Nonequilibrium fluctuations using macroscopic fluctuation theory

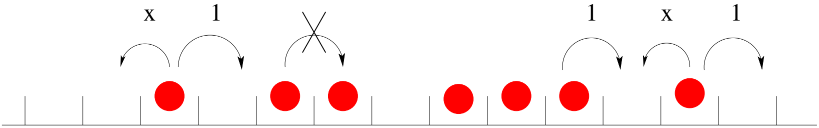
K. Mallick

Institut de Physique Théorique, CEA Saclay (France)

KITP, Santa Barbara, February 26, 2016

(Joint work with P. Krapivsky and T. Sadhu)

A Paradigm of non-equilibrium behaviour: ASEP



Asymmetric Exclusion Process. A Minimal Model for non-equilibrium Statistical Mechanics.

- EXCLUSION: Hard core-interaction; at most 1 particle per site.
- ASYMMETRIC: External driving; breaks detailed-balance
- PROCESS: Stochastic Markovian dynamics; no Hamiltonian.

The ASEP appears as a building block in many realistic models of 1d transport and is studied extensively in probability, combinatorics, statistical physics...

ASEP was invented in 1968 by molecular biologists.

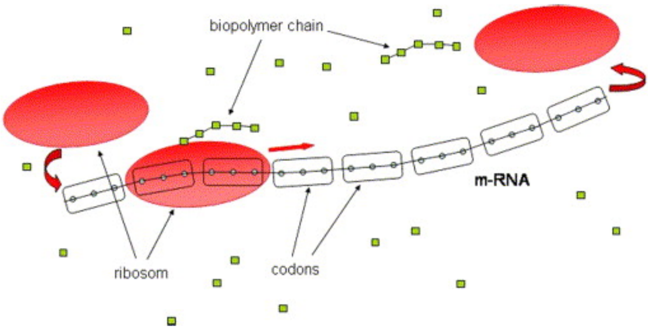
ORIGINS

- Interacting Brownian Processes (Spitzer, Harris, Liggett).
- Driven diffusive systems (Katz, Lebowitz and Spohn).
- Transport of Macromolecules through thin vessels.
  Motion of RNA templates.
- Hopping conductivity in solid electrolytes.
- Directed Polymers in random media. Reptation models.
- Interface dynamics. KPZ equation

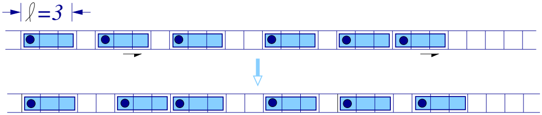
SOME RECENT APPLICATIONS

- Traffic flow.
- Sequence matching.
- Brownian motors.

An Elementary Model for Protein Synthesis



C. T. MacDonald, J. H. Gibbs and A.C. Pipkin, Kinetics of biopolymerization on nucleic acid templates, *Biopolymers* (1968).



Single-file Diffusion

Anomalous diffusion in SEP

Consider the Symmetric Exclusion Process on an infinite one-dimensional line with a finite density ρ of particles.

Suppose that we tag and observe a particle that was initially located at site 0 and monitor its position X_t with time.

On the average $\langle X_t \rangle = 0$ but how large are its fluctuations?

- If the particles were non-interacting (no exclusion constraint), each particle would diffuse normally $\langle X_t^2 \rangle = Dt$.

Anomalous diffusion in SEP

Consider the Symmetric Exclusion Process on an infinite one-dimensional line with a finite density ρ of particles.

Suppose that we tag and observe a particle that was initially located at site 0 and monitor its position X_t with time.

On the average $\langle X_t \rangle = 0$ but how large are its fluctuations?

- If the particles were non-interacting (no exclusion constraint), each particle would diffuse normally $\langle X_t^2 \rangle = Dt$.
- Because of the exclusion condition, a particle displays an anomalous diffusive behaviour:

$$\langle X_t^2 \rangle = 2\frac{1-\rho}{\rho}\sqrt{\frac{Dt}{\pi}} \qquad \text{(Arratia, 1983)}$$

Single-File Diffusion is an important model in soft-condensed matter; for example, ion transport through cell membranes (cf. experiments by C. Bechinger).

