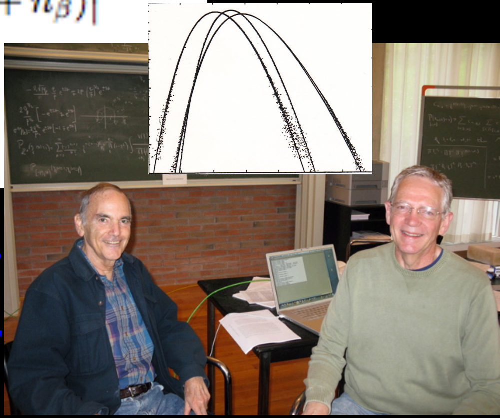
Mathematical sleight of hand:
Airy, Cauchy, Fredholm…



$e^{n^3} = \int_{-\infty}^{\infty} dy\, Ai(y)\, e^{ny}$

AIRY

Tour de
force;
crunch….
F=lnZ
TW PDF



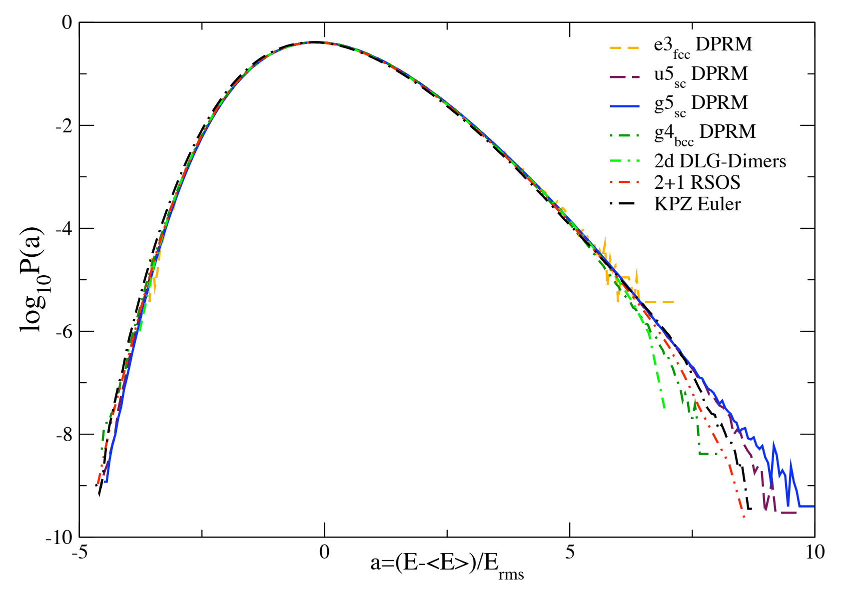
2+1 KPZ

Universality Class....

1^{st} Pass: Simple, Asymmetric Height PDFs...

(unit variance, zero mean)

skewness s=0.424
kurtosis k=0.346

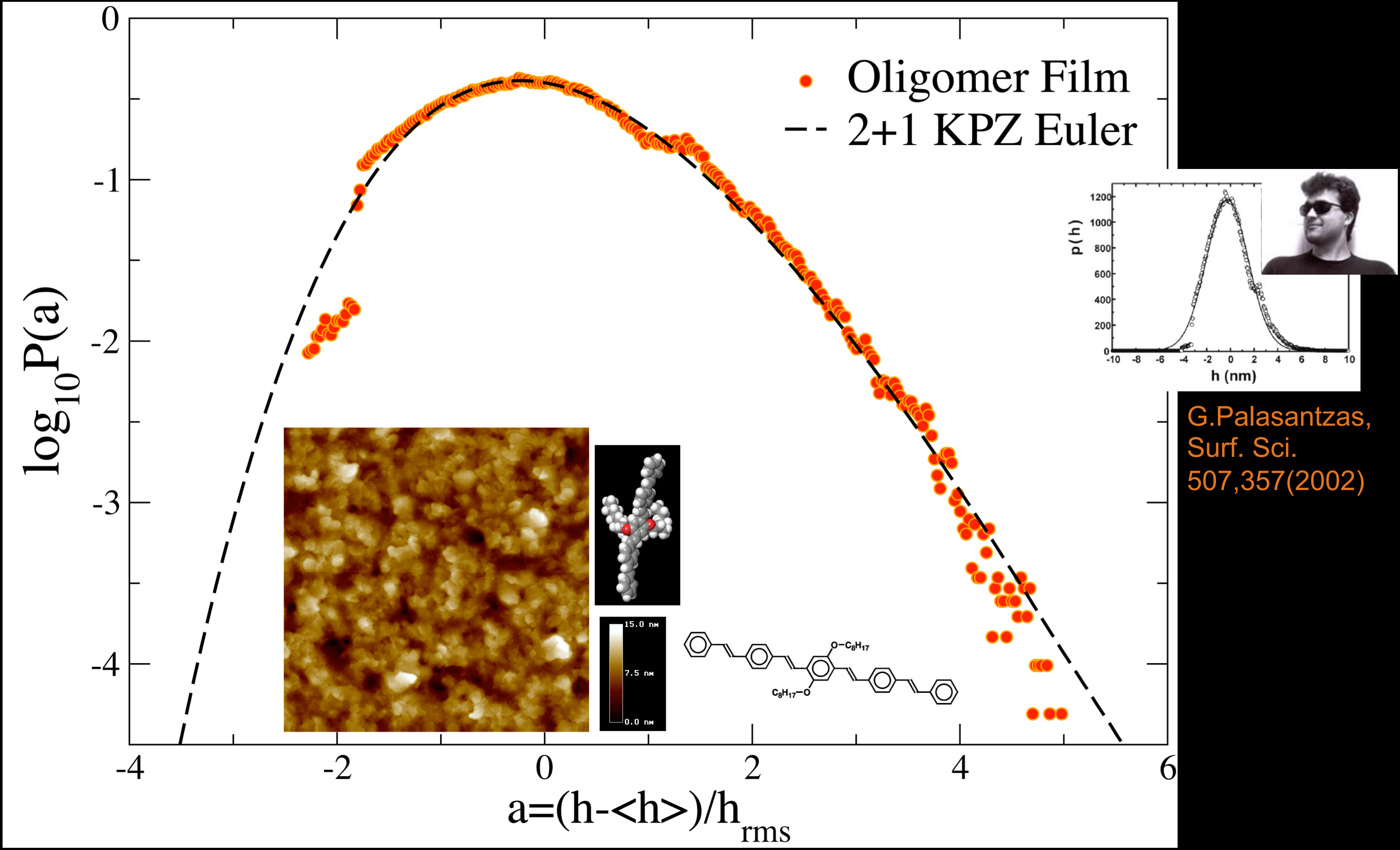


THH- PRL**109**,170602(2012)

1+1 KPZ TW-GOE:
s=0.2935
k=0.1652

2+1 KPZ CLASS HD:
Thin Film Expt-

*Almeida-PRB89,045309(2014)
THH/GP-EPL105,50001(2014)



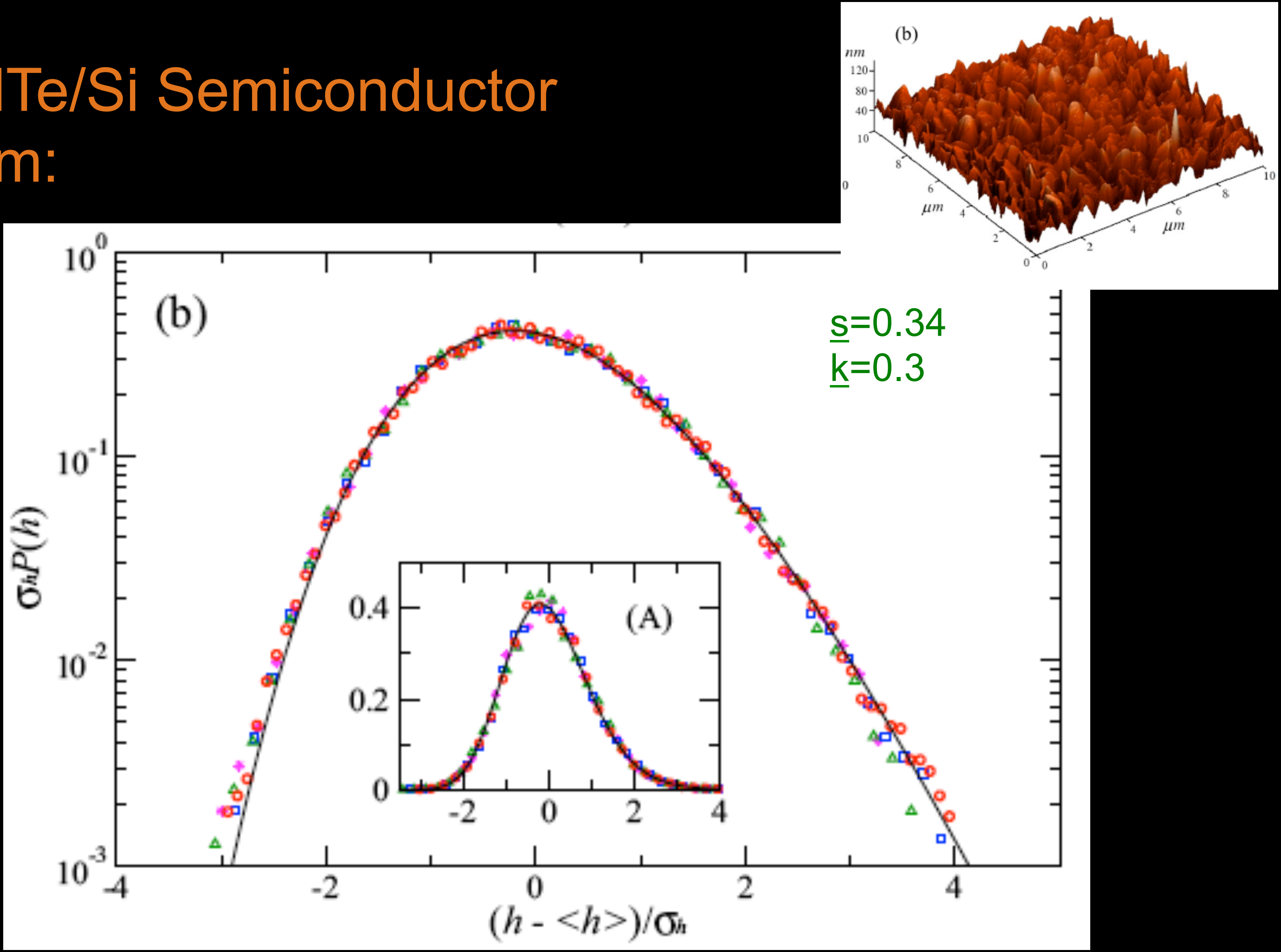
G.Palasantzas,
Surf. Sci.
507,357(2002)

2+1 KPZ CLASS HD:

Thin Film Expt

*Almeida-PRB89,045309(2014)

CdTe/Si Semiconductor Film:



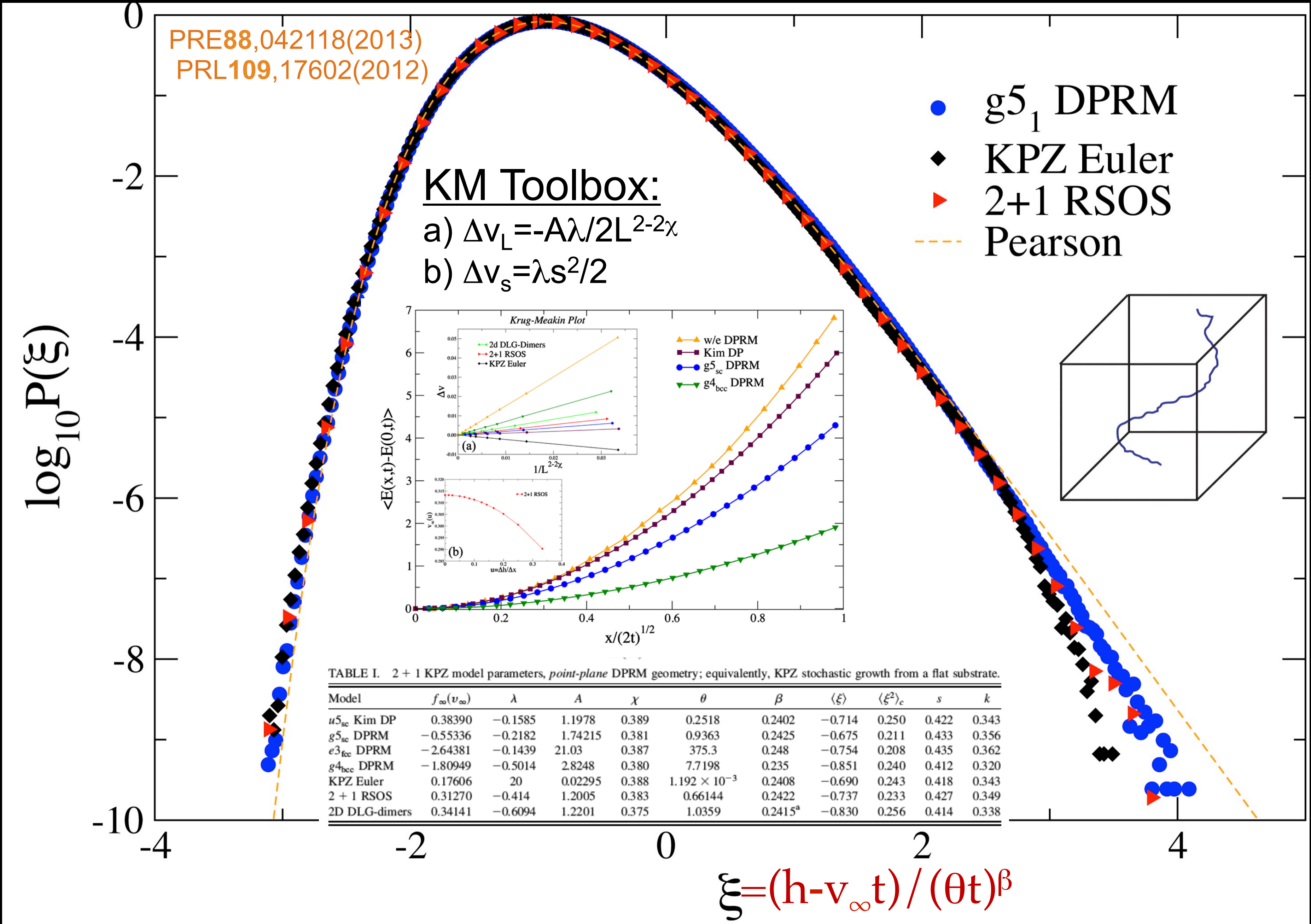
<u>s</u>=0.34
<u>k</u>=0.3

2+1 KPZ

Universal Limit Distributions*

A set piece....

2+1 KPZ CLASS: Limit Distribution*



PRE**88**,042118(2013)
PRL**109**,17602(2012)

KM Toolbox:

a) $\Delta v_L=-A\lambda/2L^{2-2\chi}$

b) $\Delta v_s=\lambda s^2/2$

TABLE I. 2 + 1 KPZ model parameters, *point-plane* DPRM geometry; equivalently, KPZ stochastic growth from a flat substrate.

Model	$f_\infty(v_\infty)$	λ	A	χ	θ	β	$\langle\xi\rangle$	$\langle\xi^2\rangle_c$	s	k
$u5_{sc}$ Kim DP	0.38390	−0.1585	1.1978	0.389	0.2518	0.2402	−0.714	0.250	0.422	0.343
$g5_{sc}$ DPRM	−0.55336	−0.2182	1.74215	0.381	0.9363	0.2425	−0.675	0.211	0.433	0.356
$e3_{fcc}$ DPRM	−2.64381	−0.1439	21.03	0.387	375.3	0.248	−0.754	0.208	0.435	0.362
$g4_{bcc}$ DPRM	−1.80949	−0.5014	2.8248	0.380	7.7198	0.235	−0.851	0.240	0.412	0.320
KPZ Euler	0.17606	20	0.02295	0.388	1.192×10^{-3}	0.2408	−0.690	0.243	0.418	0.343
2 + 1 RSOS	0.31270	−0.414	1.2005	0.383	0.66144	0.2422	−0.737	0.233	0.427	0.349
2D DLG-dimers	0.34141	−0.6094	1.2201	0.375	1.0359	0.2415^a	−0.830	0.256	0.414	0.338

3d Radial/pt-pt KPZ Limit Distribution:

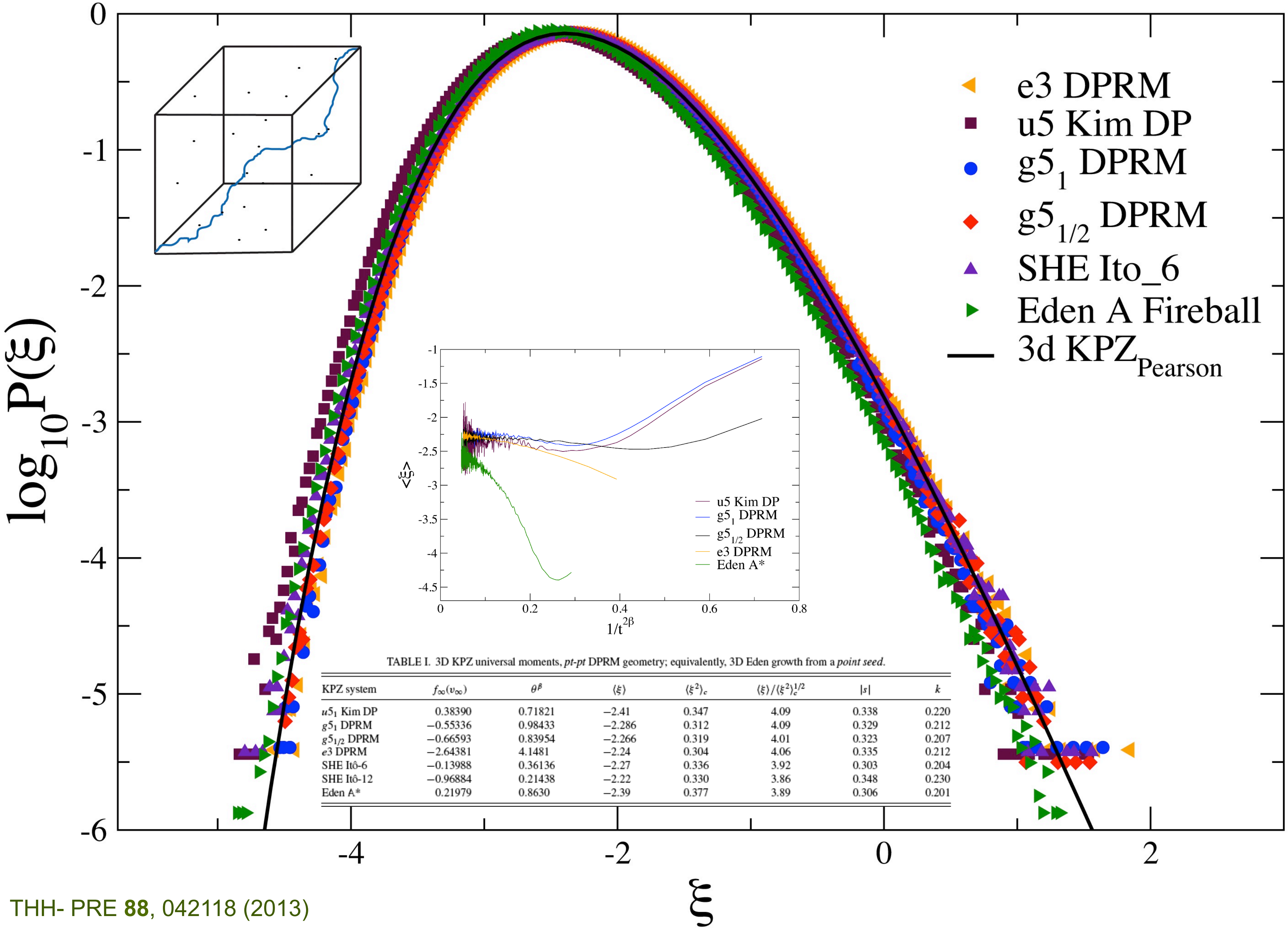


TABLE I. 3D KPZ universal moments, *pt-pt* DPRM geometry; equivalently, 3D Eden growth from a *point seed*.

KPZ system	$f_\infty(v_\infty)$	θ^β	$\langle\xi\rangle$	$\langle\xi^2\rangle_c$	$\langle\xi\rangle/\langle\xi^2\rangle_c^{1/2}$	$\lvert s\rvert$	k
$u5_1$ Kim DP	0.38390	0.71821	−2.41	0.347	4.09	0.338	0.220
$g5_1$ DPRM	−0.55336	0.98433	−2.286	0.312	4.09	0.329	0.212
$g5_{1/2}$ DPRM	−0.66593	0.83954	−2.266	0.319	4.01	0.323	0.207
$e3$ DPRM	−2.64381	4.1481	−2.24	0.304	4.06	0.335	0.212
SHE Itô-6	−0.13988	0.36136	−2.27	0.336	3.92	0.303	0.204
SHE Itô-12	−0.96884	0.21438	−2.22	0.330	3.86	0.348	0.230
Eden A*	0.21979	0.8630	−2.39	0.377	3.89	0.306	0.201

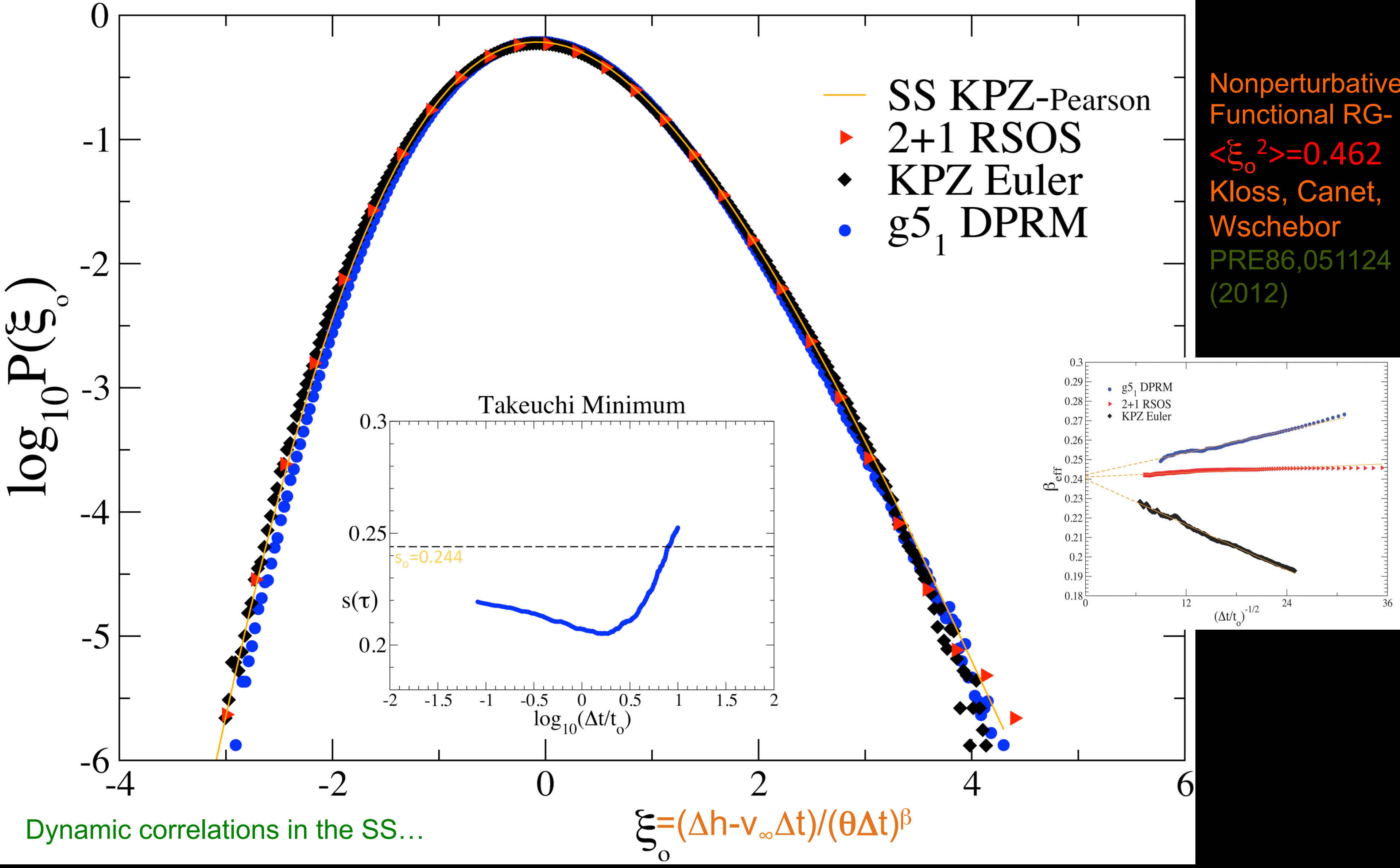
THH- PRE **88**, 042118 (2013)

