Determinantal structures in the O'Connell-Yor polymer model

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Joint work with Tomohiro Sasamoto

Ref: arXiv:1506.05548

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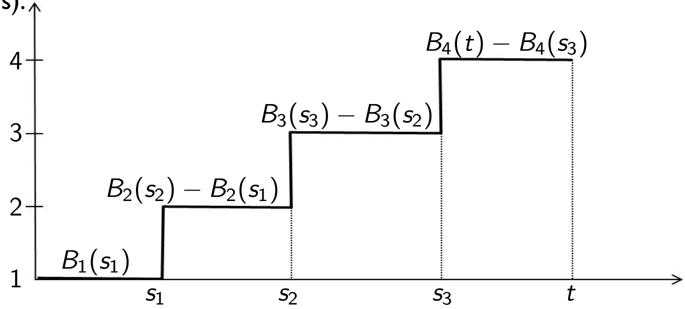
The O'Connell-Yor polymer

- It is introduced by O'Connell-Yor(2001). A typical model of the directed polymer in random environment in two dimension (one discrete + one continuous).



- The polymer partition function ($B_j(x,t) = B_j(t) - B_j(s),\ j = 1, \cdots, N$)

$$Z_N(t) = \int_{0<s_1<\cdots<s_{N-1}<t} e^{\beta(B_1(s_1)+B_2(s_1,s_2)+\cdots+B_N(s_{N-1},t))} ds_1 \cdots ds_{N-1}$$

$\beta = 1/k_B T$: inverse temperature

The zero-temperature limit $\beta \to \infty$

- First we focus on the zero-temperature case ($\beta \to \infty$). In this limit, the polymer free energy $F_N(t)$ becomes

$$-F_N(t) := \frac{1}{\beta} \log Z_N(t) = \frac{1}{\beta} \log \int_{0=s_0<s_1<\cdots<s_N=t} e^{\beta \sum_{j=1}^{N}(B_j(t_j)-B_j(t_{j-1})))} ds_1 \cdots ds_{N-1}$$

$$\to f_N(t) = \max_{0=s_0<s_1<\cdots<s_N=t} \sum_{i=1}^{N} (B_i(s_i) - B_i(s_{i-1}))$$

- Baryshnikov (2001) and Gravner-Tracy-Widom (2001) found that $f_N(t)$ has the same law as the largest eigenvalue of the Gaussian Unitary Ensemble(GUE) in the random matrix theory:

$$\mathrm{Prob}(f_N(t) < s) = \int_{-\infty}^{s} dx_1 \cdots dx_N P_{\mathrm{GUE}}(x_1, \cdots, x_N; t).$$

$$P_{\mathrm{GUE}}(x_1, \cdots, x_N; t) = \prod_{j=1}^{N} \frac{e^{-x_j^2/2t}}{j! t^{j-1} \sqrt{2\pi t}} \cdot \prod_{1 \le j < k \le N} (x_k - x_j)^2$$

The Whittaker measure

- In the finite temperature case, O'Connell(2010) first obtained the following relation using $\Psi_\theta(x)$: the Whittaker function.

$$\mathrm{Prob}(F_N(t) < s) = \int_{-\infty}^{s} dx_1 \cdots dx_N P_{\mathrm{Wh}}(x_1, \cdots, x_N; t)$$

$$P_{\mathrm{Wh}}(x_1, \cdots, x_N; t) = \Psi_0(\beta x) \int_{i\mathbb{R}} d\lambda \Psi_{-\lambda/\beta}(\beta x) e^{\sum_{i=1}^{N} \lambda_i^2 t/2} s_N(\lambda/\beta),$$

$$s_N(\lambda) = \frac{1}{(2\pi i)^N N!} \prod_{i<j} \frac{\sin \pi(\lambda_i - \lambda_j)}{\pi} \prod_{i>j} (\lambda_i - \lambda_j) \text{ : Sklyanin measure}$$

- The Whittaker function is a generalization of the Vandermonde determinant $\prod_{1\le j<k\le N}(x_k - x_j)$. But it does not inherit the determinant structure.

Fredholm determinant representation

- Borodin-Corwin (2011) obtained

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}}\right)=\det\left(1+L\right)_{L^2(C_0)}$$

where C_0 denotes the contour enclosing only the origin positively with radius $r<\beta/2$ and the kernel $L(v,v';t)$ is written as

$$L(v,v';t)=\frac{1}{2\pi i}\int_{i\mathbb{R}+\delta}dw\,\frac{\pi/\beta}{\sin(v'-w)/\beta}\frac{w^N e^{w^2t/2-wu}}{v'^N e^{v'^2t/2-v'u}}\frac{1}{w-v}\frac{\Gamma(1+v'/\beta)^N}{\Gamma(1+w/\beta)^N}.$$

Here δ satisfies the condition $r<\delta<\beta-r$.

- Now many generalized models which has the Fredholm determinant representations have been discovered. (Integrable probability)
- not a determinantal point process (except the Schur case).
- determinantal structures behind the nondeterminantal point processes. Katori (2010-): relation to the complex Brownian motion and Martingale

The Kardar-Parisi-Zhang equation (1986)

- $h(T,X)$: height at time $T(>0)$ and position $X(\in \mathbb{R})$,

$$\frac{\partial}{\partial T}h(T,X) = \frac{1}{2}\frac{\partial^2}{\partial X^2}h(T,X) + \frac{1}{2}\left(\frac{\partial}{\partial X}h(T,X)\right)^2 + \eta(T,X).$$

- Sasamoto-Spohn (2010) and Amir-Corwin-Quastel(2010) obtained the hight distribution function. with narrow wedge initial data

$$\text{Prob}\left(h(t,0) - \frac{t}{24} \geq \gamma_t s\right),\ \gamma_t = \left(\frac{t}{2}\right)^{\frac{1}{3}}$$

  In the limit $t \to \infty$, it becomes the GUE Tracy-Widom distribution.
- Takeuchi-Sano (2010-): Experiments (turbulent liquid crystal)

