Asymptotics in periodic TASEP with step and flat initial conditions

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Overview

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   - Our results
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   - Transition Probability
   - Distribution of the k-th particle
   - Fredholm determinant formula

Periodic TASEP

we consider the following periodic TASEP model on the configuration space
$\mathfrak{X}_N(M) = \{(x_1, x_2 \cdots , x_N); x_i \in \mathbb{Z}, x_1 < x_2 < \cdots < x_N < x_1 + M\}$.
Each particle has an independent clock which will ring after an exponential waiting time with parameter 1. Once a clock rings, it will be reset. And the corresponding particle moves to the right by 1 if the resulting configuration is still in $\mathfrak{X}_N(M)$, otherwise it does not move.

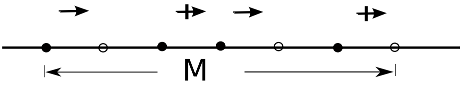


Figure : Illustration of periodic TASEP

Macroscopic picture

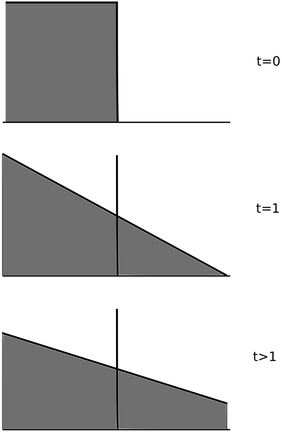


Figure : Evolution of the density for step initial condition and $\rho = 1/2$

Let the ratio $N/M = \rho$ fixed. We are interested in the fluctuations of $x_j(t)$ for both step and flat initial conditions as $M, N, t \to \infty$, where $j = [\alpha N]$ for fixed $\alpha \in (0, 1)$.

Intuition from LPP

W_{11}	W_{12}	W_{13}	W_{14}	W_{15}	W_{16}	W_{17}	W_{18}	W_{19}
W_{21}	W_{22}	W_{23}	W_{24}	W_{25}	W_{26}	W_{27}	W_{28}	W_{29}
W_{31}	W_{32}	W_{33}	W_{34}	W_{35}	W_{36}	W_{37}	W_{38}	W_{39}
W_{41}	W_{42}	W_{43}	W_{44}	W_{45}	W_{46}	W_{47}	W_{48}	W_{49}
			W_{11}	W_{12}	W_{13}	W_{14}	W_{15}	W_{16}
			W_{21}	W_{22}	W_{23}	W_{24}	W_{25}	W_{26}
			W_{31}	W_{32}	W_{33}	W_{34}	W_{35}	W_{36}
			W_{41}	W_{42}	W_{43}	W_{44}	W_{45}	W_{46}
						W_{11}	W_{12}	W_{13}
						W_{21}	W_{22}	W_{23}
						W_{31}	W_{32}	W_{33}
						W_{41}	W_{42}	W_{43}

Figure : periodic LPP

Intuition from LPP

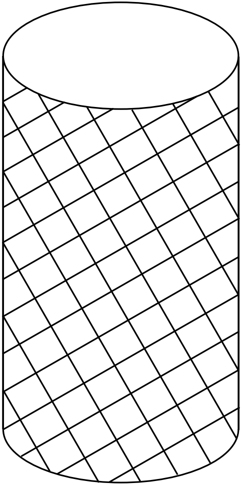


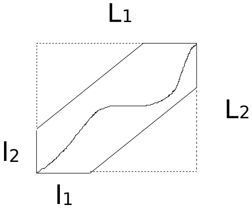
Figure : cylinder LPP

LPP with i.i.d. exponential entries

It is known that (Johansson, 1999)

$$H(L_1, L_2) \sim (\sqrt{L_1} + \sqrt{L_2})^2 + (\sqrt{L_1} + \sqrt{L_2})^{4/3} L_1^{-1/6} L_2^{-1/6} \chi_{TW}$$

where χ_{TW} is the GUE Tracy-Widom random variable. Moreover, the transversal exponent is 2/3(Johansson, Baik-Deift-McLaughlin-Miller-Zhou). In fact the probability that the longest path stays in a band of width $L_1^{2/3+\epsilon}$ is $1 - \exp(-cL_1^{2\epsilon})$ (Basu-Sidoravicius-Sly).