2+1 Stationary-State KPZ

(higher-dimensional analog Baik-Rains)

Universal Variance-
$\langle \xi_o^2 \rangle = 0.464$

Nonperturbative Functional RG-
$\langle \xi_o^2 \rangle = 0.462$
Kloss, Canet, Wschebor
PRE86,051124 (2012)

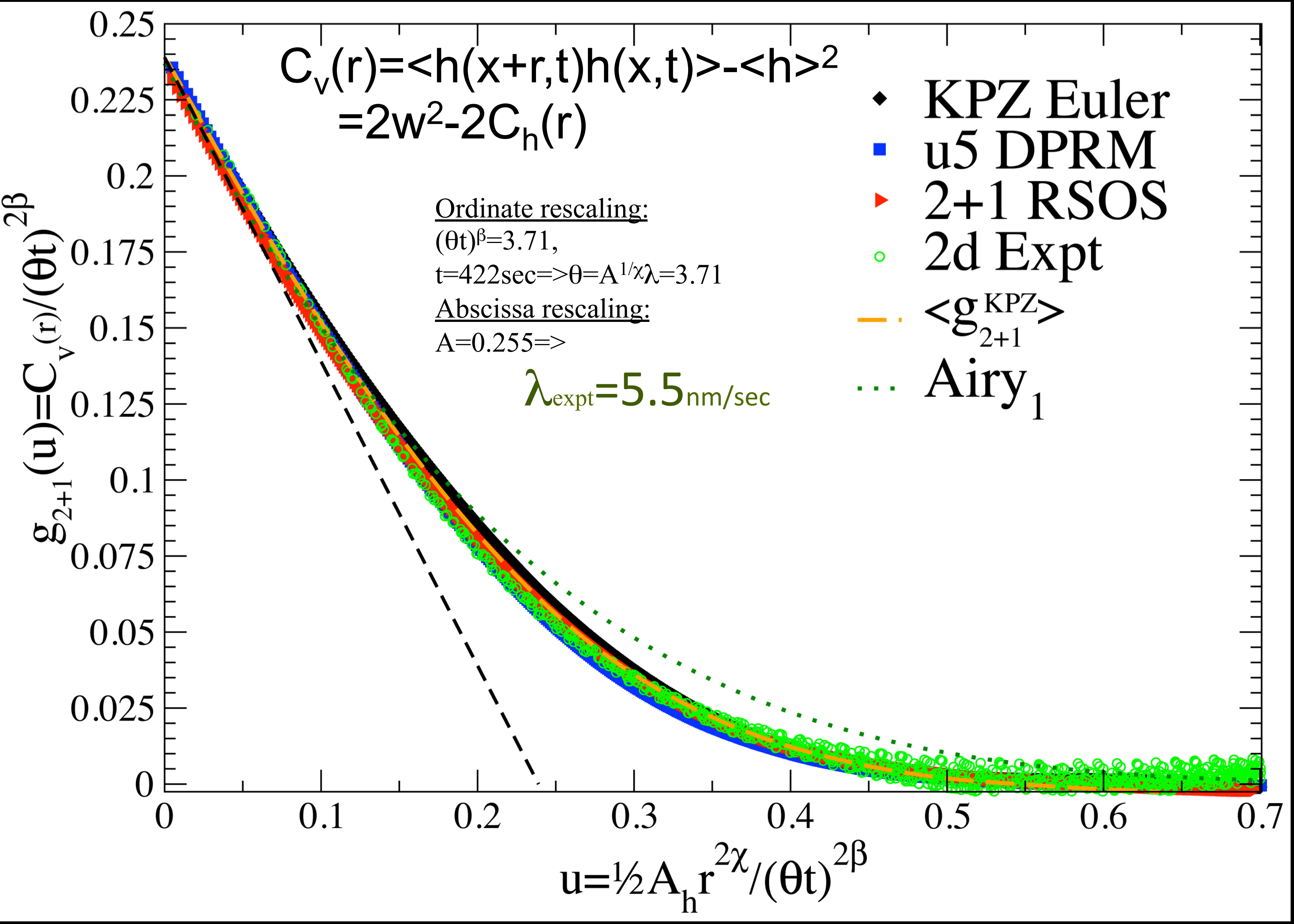


Dynamic correlations in the SS…

2+1 KPZ

Universal Correlators*

2+1 KPZ Spatial Covariance*



$$C_v(r)=\langle h(x+r,t)h(x,t)\rangle-\langle h\rangle^2$$
$$=2w^2-2C_h(r)$$

Ordinate rescaling:

$(\theta t)^\beta=3.71$,

t=422sec=>$\theta=A^{1/\chi}\lambda=3.71$

Abscissa rescaling:

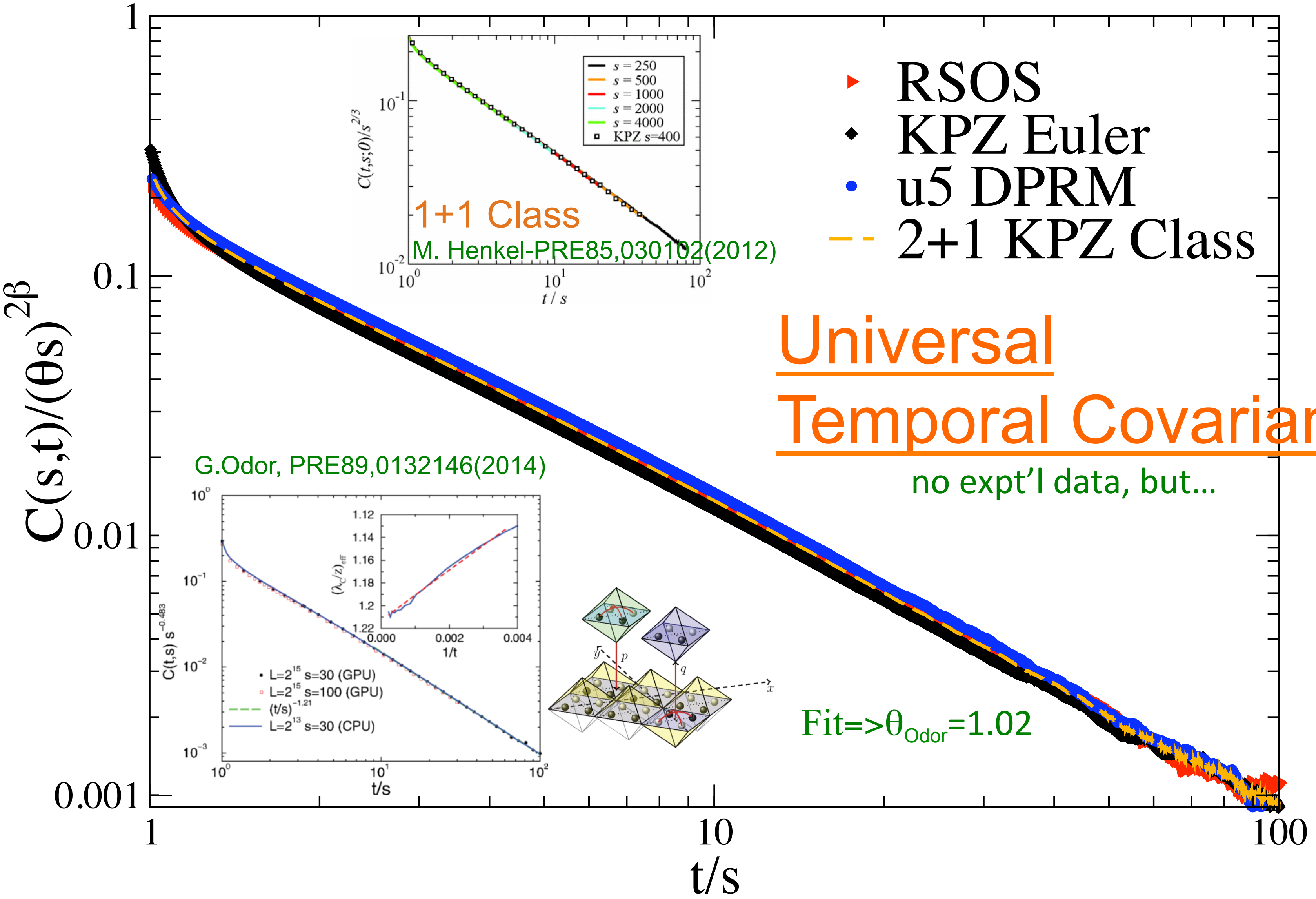
A=0.255=>

$\lambda_{expt}=5.5$nm/sec

Two-Time Autocorrelator: C(t,s)$=\langle h(t)h(s)\rangle-\langle h(t)\rangle\langle h(s)\rangle$

$=s^{2\beta}F_c(t/s)$

z2age.f: its*dt=250*0.02=>s=5; r2age.f: s=100; L=10k; nr=28



1+1 Class

M. Henkel-PRE85,030102(2012)

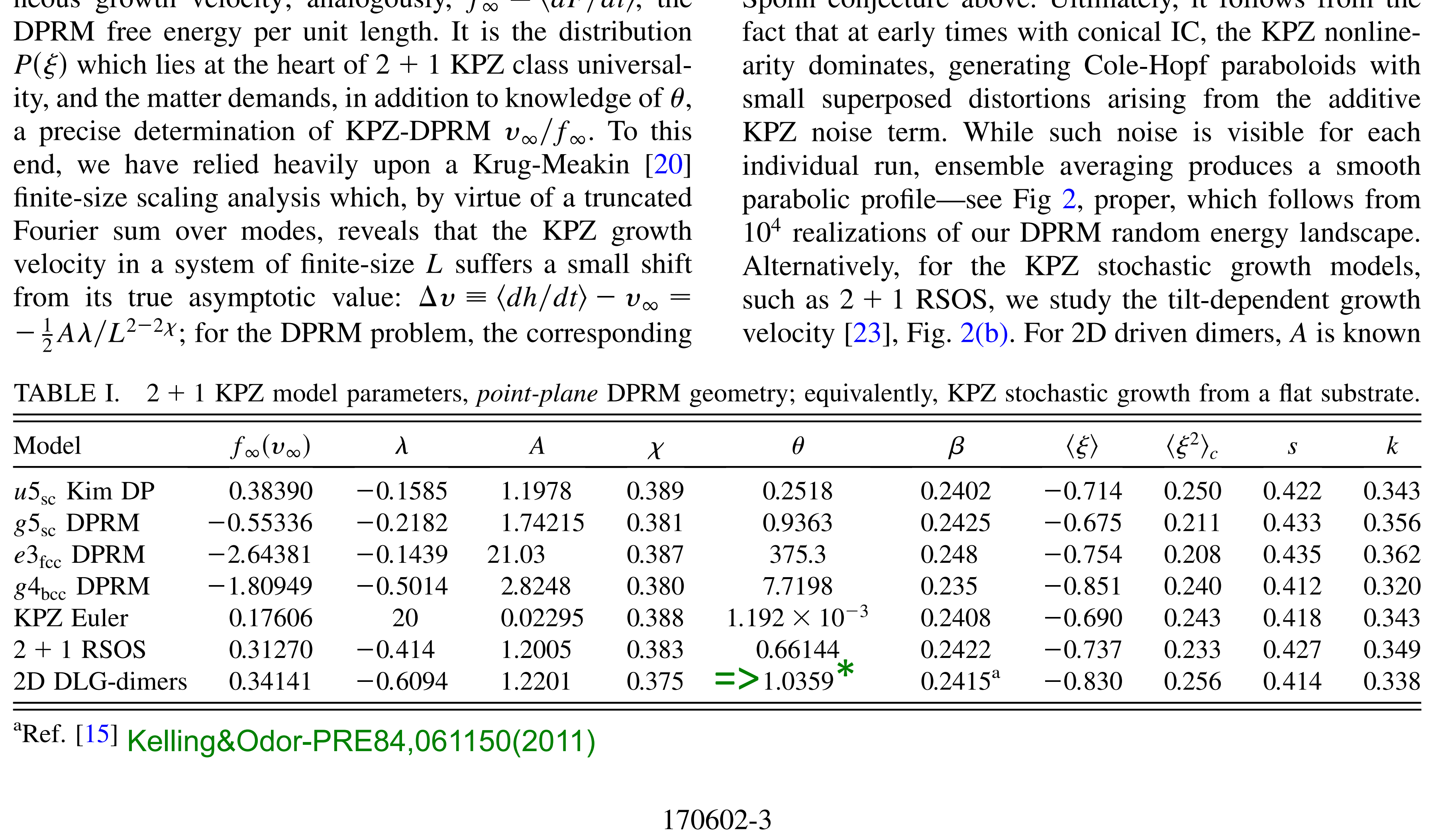
G.Odor, PRE89,0132146(2014)

Universal Temporal Covariance*

no expt'l data, but...

