Finite temperature free fermions and the Kardar-Parisi-Zhang equation at finite time

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In collaboration with

- David S. Dean (LOMA, Univ. de Bordeaux)
- Pierre Le Doussal (LPT ENS, Paris)
- Grégory Schehr (LPTMS, Univ. d'Orsay)

- d=1, finite T: Phys. Rev. Lett. 114, 110402 (2015), arXiv:1412.1590
- d>1, T=0: Europhys. Lett. 112, 60001 (2015), arXiv:1505.01543

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Acknowledgements to Christophe Salomon (LKB, ENS Paris)

Ultra-cold atoms in confining potentials

- Recent progress in the experimental manipulation of cold atoms

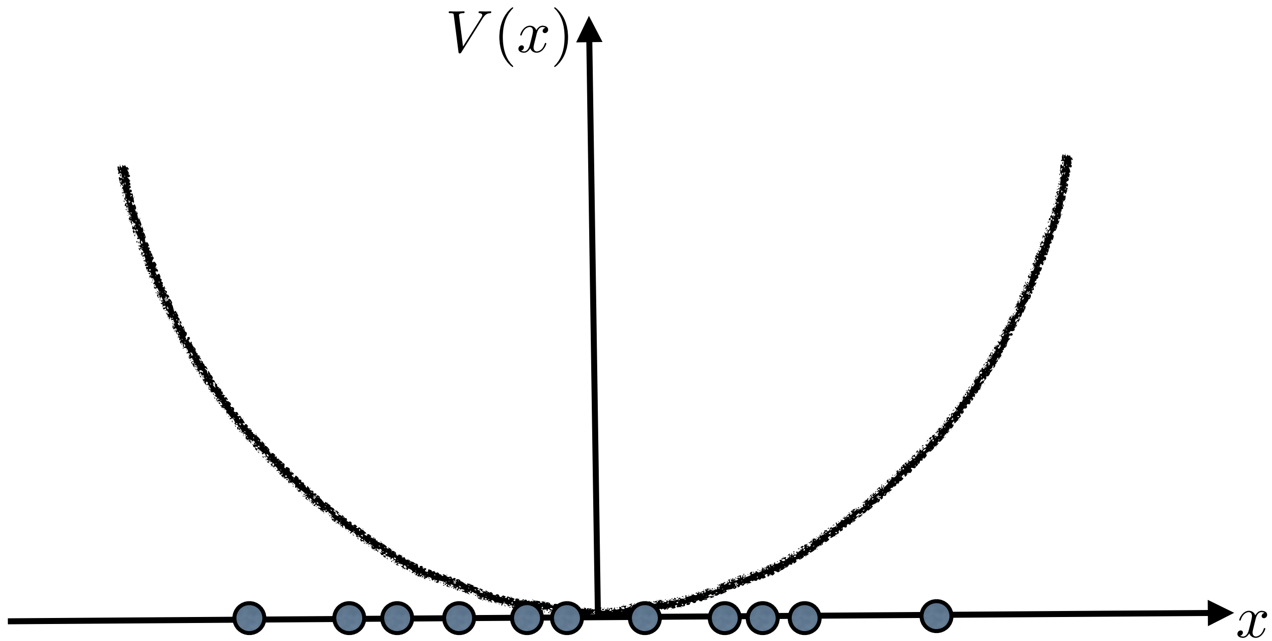
  ⇒ to investigate the interplay between quantum and thermal behaviors in many-body systems at low temperature

Ultra-cold atoms in confining potentials

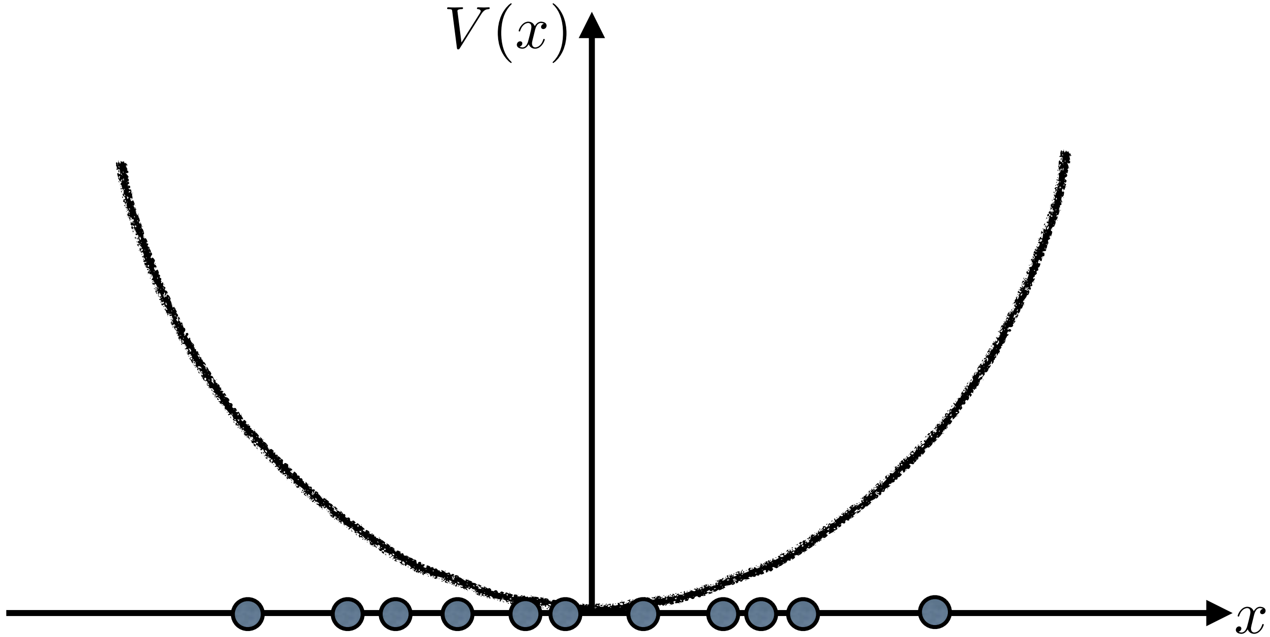
- Recent progress in the experimental manipulation of cold atoms

  → to investigate the interplay between quantum and thermal behaviors in many-body systems at low temperature

- A common feature of these experiments: presence of a confining potential that traps the atoms within a limited spatial region



Spinless free fermions in a 1d harmonic potential



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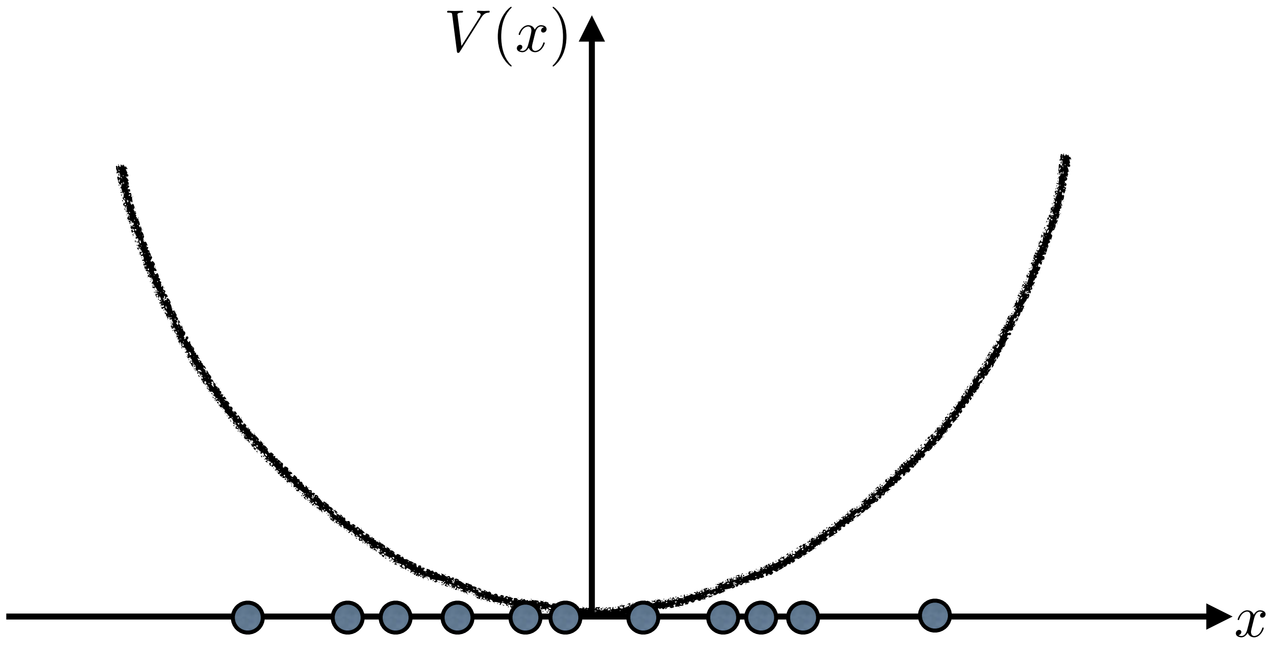

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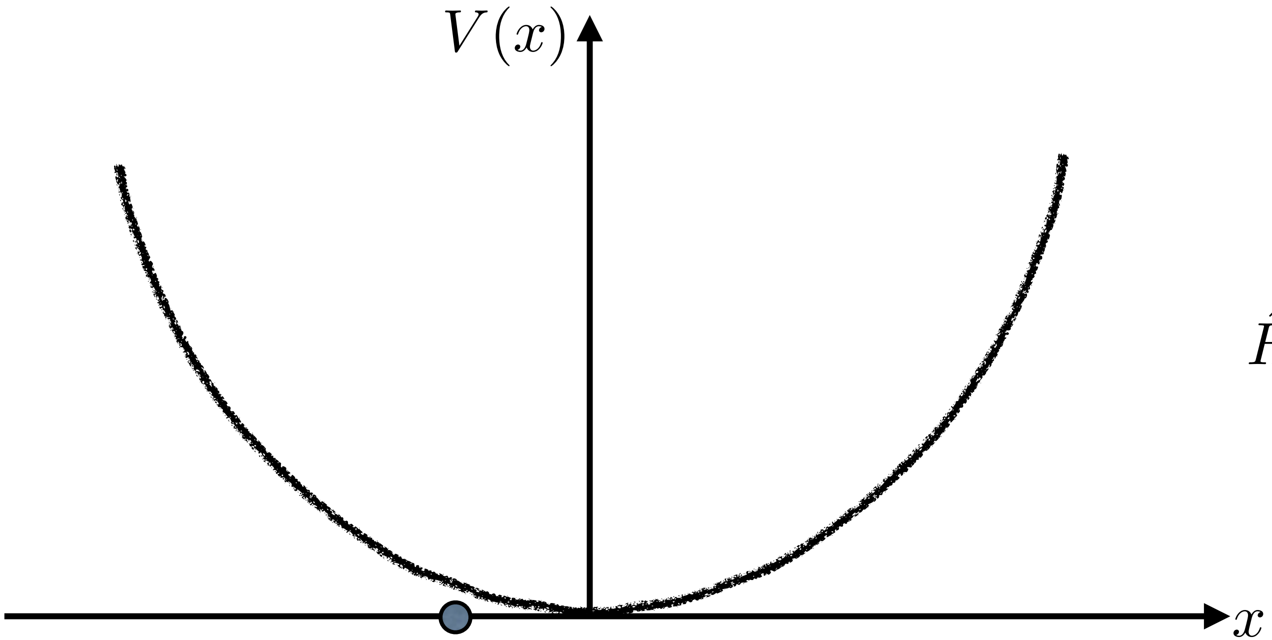
At zero temperature: connection between spinless free fermions in a harmonic trap and Random Matrix Theory (GUE)

Outline

- Free fermions in d=1 & T=0 and Random Matrix Theory (RMT)
- Free fermions in d=1 & T>0 and KPZ equation: main results
- Sketch of the derivation of our results
- Extension to higher dimensions, d>1
- Conclusion

Connection between free fermions at T=0 and RMT

- A single quantum particle in a harmonic potential



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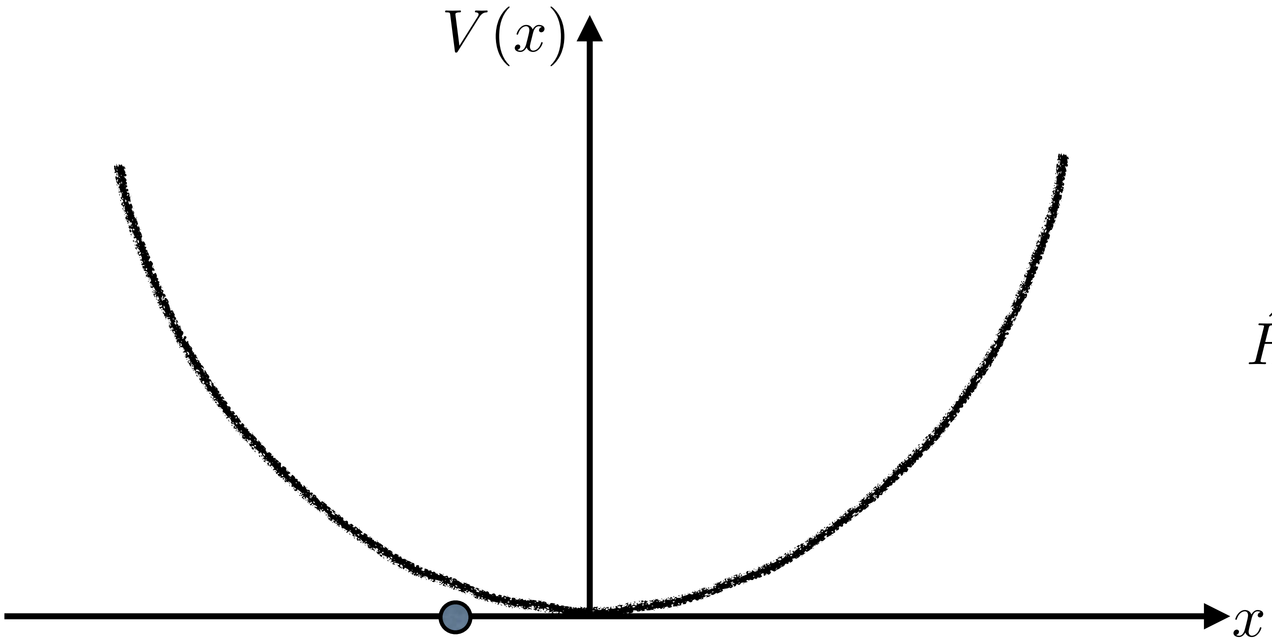

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$$\hat{H}\,\varphi_k(x) = \epsilon_k\,\varphi_k(x)$$

$$\text{with } \varphi_k(x \to \pm\infty) = 0$$

$$\varphi_k(x) = \left[\frac{\alpha}{\sqrt{\pi} 2^k k!}\right]^{1/2} e^{-\frac{\alpha^2 x^2}{2}} H_k(\alpha x)$$

$$\epsilon_k = \hbar\omega(k + 1/2) \quad , \quad \alpha = \sqrt{m\omega/\hbar}$$

$$k \in \mathbb{N}$$

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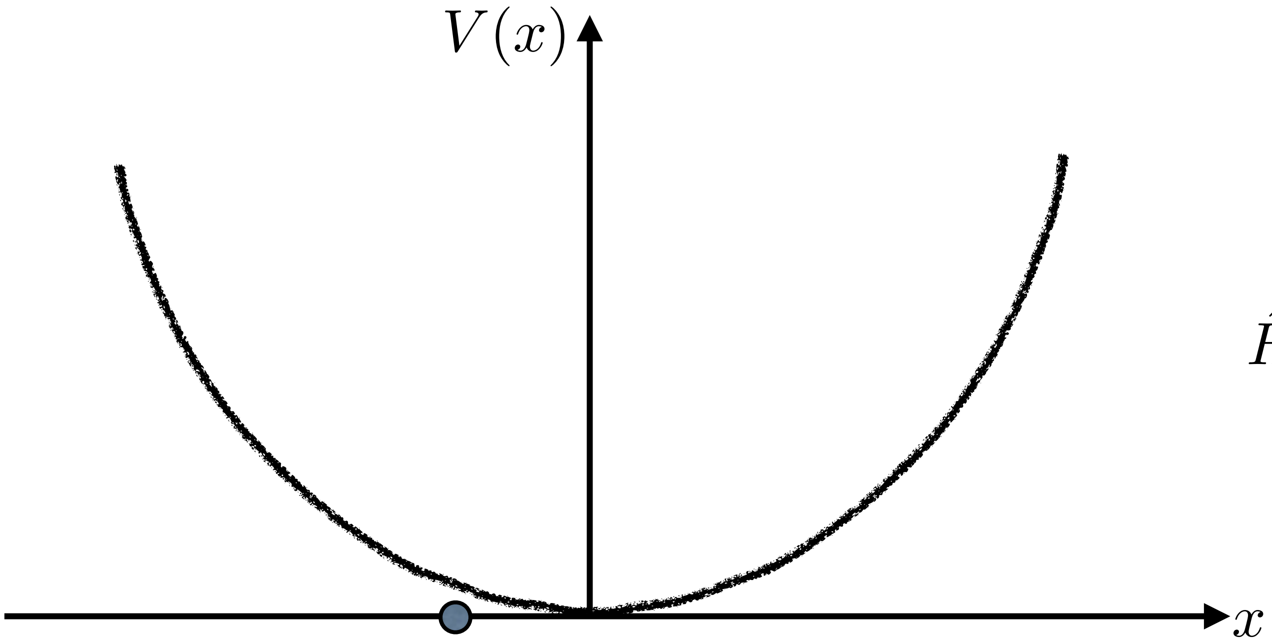