LPP with block-periodic exponential entries

$$H^{(P)}(L_1, L_2) \sim (\sqrt{L_1} + \sqrt{L_2})^2 + (\sqrt{L_1} + \sqrt{L_2})^{4/3} L_1^{-1/6} L_2^{-1/6} \chi_{TW}$$

if the period is of size $L_1^{2/3+\epsilon}$. Similar transversal result holds in this case. Therefore $H^{(P)}(L_1, L_2) \sim H(L_1, L_2)$ when the period is of size $L_1^{2/3+\epsilon}$.

Step initial condition: when $t < N^{3/2}$

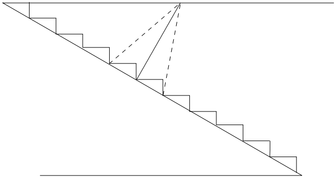


Figure : Main contribution from one single corner

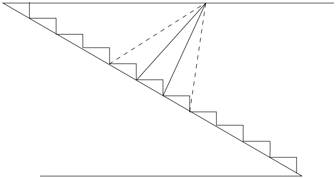


Figure : Main contribution from two corners

Theorem (J. Baik, Z. Liu)

$$\lim_{N\to\infty} \mathbb{P}\left(x_{[\alpha N]}(\tau N) \geq \mu_j N - x\tau^{1/6} N^{2/3}\right)$$
$$= \begin{cases} F_{GUE}(\gamma_{j-1}x)F_{GUE}(\gamma_j x), & \tau = \tau_j, \\ F_{GUE}(\gamma_j x), & \tau_j < \tau < \tau_{j+1}. \end{cases}$$

$$\tau_j = \frac{(\sqrt{j-\alpha} + \sqrt{j+1-\alpha})^2}{4\rho^2}$$
$$\mu_j = \tau - 2\sqrt{(j+1-\alpha)\tau} + j\rho^{-1},$$
$$\gamma_j = (j+1-\alpha)^{1/6}(\sqrt{\tau} - \sqrt{j+1-\alpha})^{-2/3}.$$

Step initial condition: when $t < N^{3/2}$

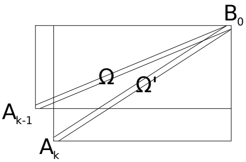


Figure : The two paths are asymptotically independent

This result is valid until $t = O(N^{3/2-\epsilon})$. $t = N^{3/2}$ is the so-called relaxation scale, which was first predicted by Gwa and Spohn (1992).

Step initial condition: when $t = N^{3/2}$

Theorem (J. Baik, Z. Liu)

Suppose $\rho \in (0,1)$, $\gamma \in \mathbb{R}$ and $\tau > 0$ are all fixed. We have

$$\mathbb{P}\left(\frac{x_{k_N}(t_N) - x_{k_N}(0) - (1-\rho)t_N + (\rho^{-1}-1)(N-k_N)}{\sqrt{2}\rho(1-\rho)^{-1/2}N^{1/2}} \geq -x \right) \to F(x;\tau,\gamma)$$

where $1 \leq k_N \leq N$, and

$$t_N = \frac{1}{\rho^2} N \left(\left[\frac{\tau}{\sqrt{1-\rho}} N^{1/2} \right] + \gamma \right) - \frac{1}{\rho^2} k_N.$$

Remark: Recently Prolhac obtained a similar result $\rho = \frac{1}{2}$.