Fit=>θ_{Odor}=1.02

Fri Jan 3 08:30:55 2014



ncous growth velocity, analogously, $f_\infty = \langle dF/dt \rangle$, the DPRM free energy per unit length. It is the distribution $P(\xi)$ which lies at the heart of 2 + 1 KPZ class universality, and the matter demands, in addition to knowledge of θ, a precise determination of KPZ-DPRM v_∞ / f_∞. To this end, we have relied heavily upon a Krug-Meakin [20] finite-size scaling analysis which, by virtue of a truncated Fourier sum over modes, reveals that the KPZ growth velocity in a system of finite-size L suffers a small shift from its true asymptotic value: $\Delta v \equiv \langle dh/dt \rangle - v_\infty = -\frac{1}{2} A\lambda / L^{2-2\chi}$; for the DPRM problem, the corresponding

Spohn conjecture above. Ultimately, it follows from the fact that at early times with conical IC, the KPZ nonlinearity dominates, generating Cole-Hopf paraboloids with small superposed distortions arising from the additive KPZ noise term. While such noise is visible for each individual run, ensemble averaging produces a smooth parabolic profile—see Fig 2, proper, which follows from 10^4 realizations of our DPRM random energy landscape. Alternatively, for the KPZ stochastic growth models, such as 2 + 1 RSOS, we study the tilt-dependent growth velocity [23], Fig. 2(b). For 2D driven dimers, A is known

TABLE I. 2 + 1 KPZ model parameters, *point-plane* DPRM geometry; equivalently, KPZ stochastic growth from a flat substrate.

Model	$f_\infty(v_\infty)$	λ	A	χ	θ	β	$\langle \xi \rangle$	$\langle \xi^2 \rangle_c$	s	k
$u5_{sc}$ Kim DP	0.38390	−0.1585	1.1978	0.389	0.2518	0.2402	−0.714	0.250	0.422	0.343
$g5_{sc}$ DPRM	−0.55336	−0.2182	1.74215	0.381	0.9363	0.2425	−0.675	0.211	0.433	0.356
$e3_{fcc}$ DPRM	−2.64381	−0.1439	21.03	0.387	375.3	0.248	−0.754	0.208	0.435	0.362
$g4_{bcc}$ DPRM	−1.80949	−0.5014	2.8248	0.380	7.7198	0.235	−0.851	0.240	0.412	0.320
KPZ Euler	0.17606	20	0.02295	0.388	1.192×10^{-3}	0.2408	−0.690	0.243	0.418	0.343
2 + 1 RSOS	0.31270	−0.414	1.2005	0.383	0.66144	0.2422	−0.737	0.233	0.427	0.349
2D DLG-dimers	0.34141	−0.6094	1.2201	0.375	=>1.0359*	0.2415[a]	−0.830	0.256	0.414	0.338

[a] Ref. [15] Kelling&Odor-PRE84,061150(2011)

Devil in the details...

λ, A, $\theta = A^{1/\chi}\lambda$

Challenge Problem #1: Diamonds in the Rough...

Exact RR known for limit distribution!

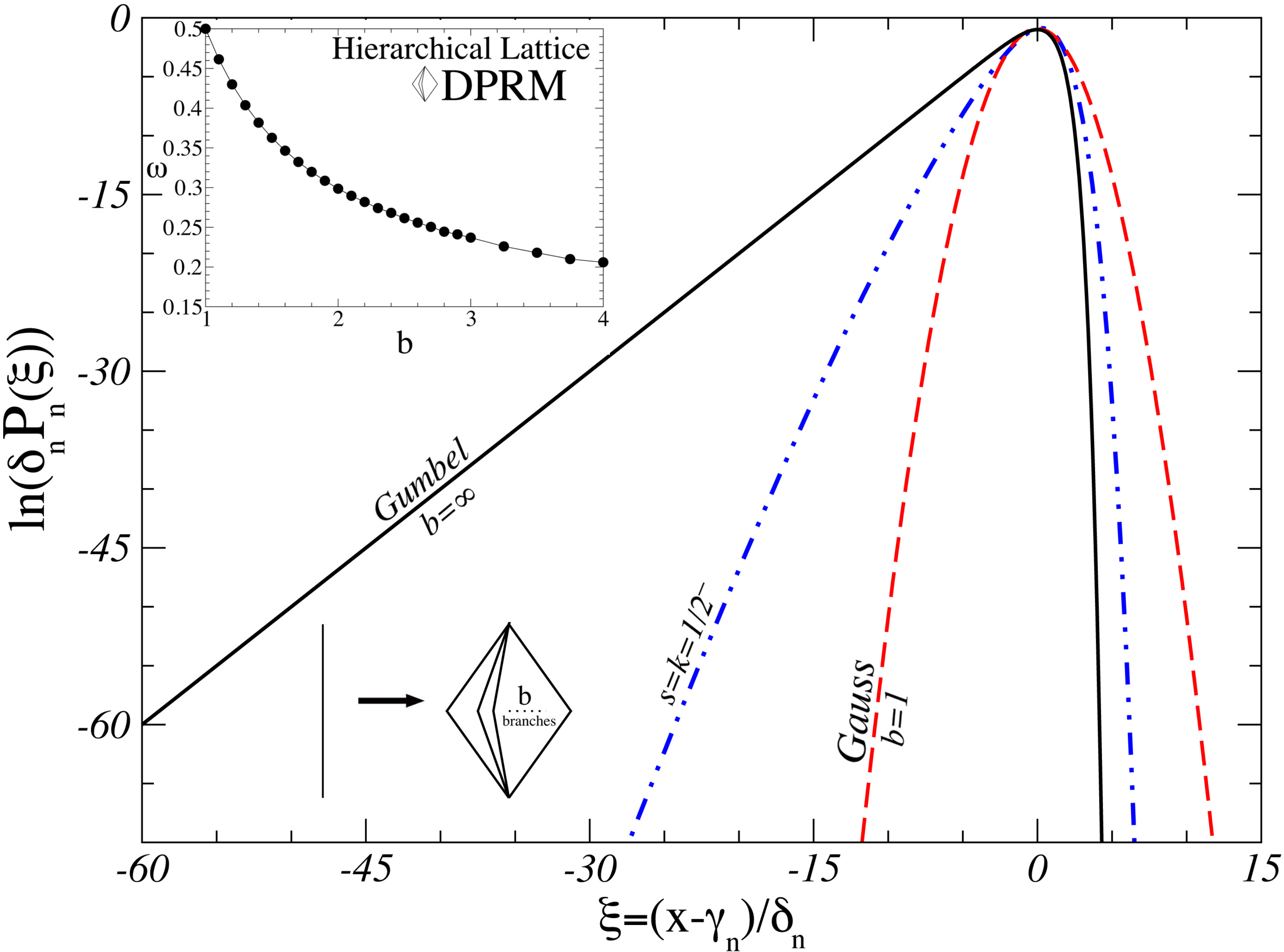
a) Solution? Your choice of b.

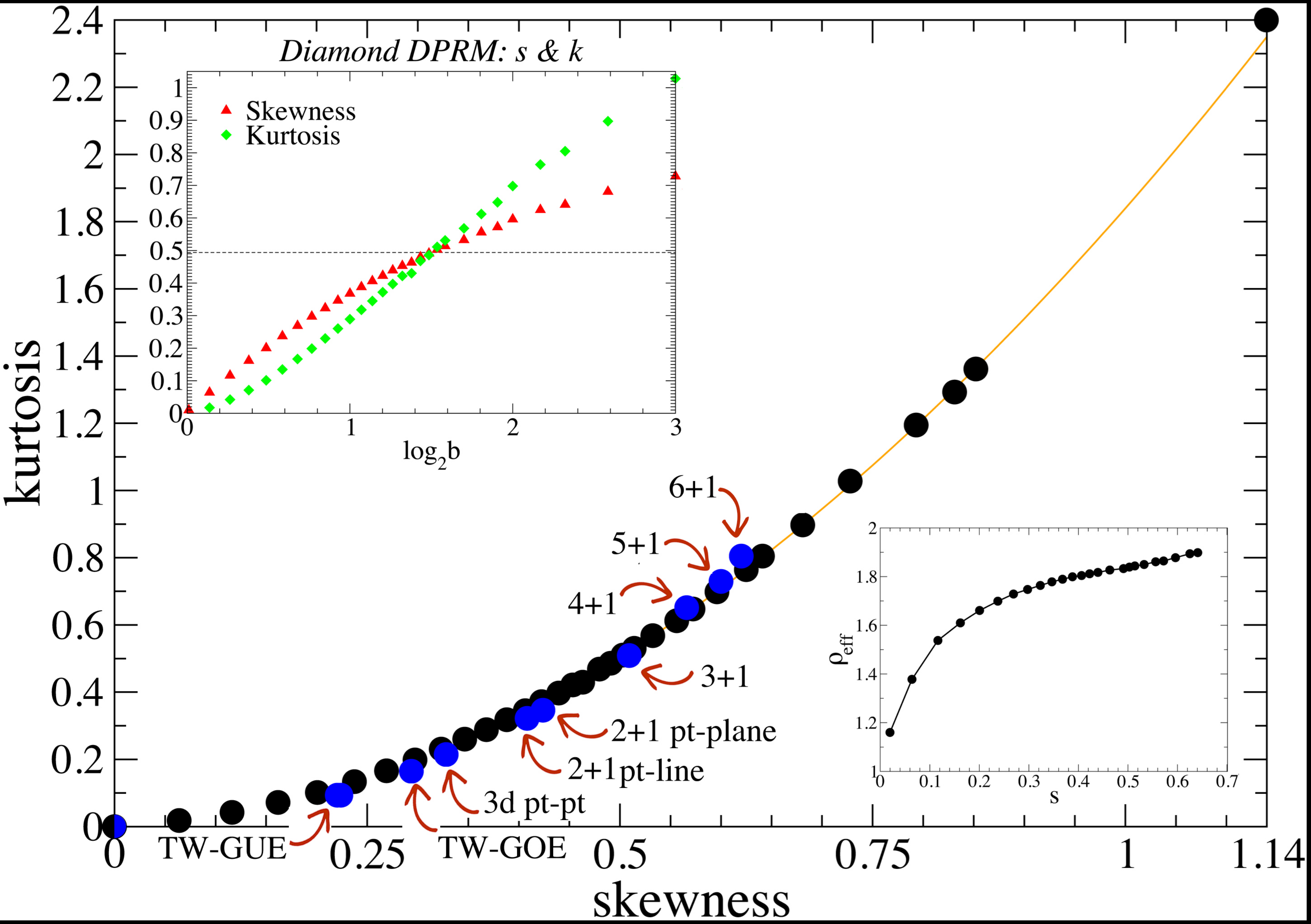
b) d-dependent exponent: $\beta=\beta(b)$

c) LD Tails [MG Conjecture*]
Long: $\eta=1/(1-\beta)$
Short: $\eta'=d_{eff}\,\eta$ w/ $d_{eff}=1+\ln_2 b$

*JSTAT(2008) P01008.

n.b., Here, T=0; for $b>b_c=2$, FINITE TEMP phase transition



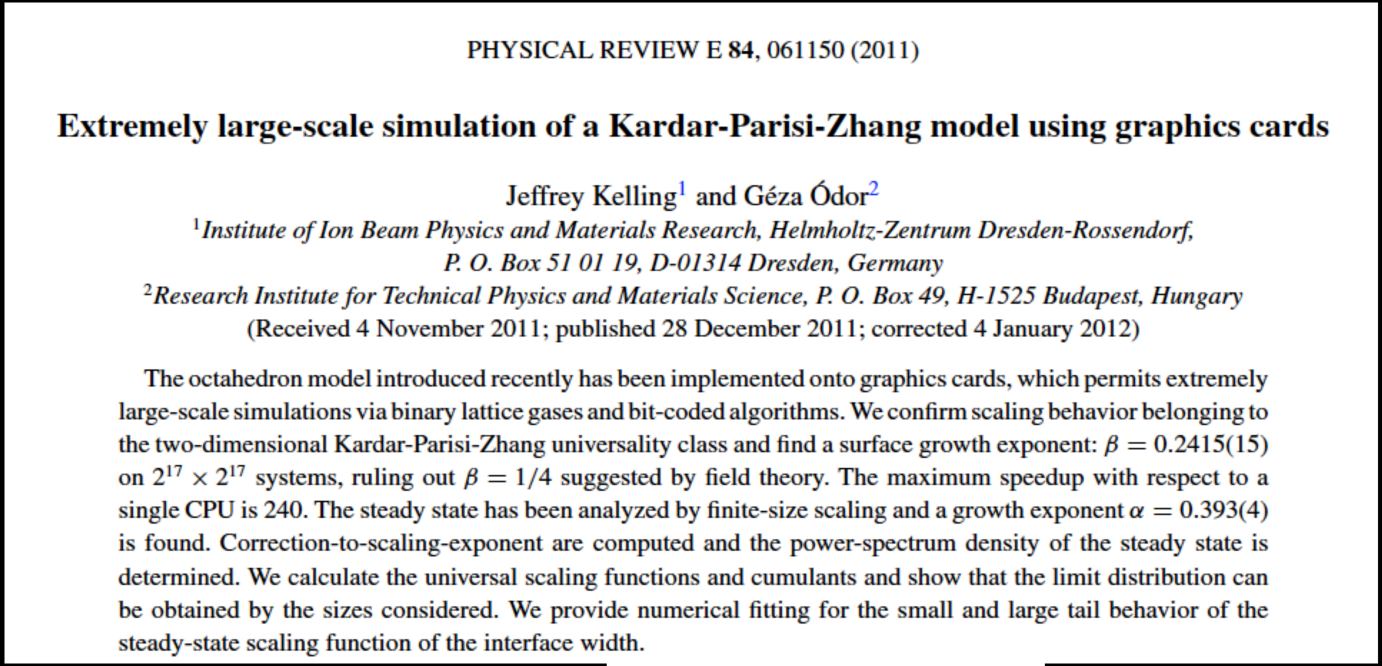


d) s=s(b), k=k(b)-
Parametric Rep-> $k \approx 2s^2$?!!