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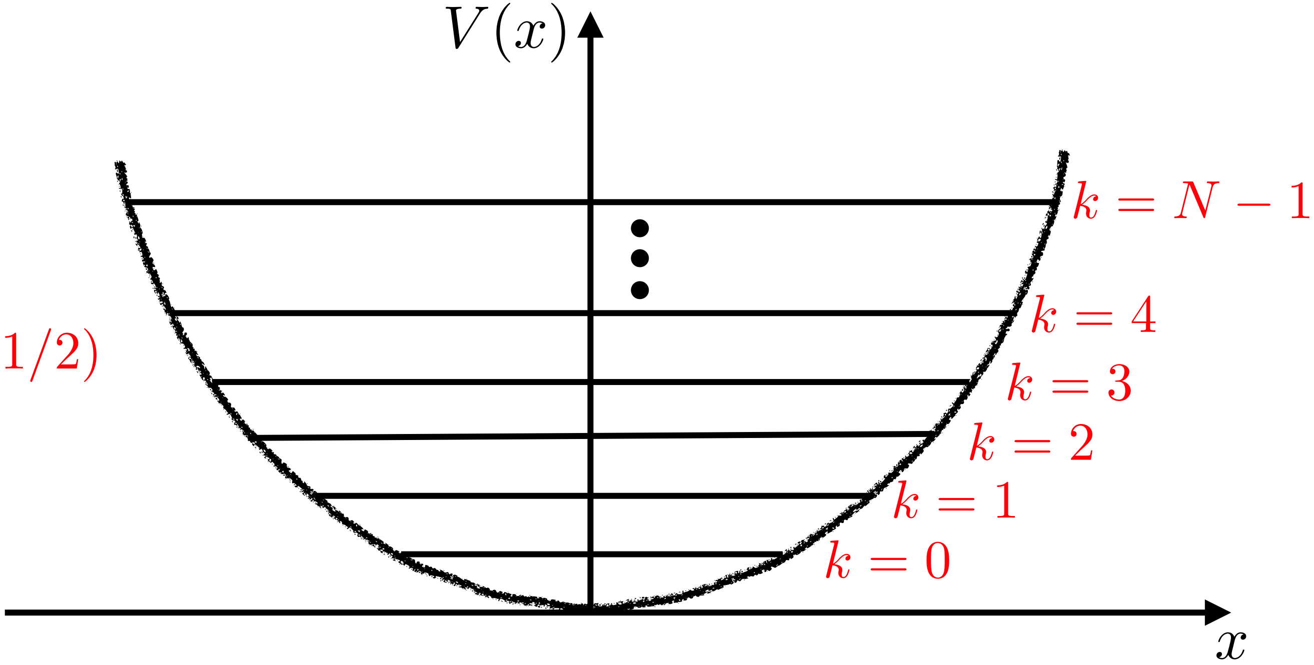
N spinless free fermions in a 1d harmonic trap at T=0



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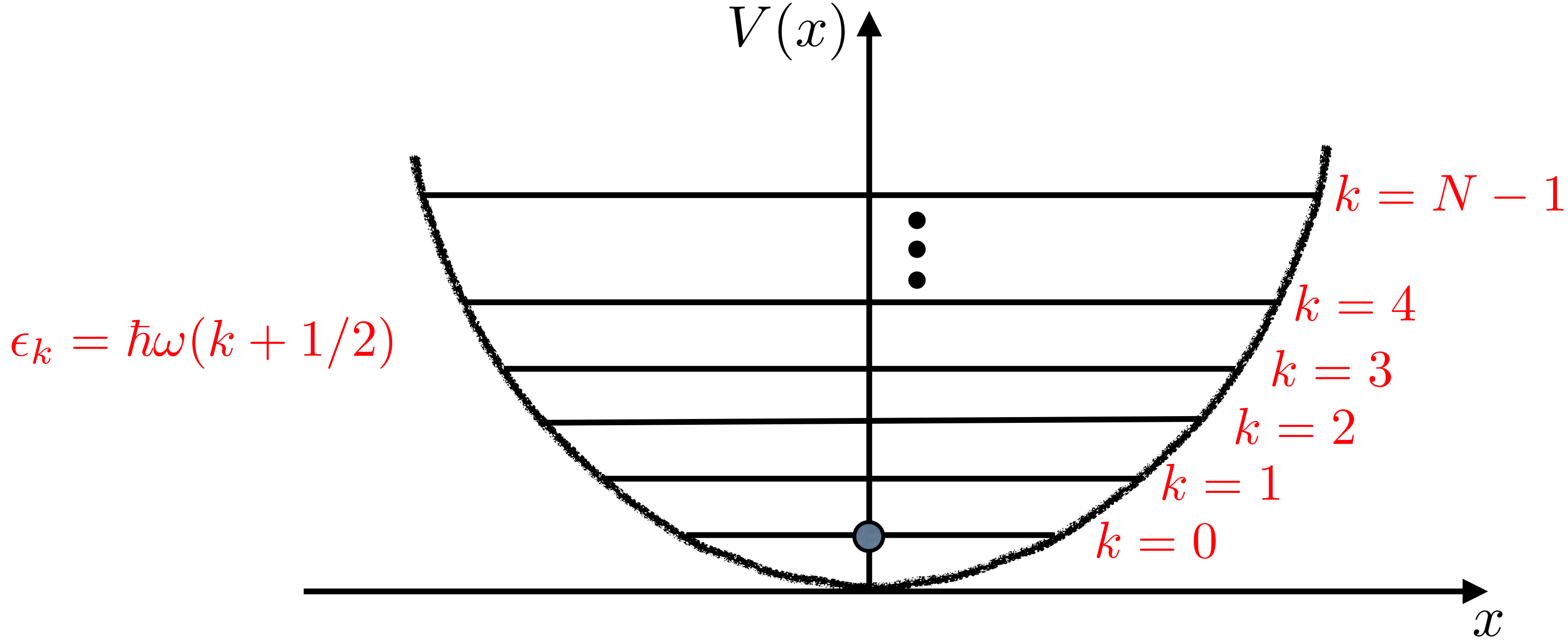

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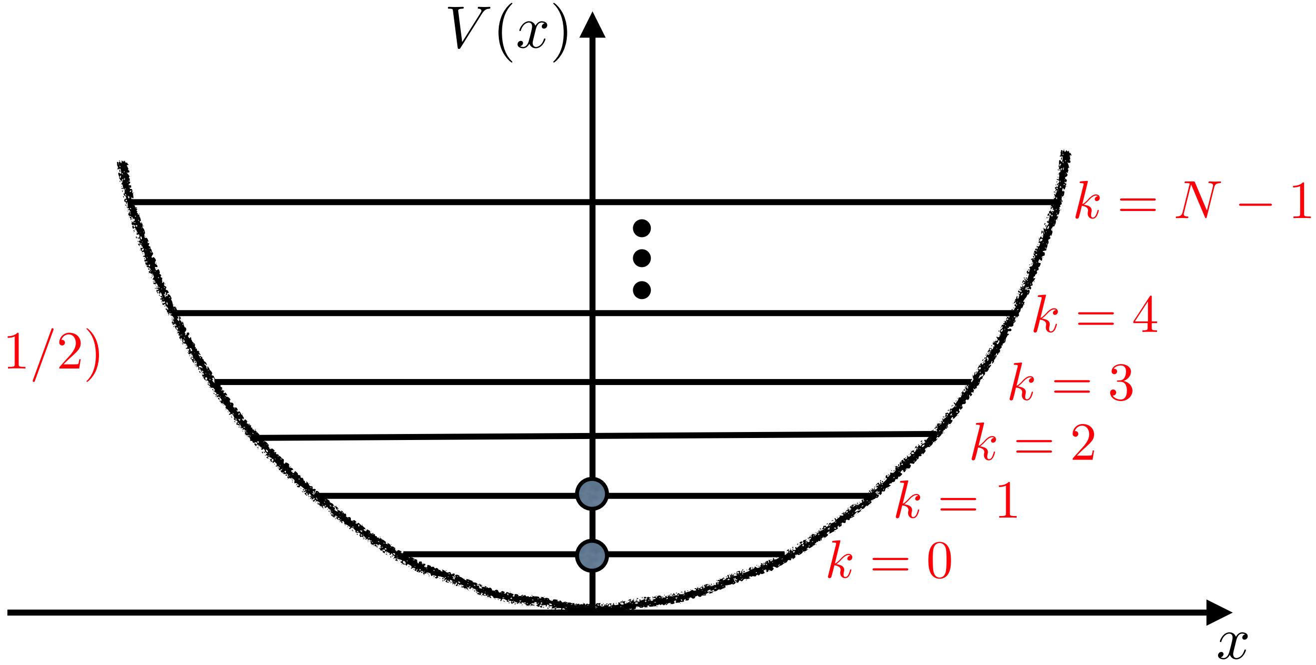

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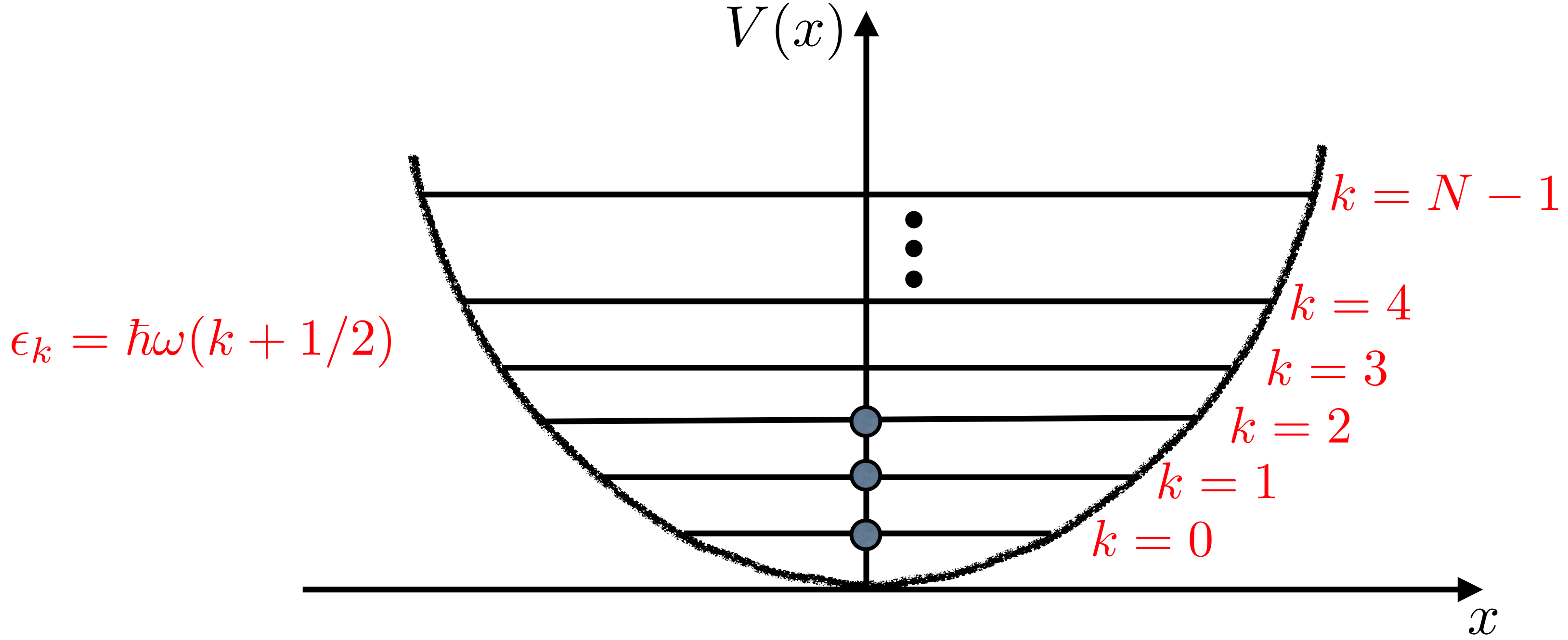

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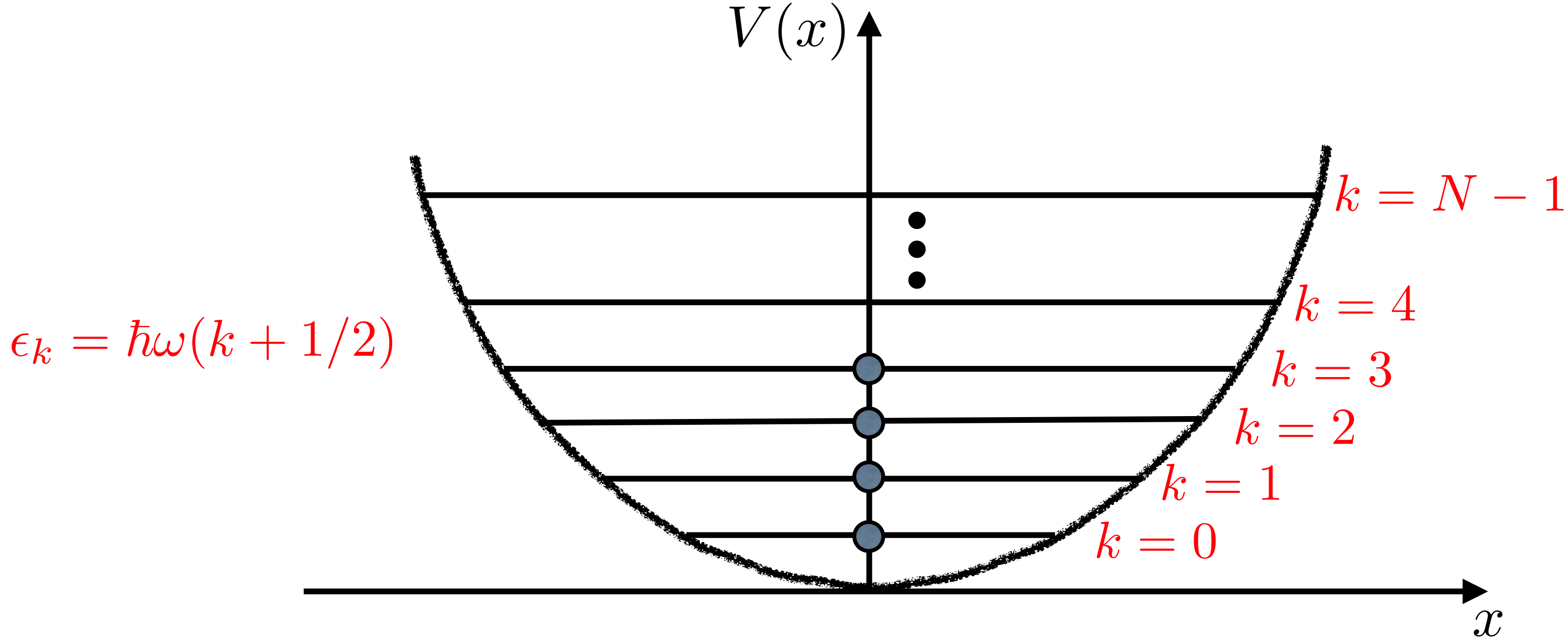

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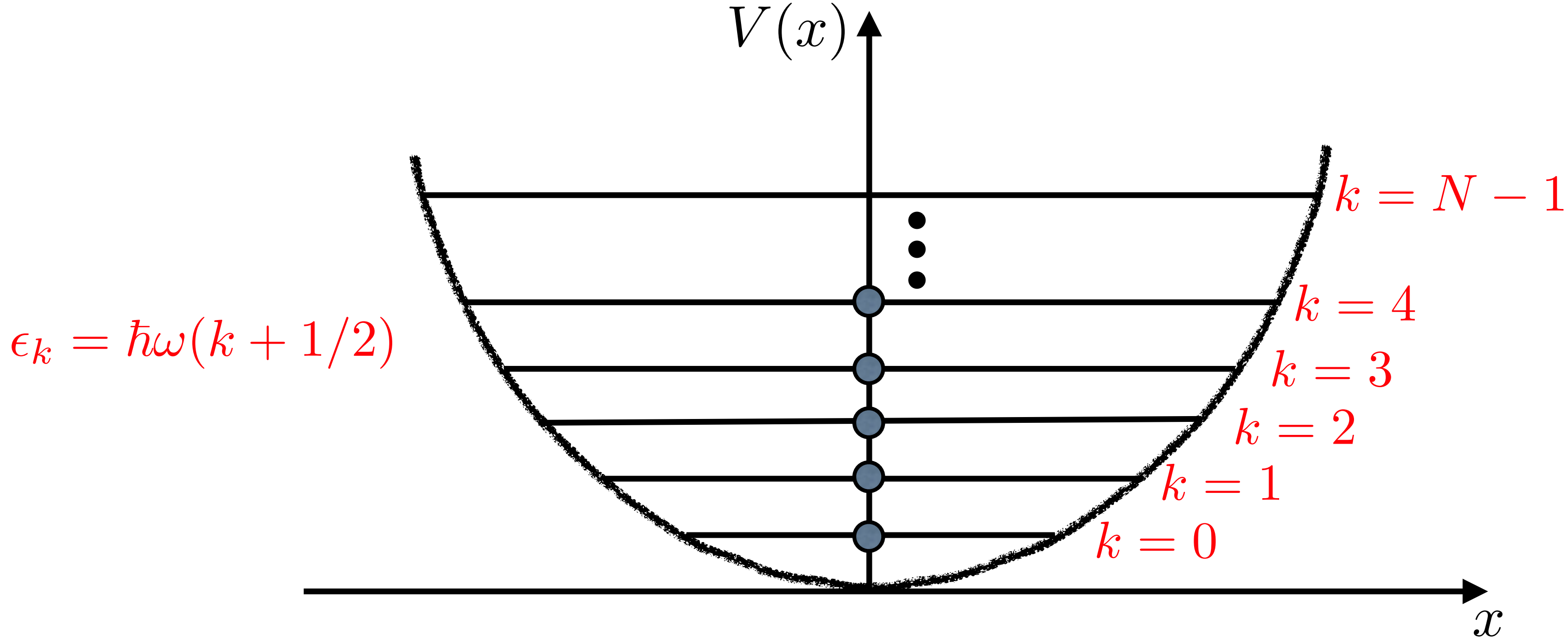