ASEP on the infinite line

Consider now the Asymmetric Exclusion Process on an infinite one-dimensional line with a finite density ρ of particles. Allowed jumps are performed with rate 1 towards the right and rate x towards the left.

- A tagged particle will display a normal diffusive behaviour

$$\langle X_t^2 \rangle - \langle X_t \rangle^2 \simeq \mathcal{D}_0 t \quad \text{with} \quad \mathcal{D}_0 = (1-x)(1-\rho)$$

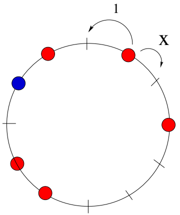
if we take the averages with respect to the initial condition (at density ρ) and with respect to the history of the process (A. De Masi and P. Ferrari, 1985). This is the **annealed average**.

- If we consider the **quenched average** with **fixed** initial condition

$$\lim_{t \to \infty} \frac{\langle X_t^2 \rangle - \langle X_t \rangle^2}{t} = 0$$

It has been proved rigorously that the quenched diffusion constant vanishes. More precisely: $\langle X_t^2 \rangle - \langle X_t \rangle^2 \sim t^{2/3}$.

Tracer on a ring I



Consider an ASEP on a finite ring with asymmetry parameter x. In the long time limit, a tagged particle undergoes normal diffusion with Diffusion Constant:

$$D = \lim_{t\to\infty} \frac{\langle Y_t^2\rangle - \langle Y_t\rangle^2}{t} = (1-x)\frac{2L}{L-1}\sum_{k>0} k^2 \frac{C_L^{N+k}}{C_L^N}\frac{C_L^{N-k}}{C_L^N}\left(\frac{1+x^k}{1-x^k}\right)$$

- Symmetric case $x = 1$:

$$D = 2\frac{L-N}{N(L-1)} \simeq 2\frac{1-\rho}{L\rho}$$

where $\rho = N/L$ is the density. The diffusion constant vanishes as $1/L$

Tracer on a ring II

This leads by finite-size scaling to the $t^{1/4}$ behaviour of SEP on the infinite line. We write, taking into account that the dynamical exponent of SEP is $z = 2$,

$$\langle X_t^2 \rangle - \langle X_t \rangle^2 \simeq L^{2\chi} \Phi\left(\frac{t}{L^2}\right)$$

Taking $t \to \infty$ and L finite: $\chi = 1/2$ and $\Phi(u) \simeq 2\frac{1-\rho}{\rho} u$ when $u \to \infty$.
For $L \to \infty$ and t finite, we must have $\Phi(u) \sim u^{1/2}$ when $u \to 0$,

$$\langle X_t^2 \rangle - \langle X_t \rangle^2 \sim t^{1/2}$$

- Asymmetric case $x < 1$:

$$D \simeq (1-x)\frac{\sqrt{\pi}(1-\rho)^{3/2}}{2\rho^{1/2}}\frac{1}{\sqrt{L}}$$

For ASEP, the dynamical exponent is $z = 3/2$ and Finite-Size scaling implies

$$\langle X_t^2 \rangle - \langle X_t \rangle^2 \sim t^{2/3}$$

(Recall quenched average on the infinite line.)

Open questions about SEP on the infinite line

In present talk, we shall focus on SEP on the infinite line.

- We are interested in the quantitative behaviour of the higher cumulants of X_t. Can we calculate the cumulant generating function $\log[\langle e^{\lambda X_T} \rangle]$ or the full distribution of X_t?
- What is the robustness of the $t^{1/4}$? Can tagged particle statistics be determined for more general systems, *without having to use integrability or rely on some combinatorial trick*?
- What is the influence of the initial setting?
- Statistical properties of the tagged particle trajectory? Multiple-time correlations?

Macroscopic Fluctuation Theory: Fundamental Formula

Study the system at a coarse-grained hydrodynamical level.