*Fate of d=∞ KPZ?

Challenge Problem #2: $\beta_{2+1}^{KPZ}<1/4$

a) Exact value?



PHYSICAL REVIEW E **84**, 061150 (2011)

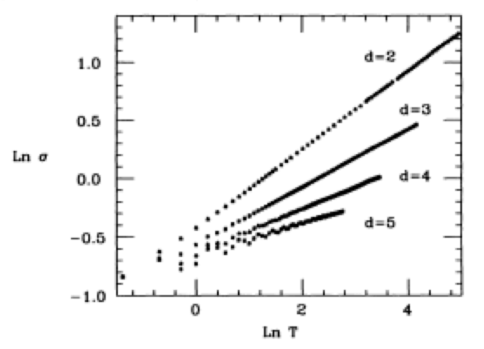
Extremely large-scale simulation of a Kardar-Parisi-Zhang model using graphics cards

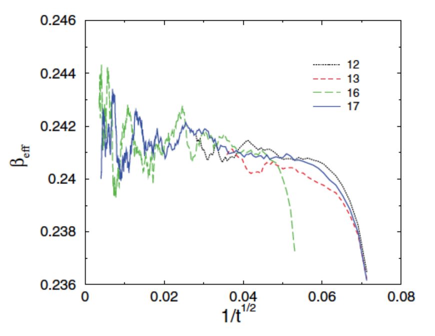
Jeffrey Kelling[1] and Géza Ódor[2]
[1]*Institute of Ion Beam Physics and Materials Research, Helmholtz-Zentrum Dresden-Rossendorf, P. O. Box 51 01 19, D-01314 Dresden, Germany*
[2]*Research Institute for Technical Physics and Materials Science, P. O. Box 49, H-1525 Budapest, Hungary*

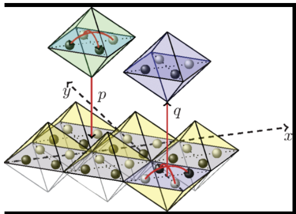
(Received 4 November 2011; published 28 December 2011; corrected 4 January 2012)

The octahedron model introduced recently has been implemented onto graphics cards, which permits extremely large-scale simulations via binary lattice gases and bit-coded algorithms. We confirm scaling behavior belonging to the two-dimensional Kardar-Parisi-Zhang universality class and find a surface growth exponent: $\beta = 0.2415(15)$ on $2^{17} \times 2^{17}$ systems, ruling out $\beta = 1/4$ suggested by field theory. The maximum speedup with respect to a single CPU is 240. The steady state has been analyzed by finite-size scaling and a growth exponent $\alpha = 0.393(4)$ is found. Correction-to-scaling-exponent are computed and the power-spectrum density of the steady state is determined. We calculate the universal scaling functions and cumulants and show that the limit distribution can be obtained by the sizes considered. We provide numerical fitting for the small and large tail behavior of the steady-state scaling function of the interface width.

*GOAL= Bury the KK conjecture:
$\beta=1/(d+1)$





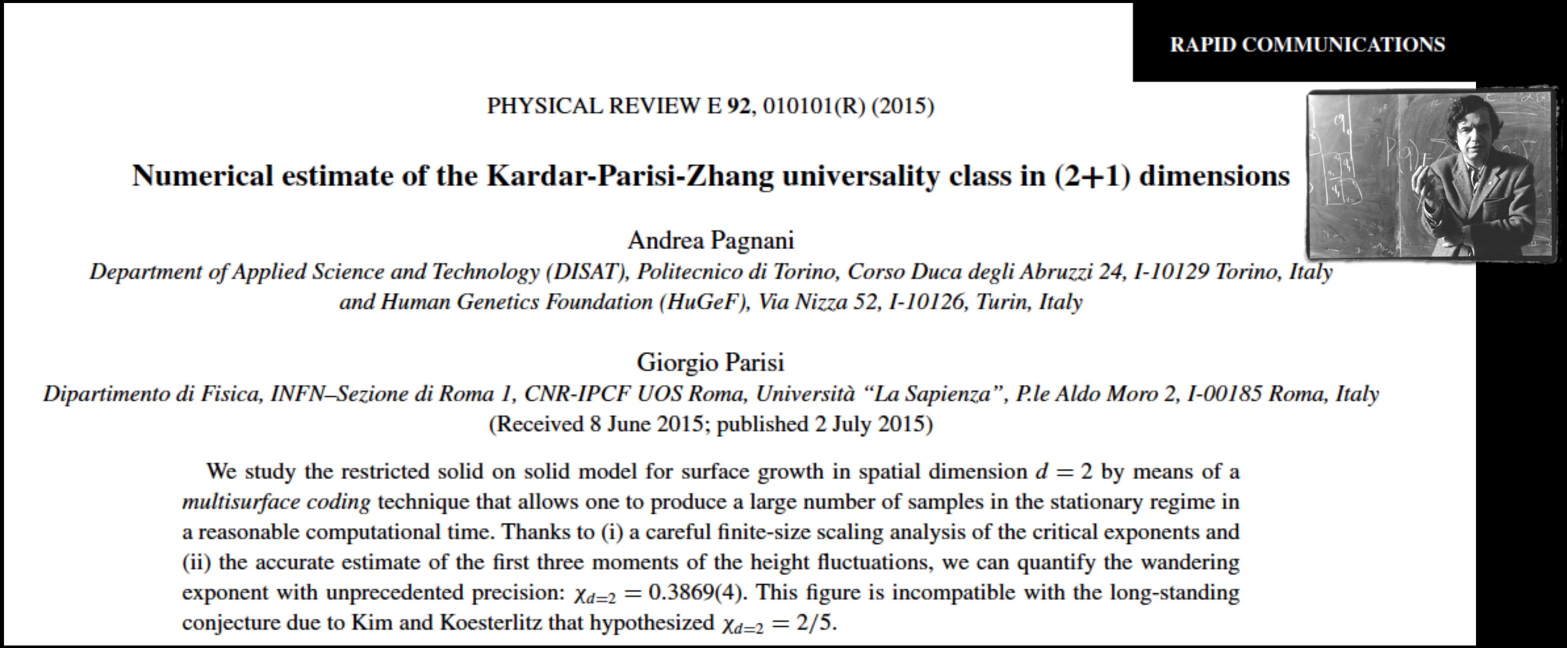


Kim & Kosterlitz, PRL 62 2289 (1989); n.b., M. Lassig-PRL 80, 2366 (1998) indicates field-theoretic $\beta=1/4$....

Challenge Problem #2: $\alpha^{KPZ}_{2+1} < 2/5$ ← *Kim-Kosterlitz value*

a) Exact value?

Naaah-
Too hard...



RAPID COMMUNICATIONS

PHYSICAL REVIEW E **92**, 010101(R) (2015)

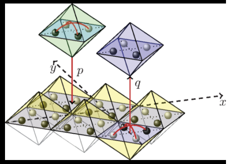
Numerical estimate of the Kardar-Parisi-Zhang universality class in (2+1) dimensions

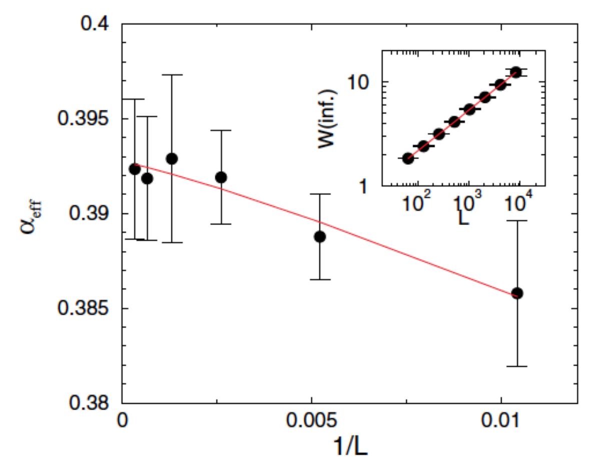
Andrea Pagnani
Department of Applied Science and Technology (DISAT), Politecnico di Torino, Corso Duca degli Abruzzi 24, I-10129 Torino, Italy
and Human Genetics Foundation (HuGeF), Via Nizza 52, I-10126, Turin, Italy

Giorgio Parisi
Dipartimento di Fisica, INFN–Sezione di Roma 1, CNR-IPCF UOS Roma, Università "La Sapienza", P.le Aldo Moro 2, I-00185 Roma, Italy
(Received 8 June 2015; published 2 July 2015)

We study the restricted solid on solid model for surface growth in spatial dimension $d = 2$ by means of a *multisurface coding* technique that allows one to produce a large number of samples in the stationary regime in a reasonable computational time. Thanks to (i) a careful finite-size scaling analysis of the critical exponents and (ii) the accurate estimate of the first three moments of the height fluctuations, we can quantify the wandering exponent with unprecedented precision: $\chi_{d=2} = 0.3869(4)$. This figure is incompatible with the long-standing conjecture due to Kim and Koesterlitz that hypothesized $\chi_{d=2} = 2/5$.