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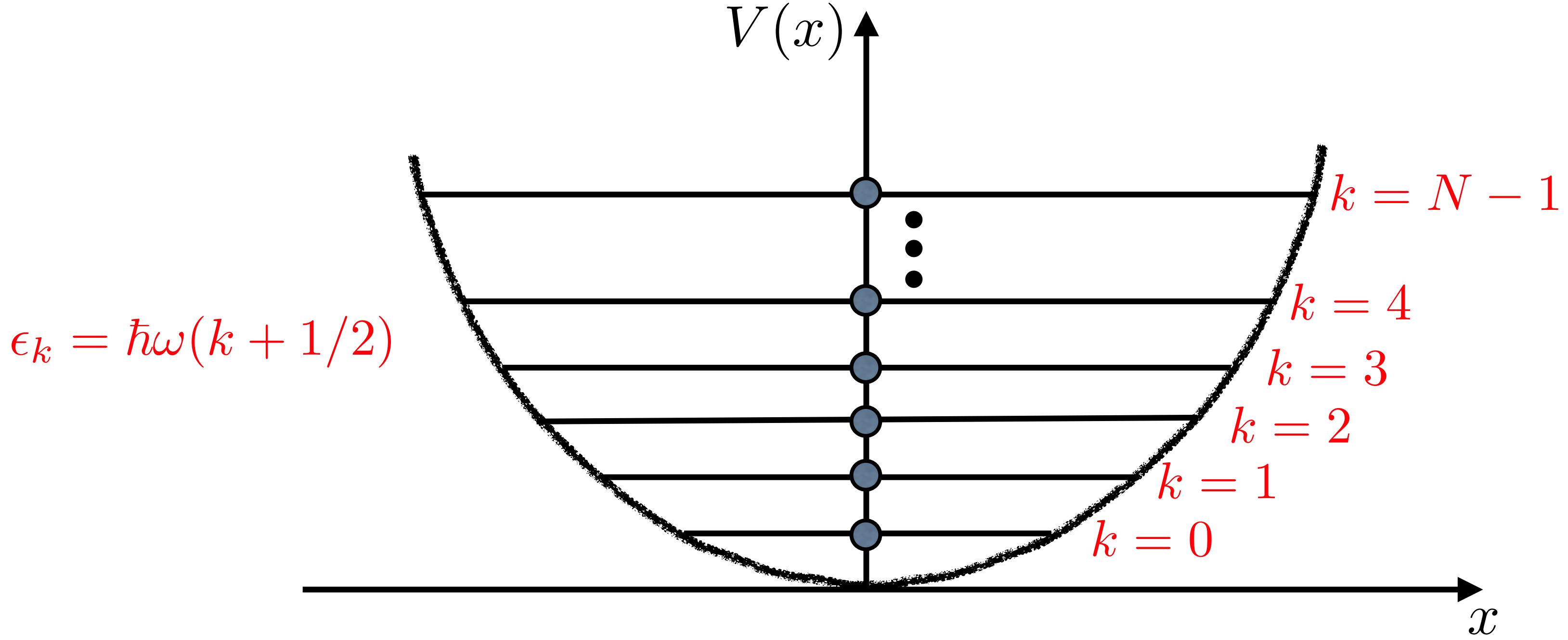

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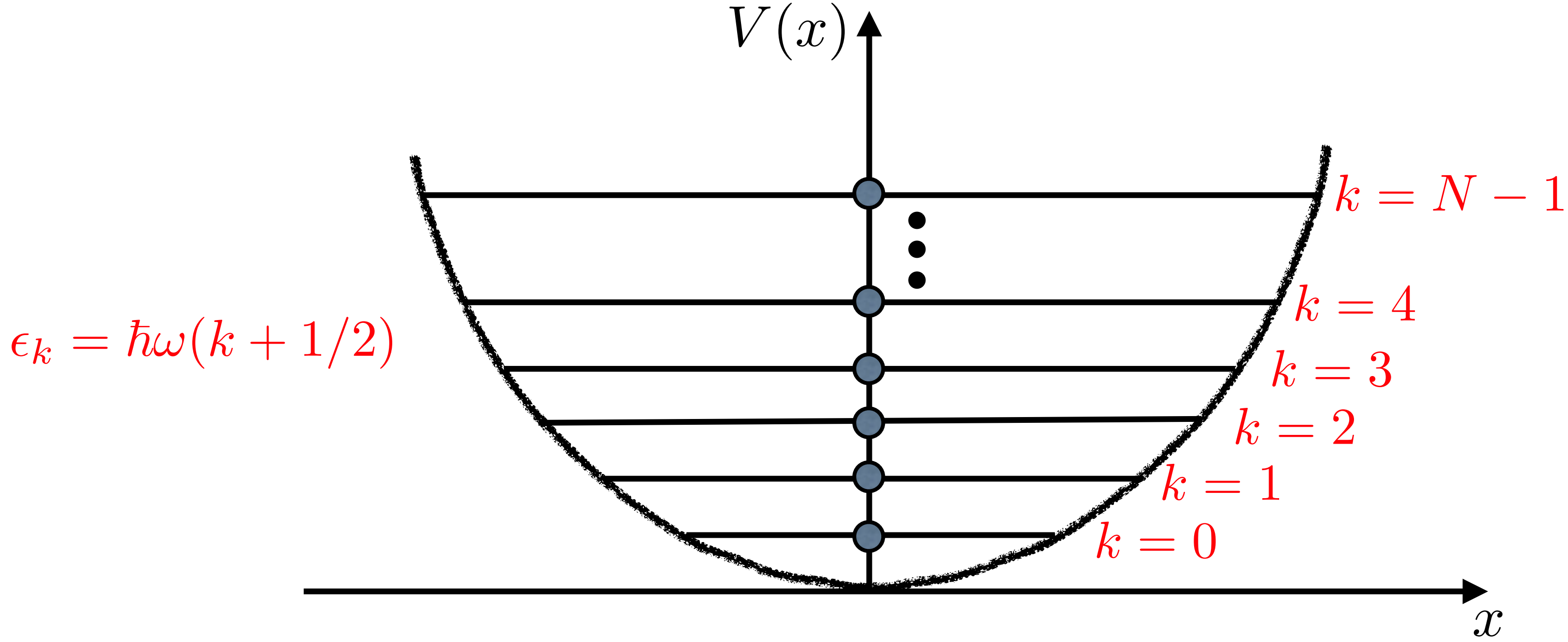

- The N-particle wave function is given by a $N \times N$ Slater determinant

$$\Psi_0(x_1, x_2, \cdots, x_N) = \frac{1}{\sqrt{N!}} \det[\varphi_i(x_j)] \qquad \begin{array}{l} 0 \leq i \leq N-1 \\ 1 \leq j \leq N \end{array}$$

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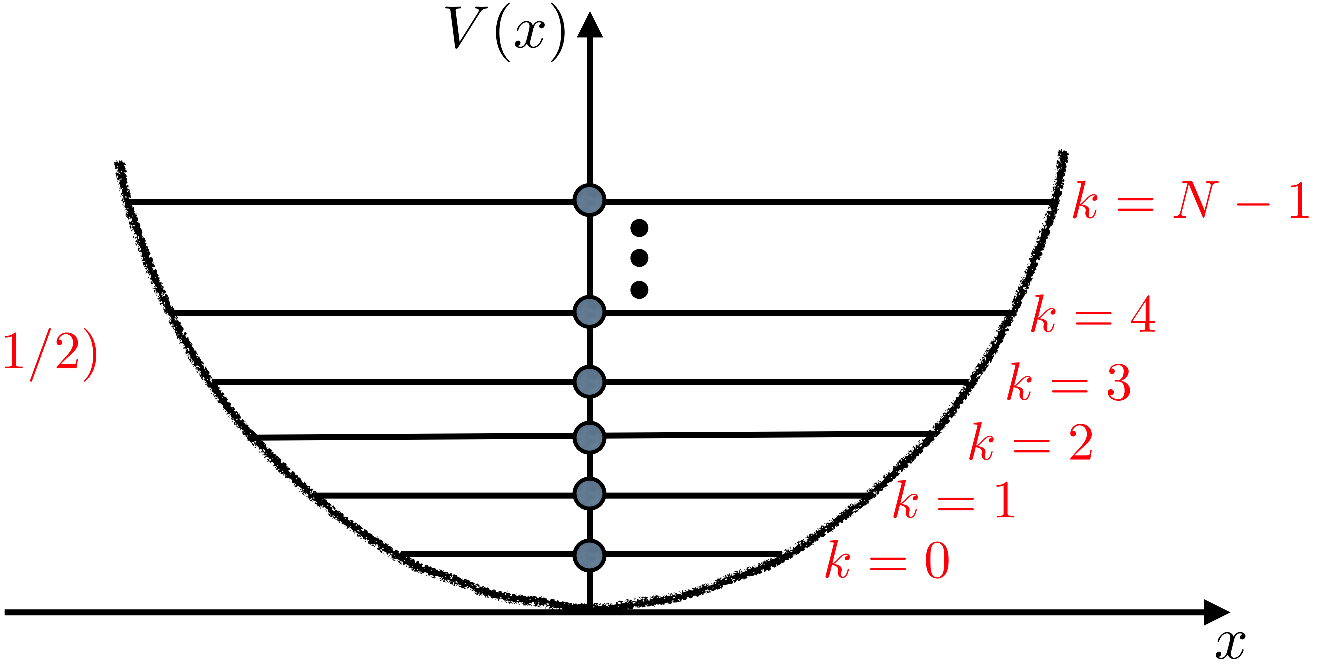

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Ground state energy $E_0 = \sum_{k=0}^{N-1} \epsilon_k = \frac{N^2}{2}$

Connection between free fermions at T=0 and RMT

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Hermite polynomial of degree i

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- Let M be a $N \times N$ random Hermitian matrix with Gaussian (complex) entries. The PDF of the (real) eigenvalues $\lambda_i' s$ is given by

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$\Rightarrow$ The positions of the free fermions behave statistically like the eigenvalues of GUE random matrices

$$(\alpha\, x_1, \alpha\, x_2, \cdots \alpha\, x_N) \stackrel{d}{=} (\lambda_1, \lambda_2, \cdots, \lambda_N)$$

Properties of fermions in a 1d harmonic trap at T=0

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$\Rightarrow$ The spatial properties of free fermions in a harmonic trap at T=0 can directly be obtained from the known results in RMT

Eisler '13/Marino, S. N. M., Schehr, Vivo '14/Calabrese, Le Doussal, S. N. M. '15

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- Average density of free fermions: Wigner semi-circle law

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for $N \gg 1$ $\rho_N(x, T=0) \approx \frac{\alpha}{\sqrt{N}} f_W\left(\frac{\alpha\, x}{\sqrt{N}}\right)$, $\boxed{f_W(z) = \frac{1}{\pi}\sqrt{2 - z^2}}$

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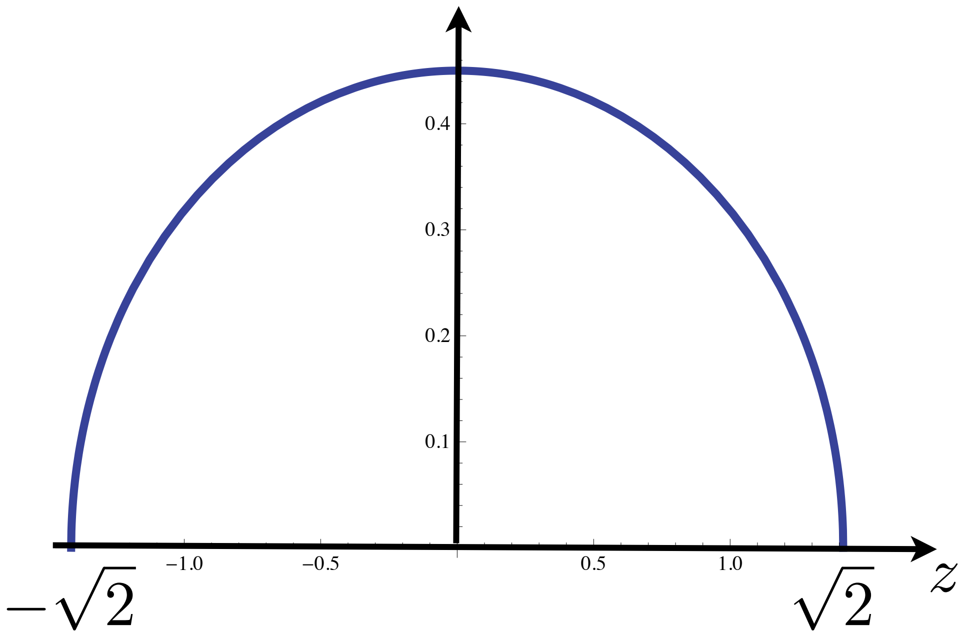
See also Local Density (or Thomas-Fermi) Approx. in the literature on fermions



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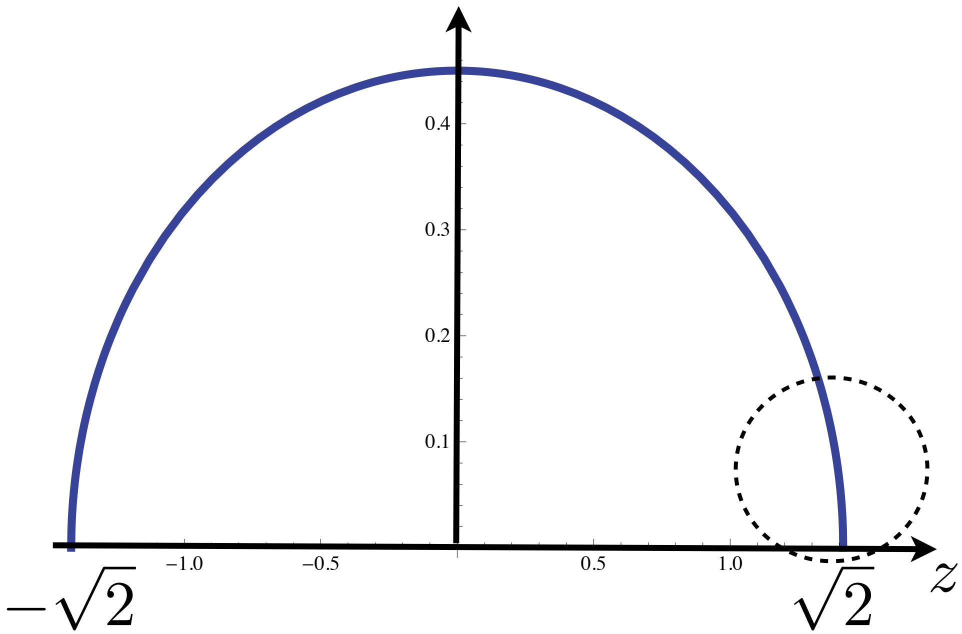

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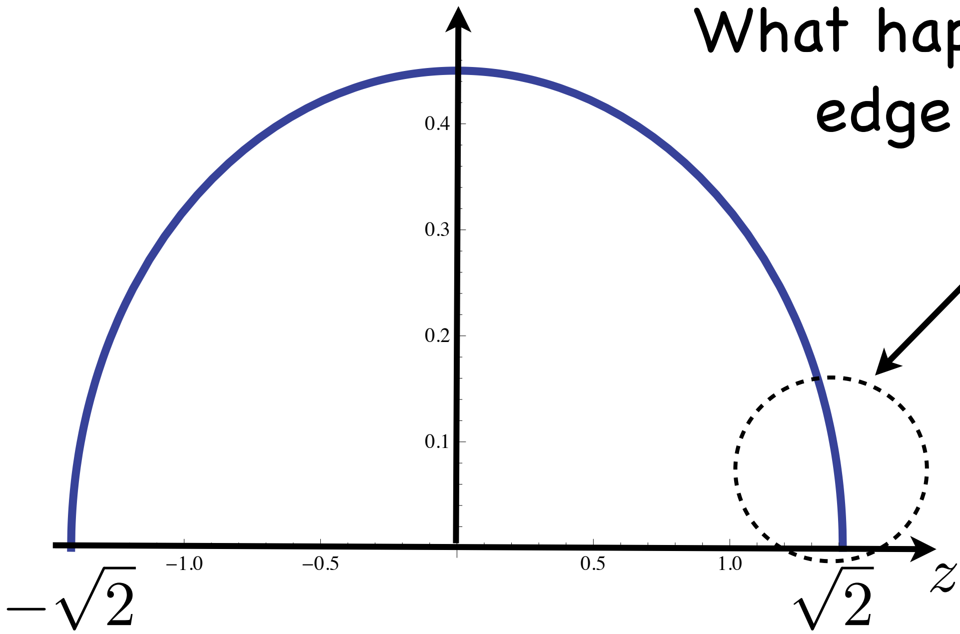

What happens close to the edge for finite N ?

Properties of fermions in a 1d harmonic trap at T=0

- Edge density of free fermions — Bowick, Brézin '91/Forrester '93

$$\rho_N(x) \approx \frac{1}{N w_N} F_1\left(\frac{x - \sqrt{2N}/\alpha}{w_N}\right)$$

with $w_N = \dfrac{N^{-1/6}}{\sqrt{2}\,\alpha}$ and $F_1(z) = [Ai'(z)]^2 - z[Ai(z)]^2$

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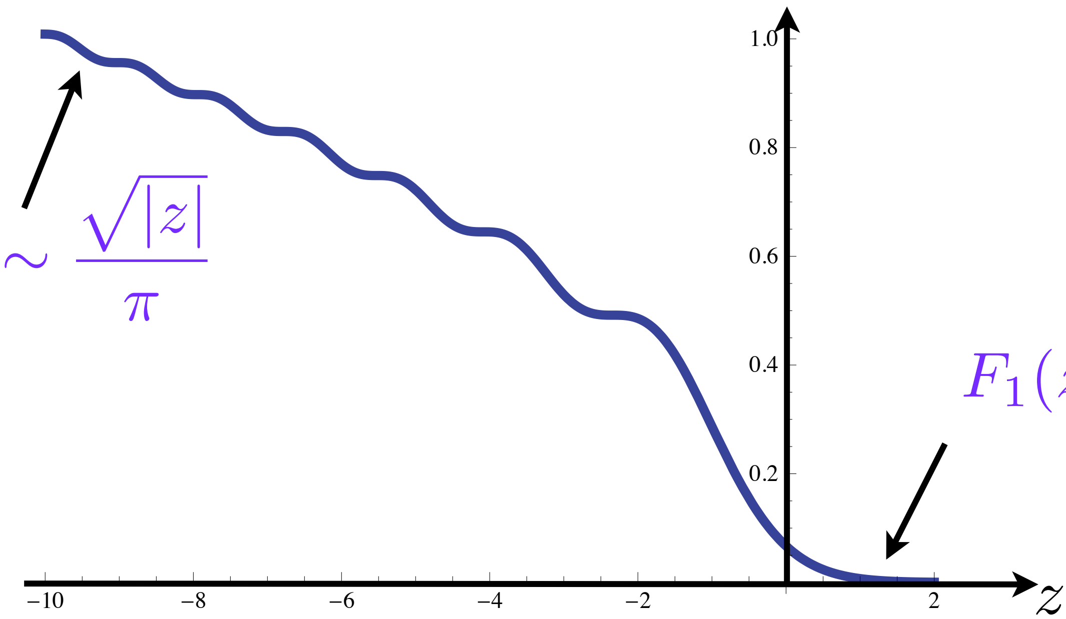
$$F_1(z) \sim \frac{\sqrt{|z|}}{\pi}$$



$$F_1(z) \sim e^{-\frac{4}{3}z^{3/2}}$$

Fermions in a 1d harmonic trap at T=0: kernel

- Higher order correlations

e.g., 2-point correlation function: $R_2(y,z) = \sum_{i \neq j} \langle \delta(y - x_i) \delta(z - x_j) \rangle$