The function $F(x;\tau,\gamma)$ is a periodic in γ. The explicit formula is given by

$$F(x;\tau,\gamma) := \oint_{|\mathfrak{s}|=r} e^{A(\mathfrak{s},\tau,x)} \det(1+\mathcal{K})|_{l^2(\mathfrak{R}_{\mathfrak{s},L})} \frac{d\mathfrak{s}}{2\pi i\mathfrak{s}},$$

where r is a constant in $(0,1)$, $\mathfrak{H}$ is a function independent of γ and τ, and $A(\mathfrak{s},\tau,x) = \mathfrak{H}(\mathfrak{s}) - 2c\tau\mathrm{Li}_{5/2}(\mathfrak{s}) - cx\mathrm{Li}_{3/2}(\mathfrak{s})$.

$\det(1+\mathcal{K})$ is the Fredholm determinant acting on $l^2(\mathfrak{R}_{s,L})$ with kernel $\mathcal{K}(\xi_1,\xi_2)$

$$\sum_{\eta\in\mathfrak{R}_{\mathfrak{s},R}} \frac{e^{\phi\left(-\frac{2\sqrt{2}\tau}{3},\gamma,\frac{x}{2};\xi_1\right)+\phi\left(-\frac{2\sqrt{2}\tau}{3},\gamma,\frac{x}{2};\xi_2\right)-2\phi\left(-\frac{2\sqrt{2}\tau}{3},\gamma,\frac{x}{2};\eta\right)}}{4\sqrt{\xi_1\xi_2}\eta(\xi_1-\eta)(\xi_2-\eta)},$$

where

$$\mathfrak{R}_{\mathfrak{s},L} := \{\xi;\, e^{-\xi^2}=\mathfrak{s}, \Re\xi<0\}, \quad \mathfrak{R}_{\mathfrak{s},R} := \{\xi;\, e^{-\xi^2}=\mathfrak{s}, \Re\xi>0\},$$

$$\phi(d_3,d_2,d_1;\xi) := \begin{cases} \mathfrak{h}_R(\mathfrak{s},\xi)+\frac{1}{2}(d_3\xi^3+d_2\xi^2+d_1\xi), & \xi\in\mathfrak{R}_{\mathfrak{s},L}, \\ -\mathfrak{h}_L(\mathfrak{s},\xi)+\frac{1}{2}(d_3\xi^3+d_2\xi^2+d_1\xi), & \xi\in\mathfrak{R}_{\mathfrak{s},R}, \end{cases}$$

$$\mathfrak{h}_L(\mathfrak{s},\xi) := -2\int_{-i\infty}^{i\infty} \frac{\log(-\eta+\xi)\eta}{\mathfrak{s}^{-1}e^{-\eta^2}-1}\frac{d\eta}{2\pi i},$$

$$\mathfrak{h}_R(\mathfrak{s},\xi) := 2\int_{-i\infty}^{i\infty} \frac{\log(\eta-\xi)\eta}{\mathfrak{s}^{-1}e^{-\eta^2}-1}\frac{d\eta}{2\pi i}.$$

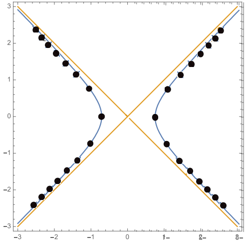


Figure : The nodes set $\mathfrak{R}_{s,L}$ and $\mathfrak{R}_{s,R}$

Remarks on $F(\tau,\gamma;x)$

- Periodicity on γ: $F(\tau,\gamma;x) = F(\tau,\gamma+1;x)$.
- Comparison with F_{GUE}

$$F_{GUE}(x) = \det\left(1+\mathcal{K}\right)\big|_{L^2(\Sigma_L)} \tag{1}$$

with

$$\mathcal{K}(\xi_1,\xi_2) := \int_{\Sigma_R} \frac{e^{\left(-\frac{\xi_1^3}{6}+\frac{x\xi_1}{2}\right)+\left(-\frac{\xi_2^3}{6}+\frac{x\xi_2}{2}\right)-2\left(-\frac{\eta^3}{6}+\frac{x\eta}{2}\right)}}{(\xi_1-\eta)(\xi_2-\eta)}\frac{d\eta}{2\pi i} \tag{2}$$

where Σ_L is a contour from $e^{-2\pi/3}\infty$ to $e^{2\pi/3}\infty$ and $\Sigma_R = -\Sigma_L$.