Kelling & Odor, PRE84, 061150





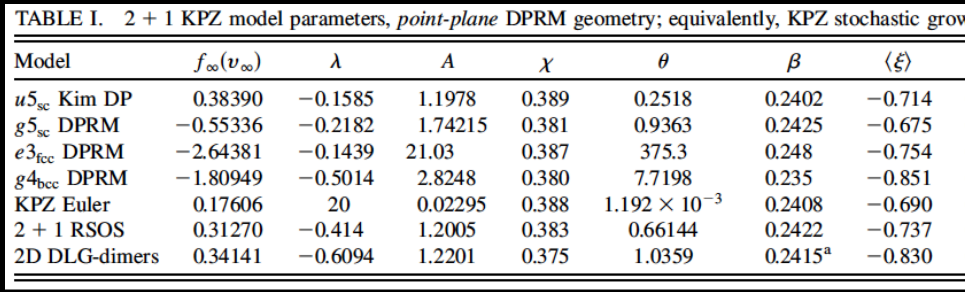


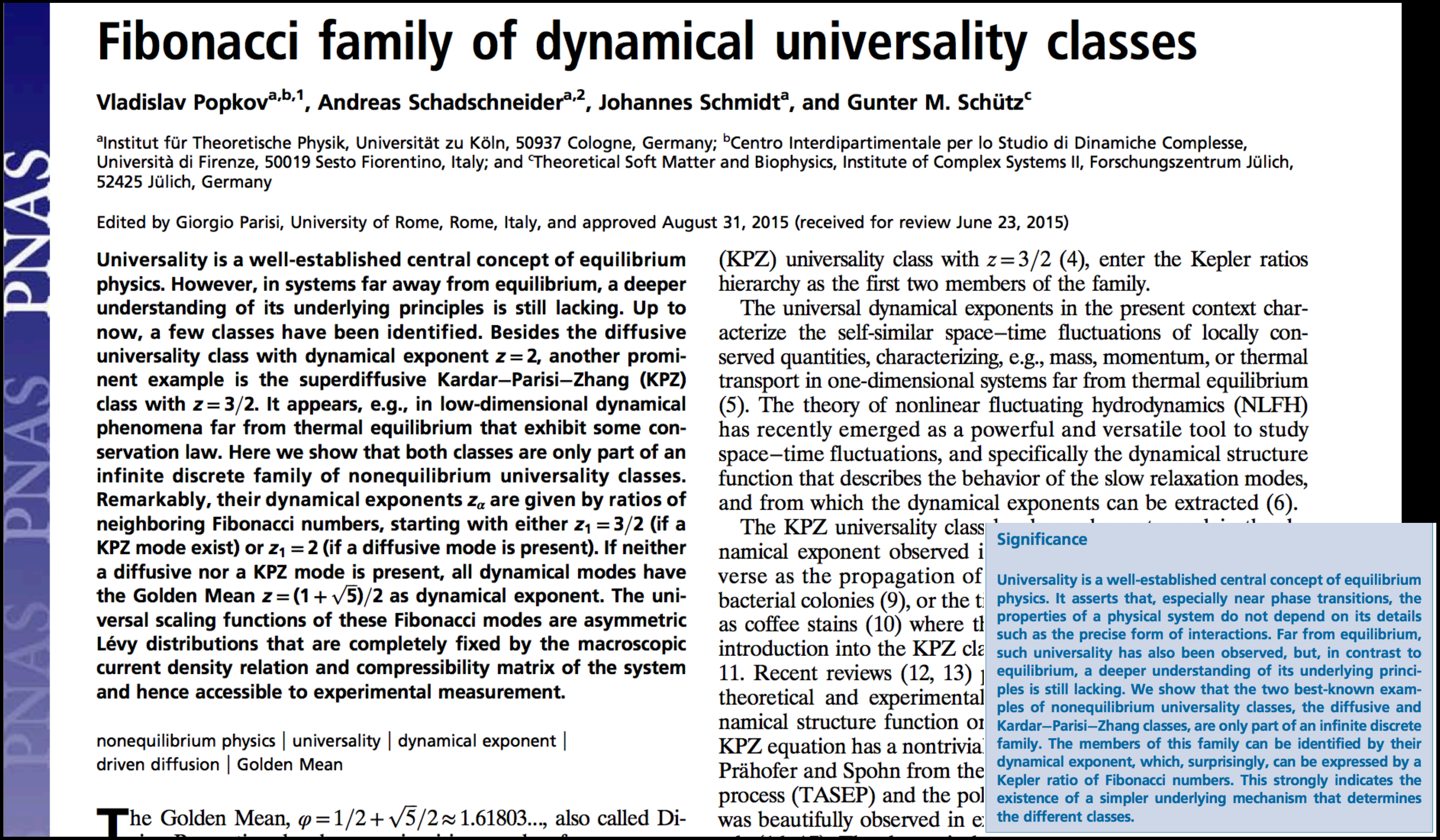
TABLE I. 2 + 1 KPZ model parameters, *point-plane* DPRM geometry; equivalently, KPZ stochastic grow

Model	$f_\infty(v_\infty)$	λ	A	χ	θ	β	$\langle\xi\rangle$
$u5_{sc}$ Kim DP	0.38390	−0.1585	1.1978	0.389	0.2518	0.2402	−0.714
$g5_{sc}$ DPRM	−0.55336	−0.2182	1.74215	0.381	0.9363	0.2425	−0.675
$e3_{fcc}$ DPRM	−2.64381	−0.1439	21.03	0.387	375.3	0.248	−0.754
$g4_{bcc}$ DPRM	−1.80949	−0.5014	2.8248	0.380	7.7198	0.235	−0.851
KPZ Euler	0.17606	20	0.02295	0.388	1.192×10^{-3}	0.2408	−0.690
2 + 1 RSOS	0.31270	−0.414	1.2005	0.383	0.66144	0.2422	−0.737
2D DLG-dimers	0.34141	−0.6094	1.2201	0.375	1.0359	0.2415[a]	−0.830

THH, PRL 109, 170602: using Krug-Meakin FSS....

How about a bound?

Challenge Problem #2, cont: z_{2+1}=0.390/0.2415=1.615



Fibonacci family of dynamical universality classes

Vladislav Popkov[a,b,1], **Andreas Schadschneider**[a,2], **Johannes Schmidt**[a], **and Gunter M. Schütz**[c]

[a]Institut für Theoretische Physik, Universität zu Köln, 50937 Cologne, Germany; [b]Centro Interdipartimentale per lo Studio di Dinamiche Complesse, Università di Firenze, 50019 Sesto Fiorentino, Italy; and [c]Theoretical Soft Matter and Biophysics, Institute of Complex Systems II, Forschungszentrum Jülich, 52425 Jülich, Germany

Edited by Giorgio Parisi, University of Rome, Rome, Italy, and approved August 31, 2015 (received for review June 23, 2015)

Universality is a well-established central concept of equilibrium physics. However, in systems far away from equilibrium, a deeper understanding of its underlying principles is still lacking. Up to now, a few classes have been identified. Besides the diffusive universality class with dynamical exponent $z=2$, another prominent example is the superdiffusive Kardar–Parisi–Zhang (KPZ) class with $z=3/2$. It appears, e.g., in low-dimensional dynamical phenomena far from thermal equilibrium that exhibit some conservation law. Here we show that both classes are only part of an infinite discrete family of nonequilibrium universality classes. Remarkably, their dynamical exponents z_α are given by ratios of neighboring Fibonacci numbers, starting with either $z_1=3/2$ (if a KPZ mode exist) or $z_1=2$ (if a diffusive mode is present). If neither a diffusive nor a KPZ mode is present, all dynamical modes have the Golden Mean $z=(1+\sqrt{5})/2$ as dynamical exponent. The universal scaling functions of these Fibonacci modes are asymmetric Lévy distributions that are completely fixed by the macroscopic current density relation and compressibility matrix of the system and hence accessible to experimental measurement.

nonequilibrium physics | universality | dynamical exponent | driven diffusion | Golden Mean

The Golden Mean, $\varphi=1/2+\sqrt{5}/2\approx 1.61803...$, also called Di-

(KPZ) universality class with $z=3/2$ (4), enter the Kepler ratios hierarchy as the first two members of the family.

The universal dynamical exponents in the present context characterize the self-similar space–time fluctuations of locally conserved quantities, characterizing, e.g., mass, momentum, or thermal transport in one-dimensional systems far from thermal equilibrium (5). The theory of nonlinear fluctuating hydrodynamics (NLFH) has recently emerged as a powerful and versatile tool to study space–time fluctuations, and specifically the dynamical structure function that describes the behavior of the slow relaxation modes, and from which the dynamical exponents can be extracted (6).

The KPZ universality class
namical exponent observed i
verse as the propagation of
bacterial colonies (9), or the t
as coffee stains (10) where t
introduction into the KPZ cla
11. Recent reviews (12, 13)
theoretical and experimental
namical structure function o
KPZ equation has a nontrivia
Prähofer and Spohn from the
process (TASEP) and the pol
was beautifully observed in e

Significance

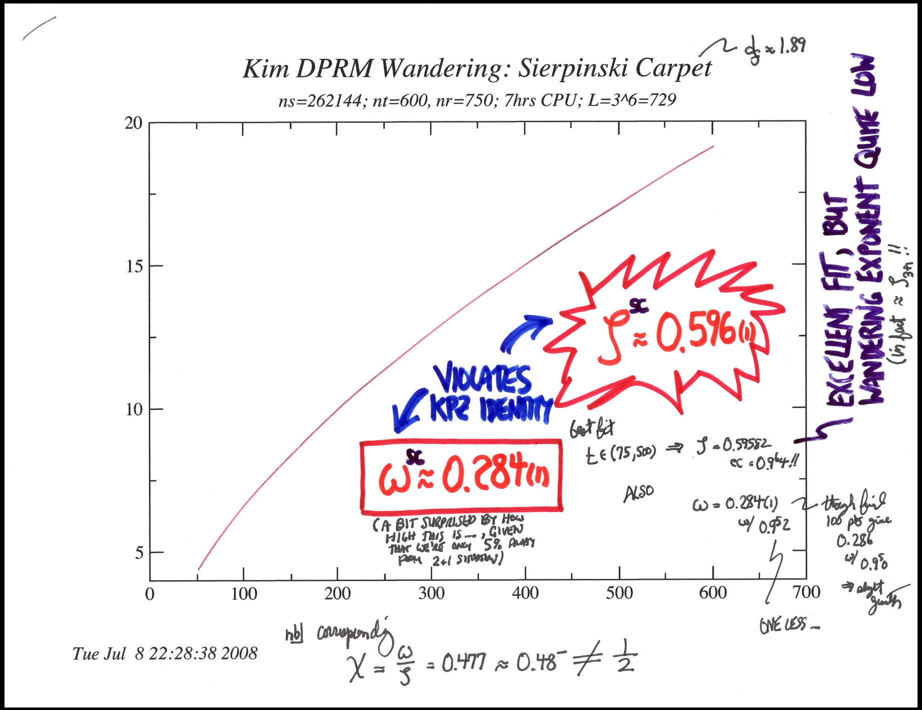
Universality is a well-established central concept of equilibrium physics. It asserts that, especially near phase transitions, the properties of a physical system do not depend on its details such as the precise form of interactions. Far from equilibrium, such universality has also been observed, but, in contrast to equilibrium, a deeper understanding of its underlying principles is still lacking. We show that the two best-known examples of nonequilibrium universality classes, the diffusive and Kardar–Parisi–Zhang classes, are only part of an infinite discrete family. The members of this family can be identified by their dynamical exponent, which, surprisingly, can be expressed by a Kepler ratio of Fibonacci numbers. This strongly indicates the existence of a simpler underlying mechanism that determines the different classes.

1,1,2,3,5,8,...

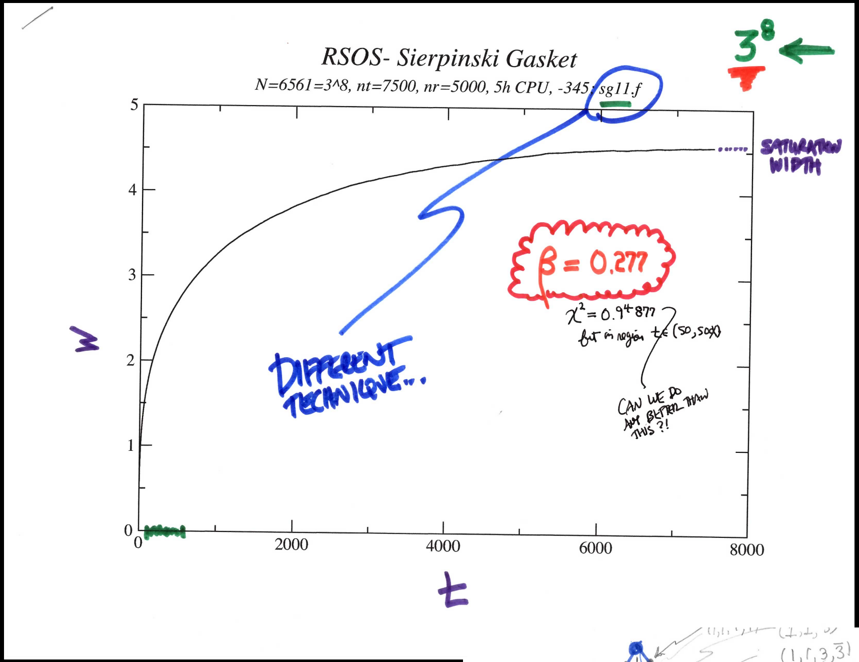
multi-lane KPZ: Golden Ratio?

Challenge Problem #3: Fractal Substrates...

DPRM:



RSOS:



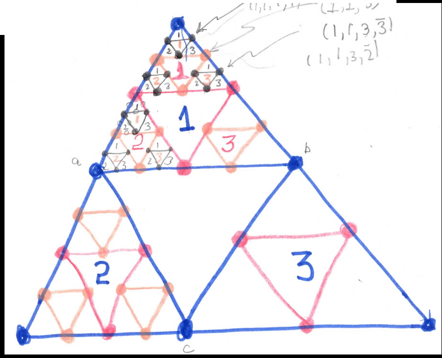
$d_f = \ln 3/\ln 2 = 1.585$

$\beta = 0.27$

$\alpha = 0.48$

$z = \alpha/\beta = 1.78$

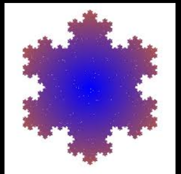
$z_{rw} = 2.32$



Definitely not KPZ!