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n-point correlation function

$$R_n(x_1,\cdots,x_n) = \frac{N!}{(N-n)!}\int_{-\infty}^{\infty} dx_{n+1}\cdots\int_{-\infty}^{\infty} dx_N |\Psi_0(x_1,\cdots,x_n,x_{n+1},\cdots,x_N)|^2$$

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kernel

in particular, the average density is given by $\rho_N(x) = \frac{1}{N} K_N(x,x)$

Limiting form of the kernel for trapped fermions at T=0

- Bulk limit: when x & y are far from the edge and

  and $|x-y| \sim \dfrac{1}{N\rho_N(x)} \equiv$ inter-particle distance

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Sine-kernel

Limiting form of the kernel for trapped fermions at T=0

- Bulk limit: when x & y are far from the edge and

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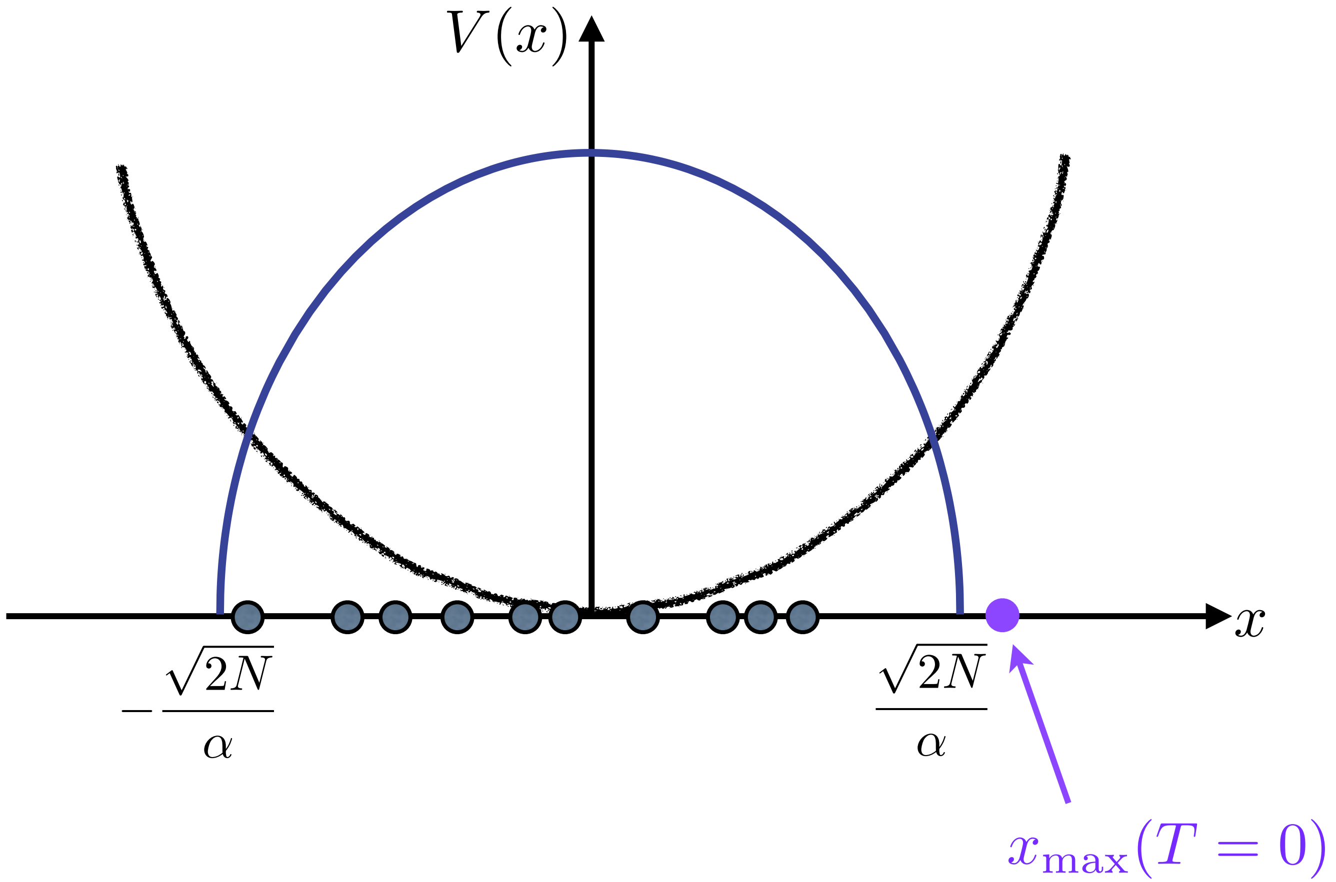
- Edge scaling limit: for x & y close to the edge $r_{\text{edge}} = \sqrt{2N}/\alpha$

$$K_N(x,y) \approx \frac{1}{w_N}\mathcal{K}_{\text{edge}}\left(\frac{x-r_{\text{edge}}}{w_N}, \frac{y-r_{\text{edge}}}{w_N}\right)\ ,\ w_N = \frac{N^{-1/6}}{\sqrt{2}\alpha}$$

$$\boxed{\mathcal{K}_{\text{edge}}(a,b) = \frac{Ai(a)Ai'(b) - Ai'(a)Ai(b)}{a-b}}$$

Airy-kernel

Position of the rightmost fermion at T=0



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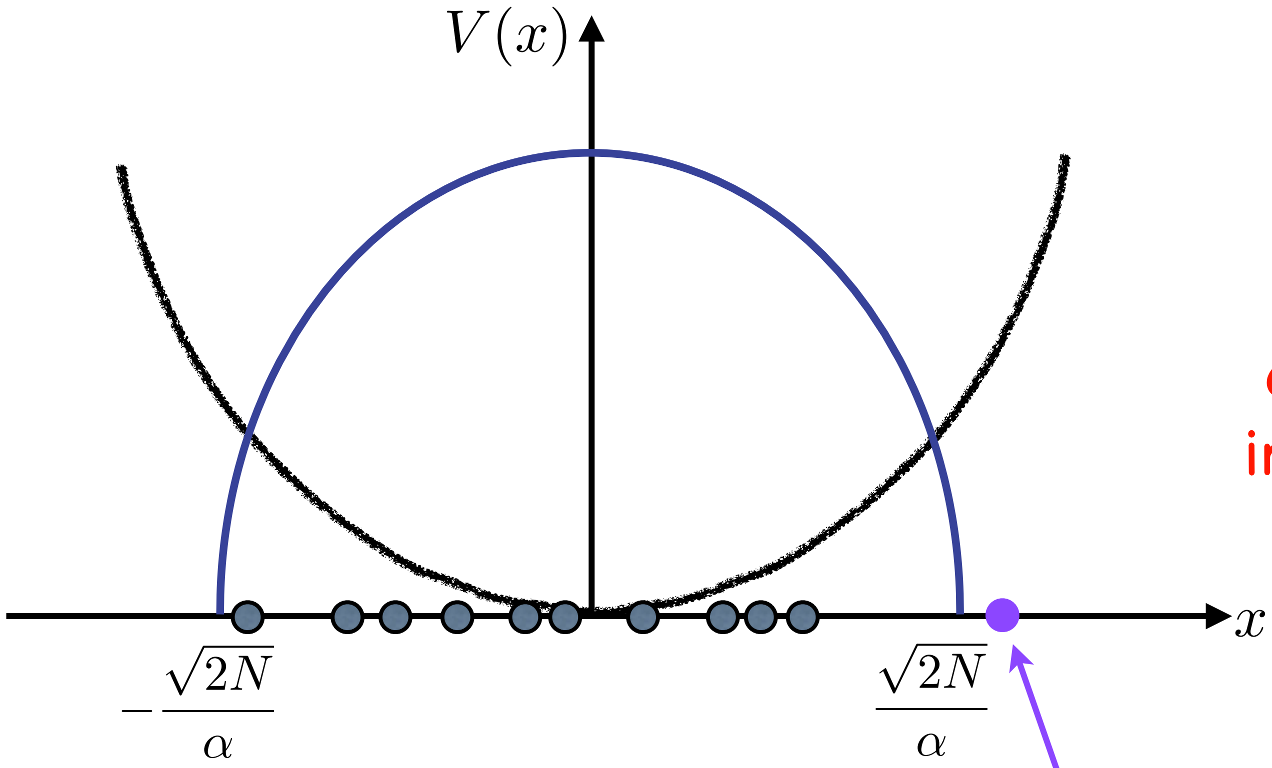

largest eigenvalue
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from the connection with RMT: $x_{\max}(T=0) \stackrel{d}{=} \frac{\lambda_{\max}}{\alpha}$

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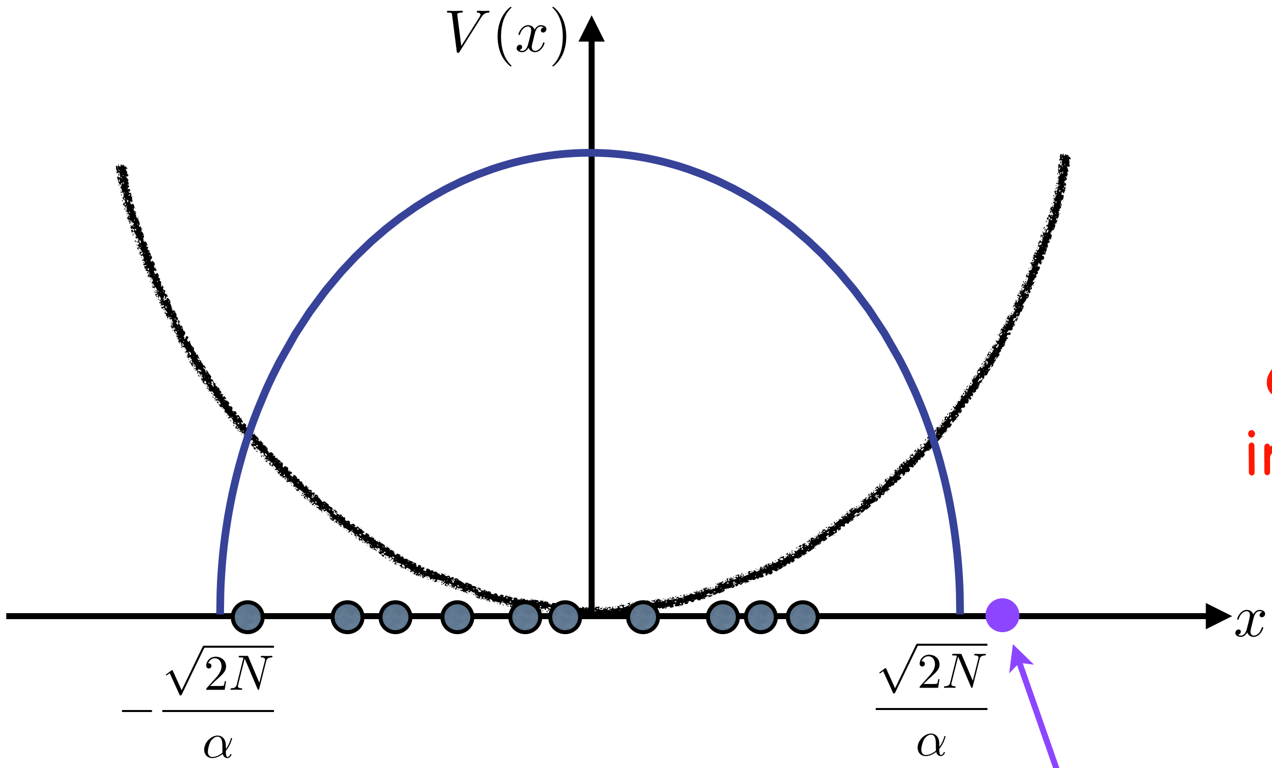

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⇒ fluctuations of $x_{\max}(T=0)$ are governed by the Tracy-Widom distribution for GUE

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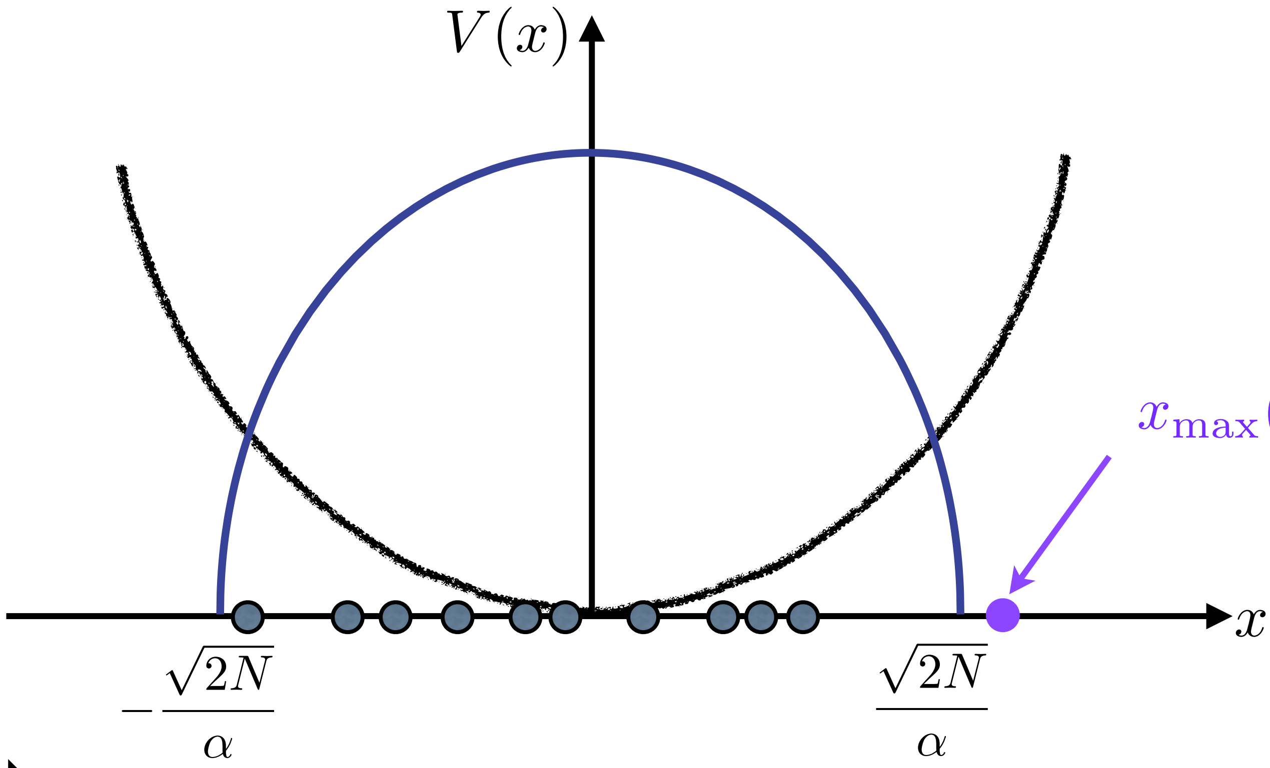

$$r_{\rm edge} = \sqrt{2N}/\alpha$$

$$w_N = \frac{N^{-1/6}}{\sqrt{2}\,\alpha}$$

$x_{\max}(T=0)$

fluctuations of $x_{\max}(T=0)$ are governed by the Tracy-Widom distribution for GUE

$$\Pr.(x_{\max}(T=0) \leq M) \approx \mathcal{F}_2\left(\frac{M - r_{\rm edge}}{w_N}\right)$$

$$\mathcal{F}_2(\xi) = \det(I - P_\xi K_{\rm edge} P_\xi)\ ,\ K_{\rm edge}(a,b) = \frac{Ai(a)Ai'(b) - Ai'(a)Ai(b)}{a-b}$$

Fredholm determinant Airy-kernel

Fermions in a 1d confining trap at T=0: summary

- Form a determinantal process (c.f. GUE for a harmonic well)

- Bulk scaling limit: Sine-kernel

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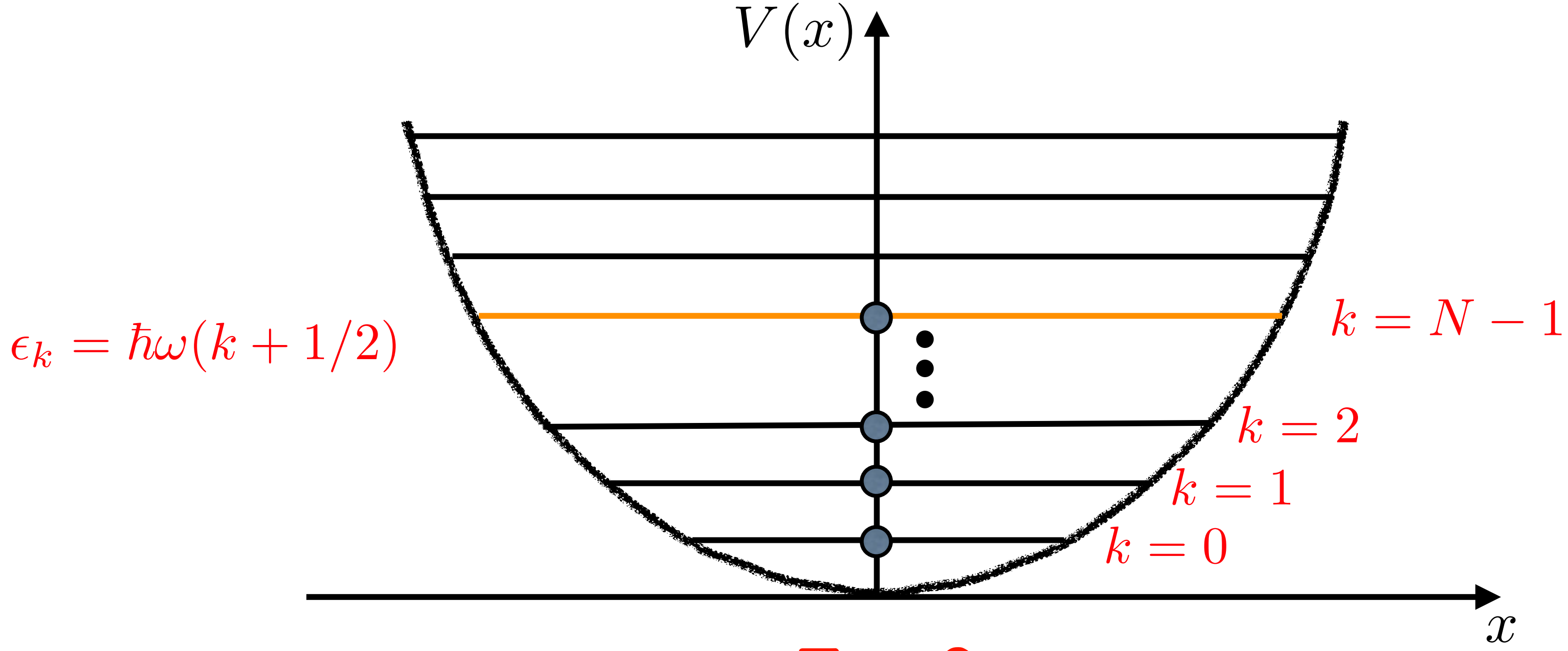
$V(x) \sim |x|^p$ with a single minimum Eisler '13

What happens at finite temperature

$T > 0$?