For a weakly-driven diffusive system, the probability to observe a current $j(x,t)$ and a density profile $\rho(x,t)$ during a time T takes a large deviation form:

$$\mathrm{Pr}\{j(x,t),\rho(x,t)\} \sim \mathrm{e}^{-\,S_{MFT}(j,\rho)}$$

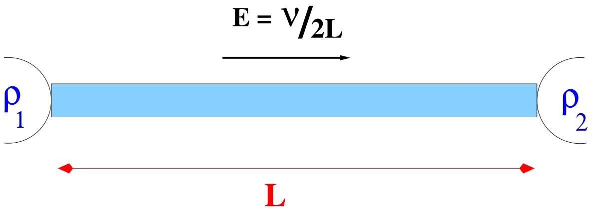
where

$$S_{MFT}(j,\rho) = \int_0^T dt \int_{-\infty}^{+\infty} \frac{\left(j - \nu\sigma(\rho) + D(\rho)\nabla\rho\right)^2 dx}{2\sigma(\rho)}$$

with the constraint: $\partial_t \rho = -\nabla . j$ (L. Bertini, D. Gabrielli, A. De Sole, G. Jona-Lasinio and C. Landim).

The transport coefficients $D(\rho)$ (Diffusivity) and $\sigma(\rho)$ (Conductivity) carry the relevant information from the microscopic level to the macroscopic stage. *They must be calculated using the microscopic dynamical rules.*

The Hydrodynamic Limit: deterministic case



Starting from the microscopic level, define local density $\rho(x, t)$ and current $j(x, t)$ with macroscopic space-time variables $x = i/L, t = s/L^2$ (diffusive scaling).
The average hydrodynamic evolution of the system is given by:

$$\partial_t \rho(x, t) = -\nabla J(x, t) \quad \text{with} \quad J = -D(\rho)\nabla\rho + \nu\sigma(\rho)$$

How can Fluctuations be taken into account?

Fluctuating Hydrodynamics

Let Y_t be the integrated current of particles transferred from the left reservoir to the right reservoir during time t.

- $\lim_{t\to\infty} \frac{\langle Y_t \rangle}{t} = D(\rho)\frac{\rho_1 - \rho_2}{L} + \sigma(\rho)\frac{\nu}{L}$ for $(\rho_1 - \rho_2)$ small
- $\lim_{t\to\infty} \frac{\langle Y_t^2 \rangle}{t} = \frac{\sigma(\rho)}{L}$ for $\rho_1 = \rho_2 = \rho$ and $\nu = 0$.

Then, the equation of motion is obtained as:

$$\partial_t \rho = -\partial_x j \quad \text{with} \quad j = -D(\rho)\nabla\rho + \nu\sigma(\rho) + \sqrt{\sigma(\rho)}\xi(x,t)$$

where $\xi(x,t)$ is a Gaussian white noise with variance

$$\langle \xi(x',t')\xi(x,t) \rangle = \frac{1}{L}\delta(x-x')\delta(t-t')$$

For the symmetric exclusion process, the 'phenomenological' coefficients are given by

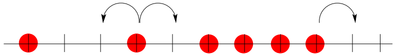
$$D(\rho) = 1 \quad \text{and} \quad \sigma(\rho) = 2\rho(1-\rho)$$

Values of Diffusivity and Conductivity

- Independent particles: $D = 1$, $\sigma = 2\rho$
- Simple Exclusion Process: $D_{\rm SEP} = 1$, $\sigma_{\rm SEP} = 2\rho(1-\rho)$
- Kipnis-Marchioro-Presutti model: $D_{\rm KMP} = 1$, $\sigma_{\rm KMP} = 2\rho^2$
- Repulsion Process: Hops increasing the number of nearest neighbourg pairs are forbidden:

$$D_{\rm RP} = \begin{cases} \frac{1}{(1-\rho)^2} & \text{if } 0 < \rho < \frac{1}{2} \\ \frac{1}{\rho^2} & \text{if } \frac{1}{2} < \rho < 1 \end{cases} \qquad \sigma_{\rm RP} = \begin{cases} \frac{2\rho(1-2\rho)}{1-\rho} & \text{if } 0 < \rho < \frac{1}{2} \\ \frac{2(1-\rho)(2\rho-1)}{\rho} & \text{if } \frac{1}{2} < \rho < 1 \end{cases}$$