A Fredholm determinant in the KPZ equation

- Dotsenko (2010) and Calabrese-Le Doussal-Rosso (2010) focused on the Laplace transform of the exponential height

$$\left\langle e^{-\zeta e^{h(t,x)}} \right\rangle \left(" = " \sum_{N=0}^{\infty} \frac{-\zeta^N}{N!} \langle e^{Nh(t,x)} \rangle \right)$$

  and obtained a Fredholm determinant representation using the replica method.
  $(\tilde{h}(T,Y) = \frac{h(T, 2\gamma_T^2 Y) + \gamma_T^3/12}{\gamma_T} + Y^2)$

$$\mathbb{E}\left(e^{-e^{\gamma_T(\tilde{h}(T,Y)-s)}} \right) = \det\left(1 - \mathcal{K}_{\mathsf{KPZ}}\right)_{L^2(\mathbb{R})},$$

$$\mathcal{K}_{\mathsf{KPZ}}(\xi_1, \xi_2) = \frac{e^{\gamma_T(\xi_1 - s)}}{e^{\gamma_T(\xi_1 - s)} + 1} \int_0^\infty d\lambda \, \mathrm{Ai}(\xi_1 + \lambda) \mathrm{Ai}(\xi_2 + \lambda).$$

- Dean-Le Doussal-Majumdar-Schehr (2015): A relation to fimite-temperature free Fermion.

Main result

T.I.-T.Sasamoto: arXiv:1506.05548

$$\mathbb{E}\left[\exp\left(-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}\right)\right] = \int_{\mathbb{R}^N}\prod_{j=1}^{N} dx_j f_F(x_j-u)\cdot W(x_1,\cdots,x_N;t),$$

$$f_F(x)=\frac{1}{e^{\beta x}+1}\ \text{: the Fermi distribution function}$$

$$W(x_1,\cdots,x_N;t)=\prod_{j=1}^{N}\frac{1}{j!}\cdot\prod_{1\le j<k\le N}(x_k-x_j)\cdot\det\left(\psi_{k-1}(x_j;t)\right)_{j,k=1}^{N},$$

$$\psi_k(x;t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dw e^{-iwx-w^2t/2}\frac{(iw)^k}{\Gamma\left(1+iw/\beta\right)^N}.$$

- $W(x;t)dx$: a signed measure
- A product of the two determinants
  Random matrix techniques can be applied.

The zero-temperature limit $\beta \to \infty$

Now we consider the zero-temperature limit($\beta \to \infty$) of our relations

$$\mathbb{E}\left[\exp\left(-\frac{e^{-\beta u} Z_N(t)}{\beta^{2(N-1)}}\right)\right] = \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_j f_F(x_j - u) \cdot W(x_1, \cdots, x_N; t),$$

$$W(x_1, \cdots, x_N; t) = \prod_{j=1}^{N} \frac{1}{j!} \cdot \prod_{1 \le j < k \le N} (x_k - x_j) \cdot \det\left(\psi_{k-1}(x_j; t)\right)_{j,k=1}^{N}$$

- In the zero-temperature limit, they go to the result by Baryshnikov(2001), Gravner-Tracy-Widom(2001)

$$\mathrm{Prob}(-f_N(t) < s) = \int_{-\infty}^{s} dx_1 \cdots dx_N P_{\mathrm{GUE}}(x_1, \cdots, x_N; t).$$

$$f_N(t) = \max_{0=s_0<s_1<\cdots<s_N=t} \sum_{i=1}^{N} (B_i(s_i) - B_i(s_{i-1}))$$

$$P_{\mathrm{GUE}}(x_1, \cdots, x_N; t) = \prod_{j=1}^{N} \frac{e^{-x_j^2/2t}}{j!\, t^{j-1}\sqrt{2\pi t}} \cdot \prod_{1 \le j < k \le N} (x_k - x_j)^2$$

O'Connell's representation revisited

- O'Connell (2010): A determinantal representation

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}}\right) = \int_{(i\mathbb{R}-\epsilon)^N} \prod_{j=1}^{N} \frac{d\lambda_j}{\beta} e^{-u\lambda_j+\lambda_j^2 t/2} \Gamma\left(-\frac{\lambda_j}{\beta}\right)^N \cdot s_N\left(\frac{\lambda}{\beta}\right)$$

$$s_N(\lambda) = \frac{1}{(2\pi i)^N N!} \prod_{i<j} \frac{\sin \pi(\lambda_i - \lambda_j)}{\pi} \prod_{i>j} (\lambda_i - \lambda_j) \text{ : Sklyanin measure}$$

- We rewrite the above expression and obtained the following.
Proposition (arXiv:1506.05548)

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}}\right) = \int_{-\infty}^{\infty} \prod_{\ell=1}^{N} dt_\ell f_F(t_\ell - u) \cdot \det\left(F_{jk}(t_j; t)\right)_{j,k=1}^{N}$$

$$F_{jk}(x;t) = \int_{i\mathbb{R}-\epsilon} \frac{d\lambda}{2\pi i} \frac{e^{-\lambda x+\lambda^2 t/2}}{\Gamma\left(\frac{\lambda}{\beta}+1\right)^N} \left(\frac{\pi}{\beta}\cot\frac{\pi\lambda}{\beta}\right)^{j-1} \lambda^{k-1}.$$

Brownian particle systems with reflection interactions

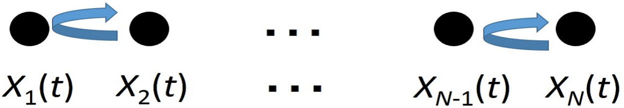
- In the zero-temperature limit, we have

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u} Z_N(t)}{\beta^{2(N-1)}}}\right) = \int_{-\infty}^{\infty} \prod_{\ell=1}^{N} dt_\ell f_F(t_\ell - u) \cdot \det\left(F_{jk}(t_j;t)\right)_{j,k=1}^{N}$$

$$\xrightarrow[\beta\to\infty]{} \mathrm{Prob}\left(f_N(t) \le u\right) = \int_{-\infty}^{u} \prod_{\ell=1}^{N} dx_\ell \cdot \det\left(\mathcal{F}_{k-j}(x_j;t)\right)_{i,j=1}^{N}$$

$$\mathcal{F}_n(x,t) = \int_{i\mathbb{R}-\epsilon} \frac{d\lambda}{2\pi i} \frac{e^{-\lambda x + \lambda^2 t/2}}{x^n}, \quad n \in \mathbb{Z}.$$

- Warren(2007) found that $\det\left(\mathcal{F}_{k-j}(x_j;t)\right)$ describes the transition density of the N-Brownian particle system where $i+1$th particle is reflected from ith and $f_N(t) \sim X_N(t)$.