N free fermions in 1d-harmonic trap at T > 0



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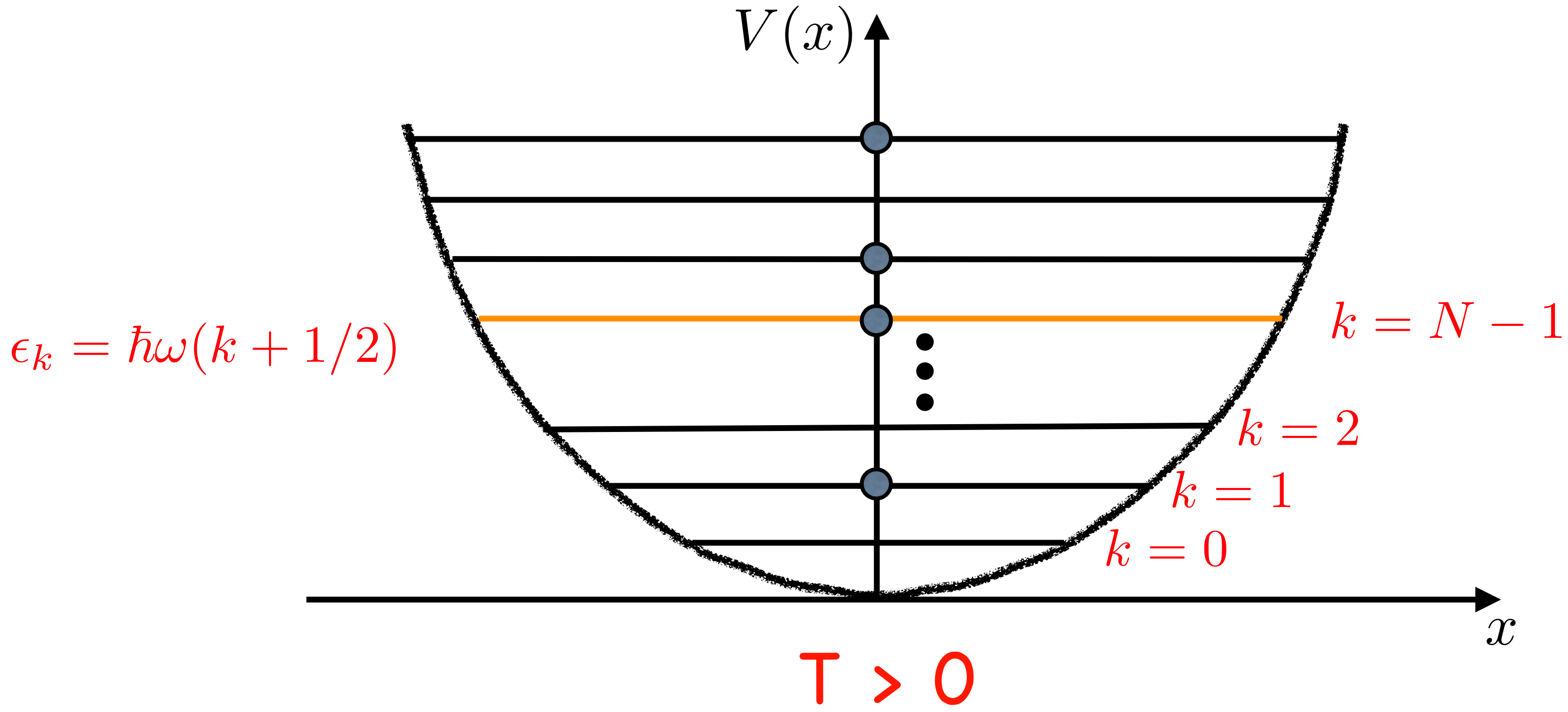

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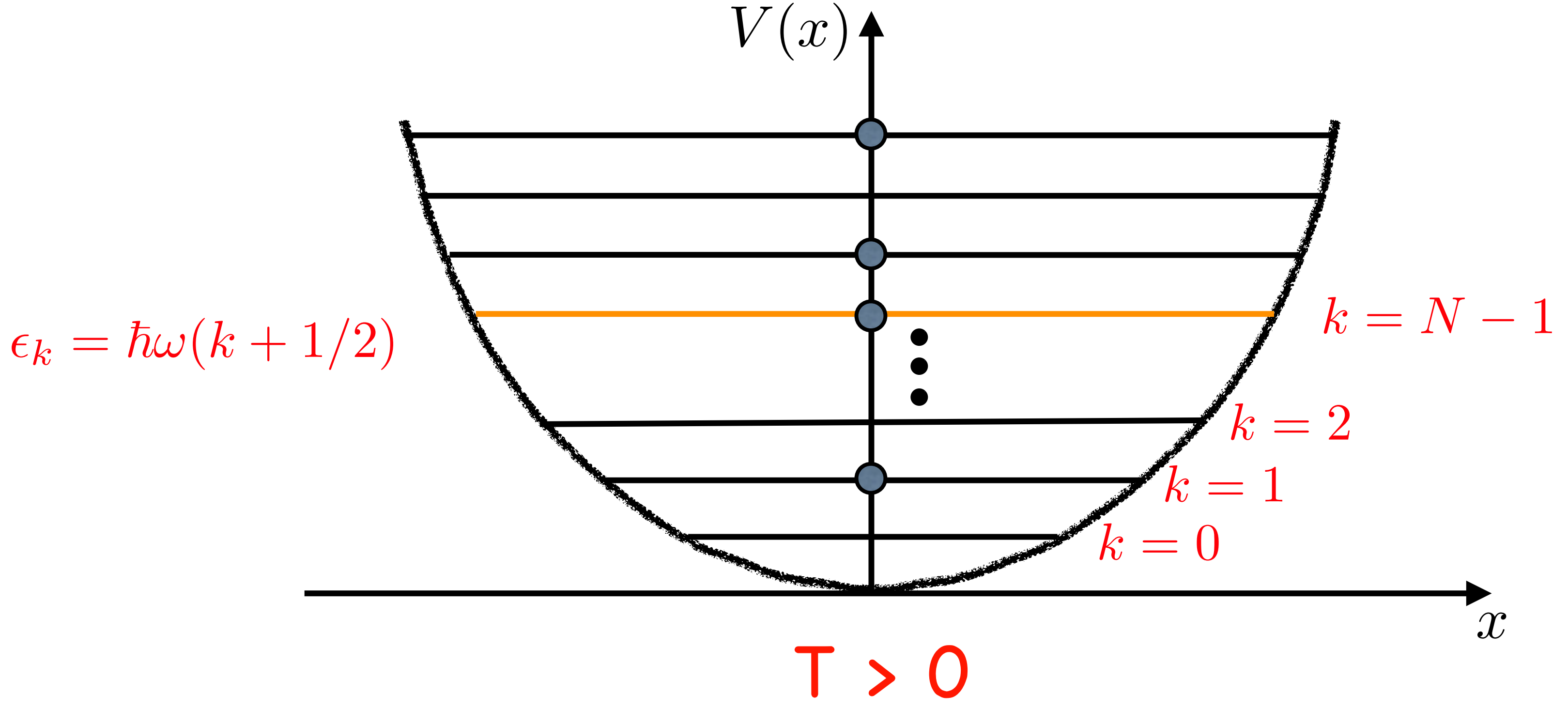

- Probability density function (PDF) of the positions $x_i's$

$$P_{\text{joint}}(x_1, \cdots x_N) = \frac{1}{N! Z_N(\beta)} \sum_{k_1 < \cdots < k_N} \left[\det_{1 \leq i,j \leq N} (\varphi_{k_i}(x_j)) \right]^2 e^{-\beta(\epsilon_{k_1} + \cdots + \epsilon_{k_N})}$$

$$Z_N(\beta) = \sum_{k_1 < k_2 < \cdots < k_N} e^{-\beta(\epsilon_{k_1} + \epsilon_{k_2} + \cdots + \epsilon_{k_N})} \quad \& \quad \varphi_k(x) = \left[\frac{\alpha}{\sqrt{\pi} 2^k k!} \right]^{1/2} e^{-\frac{\alpha^2 x^2}{2}} H_k(\alpha x)$$

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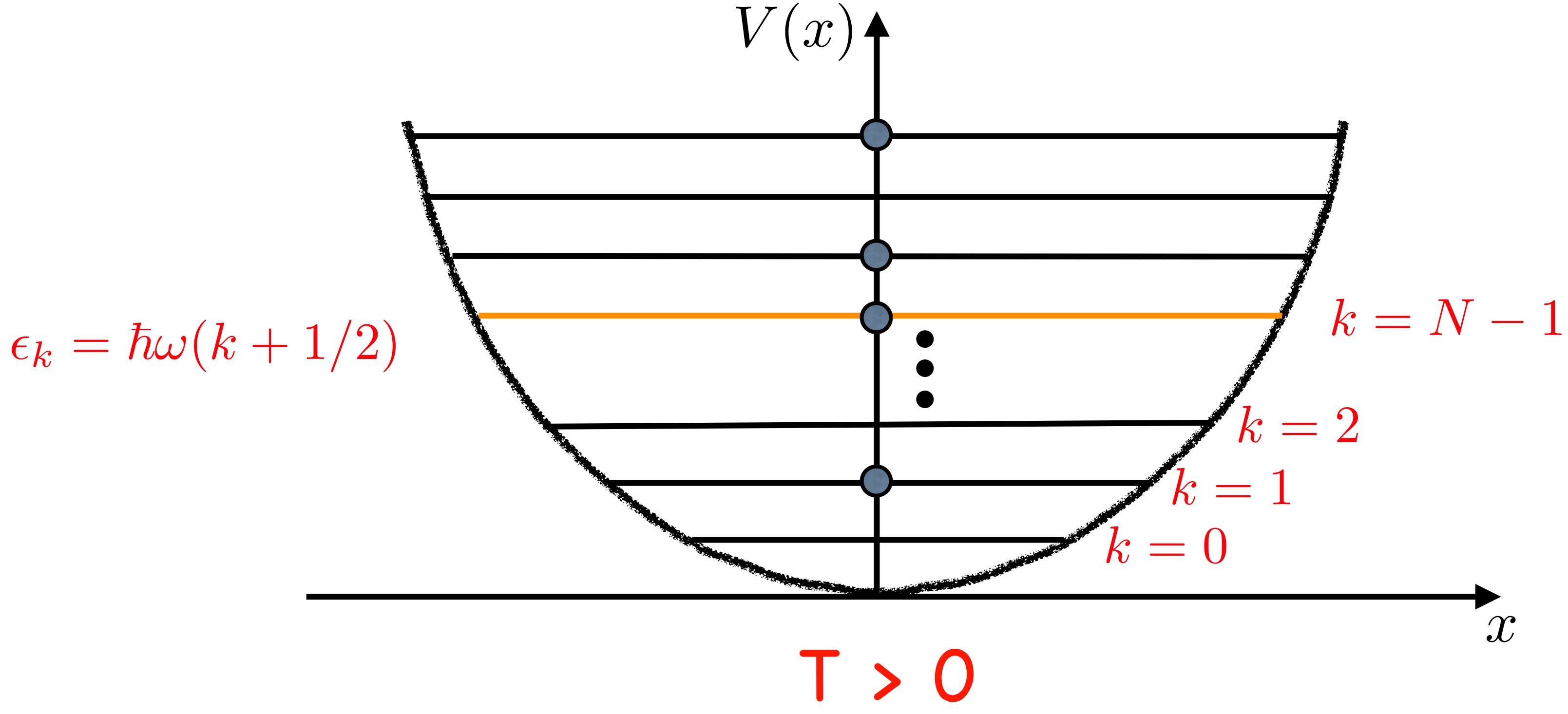

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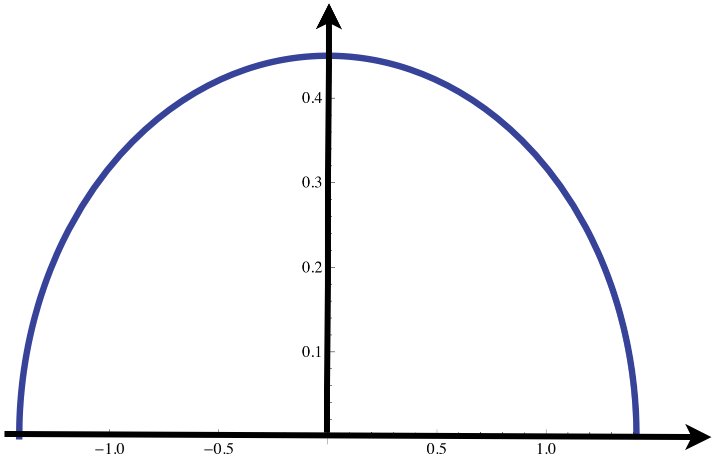
$$T = 0$$



$$\rho_N(x,T\to 0) \approx \frac{\alpha}{\sqrt{N}} f_W\left(\frac{\alpha\, x}{\sqrt{N}}\right) \; ,$$

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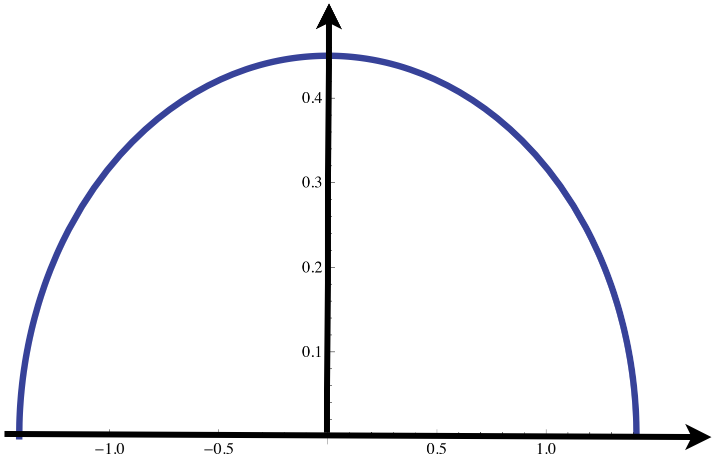

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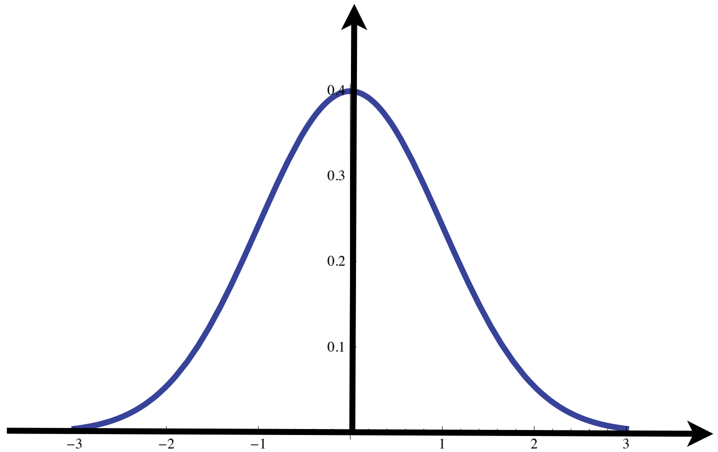

$$\rho_N(x, T\to\infty) \sim \sqrt{\frac{\beta\, m\, \omega^2}{2\pi}}\, \exp\left[-\frac{\beta}{2}\, m\, \omega^2\, x^2\right]$$

$$\beta = 1/T$$

Boltzmann-Gibbs

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Interpolation
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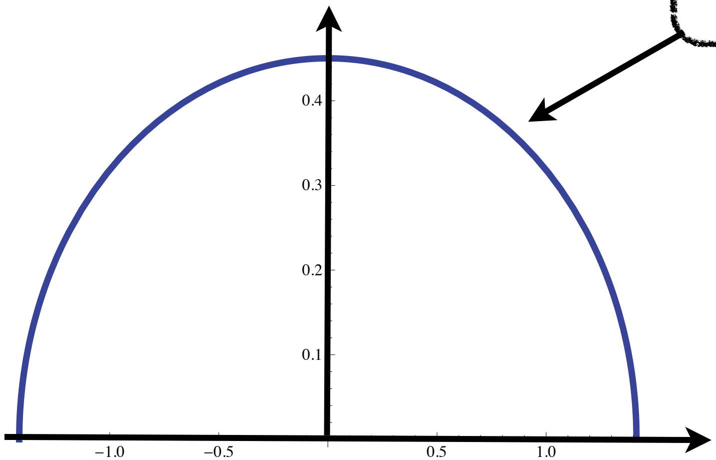

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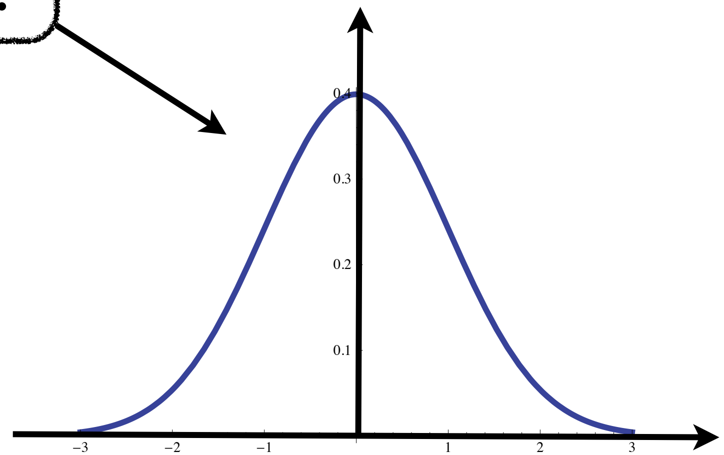

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Dean, Le Doussal, S. N. M., Schehr '14

$$\rho_N(x,T) \sim \frac{\alpha}{\sqrt{N}} R\left(\frac{N\hbar\omega}{T} = y, x\sqrt{\frac{m\,\omega^2}{2\,T}} = z\right),$$

$$R(y,z) = -\frac{1}{\sqrt{2\pi\,y}}\,\mathrm{Li}_{1/2}\left(-\left(e^y-1\right)\,e^{-z^2}\right) \qquad \mathrm{Li}_n(x) = \sum_{k=1}^{\infty}\frac{x^k}{k^n}$$

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See also Local Density (or Thomas-Fermi) Approx. in the literature on fermions