- Exclusion Process with Avalanches: $D_{\rm EPA} = \frac{1}{(1-2\rho)^3}$, $\sigma_{\rm EPA} = \frac{2\rho(1-\rho)}{(1-2\rho)^3}$



Katz-Lebowitz-Spohn model (Driven Ising Model)

The Katz-Lebowitz-Spohn model is a driven lattice gas where the hopping rates depend on the neighbouring sites:

$$0100 \underset{1+\delta}{\overset{1+\delta}{\leftrightarrows}} 0010 \quad 1101 \underset{1-\delta}{\overset{1-\delta}{\leftrightarrows}} 1011 \quad 1100 \underset{1+\epsilon}{\overset{1-\epsilon}{\leftrightarrows}} 1010 \quad 0101 \underset{1-\epsilon}{\overset{1+\epsilon}{\leftrightarrows}} 0011$$

$$\sigma_{\mathrm{KLS}} = 2\frac{\lambda(\rho)[1+\delta(1-2\rho)]-2\epsilon\sqrt{\rho(1-\rho)}}{\lambda(\rho)^3} \text{ with } \lambda(\rho) = \frac{1+\sqrt{1-8\epsilon\rho(1-\rho)/(1+\epsilon)}}{2\sqrt{\rho(1-\rho)}}$$

The diffusivity is given by $D_{\mathrm{KLS}}(\rho) = \frac{1}{2}\chi(\rho)\,\sigma_{\mathrm{KLS}}(\rho)$, where $\chi(\rho)$ is obtained by eliminating the parameter h between the two equations:

$$\chi = \frac{1}{4}\,\frac{1+\epsilon}{1-\epsilon}\,\frac{\cosh h}{\left(\sinh^2 h + \frac{1+\epsilon}{1-\epsilon}\right)^{3/2}}$$

$$\rho = \frac{1}{2}\left(1 + \frac{\sinh h}{\sqrt{\sinh^2 h + \frac{1+\epsilon}{1-\epsilon}}}\right)$$

(Y. Kafri et al., 2013)

Tagged particle as a macroscopic observable

How to write the position X_T of the Tagged Particle macroscopically? In Single-File Diffusion, particles can not overtake, *i.e.* the ordering of the particle is conserved:

$$\int_{X_T}^{+\infty} \rho(x,t) = \int_0^{+\infty} \rho(x,0)$$

This defined the functional $X_T[\rho]$, whose statistics we can study by MFT.

$$\langle \mathrm{e}^{\lambda \mathrm{X_T}} \rangle = \int \mathcal{D}\rho_0(x)\mathcal{P}[\rho_0] \int \mathcal{D}\rho(x,t)\mathcal{D}j(x,t)\mathrm{e}^{\lambda \mathrm{X_T}[\rho] - \mathrm{S_{MFT}}[\mathrm{j},\rho]}\delta(\partial_{\mathrm{t}}\rho + \nabla.\mathrm{j})$$

The initial profile ρ_0, distributed according to $\mathcal{P}[\rho_0]$ can be fixed (quenched) or fluctuate w.r.t. some chosen measure (annealed).

Scaling shows that the effective action grows as $\sqrt{T} \to$ Saddle-Point.

The calculation becomes an optimization problem: Find the optimal path (j^*, ρ^*) that generates a given fluctuation of X_T.