Warren's approach

- Warren (2007): The zero-temperature case

$$\mathrm{Prob}\left(f_N(t) \le u\right)$$

$$= \int_{-\infty}^{u} \prod_{\ell=1}^{N} dx_1^{(\ell)} \cdot \det\left(\mathcal{F}_{k-j}(x_1^{(j)};t)\right)_{j,k=1}^{N}$$

$$X_1^{(3)} \geq X_1^{(2)} \geq X_1^{(1)}$$

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$$\begin{array}{ccccc} X_1^{(3)} & & X_2^{(3)} & & X_3^{(3)} \\ & \rotatebox{-45}{$\le$} \; \rotatebox{45}{$\le$} & & \rotatebox{-45}{$\le$} \; \rotatebox{45}{$\le$} & \\ & X_1^{(2)} & & X_2^{(2)} & \\ & & \rotatebox{-45}{$\le$} \; \rotatebox{45}{$\le$} & & \\ & & X_1^{(1)} & & \end{array}$$

$$\mathcal{Q}_{\mathrm{GT}}(\underline{x}_N;t)$$

- Warren (2007) introduced the reflected Brownian motions on the Gelfand-Tsetlin cone whose transition density $\mathcal{Q}_{\mathrm{GT}}(\underline{x}_N;t)$ is expressed as

$$\mathcal{Q}_{\mathrm{GT}}(\underline{x}_N;t) = \prod_{1\le i<j\le N}\left(x_i^{(N)} - x_j^{(N)}\right) \cdot \prod_{k=1}^{N} \frac{\exp\left(-x_k^{(N)2}/2t\right)}{t^{k-1}\sqrt{2\pi t}} \cdot 1_{\mathrm{GT}}(\underline{x}_N).$$

where $1_{\mathrm{GT}}(\underline{x}_k)$ represents the indicator function on GT cone.

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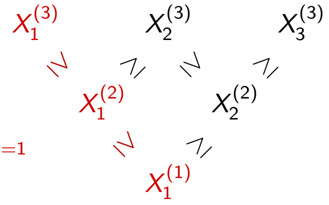

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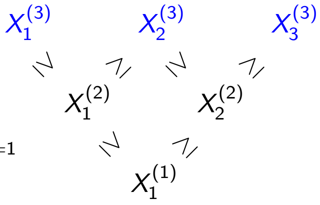

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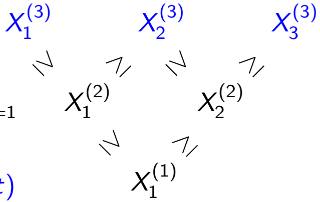
$$= \int_{-\infty}^{u} \prod_{\ell=1}^{N} dx_\ell^{(N)} \cdot P_{\rm GUE}(x_1^{(N)}, \cdots, x_N^{(N)}; t)$$



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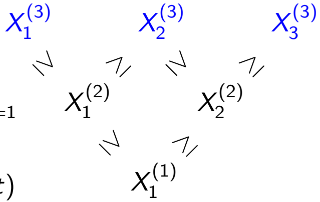

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where $1_{\mathrm{GT}}(\underline{x}_k)$ represents the indicator function on GT cone.

Warren's approach

- Warren (2007) obtained the following relations
  Proposition (Warren(2007))

$$\int_{\mathbb{R}^{N(N-1)/2}} dA_1 \mathcal{Q}_{GT}(\underline{x}_N; t) = \det\left(\mathcal{F}_{k-j}(x_1^{(j)}; t)\right)_{j,k=1}^{N} \prod_{j=1}^{N-1} 1_{>0}(x_1^{(j+1)} - x_1^{(j)}),$$

$$\int_{\mathbb{R}^{N(N-1)/2}} dA_2 \mathcal{Q}_{GT}(\underline{x}_N; t) = N! P_{\mathrm{GUE}}(x_1^{(N)}, \cdots, x_N^{(N)}; t) \prod_{j=1}^{N-1} 1_{>0}(x_j^{(N)} - x_{j+1}^{(N)}),$$

  where the function $1_{>0}(x)$ is the step function and dA_1 and dA_2 are defined as

$$dA_1 = \prod_{2\le i\le j\le N} dx_i^{(j)}, \; dA_2 = \prod_{1\le i\le j\le N-1} dx_i^{(j)}.$$

- From this relations, the equality

$$\int_{-\infty}^{u} \prod_{\ell=1}^{N} dx_1^{(\ell)} \cdot \det\left(\mathcal{F}_{k-j}(x_1^{(j)}; t)\right)_{j,k=1}^{N} = \int_{-\infty}^{u} \prod_{\ell=1}^{N} dx_\ell^{(N)} \cdot P_{\mathrm{GUE}}(x_1^{(N)}, \cdots, x_N^{(N)}; t)$$

  is established

Warren's approach

- Warren (2007): The zero-temperature case

$$\begin{aligned}
&\text{Prob}\left(f_N(t) \le u\right) \\
&= \int_{-\infty}^{u} \prod_{\ell=1}^{N} dx_1^{(\ell)} \cdot \det\left(\mathcal{F}_{k-j}(x_1^{(j)};t)\right)_{j,k=1}^{N} \\
&= \int_{-\infty}^{u} \prod_{\ell=1}^{N} dx_\ell^{(N)} \cdot P_{\text{GUE}}(x_1^{(N)},\cdots,x_N^{(N)};t)
\end{aligned}$$