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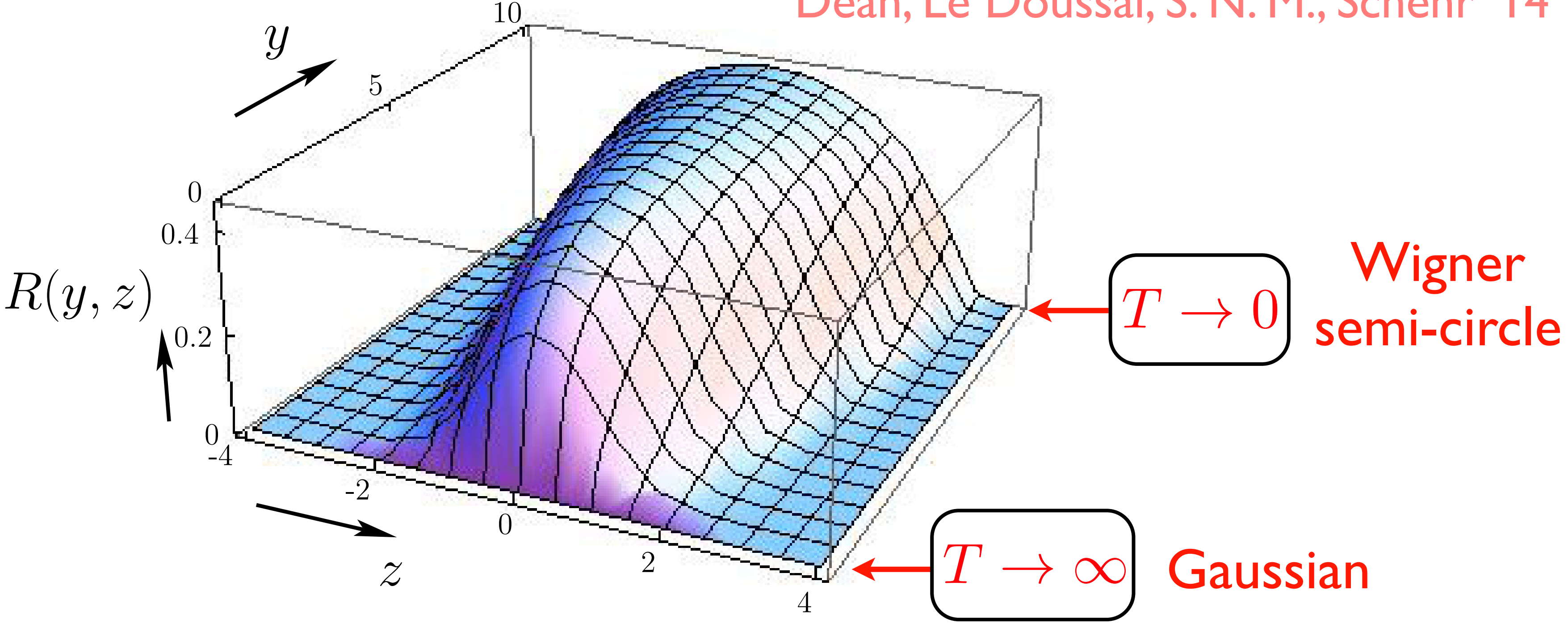
Dean, Le Doussal, S. N. M., Schehr '14



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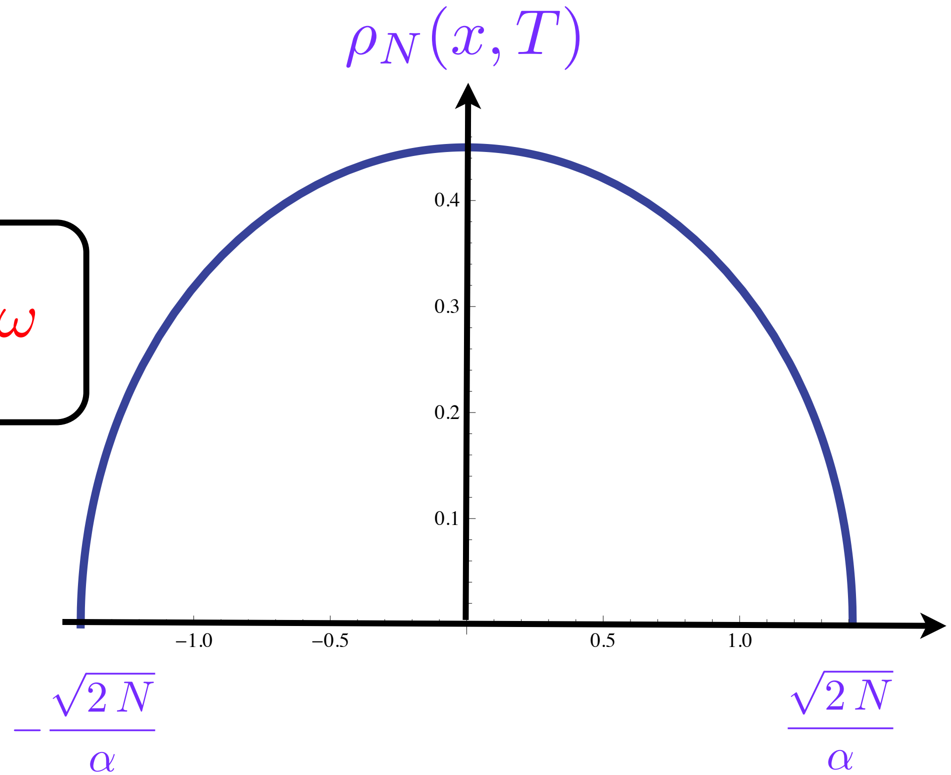

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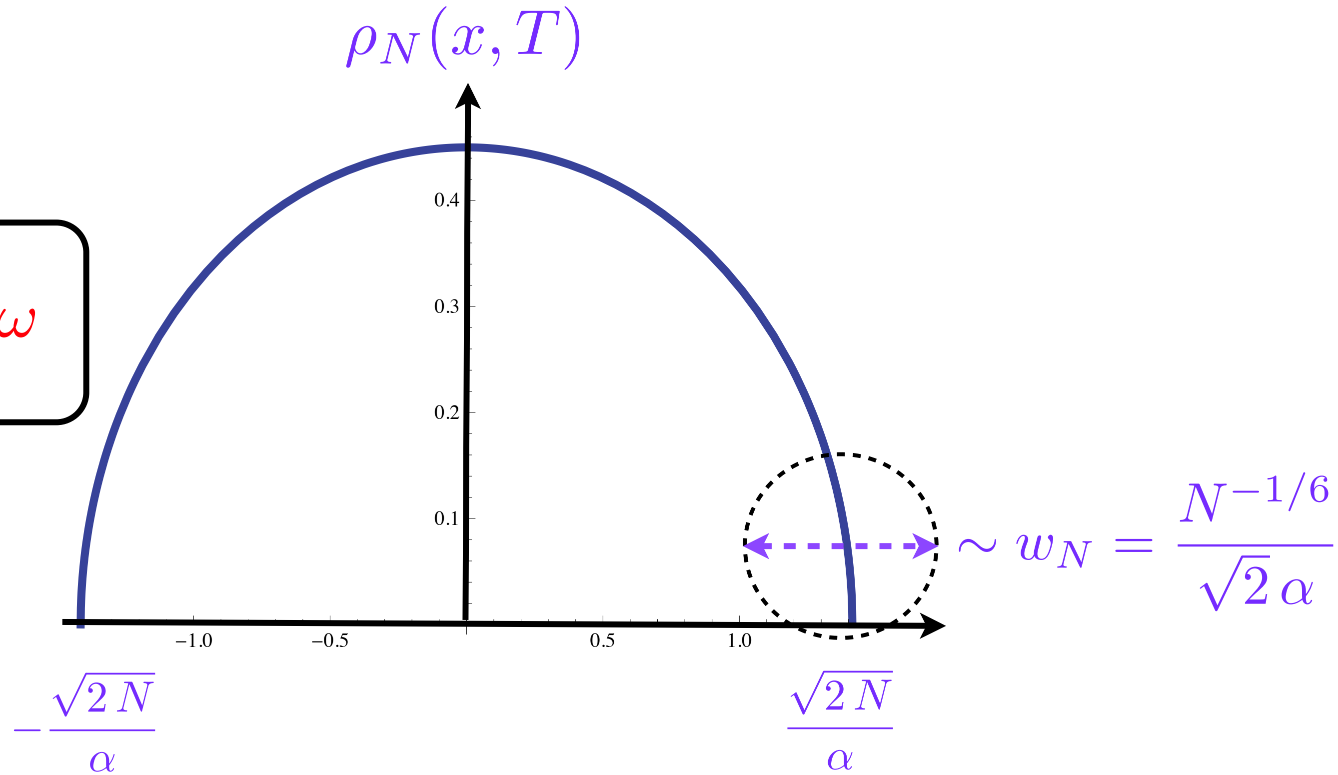

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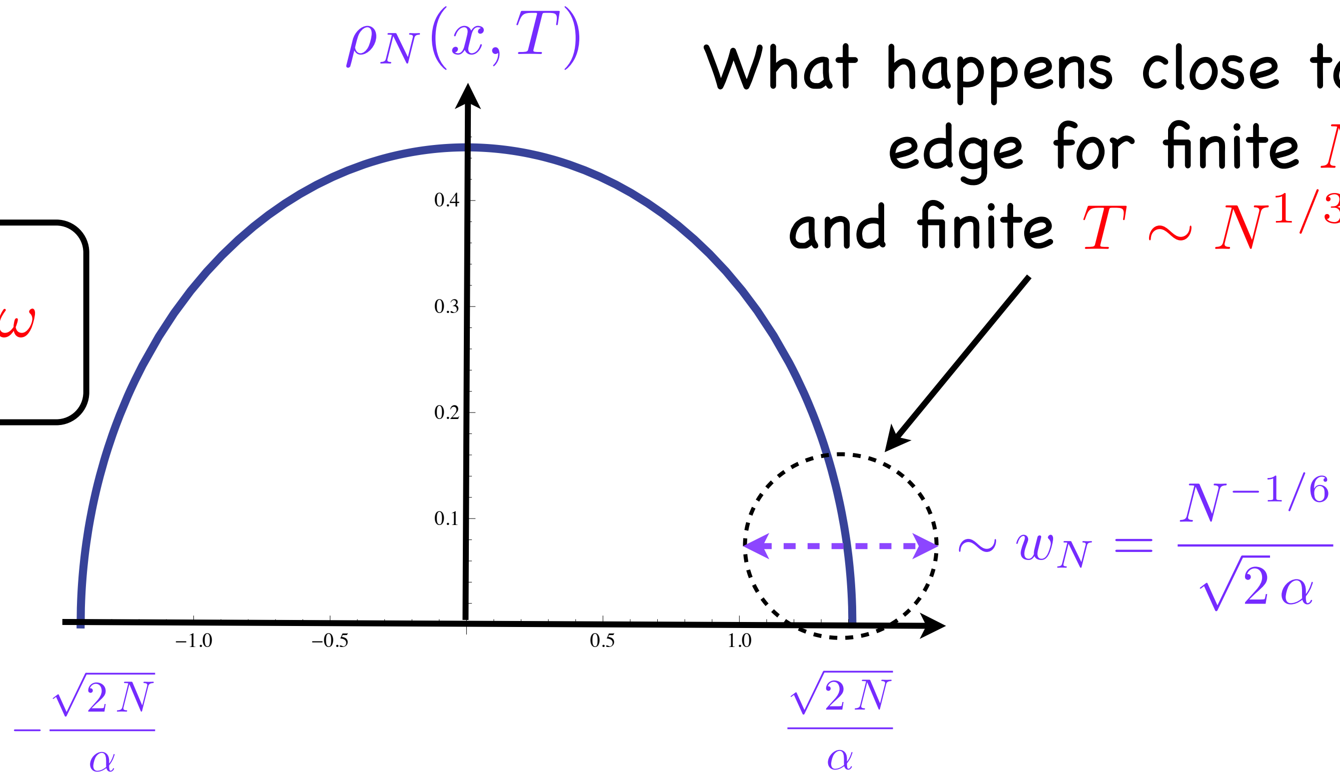

What happens close to the edge for finite N and finite $T \sim N^{1/3}$?

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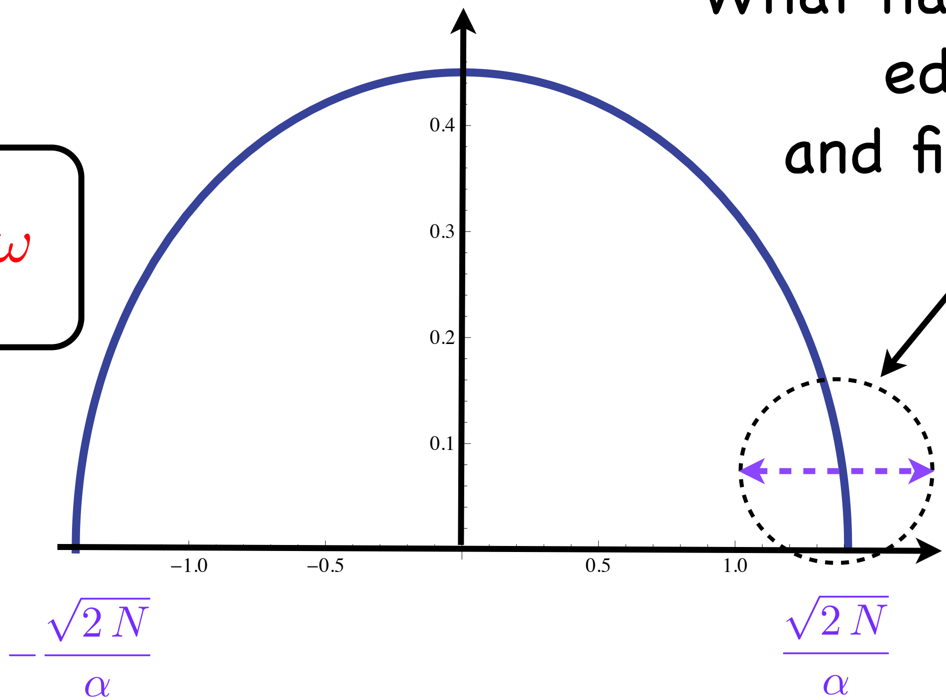
$\rho_N(x,T)$



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$$\sim w_N = \frac{N^{-1/6}}{\sqrt{2}\,\alpha}$$

$$\boxed{T \sim b^{-1} N^{1/3} \hbar\omega}$$

$$\rho_N(x,T) \approx \frac{1}{N w_N} F_1\left(\frac{x - \sqrt{2N}/\alpha}{w_N}\right)$$

$$\boxed{F_1(z) = \int_{-\infty}^{\infty} \frac{[Ai(z+u)]^2}{e^{-b\,u}+1}\, du}$$

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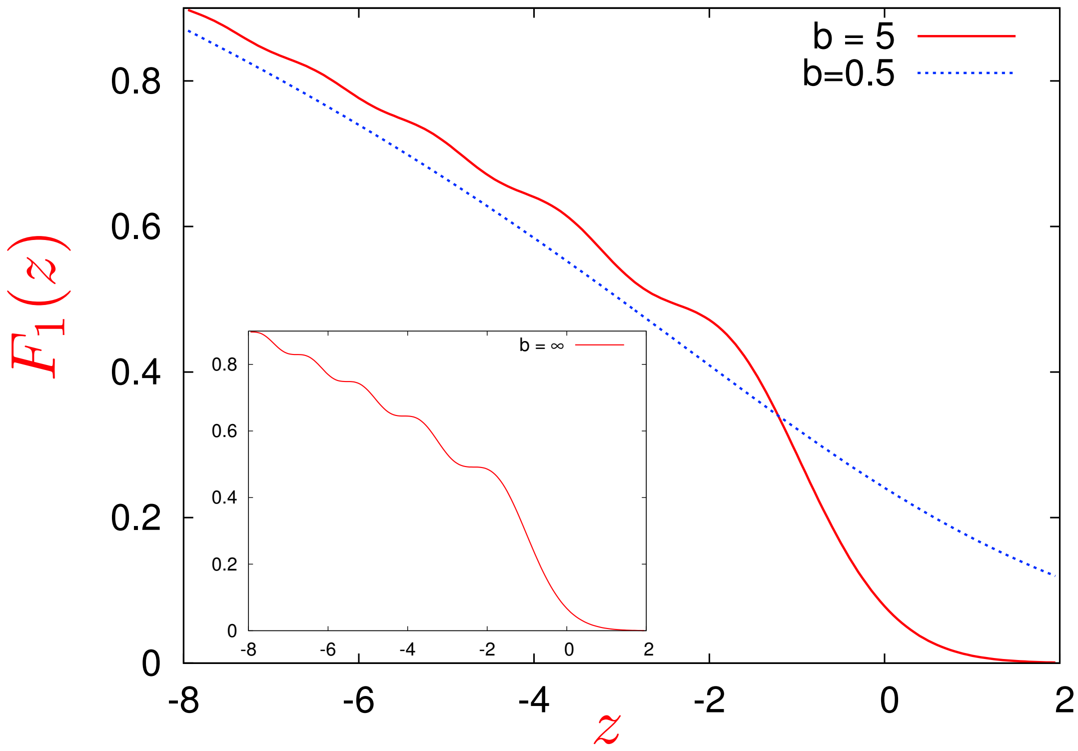