M. F. T. Equations

Evaluating the effective action at the saddle-point (j^*, ρ^*) gives

$$\langle \mathrm{e}^{\lambda \mathrm{X_T}} \rangle \simeq \mathrm{e}^{\sqrt{4\mathrm{T}}\mu(\lambda)}$$

$\sqrt{4T}\mu(\lambda)$ being the cumulant generating function: $\mu(\lambda) = \sum_n \frac{\lambda^n}{n!} \frac{\langle X_T^n \rangle_c}{\sqrt{4T}}$

M. F. T. Equations

Evaluating the effective action at the saddle-point (j^*, ρ^*) gives

$$\langle \mathrm{e}^{\lambda X_T} \rangle \simeq \mathrm{e}^{\sqrt{4T}\mu(\lambda)}$$

$\sqrt{4T}\mu(\lambda)$ being the cumulant generating function: $\mu(\lambda) = \sum_n \frac{\lambda^n}{n!} \frac{\langle X_T^n \rangle_c}{\sqrt{4T}}$

The optimization is performed by solving Euler-Lagrange equations, better reformulated as a Hamiltonian structure in terms of two conjugate variables (p, q) that satisfy coupled PDE's (setting $\nu = 0$):

$$\begin{aligned} \partial_t q &= \partial_x [D(q)\partial_x q] - \partial_x [\sigma(q)\partial_x p] \\ \partial_t p &= -D(q)\partial_{xx} p - \frac{1}{2}\sigma'(q)(\partial_x p)^2 \end{aligned}$$

where $q(x, t)$ is the optimal density-field and $p(x, t)$ is the conjugate field with Hamiltonian: $H[p, q] = -D(q)\partial_x q \partial_x p + \frac{\sigma(q)}{2}(\partial_x p)^2$
The parameter λ appears through the boundary conditions at $t = 0$ and $t = T$.

Optimal paths

- Path of least action

$$\partial_t q = -\frac{\delta H}{\delta p} \qquad \text{and} \qquad \partial_t p = \frac{\delta H}{\delta q}$$

- Boundary conditions

$$p(x, T) = \lambda \left[\frac{\delta X_T}{\delta q(x, T)}\right]$$

Quenched	Annealed
$q(x,0) = \rho$	$p(x,0) = -\lambda \left[\frac{\delta X_T}{\delta q(x,0)}\right] + \frac{\delta F}{\delta q(x,0)}$

The distinction between annealed and quenched comes from the boundary conditions.

A Formula for the variance

In the general case, the MFT equations can not be solved analytically but a perturbative approach w.r.t. λ is possible, providing us with the first few cumulants of X_T.

- Quenched case:

$$\langle X_T^2 \rangle = \frac{\sigma(\rho)}{\rho^2}\sqrt{\frac{T}{\pi D(\rho)}}$$

- Annealed case:

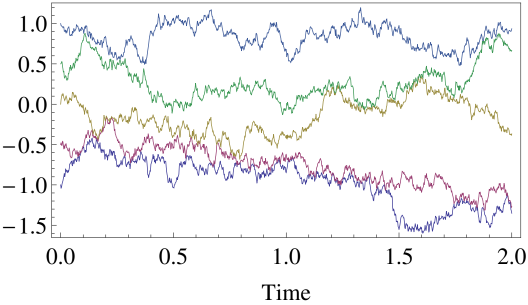
$$\langle X_T^2 \rangle = \frac{\sigma(\rho)}{\rho^2}\sqrt{\frac{\mathbf{2}T}{\pi D(\rho)}}$$

Note the *everlasting effect* of the initial conditions.
For SEP, we also obtain a formula for the 4th cumulant:

$$\langle X_T^4 \rangle_c = \frac{[1-\rho][1-\left(4-(8-3\sqrt{2})\rho\right)(1-\rho)+\frac{12}{\pi}(1-\rho)^2]}{\rho^3}\sqrt{\frac{4T}{\pi}}$$

Interacting Brownian Motions

A special case of Single-File diffusion is a system of Interacting Brownian Motions with hard-core reflection. It can be obtained as the limit of SEP in a continuous space with point-particles.



F. Spitzer, *Adv. Math.* (1970).
In this case: $D = 1$, $\sigma = 2\rho$. The MFT equations can be solved.

MFT equations for point-like particles

- Path of least-action

$$\partial_t p + \partial_{xx} p = -(\partial_x p)^2$$
$$\partial_t q - \partial_{xx} q = -\partial_x (2q\partial_x p)$$

- Boundary condition (Quenched)

$$q(x,0) = \rho$$
$$p(x,T) = B\ \Theta(x - X_T) \qquad \text{with} \qquad B = \frac{\lambda}{q(X_T,T)}$$

Note that the boundary condition depends on the solution.