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$$\mathcal{Q}_{\text{GT}}(\underline{x}_N;t)$$

- Warren (2007) introduced the stochastic process on the Gelfand-Tsetlin cone
- General β

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u} Z_N(t)}{\beta^{2(N-1)}}}\right) = \int_{-\infty}^{\infty} \prod_{\ell=1}^{N} dt_\ell f(t_\ell - u) \cdot \det\left(F_{jk}(t_j;t)\right)_{j,k=1}^{N}$$

Is there a similar structure to $\mathcal{Q}_{\text{GT}}(\underline{x}_N;t)$ and the top marginal $P_{\text{GUE}}(x,t)$?

Warren's approach

- Warren (2007): The zero-temperature case

$$
\begin{aligned}
&\mathrm{Prob}\left(f_N(t) \le u\right) \\
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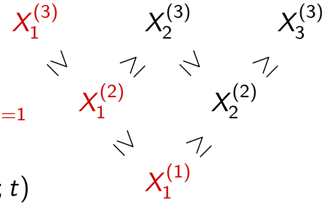
\end{aligned}
$$



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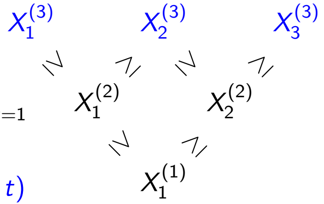

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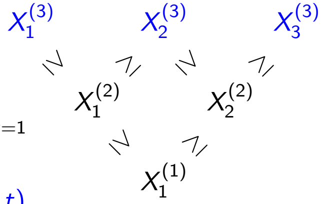
$$=\int_{-\infty}^{u}\prod_{\ell=1}^{N}dx_\ell^{(N)}\cdot P_{\mathrm{GUE}}(x_1^{(N)},\cdots,x_N^{(N)};t)$$



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$$=\int_{-\infty}^{\infty}\prod_{j=1}^{N}dx_j f(x_j-u)\cdot W(x_1,\cdots,x_N;t)$$

Is there a similar structure to $\mathcal{Q}_{\mathrm{GT}}(\underline{x}_N;t)$ and the top marginal $P_{\mathrm{GUE}}(x,t)$?

A determinantal weight on $\mathbb{R}^{N(N+1)/2}$

- We define a measure $R_u(\underline{x}_N;t)d\underline{x}_N$ by

$$R_u(\underline{x}_N;t)=\prod_{1\le i\le j\le N} f_i(x_i^{(j)}-x_{i-1}^{(j-1)})\cdot\det\left(F_{1i}(x_j^{(N)};t)\right)_{i,j=1}^{N},$$

$$x_0^{(j-1)}=u,\ \ f_i(x)=\begin{cases} f_F(x)=1/(e^{\beta x}+1), & i=1,\text{Fermi}\\ f_B(x)=1/(e^{\beta x}-1), & i\ge 2,\text{Bose}\end{cases}$$

- **Theorem** (arXiv: 1506.05548)

$$\begin{aligned}&\int_{\mathbb{R}^{\frac{N(N+1)}{2}}} d\underline{x}_N R_u(\underline{x}_N;t)\\ &=\int_{\mathbb{R}^N}\prod_{j=1}^{N} dx_1^{(j)} f_F\left(x_1^{(j)}-u\right)\cdot\det\left(F_{jk}(x_1^j;t)\right)_{j,k=1}^{N}\\ &=\int_{\mathbb{R}^N}\prod_{j=1}^{N} dx_j^{(N)} f_F\left(x_j^{(N)}-u\right)\cdot W\left(x_1^{(N)},\cdots,x_N^{(N)};t\right),\end{aligned}$$

$N = 2$ case

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}}\right) = \int_{\mathbb{R}^2} dx_1 dx_2 f_F(x_1 - u) f_F(x_2 - u) \begin{vmatrix} F_{11}(x_2) & F_{12}(x_2) \\ F_{21}(x_1) & F_{22}(x_1) \end{vmatrix}$$

$f_F(x) = 1/(e^{\beta x} + 1)$: the Fermi distribution function,

$$F_{jk}(x;t) = \int_{i\mathbb{R}-\epsilon} \frac{d\lambda}{2\pi i} \frac{e^{-\lambda x + \lambda^2 t/2}}{\Gamma\left(\frac{\lambda}{\beta}+1\right)^N} \left(\frac{\pi}{\beta}\cot\frac{\pi\lambda}{\beta}\right)^{j-1} \lambda^{k-1}.$$

$N = 2$ case

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}}\right) = \int_{\mathbb{R}^2} dx_1 dx_2 f_F(x_1-u) f_F(x_2-u) \begin{vmatrix} F_{11}(x_2) & F_{12}(x_2) \\ F_{21}(x_1) & F_{22}(x_1) \end{vmatrix}$$

$$\stackrel{?}{=} \int_{\mathbb{R}^2} dy_1 dy_2 f_F(y_1-u) f_F(y_2-u) \frac{(y_1-y_2)}{2} \begin{vmatrix} F_{11}(y_1) & F_{11}(y_2) \\ F_{12}(y_1) & F_{12}(y_2) \end{vmatrix}$$

$f_F(x) = 1/(e^{\beta x}+1)$: the Fermi distribution function,

$$F_{jk}(x;t) = \int_{i\mathbb{R}-\epsilon} \frac{d\lambda}{2\pi i} \frac{e^{-\lambda x + \lambda^2 t/2}}{\Gamma\left(\frac{\lambda}{\beta}+1\right)^N} \left(\frac{\pi}{\beta}\cot\frac{\pi\lambda}{\beta}\right)^{j-1} \lambda^{k-1}.$$

$$W(y,t) = \frac{(y_1-y_2)}{2} \begin{vmatrix} F_{11}(y_1) & F_{11}(y_2) \\ F_{12}(y_1) & F_{12}(y_2) \end{vmatrix}, \quad (\psi_{k-1}(y;t) = F_{1k}(y;t))$$