- The parameter γ only appears in the Fredholm determinant.

Where the periodicity comes from

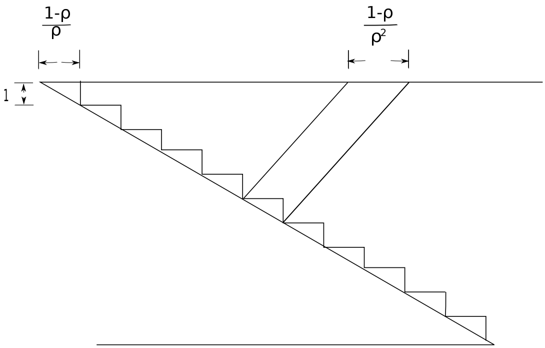


Figure : Illustration of the periodicity

γ is the parameter containing the information of the location between two periods.

Flat initial condition

Similar result holds for the periodic TASEP model with flat initial condition $y_j = j\Delta$ and $M = N\Delta$.

$$\mathbb{P}\left(\frac{x_{k_N}(t_N) - x_{k_N}(0) - (1-\rho)t_N}{\sqrt{2}\rho(1-\rho)^{-1/2}N^{1/2}} \geq -x\right) \to F(x;\tau)$$

where $t_N = \frac{1}{\rho^2\sqrt{1-\rho}}N^{3/2}$, and $F(x;\tau)$ has similar structure with $F(x;\tau,\gamma)$. The explicit formula is given by

$$\int_{|\mathfrak{s}|=\mathfrak{r}} \exp\left(\frac{1}{2}\mathfrak{H}(\mathfrak{s}) - 2c\tau \mathrm{Li}_{5/2}(\mathfrak{s}) - cx\mathrm{Li}_{3/2}(\mathfrak{s}) - \frac{1}{2}\mathfrak{h}_R(\mathfrak{s},0)\right)$$

$$\det\left(1+\tilde{\mathcal{K}}\right)\Big|_{l^2(\mathfrak{R}_{\mathfrak{s},L})} \frac{d\mathfrak{s}}{2\pi i\mathfrak{s}},$$

$$\tilde{\mathcal{K}}(\xi,\eta) := \frac{e^{\mathfrak{h}_R(\mathfrak{s},\xi)+\mathfrak{h}_L(\mathfrak{s},-\eta)-\frac{2\sqrt{2}}{3}\tau\xi^3+x\xi-\frac{2\sqrt{2}}{3}\tau\eta^3+x\eta}}{-2\xi(\xi+\eta)}. \qquad (3)$$

Transition probability for periodic TASEP

Theorem [J. Baik, Z. Liu]

Denote $P_Y(X;t)$ the transition probability. Then for any $X, Y \in \mathfrak{X}_n(M)$, we have

$$P_Y(X;t) = \oint_{|s|=r} \det\left[\frac{1}{M}\sum_{z\in R_s}\frac{z^{j-i+1}(z+1)^{-x_i+y_j+i-j}e^{tz}}{z+\rho}\right]_{i,j=1}^{N}\frac{ds}{2\pi i s},$$

where R_s is the set of all roots of $z^N(z+1)^{M-N} = s^M$, and $r > 0$ is a small constant.