- How to solve?

  Canonical change of variables: $P = e^p$ and $Q = qe^{-p}$

$$\partial_t P + \partial_{xx} P = 0 \qquad \text{and} \qquad \partial_t Q - \partial_{xx} Q = 0$$

Solution Procedure

- Step 1 Solve for p and q treating X_T and B as parameter.

- Step 2 Determine X_T self-consistently via

$$\int_{X_T}^{\infty} dz\ q(z,T) = \int_0^{\infty} dz\ q(z,0)$$

- Step 3 Determine B from minimization of Action

$$\frac{\mu_T(\lambda, B)}{dB} = 0$$

A Tracer Statistics: annealed case

For Interacting Brownian Motions, the full statistics of the tracer position, X_t, can be determined. The function $\mu(\lambda)$ is known through a parametric representation:

$$\begin{aligned}
\mu(\lambda) &= \left[\lambda + \rho\frac{1-e^B}{1+e^B}\right]\eta \\
\lambda &= \rho\left(1-e^{-B}\right)\left[1+\tfrac{1}{2}\left(e^B-1\right)\mathrm{erfc}(\eta)\right] \\
e^{2B} &= 1 + \frac{2\eta}{\pi^{-1/2}e^{-\eta^2} - \eta\,\mathrm{erfc}(\eta)}
\end{aligned}$$

The first few moments are given by

$$\langle X_T^2\rangle_c = \frac{2}{\rho\sqrt{\pi}}\sqrt{T},$$

$$\langle X_T^4\rangle_c = \frac{6\,(4-\pi)}{\left(\rho\sqrt{\pi}\right)^3}\sqrt{T}$$

$$\langle X_T^6\rangle_c = \frac{30\left(68-30\pi+3\pi^2\right)}{\left(\rho\sqrt{\pi}\right)^5}\sqrt{T}$$

A Tracer Statistics: quenched case

The function $\mu(\lambda)$ is even simpler in the quenched case

$$\mu(\lambda)=\sqrt{T}\rho\int_{-\infty}^{+\infty}\mathrm{dx}\log\left\{1+2\,\mathrm{erfc}(\mathrm{x})\mathrm{erfc}(-\mathrm{x})\sinh^2\left(\frac{\lambda}{2\rho}\right)\right\}$$

In both cases (annealed and quenched), the large deviation function of the tracer, defined, for $T \to \infty$, via

$$\mathrm{Prob}\left(\frac{X_T}{\sqrt{T}}=\xi\right)\sim\exp\left[-\sqrt{T}\phi(\xi)\right]$$

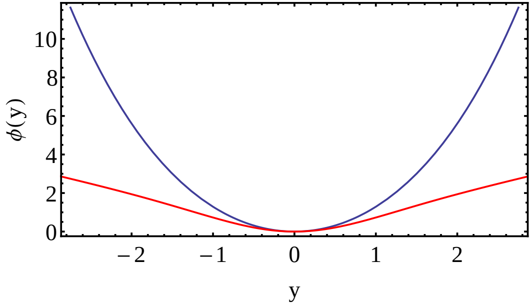
is obtained by taking the Legendre transform of $\mu(\lambda)$.

Large deviation functions

Quenched: $\frac{\phi_B(y)}{\rho} = -\int_y^\infty dx\ \log\left[1+\left(e^B-1\right)\frac{1}{2}\,\mathrm{erfc}(x)\right] - \int_{-y}^\infty dx\ \log\left[1+\left(e^{-B}-1\right)\frac{1}{2}\,\mathrm{erfc}(x)\right]$

Annealed: $\frac{\phi_B(y)}{\rho} = -\left(e^B-1\right)\int_y^\infty dx\ \frac{1}{2}\,\mathrm{erfc}(x) - \int_{-y}^\infty dx\ \left(e^{-B}-1\right)\frac{1}{2}\,\mathrm{erfc}(x)$

In both cases, B is determined from $\frac{d\phi_B(y)}{dB} = 0$.