$N = 2$ case

- To prove the relation, we introduce the function $R_u(x_1^{(1)}, x_1^{(2)}, x_2^{(2)})$,

$$R_u(x_1^{(1)}, x_1^{(2)}, x_2^{(2)}) = f_F\left(x_1^{(1)} - u\right) f_F\left(x_1^{(2)} - u\right) f_B\left(x_2^{(2)} - x_1^{(1)}\right) \begin{vmatrix} F_{11}(x_1^{(2)}) & F_{11}(x_2^{(2)}) \\ F_{12}(x_1^{(2)}) & F_{12}(x_2^{(2)}) \end{vmatrix}$$

- Two relations:

1. $\displaystyle\int_{-\infty}^{\infty} dx f_B(x-y) F_{jk}(x) = F_{j+1,k}(y),$
2. $\displaystyle\int_{-\infty}^{\infty} dy f_B(x-y) f_F(y-z) = (x-z) f_F(x-z)$

$x_1^{(2)}$ $x_2^{(2)}$

$x_1^{(1)}$

The left marginal

$x_1^{(2)}$ $x_2^{(2)}$

$$\int_{-\infty}^{\infty} dx_2^{(2)} R_u(x_1^{(1)}, x_1^{(2)}, x_2^{(2)})$$

$x_1^{(1)}$

$$= \int_{-\infty}^{\infty} dx_2^{(2)} f_F\left(x_1^{(1)} - u\right) f_F\left(x_1^{(2)} - u\right) f_B\left(x_2^{(2)} - x_1^{(1)}\right) \begin{vmatrix} F_{11}(x_1^{(2)}) & F_{11}(x_2^{(2)}) \\ F_{12}(x_1^{(2)}) & F_{12}(x_2^{(2)}) \end{vmatrix}$$

The left marginal

$x_1^{(2)}$ $x_2^{(2)}$

$x_1^{(1)}$

$$
\begin{aligned}
&\int_{-\infty}^{\infty} dx_2^{(2)} R_u(x_1^{(1)}, x_1^{(2)}, x_2^{(2)}) \\
&= \int_{-\infty}^{\infty} dx_2^{(2)} f_F\left(x_1^{(1)} - u\right) f_F\left(x_1^{(2)} - u\right) f_B\left(x_2^{(2)} - x_1^{(1)}\right) \begin{vmatrix} F_{11}(x_1^{(2)}) & F_{11}(x_2^{(2)}) \\ F_{12}(x_1^{(2)}) & F_{12}(x_2^{(2)}) \end{vmatrix} \\
&= f_F\left(x_1^{(1)} - u\right) f_F\left(x_1^{(2)} - u\right) \begin{vmatrix} F_{11}(x_1^{(2)}) & \int_{-\infty}^{\infty} dx_2^{(2)} f_B\left(x_2^{(2)} - x_1^{(1)}\right) F_{11}(x_2^{(2)}) \\ F_{12}(x_1^{(2)}) & \int_{-\infty}^{\infty} dx_2^{(2)} f_B\left(x_2^{(2)} - x_1^{(1)}\right) F_{12}(x_2^{(2)}) \end{vmatrix}
\end{aligned}
$$

The left marginal

$x_1^{(2)}$ $x_2^{(2)}$

$x_1^{(1)}$

$$
\begin{aligned}
&\int_{-\infty}^{\infty} dx_2^{(2)} R_u(x_1^{(1)}, x_1^{(2)}, x_2^{(2)}) \\
&= \int_{-\infty}^{\infty} dx_2^{(2)} f_F\left(x_1^{(1)} - u\right) f_F\left(x_1^{(2)} - u\right) f_B\left(x_2^{(2)} - x_1^{(1)}\right) \begin{vmatrix} F_{11}(x_1^{(2)}) & F_{11}(x_2^{(2)}) \\ F_{12}(x_1^{(2)}) & F_{12}(x_2^{(2)}) \end{vmatrix} \\
&= f_F\left(x_1^{(1)} - u\right) f_F\left(x_1^{(2)} - u\right) \begin{vmatrix} F_{11}(x_1^{(2)}) & \int_{-\infty}^{\infty} dx_2^{(2)} f_B\left(x_2^{(2)} - x_1^{(1)}\right) F_{11}(x_2^{(2)}) \\ F_{12}(x_1^{(2)}) & \int_{-\infty}^{\infty} dx_2^{(2)} f_B\left(x_2^{(2)} - x_1^{(1)}\right) F_{12}(x_2^{(2)}) \end{vmatrix} \\
&= f_F\left(x_1^{(1)} - u\right) f_F\left(x_1^{(2)} - u\right) \begin{vmatrix} F_{11}(x_1^{(2)}) & F_{21}(x_1^{(1)}) \\ F_{12}(x_1^{(2)}) & F_{22}(x_1^{(1)}) \end{vmatrix}
\end{aligned}
$$

The top marginal

$x_1^{(2)}$ $x_2^{(2)}$

$$\int_{-\infty}^{\infty} dx_1^{(1)} R_u(x_1^{(1)}, x_1^{(2)}, x_2^{(2)})$$

$x_1^{(1)}$

$$= \int_{-\infty}^{\infty} dx_1^{(1)} f_F\left(x_1^{(1)} - u\right) f_F\left(x_1^{(2)} - u\right) f_B\left(x_2^{(2)} - x_1^{(1)}\right) \begin{vmatrix} F_{11}(x_1^{(2)}) & F_{11}(x_2^{(2)}) \\ F_{12}(x_1^{(2)}) & F_{12}(x_2^{(2)}) \end{vmatrix}$$