$\alpha + z = z_{rw} \neq 2$

Koch:
$\beta = 1/3$?
$\alpha = 0.64$

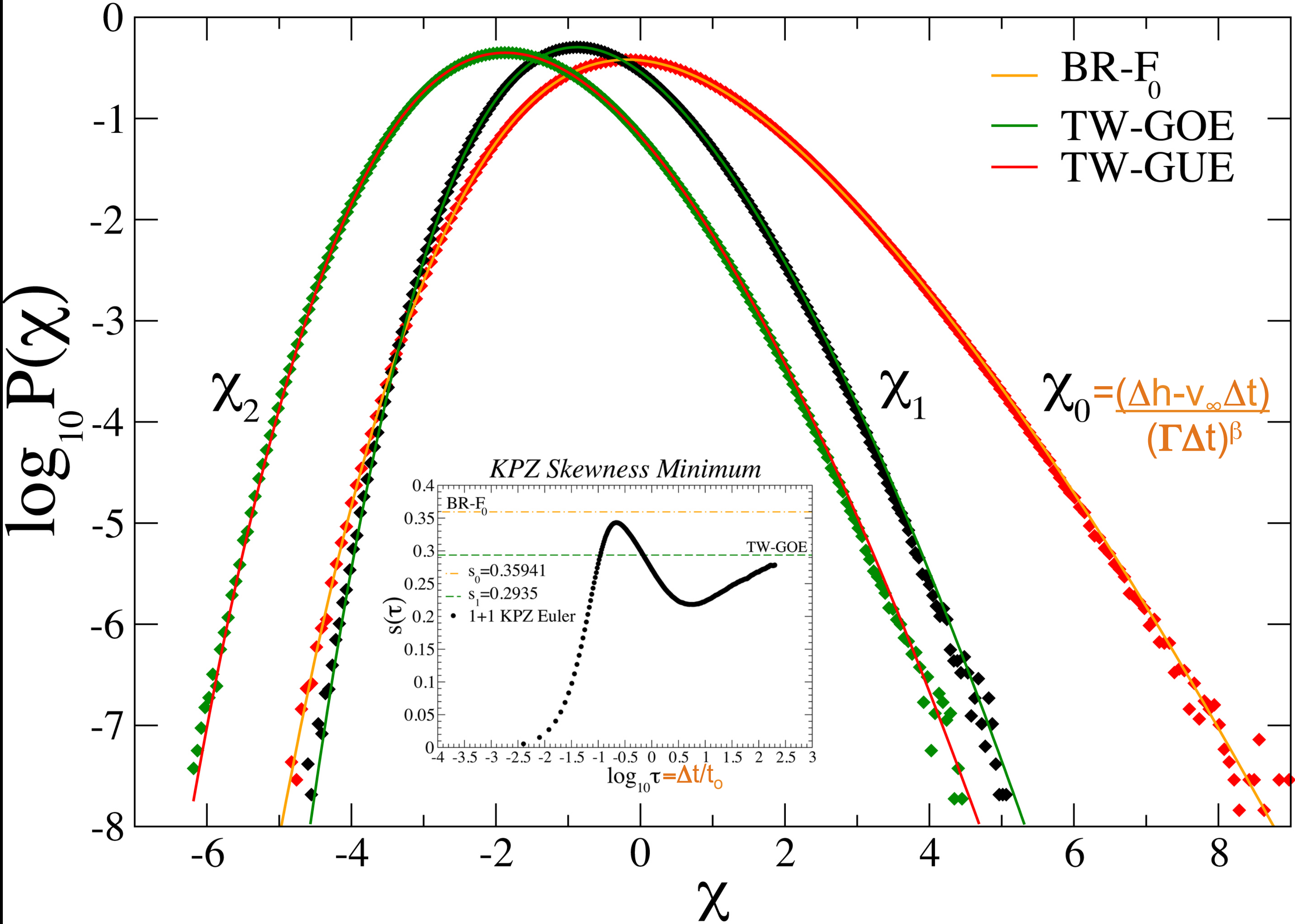


JSTAT (2015) P08016

*Extra Credit: Takeuchi Minimum

d=1+1

KPZ Eqn:

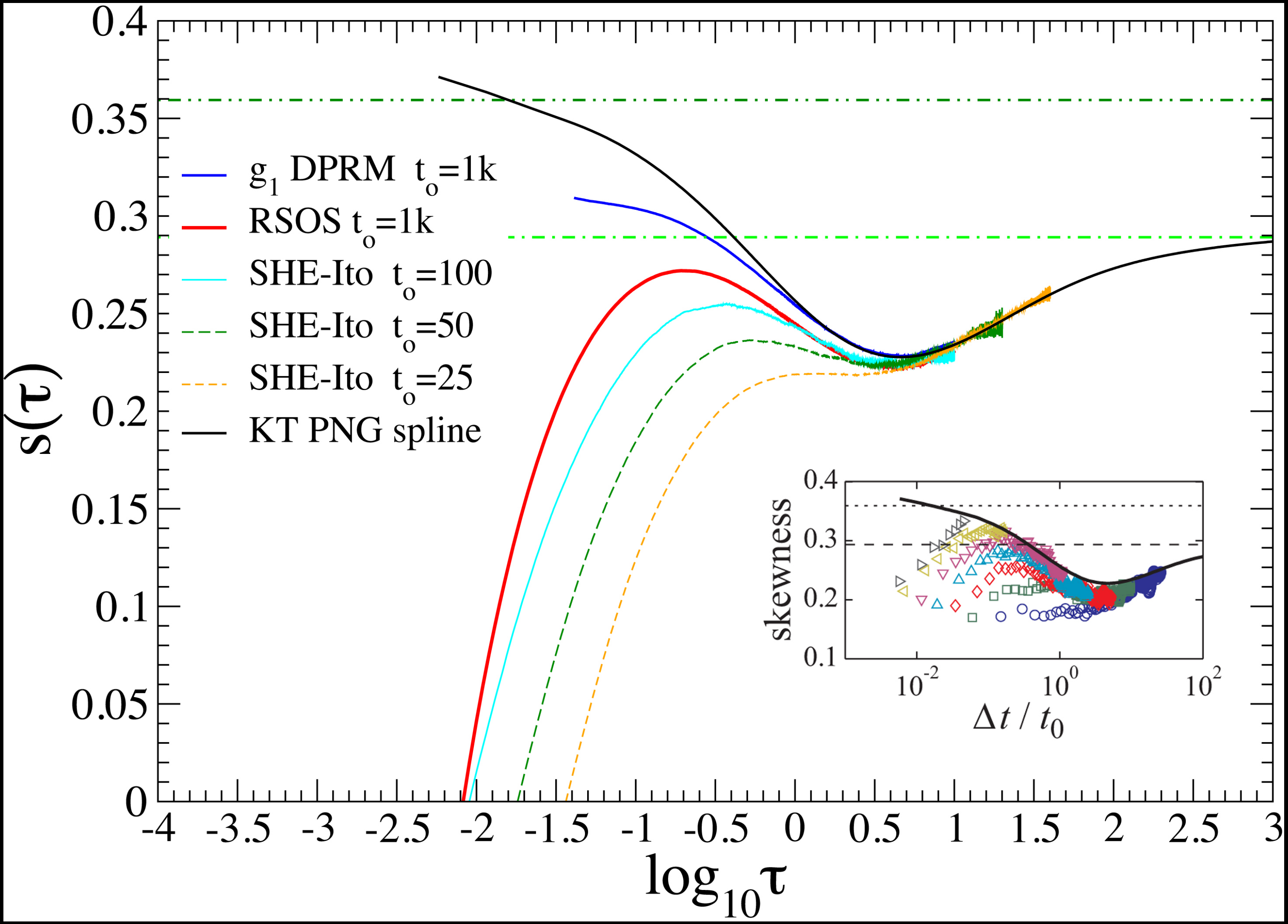


*envelope please...

Observed in Liq X-tal expts*

& universal signature**

Crossover 1+1 KPZ Stationary-State Statistics...



*PRL 110, 210604 (2013) **PRE89, 010103 (2014)-Rapid/Editor's Suggestion

2+1 KPZ Class:

3+ Universal PDFs, 2 Correlators, & KM Toolbox
=>Ripe, Rich & Ready to go…

Many Thx...

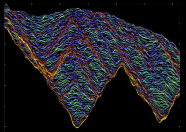
2+1 KPZ NUMERICS: THH- PRL**109**,170602 (2012)
PRE**88**,042118 (2013)
PRE**89**,010103R (2014) w/Luna Lin

2+1 KPZ Expt: Almeida-PRB**89**,045309 (2014)
Palasantzas-EPL**105**,50001 (2014)

KT

Special KPZ issue-
JSP **160**, 794 & 965 (2015)
Num/Exp Math-Spohn&Quastel

(1947-2016)

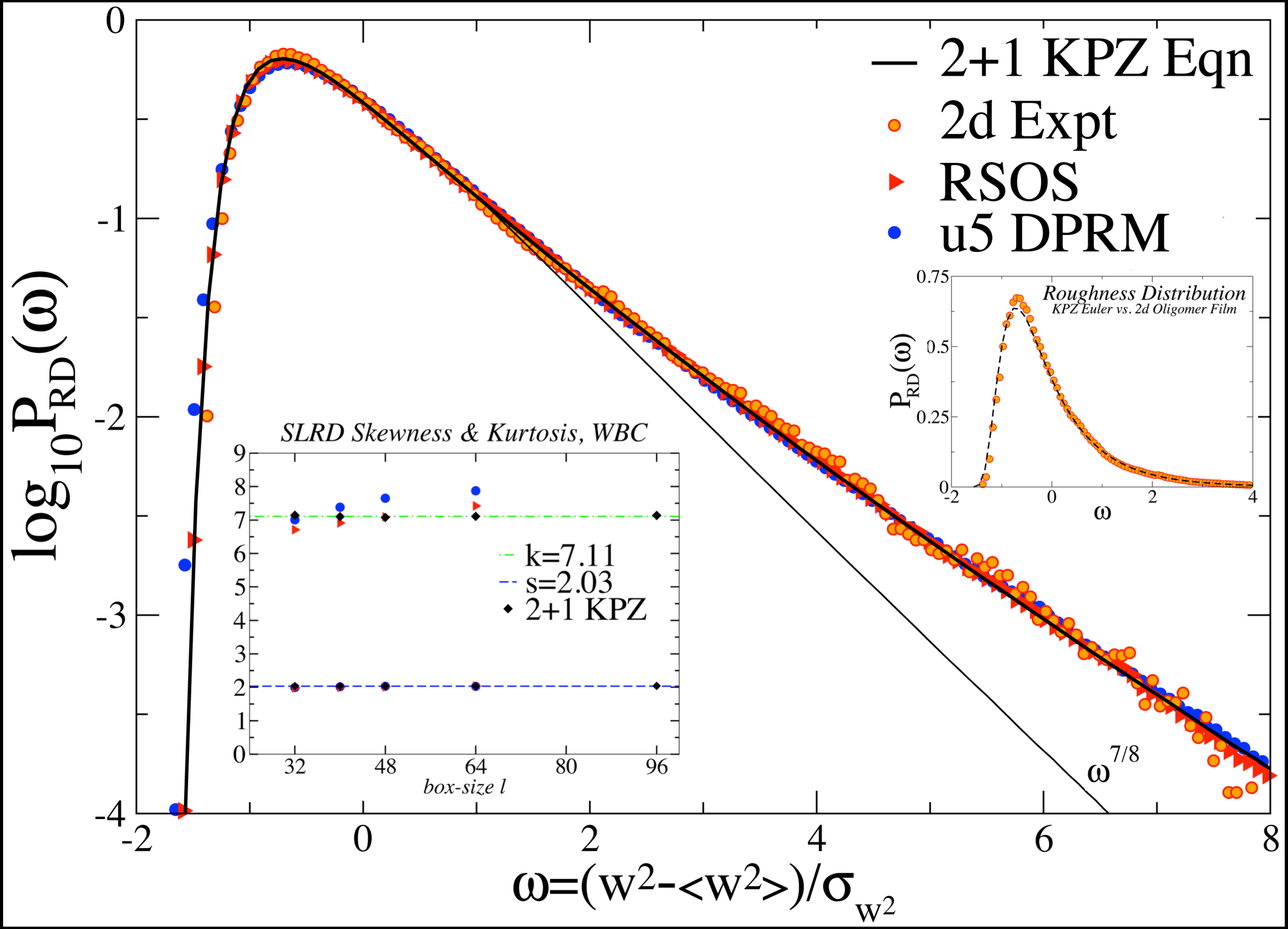


S. Atis,
a demain…

I. Squared Local Roughness Distribution:

(WBC, not PBC!)