At large y: $\phi(y) \simeq \frac{\rho}{12}|y|^3$ (Quenched) and $\phi(y) \simeq \rho|y|$ (Annealed).

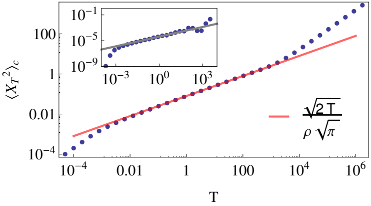
Annealed vs Quenched

- Quenched

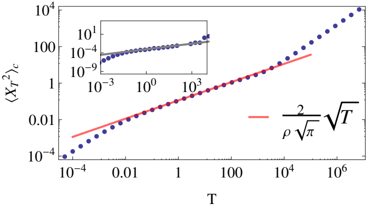
$$\langle Y_T^2 \rangle_c = \frac{\sqrt{2}}{\rho\sqrt{\pi}} \sqrt{T} \qquad \langle Y_T^4 \rangle_c = \frac{-0.04219}{\rho^3} \sqrt{T}$$

- Annealed

$$\langle Y_T^2 \rangle_c = \sqrt{2} \left[\frac{\sqrt{2}}{\rho\sqrt{\pi}} \right] \sqrt{T} \qquad \langle Y_T^4 \rangle_c = \frac{0.92495}{\rho^3} \sqrt{T}$$



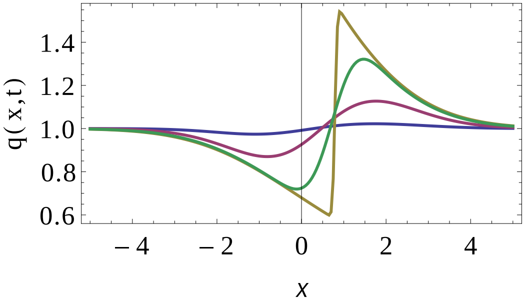
Quenched



Annealed

Shape of the optimal profiles

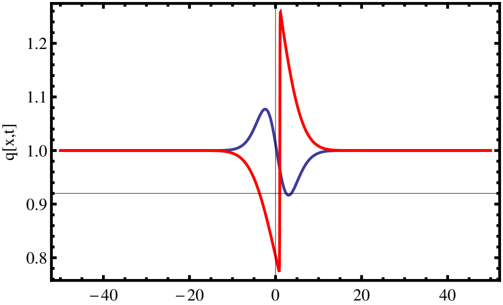
MFT provides you with the statistical properties but also with an understanding of the dynamical process leading to a given atypical fluctuation.



Quenched case

Shape of the optimal profiles

MFT provides you with the statistical properties but also with an understanding of the dynamical process leading to a given atypical fluctuation.



Annealed case

Two-time correlations

Quenched

$$\langle Y(t_1)Y(t_2)\rangle = \frac{\sigma(\rho)}{\rho^2\sqrt{\pi}D(\rho)}\ \frac{1}{2}\left[\sqrt{t_1+t_2}-\sqrt{|t_1-t_2|}\right].$$

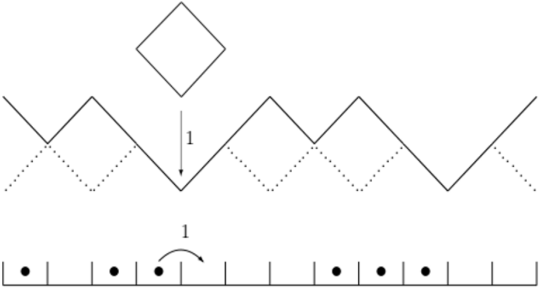
Annealed

$$\langle Y(t_1)Y(t_2)\rangle = \frac{\sigma(\rho)}{\rho^2\sqrt{\pi}D(\rho)}\ \frac{1}{2}\left[\sqrt{t_1}+\sqrt{t_2}-\sqrt{|t_1-t_2|}\right].$$

Note that the annealed result can be deduced from quenched: define $Z(t_j) = Y(t_j + T) - Y(T)$. At large T limit, $\langle Z(t_1)Z(t_2)\rangle$ yields the result for the annealed case.