The top marginal

$x_1^{(2)}$ $\qquad$ $x_2^{(2)}$

$$\int_{-\infty}^{\infty} dx_1^{(1)} R_u(x_1^{(1)}, x_1^{(2)}, x_2^{(2)})$$

$x_1^{(1)}$

$$= \int_{-\infty}^{\infty} dx_1^{(1)} f_F\left(x_1^{(1)} - u\right) f_F\left(x_1^{(2)} - u\right) f_B\left(x_2^{(2)} - x_1^{(1)}\right) \begin{vmatrix} F_{11}(x_1^{(2)}) & F_{11}(x_2^{(2)}) \\ F_{12}(x_1^{(2)}) & F_{12}(x_2^{(2)}) \end{vmatrix}$$

$$= \int_{-\infty}^{\infty} f_B\left(x_2^{(2)} - x_1^{(1)}\right) f_F\left(x_1^{(1)} - u\right) dx_1^{(1)} f_F\left(x_1^{(2)} - u\right) \begin{vmatrix} F_{11}(x_1^{(2)}) & F_{11}(x_2^{(2)}) \\ F_{12}(x_1^{(2)}) & F_{12}(x_2^{(2)}) \end{vmatrix}$$

$x_1^{(2)}$ $x_2^{(2)}$

$$\int_{-\infty}^{\infty} dx_1^{(1)} R_u(x_1^{(1)}, x_1^{(2)}, x_2^{(2)})$$

$x_1^{(1)}$

$$= \int_{-\infty}^{\infty} dx_1^{(1)} f_F\left(x_1^{(1)} - u\right) f_F\left(x_1^{(2)} - u\right) f_B\left(x_2^{(2)} - x_1^{(1)}\right) \begin{vmatrix} F_{11}(x_1^{(2)}) & F_{11}(x_2^{(2)}) \\ F_{12}(x_1^{(2)}) & F_{12}(x_2^{(2)}) \end{vmatrix}$$

$$= \int_{-\infty}^{\infty} f_B\left(x_2^{(2)} - x_1^{(1)}\right) f_F\left(x_1^{(1)} - u\right) dx_1^{(1)} f_F\left(x_1^{(2)} - u\right) \begin{vmatrix} F_{11}(x_1^{(2)}) & F_{11}(x_2^{(2)}) \\ F_{12}(x_1^{(2)}) & F_{12}(x_2^{(2)}) \end{vmatrix}$$

$$= \left(x_2^{(2)} - u\right) f_F\left(x_2^{(2)} - u\right) f_F\left(x_1^{(2)} - u\right) \begin{vmatrix} F_{11}(x_1^{(2)}) & F_{11}(x_2^{(2)}) \\ F_{12}(x_1^{(2)}) & F_{12}(x_2^{(2)}) \end{vmatrix}$$

The top marginal (symmetrization)

$$\int_{-\infty}^{\infty} dx_1^{(2)} dx_2^{(2)} dx_1^{(1)} R_u(x_1^{(1)}, x_1^{(2)}, x_2^{(2)})$$

$$= \int_{-\infty}^{\infty} dx_1^{(2)} dx_2^{(2)} f_F\left(x_2^{(2)} - u\right) f_F\left(x_1^{(2)} - u\right) \cdot \left(x_2^{(2)} - u\right) \begin{vmatrix} F_{11}(x_1^{(2)}) & F_{11}(x_2^{(2)}) \\ F_{12}(x_1^{(2)}) & F_{12}(x_2^{(2)}) \end{vmatrix}$$

$$= \int_{-\infty}^{\infty} dx_1^{(2)} dx_2^{(2)} f_F\left(x_2^{(2)} - u\right) f_F\left(x_1^{(2)} - u\right) \cdot \frac{x_2^{(2)} - x_1^{(2)}}{2} \begin{vmatrix} F_{11}(x_1^{(2)}) & F_{11}(x_2^{(2)}) \\ F_{12}(x_1^{(2)}) & F_{12}(x_2^{(2)}) \end{vmatrix}$$

$x_1^{(2)}$ $x_2^{(2)}$

$x_1^{(1)}$

$N = 2$ case (main result)

$$\int_{-\infty}^{\infty} dx_1^{(2)} dx_2^{(2)} dx_1^{(1)} R_u(x_1^{(1)}, x_1^{(2)}, x_2^{(2)})$$

$$= \int_{-\infty}^{\infty} dx_1^{(1)} dx_1^{(2)} f_F\left(x_1^{(1)} - u\right) f_F\left(x_1^{(2)} - u\right) \cdot \begin{vmatrix} F_{11}(x_1^{(2)}) & F_{21}(x_1^{(1)}) \\ F_{12}(x_1^{(2)}) & F_{22}(x_1^{(1)}) \end{vmatrix}$$

$$= \int_{-\infty}^{\infty} dx_1^{(2)} dx_2^{(2)} f_F\left(x_2^{(2)} - u\right) f_F\left(x_1^{(2)} - u\right) \frac{x_2^{(2)} - x_1^{(2)}}{2} \begin{vmatrix} F_{11}(x_1^{(2)}) & F_{11}(x_2^{(2)}) \\ F_{12}(x_1^{(2)}) & F_{12}(x_2^{(2)}) \end{vmatrix}$$

$x_1^{(2)}$ $x_2^{(2)}$

$x_1^{(1)}$

Zero-temperature limit

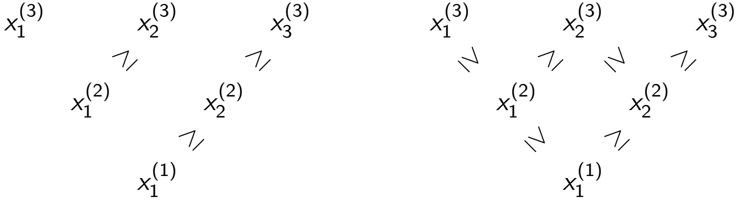
We have

$$\mathcal{R}_u(\underline{x}_N;t) := \lim_{\beta\to\infty} \det\left(F_{1i}(x_j^{(N)};t)\right)_{i,j=1}^{N} \prod_{1\le i\le j\le N} f_i(x_i^{(j)} - x_{i-1}^{(j-1)})$$

$$= \prod_{1\le i<j\le N}\left(x_i^{(N)} - x_j^{(N)}\right)\cdot \prod_{k=1}^{N} \frac{\exp -x_k^{(N)2}/2t}{t^{k-1}\sqrt{2\pi t}} \prod_{1\le j\le k\le N} 1_{>0}(x_{j-1}^{(k-1)} - x_j^{(k)}).$$