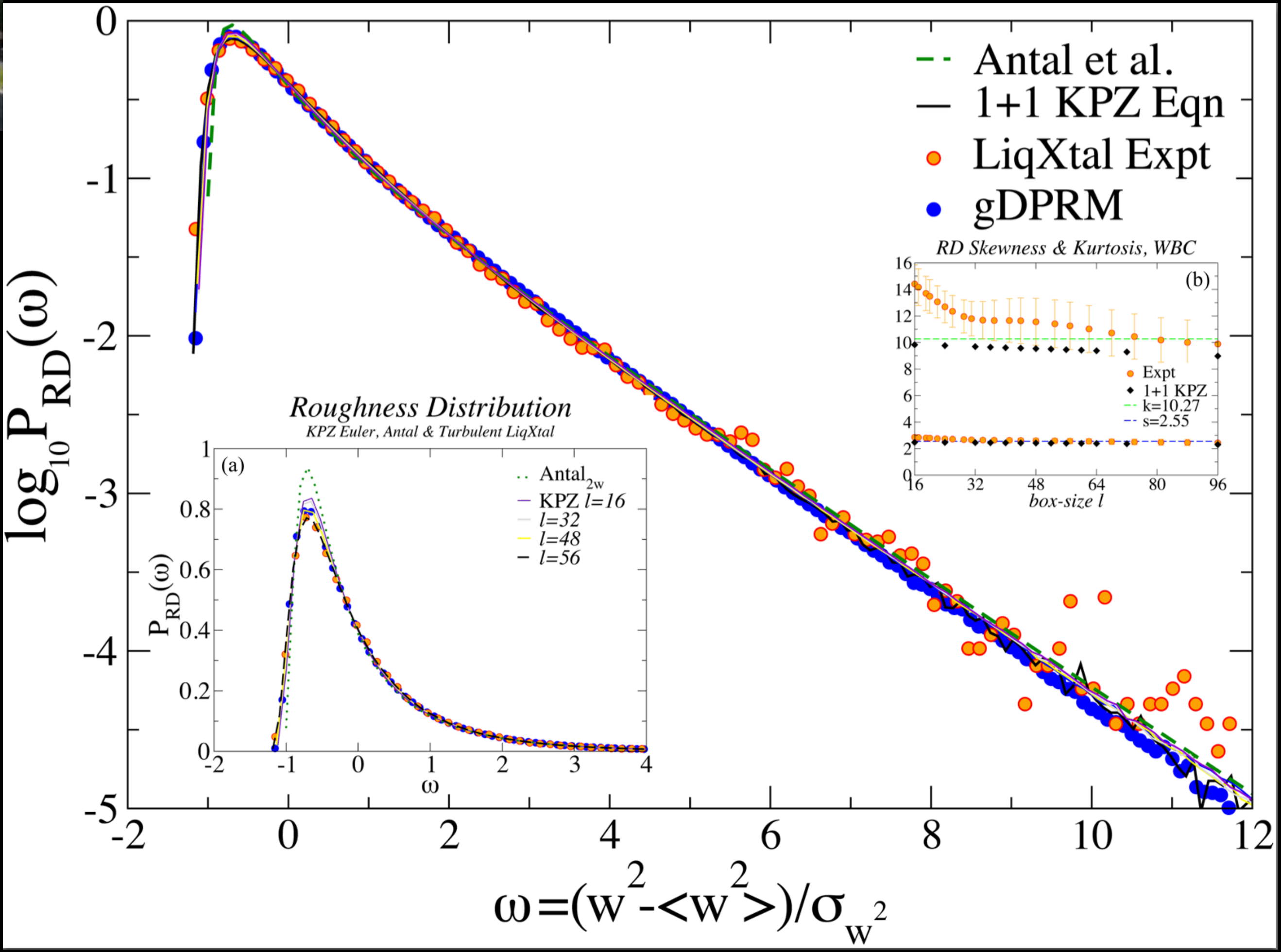
THH & Palasantzas, EPL105,50001(2014)
Almeida,(2014); Z. Racz, PRE50,3530(1994)



Check
d=1+1....

1+1 KPZ Class: SLRD

(Takeuchi & Sano- Liquid Crystal Expt vs. KPZ Euler...)



$10.27=72/7$

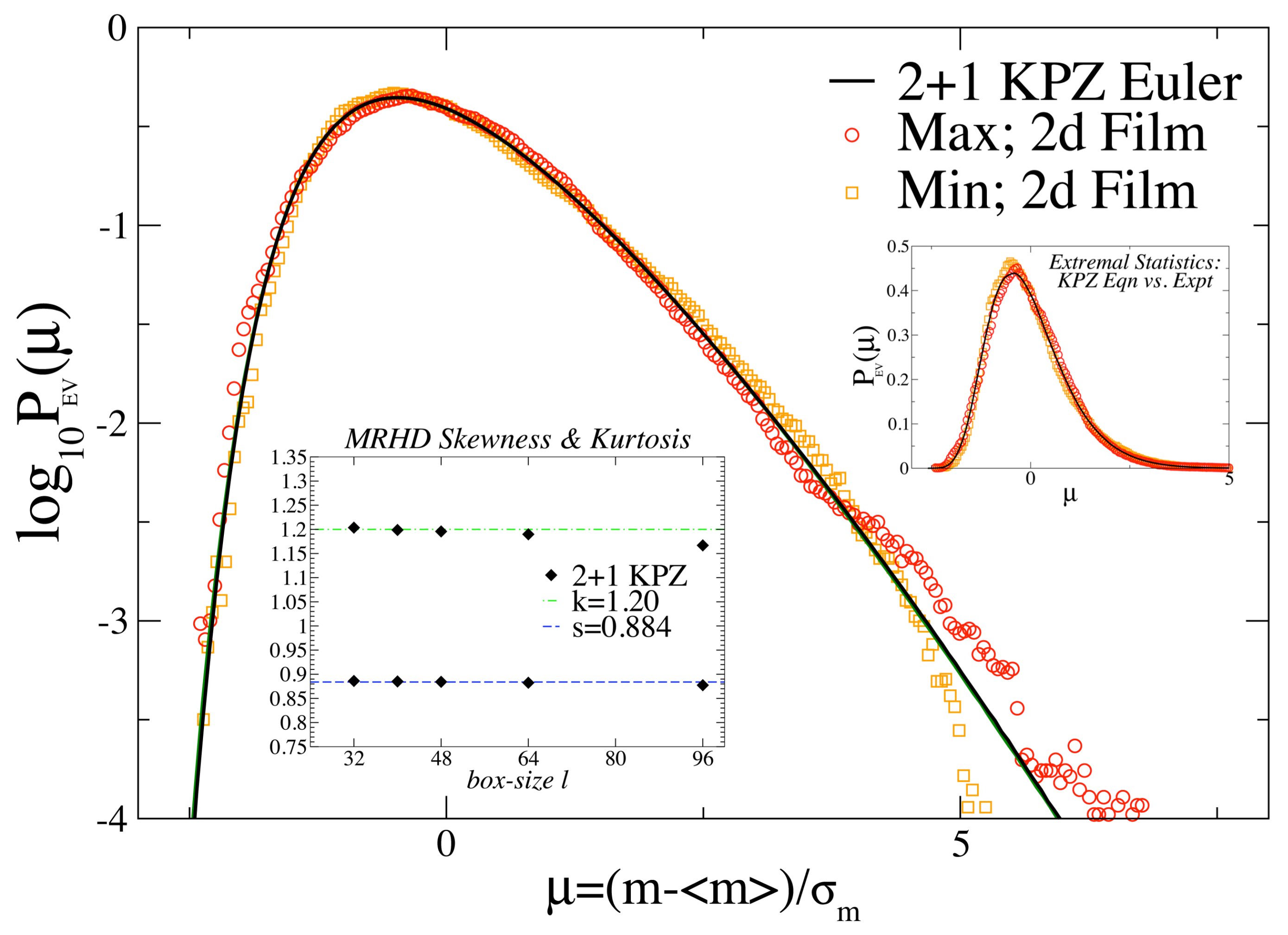
$2.55^{+}=8x5^{1/2}/7$

JSTAT (2007) P02009
-Santachiara, Rosso,
Krauth

II. Extremal Height Distributions:

(WBC)

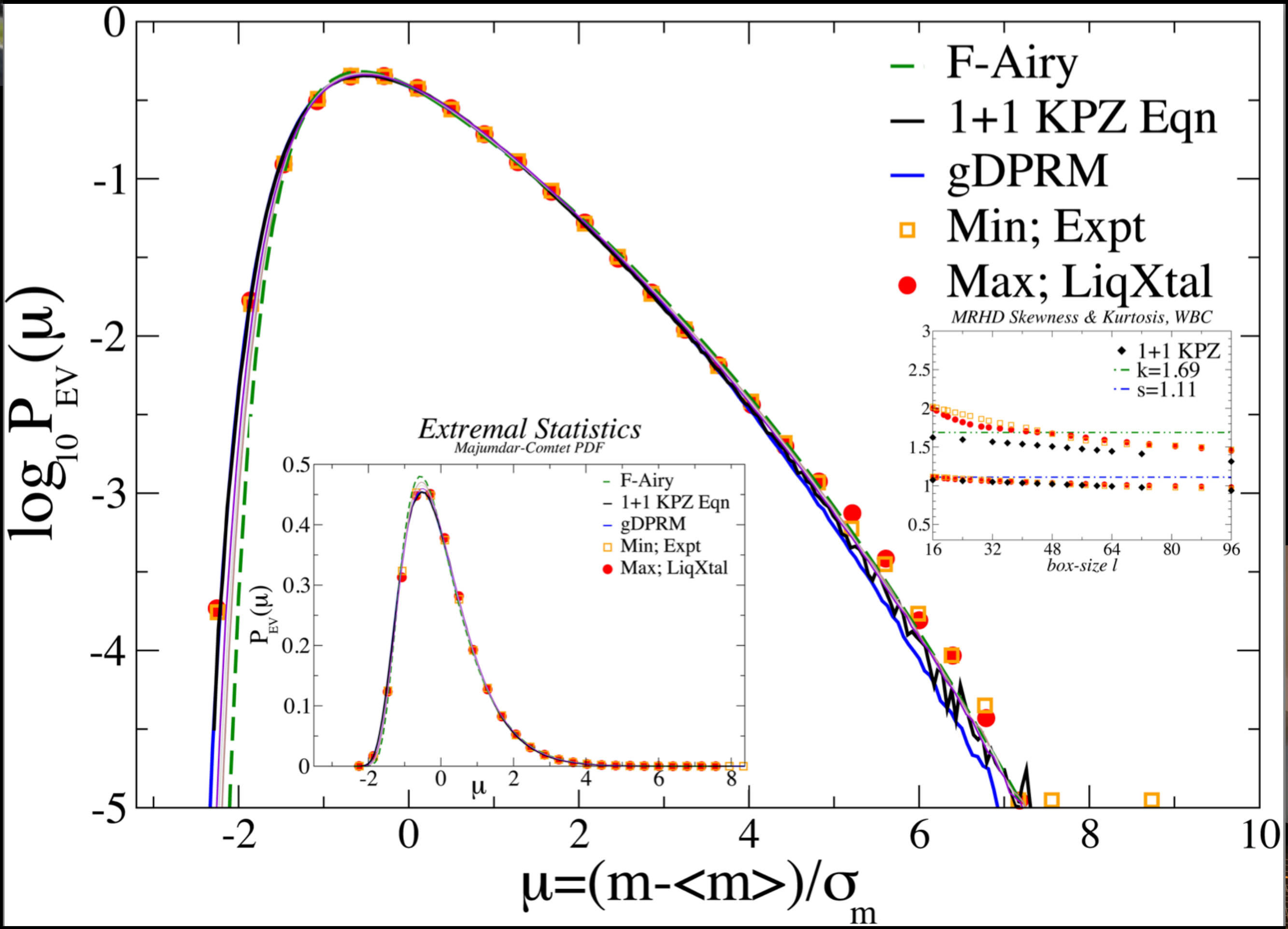
THH & Palasantzas, EPL105,50001(2014).



Check
d=1+1....

1+1 KPZ Class: MHD

(Takeuchi & Sano- Liquid Crystal Expt vs. KPZ Euler...)



Majumdar-Comtet "F-Airy" Distribution...

PRL 92, 225501 (2004)