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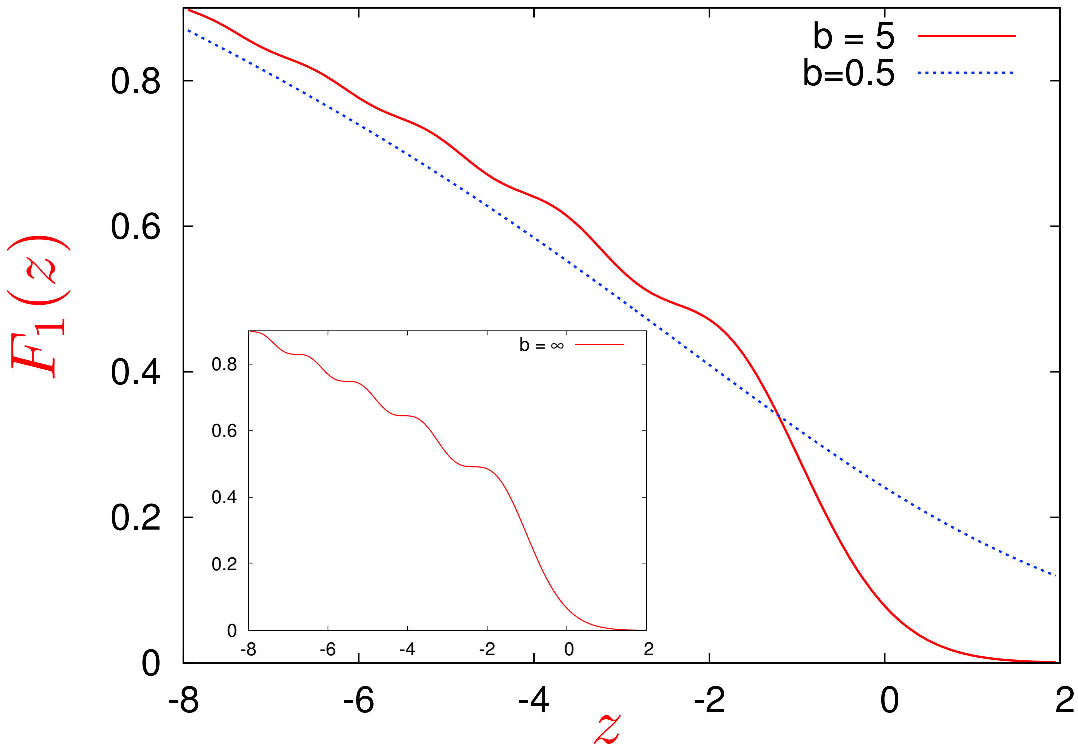

Asymptotic behaviors

$$F_1(z) \sim \begin{cases} \dfrac{\sqrt{|z|}}{\pi}\ ,\ z \to -\infty \\[2ex] \exp(-b\,z)\ ,\ z \to +\infty \end{cases}$$

Dean, Le Doussal, S. N. M., Schehr '14

Correlation kernel for N free fermions at T > 0

- For $N \gg 1$ the canonical and grand-canonical ensembles coincide

number of
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chemical potential
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where $\hat{H}\,\varphi_k = \epsilon_k\,\varphi_k$

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$$N = \sum_{k=0}^{\infty} \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$$

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see also Moshe, Neuberger, Shapiro '94/Johansson '07

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generalization of the Sine-kernel

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Correlation kernel for N free fermions for T > 0

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- Low temperature scaling limit: $N \to \infty \, , \, T \sim N^{1/3} \ll N$

when $x \,\&\, x'$ are close to the edge $r_{\rm edge} = \sqrt{2N}/\alpha$

$$K_N(x,x') \approx \frac{1}{w_N} \mathcal{K}_{\rm edge}\left(\frac{x-r_{\rm edge}}{w_N}, \frac{x'-r_{\rm edge}}{w_N}\right) \;,\; w_N = \frac{N^{-1/6}}{\sqrt{2}\alpha}$$

$$\boxed{\mathcal{K}_{\rm edge}(z_1,z_2) = \int_{-\infty}^{\infty} \frac{Ai(z_1+u)Ai(z_2+u)}{e^{-b\,u}+1} du}$$

Dean, Le Doussal, S. N. M., Schehr '14

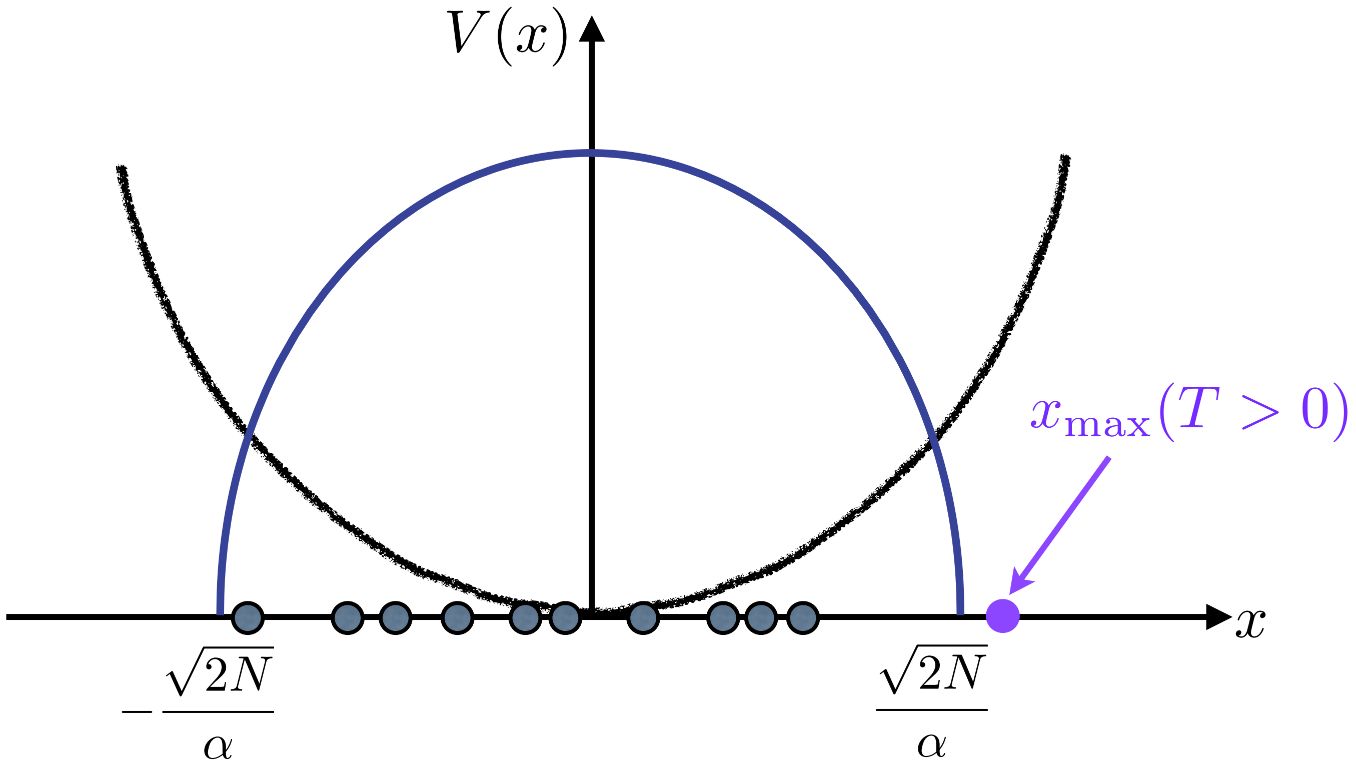
see also Johansson '07

generalization of the Airy-kernel

- Universal behavior, i.e., independent of the confining potential

$V(x) \sim |x|^p$

Position of the rightmost fermion at finite but low T



$$b = N^{1/3}\frac{\hbar\omega}{T}$$

$$r_{\text{edge}} = \sqrt{2N}/\alpha$$

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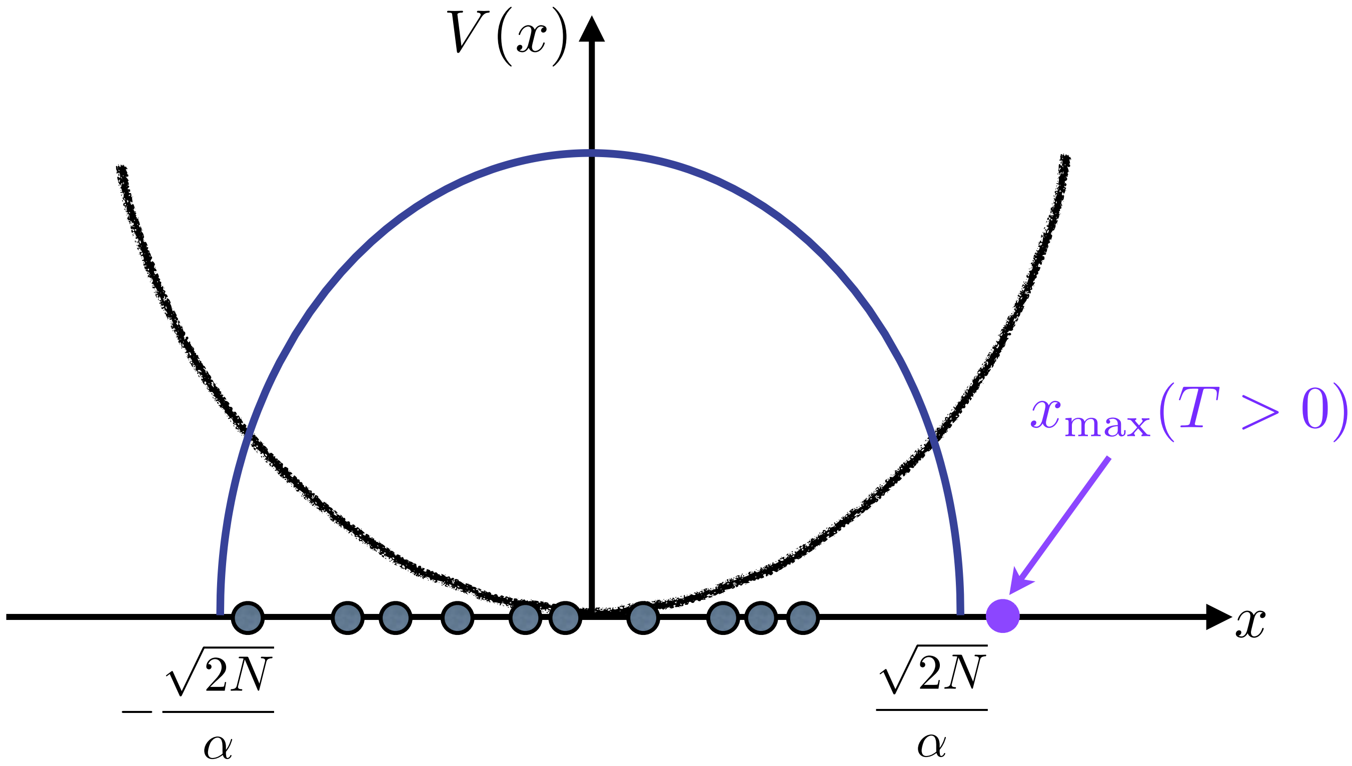

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- Fluctuations of $x_{\max}(T>0)$ — Dean, Le Doussal, S. N. M., Schehr '14

$$\Pr.(x_{\max}(T>0) \le M) \approx \mathcal{F}\left(\frac{M - r_{\rm edge}}{w_N}\right)$$

$$\mathcal{F}(\xi) = \det(I - P_\xi K_{\rm edge} P_\xi)\ ,\ K_{\rm edge}(z_1, z_2) = \int_{-\infty}^{\infty} \frac{Ai(z_1+u)Ai(z_2+u)}{e^{-b\,u}+1}\, du$$

Kardar-Parisi-Zhang (KPZ) equation at finite time

- KPZ equation in 1+1 dimensions and curved geometry

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2} (\partial_x h)^2 + \sqrt{D}\eta(x,t)$$

$$\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$$

$$h(x,t=0) = -\frac{|x|}{\delta}$$

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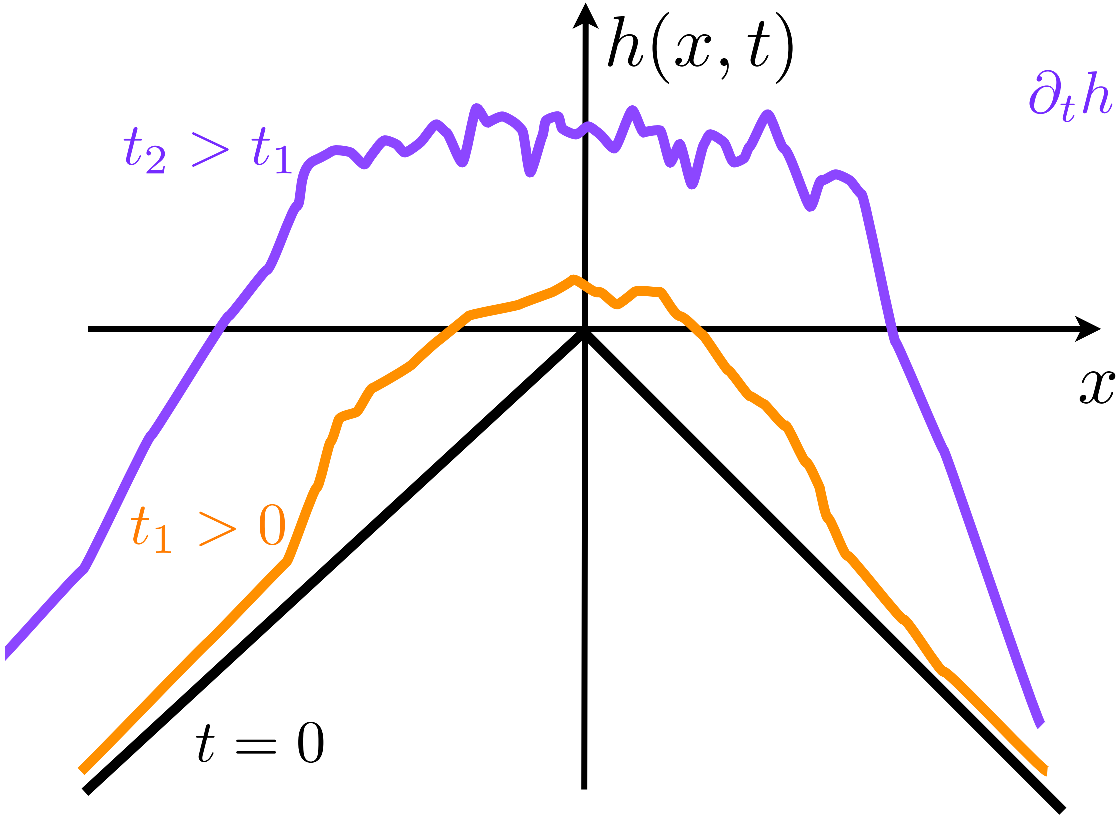

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Scaled variable:

$$\tilde{h}(0,t) = \frac{\frac{\lambda_0}{2\nu} h(0,t) + \frac{t}{12}}{\gamma_t}$$

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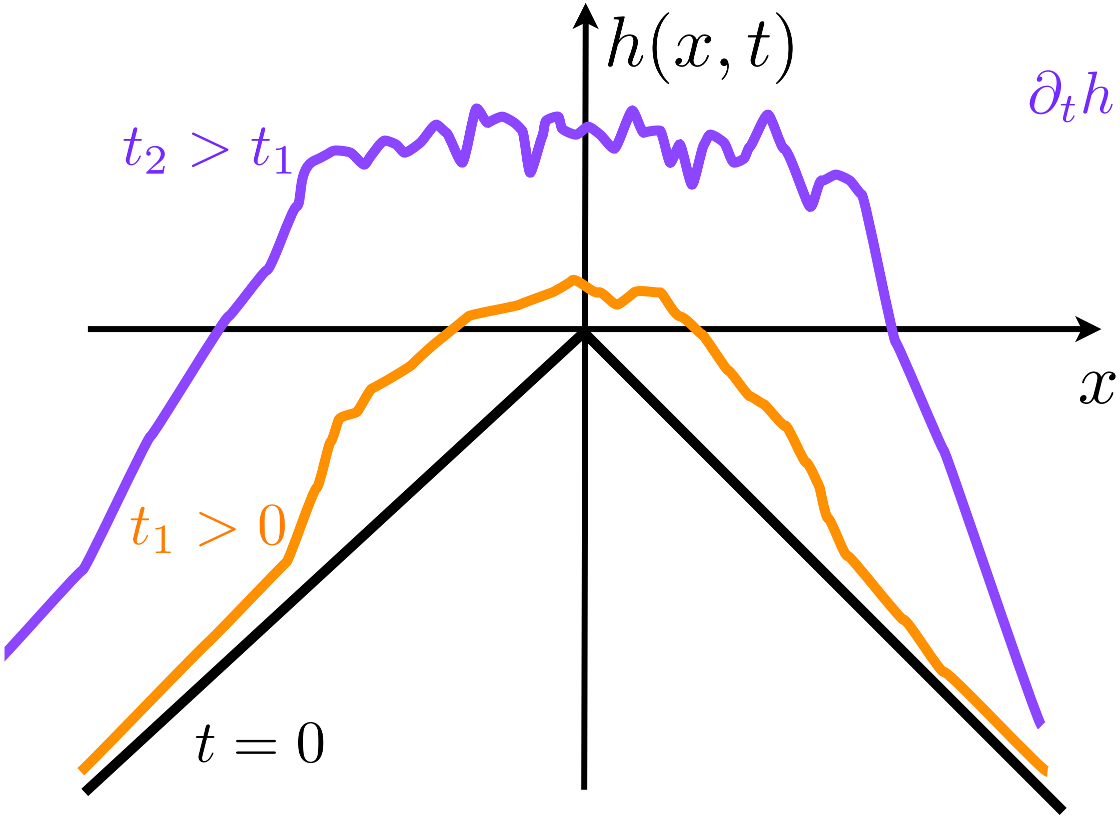

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Sasamoto, Spohn '10/Calabrese, Le Doussal, Rosso '10/Dotsenko '10/ Amir, Corwin, Quastel '11
Imamura, Sasamoto, Spohn '13

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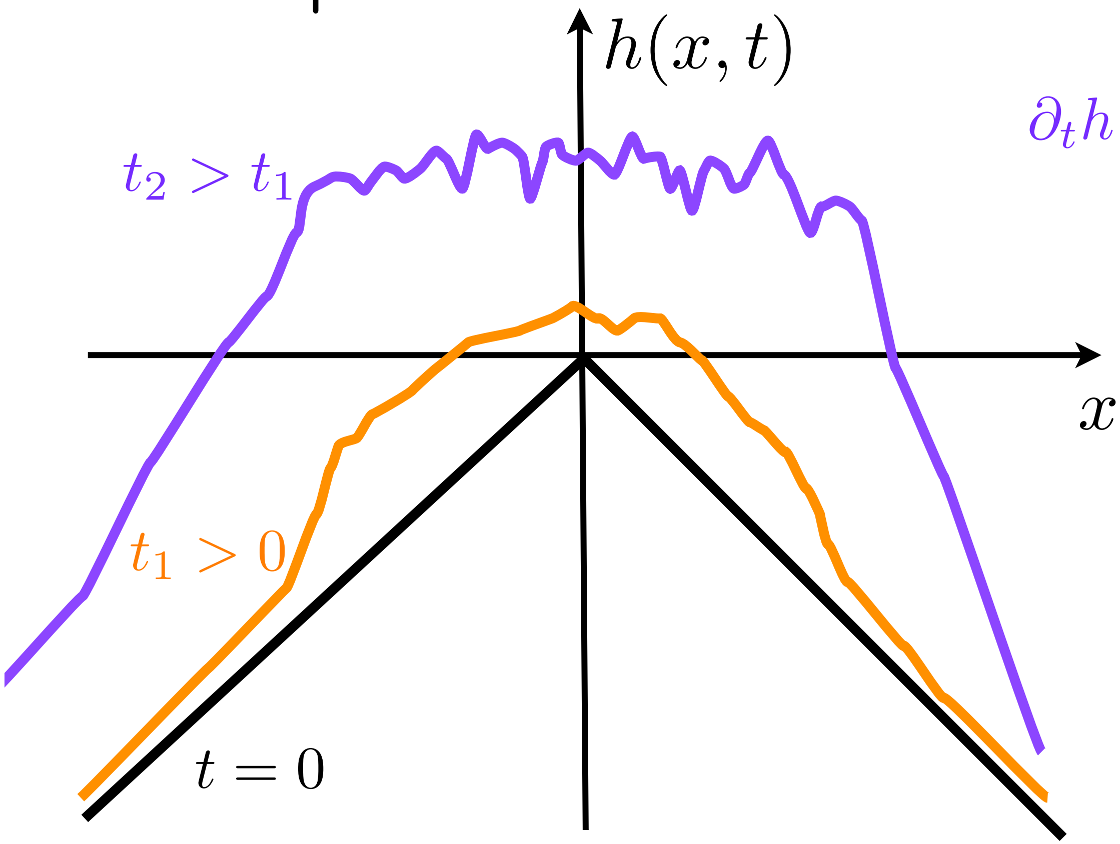