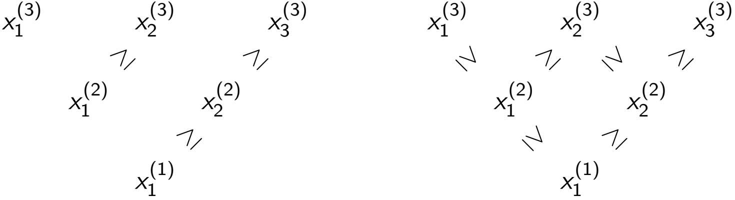
On the other hand, in Warren (2007),

$$\mathcal{Q}_{\mathrm{GT}}(\underline{x}_N;t) = \prod_{1\le i<j\le N}\left(x_i^{(N)} - x_j^{(N)}\right)\cdot \prod_{k=1}^{N} \frac{\exp\left(-x_k^{(N)2}/2t\right)}{t^{k-1}\sqrt{2\pi t}}\cdot 1_{\mathsf{GT}}(\underline{x}_N).$$



Zero-temperature limit

- Thus $\mathcal{R}_u(\underline{x}_N;t)$ and $\mathcal{Q}_{\mathrm{GT}}(\underline{x}_N;t)$ have the different supports.



- But we have

$$\int_{\mathbb{R}^{N(N+1)/2}} d\underline{x}_N \mathcal{R}_u(\underline{x}_N;t) = \int_{(-\infty,u)^{N(N+1)/2}} \mathcal{Q}_{\mathrm{GT}}(\underline{x}_N;t).$$

Dynamics of the left marginal

$X_1^{(3)}$ $X_2^{(3)}$ $X_3^{(3)}$

$X_1^{(2)}$ $X_2^{(2)}$

$X_1^{(1)}$

- $\det(F_{jk}(x_{N-j+1};t))_{j,k=1,\cdots,N}$
- The properties of $F_{jk}(x;t)$ $j,k\in\{1,2,\cdots\}$

$$\frac{\partial}{\partial t}F_{jk}(x;t)=\frac{1}{2}\frac{\partial^2}{\partial x^2}F_{jk}(x;t),$$

$$-\frac{\beta^2}{\pi^2}\int_{-\infty}^{\infty}dx_{j+1}\frac{e^{\frac{\beta}{2}(x_{j+1}-x_j)}}{e^{\beta(x_{j+1}-x_j)}-1}F_{j+1k}(x_{j+1};t)=F_{jk}(x_j,k),$$

- $\det(F_{jk}(x_{N-j+1};t))_{j,k=1,\cdots,N}$ solves the diffusion equation,

$$\frac{\partial}{\partial t}\det(F_{jk}(x_{N-j+1};t))_{j,k=1}^N=\frac{1}{2}\sum_{j=1}^N\frac{\partial^2}{\partial x_j^2}\det(F_{jk}(x_{N-j+1};t))_{j,k=1}^N,$$

with the condition ($j=1,2,\cdots,N-1$)

$$-\frac{\beta^2}{\pi^2}\int_{-\infty}^{\infty}dx_{j+1}\frac{e^{-\frac{\beta}{2}(x_{j+1}-x_j)}}{e^{\beta(x_{j+1}-x_j)}-1}\det(F_{jk}(x_{N-j+1};t))_{j,k=1}^N=0,$$

Dynamics of the left marginal (the zero-temp. limit)

- We have

$$\lim_{\beta\to\infty} \det\left(F_{j,k}(x_{N-j+1};t)\right) = \det\left(\mathcal{F}_{j-k}(x_{N-j+1};t)\right),$$

$$\mathcal{F}_n(x;t) = \int_{i\mathbb{R}-\epsilon} \frac{d\lambda}{2\pi i} \frac{e^{-\lambda x+\lambda^2 t/2}}{\lambda^n}.$$

- $\det(\mathcal{F}_{jk}(x_{N-j+1};t))_{j,k=1,\cdots,N}$ solves the diffusion equation,

$$\frac{\partial}{\partial t}\det(\mathcal{F}_{jk}(x_{N-j+1};t))_{j,k=1}^N = \frac{1}{2}\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}\det(\mathcal{F}_{jk}(x_{N-j+1};t))_{j,k=1}^N,$$

with the Neumann boundary condition ($j = 1, 2, \cdots, N-1$)

$$\frac{d}{dx_j}\det(\mathcal{F}_{jk}(x_{N-j+1};t))_{j,k=1}^N|_{x_j\to x_{j+1}} = 0.$$

- $\det(\mathcal{F}_{jk}(x_{N-j+1};t))_{j,k=1,\cdots,N}$ is the transition probability density of the N-Brownian particle system with the reflection interaction (Warren(2007)).