We followed the idea of Tracy-Widom on the formula of ASEP. We consider a system of equations of $u(X;t)$ where $(X;t) = (x_1, \cdots, x_N; t) \in \mathbb{Z}^N \times \mathbb{R}_{\geq 0}$.

$$\text{Master Equations :} \quad \frac{d}{dt} u(X;t) = \sum_{i=1}^{N} (u(X_i;t) - u(X;t))$$

where $X_i := (x_1, \cdots, x_{i-1}, x_i - 1, x_{i+1}, \cdots, x_N)$.

$$\text{Boundary Conditions 1 :} \quad u(X_i;t) = u(X;t) \text{ if } x_i = x_{i-1} + 1$$

$$\text{Boundary Conditions 2 :} \quad u(X_1;t) = u(X;t) \text{ if } x_N = x_1 + M - 1.$$

$$\text{Initial Condition :} \quad u(X;0) = \delta_Y(X), \text{ for all } X \in \mathfrak{X}_N(M).$$

Without the second boundary condition, it gives the transition probability for TASEP on the integer lattice:

$$\det\left[\oint_{|\xi|=\epsilon}(1-\xi)^{j-i}\xi^{x_i-y_j}e^{t(\xi^{-1}-1)}\frac{d\xi}{2\pi i\xi}\right]_{i,j=1}^{N}.$$

The extra boundary condition gives rise to the discreteness of the sum. The solution is constructive.

Distribution of the k-th particle

Suppose $x_j(t=0)=j-N$ is the initial configuration of the periodic TASEP. For any $1\leq k\leq N$ we have the following one point distribution of the k-th particle

$$P\left(x_k\geq a;t\right)=(-1)^{(k-1)(N+1)}$$

$$\oint_{|s|=r}\det\left[\frac{1}{M}\sum_{z\in R_s}\frac{z^{j-i+1-k}(z+1)^{-N-a+k+1}e^{tz}}{z+\rho}\right]_{i,j=1}^{N}\frac{ds}{2\pi i s^{1-(k-1)M}}.$$

Some typical ways to analyze the asymptotics of a discrete Toeplitz determinant:

- Discrete orthogonal polynomials, developed by Baik-Kriecherbauer-McLaughlin-Miller.
- Continuous orthogonal polynomials, by an identity which relates the discrete Toeplitz determinant to its continuous counterpart.
  The difficulties for this Toeplitz determinant are: the weight is complex-values, and the nodes are on a curve which is neither $\mathbb{R}$ nor $S = \{z; |z| = 1\}$.

Fredholm determinant representation

Let $R_{s,L}, R_{s,R}$ be the left and right parts of the roots set R_s.
Define $h_{s,L}(z) = \prod_{z_j \in R_{s,L}} (z - z_j)$ and $h_{s,R}(z) = \prod_{z_j \in R_{s,R}} (z - z_j)$.
If $Y = (-N+1, -N+2, \cdots, 0)$, then

$$P(x_k \geq a; t) = \int_{|s|=r} Z_N(s) \det(1 + K_L) \frac{ds}{2\pi i s^{1-(k-1)M}},$$

where $\det(1 + K_L)$ is the Fredholm determinant on $l^2(R_{s,L})$

$$K_L(z, w) = \sqrt{f(z) f(w)} \sum_{v \in R_{s,R}} \frac{f(v)^{-1}}{M^2 (z - v)(w - v)},$$

$$f(z) := \begin{cases} \dfrac{h_{s,R}(z)^2 z^{-N-k+2} (z+1)^{-a-N+k+1} e^{tz}}{z + \gamma^{-1}}, & z \in R_{s,L}, \\ \dfrac{h'_{s,R}(z)^2 z^{-N-k+2} (z+1)^{-a-N+k+1} e^{tz}}{M^2 (z + \gamma^{-1})}, & z \in R_{s,R}. \end{cases}$$

$Z_N(s)$ is a constant given by

$$Z_N(s) = (-1)^{(k-1)(N+1)} \prod_{z \in R_{s,R}} \frac{h'_{s,R}(z) z^{-N-k+2} (z+1)^{-a-N+k+1} e^{tz}}{M(z+\rho)}.$$

It turns out that when $t = O(N^{3/2})$, the leading term of $Z_N(s)$ is

$$s^{(k-1)M} \exp\left(A(\mathfrak{s}, \tau, x)\right)$$

and the leading term of the Fredholm determinant is $\det(1+\mathcal{K})$ under proper scaling.

Thank you!