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- Time-dependent generating function of the height field

$$g_t(\zeta) = \langle \exp(-e^{\gamma_t(\tilde{h}(0,t)-\zeta)})\rangle \qquad g_t(\zeta) = \det(I - P_\zeta K_{\rm edge} P_\zeta)$$

$$K_{\rm edge}(z_1,z_2) = \int_{-\infty}^{\infty} \frac{Ai(z_1+u)Ai(z_2+u)}{e^{-\gamma_t u}+1}\, du$$

Connection between fermions at finite temperature and KPZ at finite time

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- Free fermions problem: fluctuations of $x_{\max}(T>0)$; $b = N^{1/3}\hbar\omega/T$

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→ formal connection between the two problems with $1/T \Longleftrightarrow t^{1/3}$

Dean, Le Doussal, S. N. M., Schehr '14

Outline

- Free fermions in d=1 and T=0 and Random Matrix Theory (RMT)
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Average density of free fermions at T > 0

$$P_{\rm joint}(x_1, \cdots x_N) = \frac{1}{N! Z_N(\beta)} \sum_{k_1 < \cdots < k_N} \left[\det_{1 \le i,j \le N} (\varphi_{k_i}(x_j)) \right]^2 e^{-\beta(\epsilon_{k_1} + \cdots + \epsilon_{k_N})}$$

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- Exact expression for the av. density $\rho_N(x,T) = \frac{1}{N} \sum_{i=1}^{N} \langle \delta(x - x_i) \rangle$

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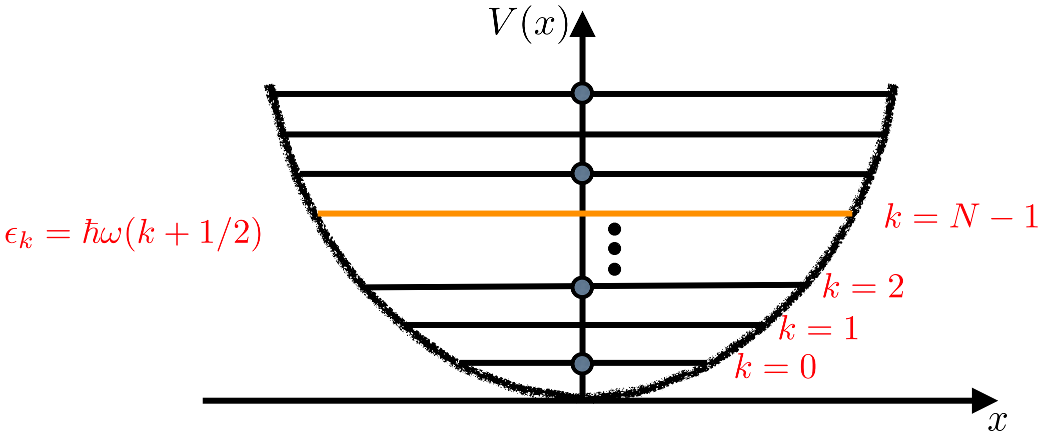
introduce the occupation numbers



$$m_k = \begin{cases} 0 \text{ , if state } k \text{ is empty} \\ 1 \text{ , if state } k \text{ is occupied} \end{cases}$$

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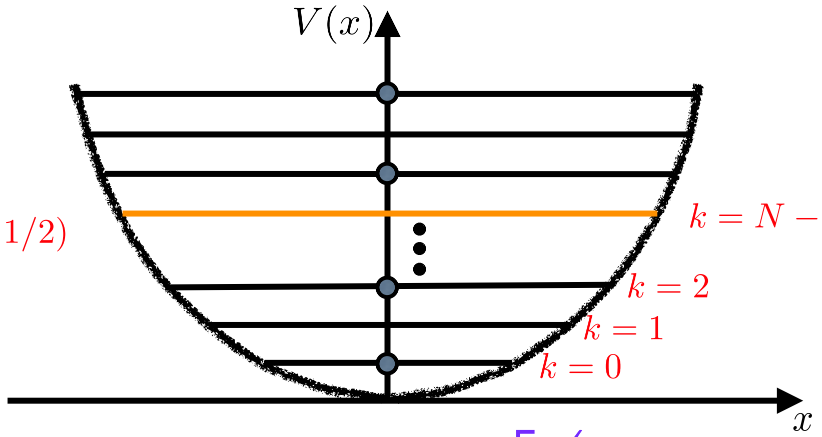

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Average density of free fermions at T > 0

$$\rho_N(x) = \frac{\mathcal{N}_N(x)}{N Z_N(\beta)}$$

with

$$\mathcal{N}_N(x) = \sum_{\{m_k\}} \left[\left(\sum_{k\geq 0} m_k (\varphi_k(x))^2 \right) e^{-\beta \sum_{k\geq 0} m_k \epsilon_k} \delta\left(\sum_{k \geq 0} m_k, N \right) \right]$$

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&= \prod_{j=0}^{\infty} [1 + z\, e^{-\beta\epsilon_j}] \sum_{k\geq 0} (\varphi_k(x))^2 \frac{z\, e^{-\beta\epsilon_k}}{1 + z\, e^{-\beta\epsilon_k}}
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similarly $$\sum_{N\geq 0} z^N Z_N(\beta) = \prod_{j=0}^{\infty} (1 + z\, e^{-\beta\epsilon_j})$$ grand-canonical partition function

Average density of free fermions at T > 0

- An exact formula (using Cauchy formula)

$$\rho_N(x) = \frac{1}{N} \frac{\oint \frac{dz}{2i\pi} z^{-(N+1)} \prod_{j=0}^{\infty}(1+z\, e^{-\beta\epsilon_j}) \sum_{k\geq 0} (\varphi_k(x))^2 \frac{z\, e^{-\beta\epsilon_k}}{1+z\, e^{-\beta\epsilon_k}}}{\oint \frac{dz}{2i\pi} z^{-(N+1)} \prod_{j=0}^{\infty}(1+z\, e^{-\beta\epsilon_j})}$$

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via saddle-point calculation

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Fermi factor

Correlation kernel for free fermions at T > 0

- Final result for the density for large N

$$\rho_N(x) \approx \frac{1}{N} \sum_{k \geq 0} (\varphi_k(x))^2 \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$$

Correlation kernel for free fermions at T > 0

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- n-point correlation functions for large N (by similar computations)

$$R_n(x_1, \cdots, x_n) \approx \det_{1 \leq i,j \leq n} K_N(x_i, x_j)$$

where the correlation kernel is given by

$$K_N(x, x') = \sum_{k=0}^{\infty} \frac{\varphi_k(x)\varphi_k(x')}{e^{\beta(\epsilon_k - \mu)} + 1} \quad \text{and} \quad N = \sum_{k=0}^{\infty} \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$$

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Free fermions in a d-dimensional harmonic trap (T=0)

- Single particle Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}\right) + \frac{1}{2}m\omega^2\left(\underbrace{x_1^2 + \cdots x_d^2}_{r^2}\right)$$

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- Global density (at T=0)

$$\boxed{\rho_N(\mathbf{x}) \approx \frac{1}{N}\left(\frac{m}{2\pi\hbar^2}\right)^{d/2}\frac{[\mu - \frac{1}{2}m\omega^2 r^2]^{d/2}}{\Gamma(d/2+1)}}$$

with $\mu \approx \hbar\omega[\Gamma(d+1)\,N]^{1/d}$

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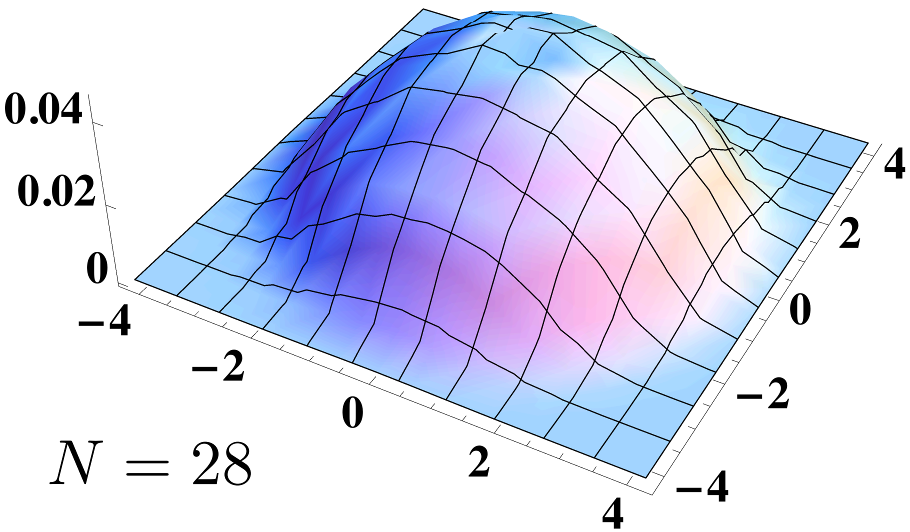
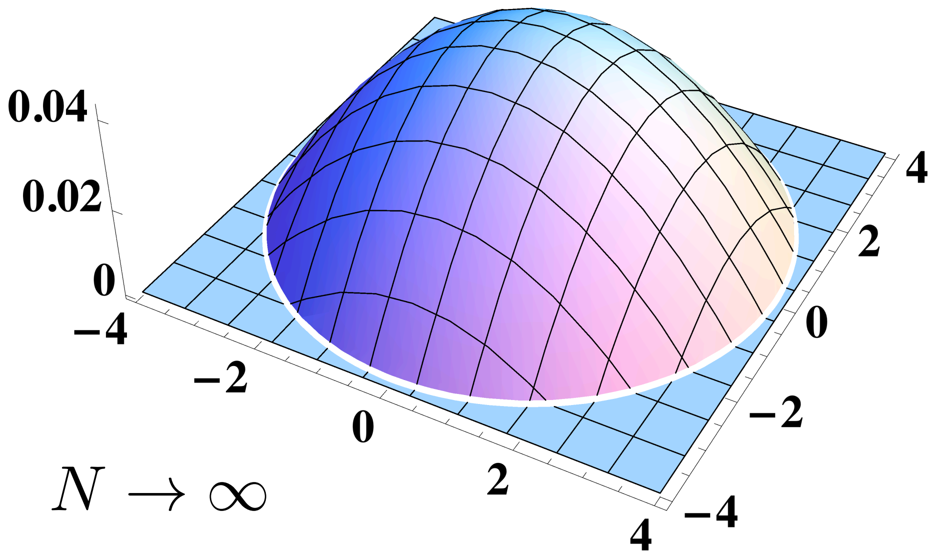
with $\mu \approx \hbar\omega[\Gamma(d+1)\, N]^{1/d}$ Dean, Le Doussal, S. N. M., Schehr '15

$d = 2$





Free fermions in a d-dimensional harmonic trap (T=0)

Dean, Le Doussal, S. N. M., Schehr '15

- Edge density of free fermions

$$\rho_{\rm edge}(\mathbf{x}) \approx \frac{1}{N}\frac{1}{w_N^d} F_d\left(\frac{r-r_{\rm edge}}{w_N}\right)$$

with $w_N = b_d\, N^{-\frac{1}{6d}}$ and $F_d(z) = \dfrac{1}{\Gamma(\frac{d}{2}+1)2^{\frac{4d}{3}}\pi^{\frac{d}{2}}}\displaystyle\int_0^\infty du\; u^{\frac{d}{2}} Ai(u+2^{2/3}\,z)$

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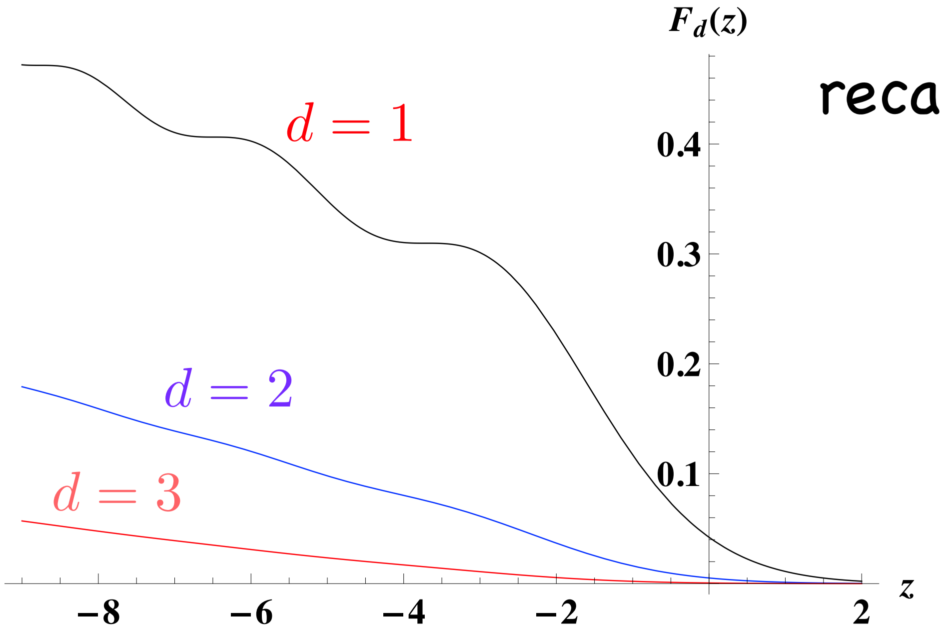

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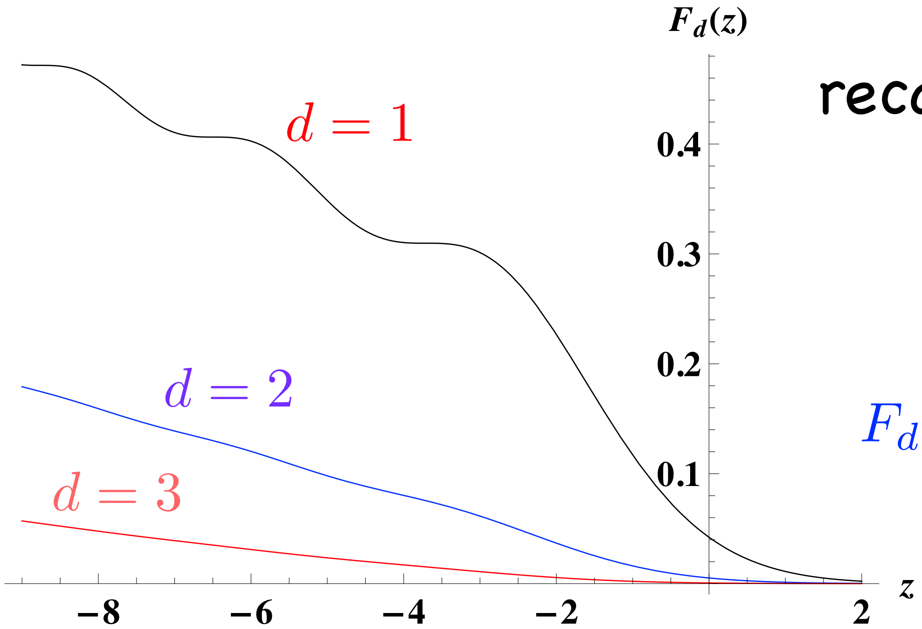

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$$F_d(z) \approx \begin{cases} (8\pi)^{-\frac{d+1}{2}}\, z^{-\frac{d+3}{4}}\, e^{-\frac{4}{3}\, z^{3/2}} \text{ as } z \to \infty \\ \dfrac{(4\pi)^{-\frac{d}{2}}}{\Gamma(d/2+1)}\, |z|^{\frac{d}{2}} \quad \text{as} \quad z \to -\infty \end{cases}$$

Free fermions in a d-dimensional harmonic trap (T=0): limiting correlation kernels

$$K_N(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{k}} \theta(E_F - \epsilon_{\mathbf{k}}) \psi_{\mathbf{k}}(\mathbf{x}) \psi_{\mathbf{k}}(\mathbf{y})$$

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$$K_N(\mathbf{x},\mathbf{y}) \approx \frac{1}{w_N^d}\mathcal{K}_{\text{edge}}\left(\frac{\mathbf{x}-\mathbf{r}_{\text{edge}}}{w_N}, \frac{\mathbf{y}-\mathbf{r}_{\text{edge}}}{w_N}\right)$$

with $$\boxed{\mathcal{K}_{\text{edge}}(\mathbf{a},\mathbf{b}) = \int \frac{d^d q}{(2\pi)^d} e^{-i\mathbf{q}\cdot(\mathbf{a}-\mathbf{b})} Ai_1\left(2^{\frac{2}{3}}q^2 + \frac{a_n+b_n}{2^{1/3}}\right)}$$

$a_n = \mathbf{a}\cdot\mathbf{r}_{\text{edge}}/r_{\text{edge}}$ and $b_n = \mathbf{b}\cdot\mathbf{r}_{\text{edge}}/r_{\text{edge}}$

$$Ai_1(z) = \int_z^\infty Ai(u)\,du$$

Outline

- Free fermions in d=1 and T=0 and Random Matrix Theory (RMT)
- Free fermions in d=1 and T>0 and KPZ equation: main results
- Sketch of the derivation of our results
- Extension to higher dimensions, d>1
- Conclusion

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«Quantum gas microscope», M. Greiner et al. PRL 2015