Dynamics of the top marginal

- The weight: $W(x;t)$. $\lim_{\beta\to\infty} W(x;t) = P_{\mathrm{GUE}}(x;t)$,

$$W(x;t) = \prod_{j=1}^{N} \frac{1}{j!} \cdot \prod_{1\le j<k\le N} (x_k - x_j) \cdot \det\left(\psi_{k-1}(x_j;t)\right)_{j,k=1}^{N},$$

$$X_1^{(3)} \qquad X_2^{(3)} \qquad X_3^{(3)}$$

$$X_1^{(2)} \qquad X_2^{(2)}$$

$$X_1^{(1)}$$

- $W(x;t)$ solves the Kolmogorov forward equation of the GUE Dyson's Brownian motion,

$$\frac{\partial}{\partial t} W(x;t) = \frac{1}{2}\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} W(x;t) - \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(\sum_{\substack{m=1\\ m\neq j}}^{N} \frac{1}{x_j - x_m}\right) W(x;t).$$

A Fredholm determinant (arXiv:1506.05548)

$$\mathbb{E}\left(\exp\left(-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}\right)\right) = \det\left(1-\bar{f}_u K\right)_{L^2(\mathbb{R})}, \quad \bar{f}_u(x) = f_F(x-u)-1$$

$$K(x,y;t) = \sum_{j=0}^{N-1} \phi_k(x;t)\psi_k(y;t)$$

$$\phi(x;t) = \frac{1}{2\pi i}\oint dv e^{vx-vt^2/2}\frac{\Gamma(v/\beta+1)^N}{v^{k+1}}$$

$$\psi_k(x;t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dw e^{-wx+w^2t/2}\frac{w^k}{\Gamma\left(1+w/\beta\right)^N}$$

- Biorthogonal relation $\int_{-\infty}^{\infty}\phi_j(x;t)\psi_k(x;t) = \delta_{j,k}$
- We can show the equivalence between our representation and Borodin-Corwin's.

$$\mathbb{E}\left(\exp\left(-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}\right)\right) = \det\left(1+L\right)_{L^2(C_0)}$$

- Borodin-Corwin-Remenik (2013)
  O'Connell's representation $\rightarrow$ the Fredholm determinant

The zero-temperature limit $\beta \to \infty$

Fredholm determinant representation

$$\mathbb{E}\left[\exp\left(-\frac{e^{-\beta u} Z_N(t)}{\beta^{2(N-1)}}\right)\right] = \det\left(1 - \bar{f}_u K\right)_{L^2(\mathbb{R})},$$

$$K(x, y; t) = \sum_{j=0}^{N-1} \phi_k(x; t) \psi_k(y; t).$$

- In the zero-temperature limit, it goes to

$$\mathrm{Prob}(f_N(t) < s) = \det\left(1 - 1_{(s,\infty)} K_{\mathrm{GUE}}\right)_{L^2(\mathbb{R})}$$

$$f_N(t) = \max_{0=s_0<s_1<\cdots<s_N=t} \sum_{i=1}^{N} (B_i(s_i) - B_i(s_{i-1}))$$

$$K_{\mathrm{GUE}}(x_1, x_2; t) = \frac{e^{-x_2^2/2t}}{\sqrt{2\pi t}} \sum_{k=0}^{N-1} \frac{H_k(x_1/\sqrt{2t}) H_k(x_2/\sqrt{2t})}{2^k k!}.$$

- $\lim_{\beta\to\infty} (2t)^{\frac{k+1}{2}} \pi^{\frac{1}{2}} e^{\frac{x^2}{2t}} \psi_k(x; t) = \lim_{\beta\to\infty} k! \left(\frac{t}{2}\right)^{-\frac{k}{2}} \phi_k(x; t) = H_k\left(\frac{x}{\sqrt{2t}}\right)$

The scaling limit to the KPZ equation

- Moreno Flores-Quastel , Corwin-Tsai (2015): The KPZ equation limit

$$F_N(t) \to h(T,X), \qquad \partial_T h = \frac{1}{2}\partial_{XX} h + \frac{1}{2}(\partial_X h)^2 + \eta$$

- By the saddle point analysis, we find in the KPZ scaling, ($\gamma_T = (T/2)^{1/3}$)

$$\lim_{N\to\infty} \psi_k(x;t)/C(N) = \lim_{N\to\infty} C(N)\phi_k(x;t) = \frac{\mathrm{Ai}(\xi-\lambda)}{\gamma_T}$$

which leads to the pointwise convergence to the KPZ kernel.

$$\bar{f}_u(x_i)K(x_i,x_j) = \bar{f}_u(x_i)\sum_{j=0}^{N-1}\phi_k(x_i;t)\psi_k(x_j;t) \to \mathcal{K}_{\mathsf{KPZ}}(\xi_i,\xi_j)$$

$$\mathcal{K}_{\mathsf{KPZ}}(\xi_i,\xi_j) = \frac{e^{\gamma_T(\xi_i - s)}}{e^{\gamma_T(\xi_i - s)}+1}\int_0^\infty d\lambda \mathrm{Ai}(\xi_i+\lambda)\mathrm{Ai}(\xi_j+\lambda)$$

- This implies that our relation recovers the relation in the KPZ equation.

$$\mathbb{E}_{\mathsf{KPZ}}\left(e^{-e^{\gamma_T(h(X,T)-s)}}\right) = \det\left(1-\mathcal{K}_{\mathsf{KPZ}}\right)$$

Summary

- We have considered the O'Connell-Yor random directed polymer

$$Z_N(t) = \int_{0<s_1<\cdots<s_{N-1}<t} e^{\beta(B_1(s_1)+B_2(s_1,s_2)+\cdots+B_N(s_{N-1},t))} ds_1 \cdots ds_{N-1}$$

- We have obtained the relation

$$\mathbb{E}\left[\exp\left(-\frac{e^{-\beta u} Z_N(t)}{\beta^{2(N-1)}}\right)\right] = \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_j f_F(x_j - u) \cdot W(x_1, \cdots, x_N; t).$$

To get it, we introduced the determinantal weight on $\mathbb{R}^{\frac{N(N+1)}{2}}$

$$R_u(\underline{x}_N; t) = \prod_{1\le i\le j\le N} f_i(x_i^{(j)} - x_{i-1}^{(j-1)}) \cdot \det\left(F_{1i}(x_j^{(N)}; t)\right)_{i,j=1}^{N}$$

This is a finite-temperature generalization of the pdf of the interacting BMs on GT cone (Warren 2007).