Melting of an Ising Crystal

ASEP as an Interface model

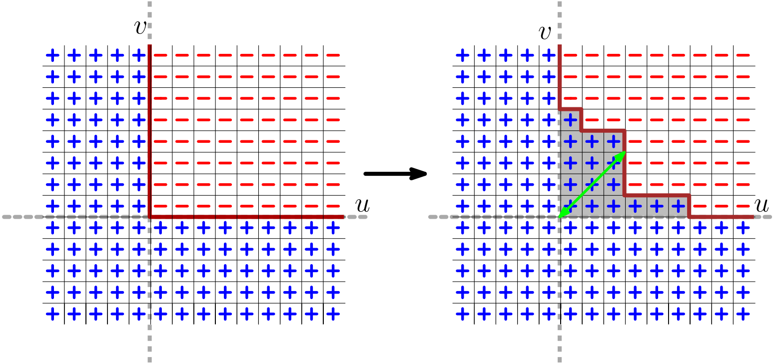


The height of an interface $h(x,t)$ satisfies the generic KPZ equation

$$\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial x^2} + \frac{\lambda}{2}\left(\frac{\partial h}{\partial x}\right)^2 + \xi(x,t)$$

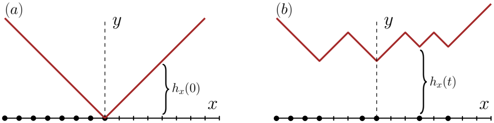
The ASEP is a discrete version of the KPZ equation in one-dimension.

Evolution of a quadrant



- Ising spin-flip dynamics at zero temperature
- Limiting shape of the interface and its fluctuations?
- Observables related to the shape:
  - Diagonal height, d_T
  - Area of the melted region, A_T.

Variational formulation



- Diagonal height: d_T = current through origin.
- Melted area: A_T = displacement of all particles.

Hydrodynamic limit:

- Fluctuating hydrodynamics

$$\partial_t \rho = \partial_x [\partial_x \rho + \sqrt{\rho(1-\rho)}\, \eta]$$

- Observables

$$d_T = \int dx\, \Theta(x)[\rho(x,T) - \rho(x,0)]$$

$$A_T = \int dx\, x\, [\rho(x,T) - \rho(x,0)]$$

Formulate as a variational problem (macroscopic fluctuation theory).

Current and Mean-Shape

The calculation of the statistics of d_T is found by using the fact that d_T is proportional to the integrated current Q_T through the origin. Then, using Derrida-Gershenfeld 2011, the cumulant generating function $\chi_T(\lambda) = \langle \exp[\lambda\, d_T] \rangle$ is found to be

$$\chi_T(\lambda) = \frac{\sqrt{T}}{\pi} \int_0^\infty d\xi \ln \left[1 + \left(e^{\lambda\sqrt{2}} - 1 \right) e^{-\xi^2} \right]$$

Besides, in the long time limit, the crystal takes a limiting average shape given by

$$\boxed{\eta = \frac{1}{\sqrt{4\pi}}\, e^{-(\xi-\eta)^2} - \frac{\xi-\eta}{\sqrt{\pi}} \int_{\xi-\eta}^{\infty} d\zeta\, e^{-\zeta^2}}$$

where $\xi = \frac{x}{\sqrt{4T}}$, and $\eta = \frac{y}{\sqrt{4T}}$.
In particular, the diagonal $x = y$ crosses the interface at $\xi = \eta = (4\pi)^{-1/2}$ and therefore $x = y = \sqrt{T/\pi}$.

Expressions of the cumulants

In the limit of a large time T, we have:

- Mean Area: $\langle A_T \rangle = T$
- Variance: $\langle A_T^2 \rangle_c = T^{3/2} \left[\frac{4}{3}\sqrt{\frac{2}{\pi}} \right]$
- Third cumulant (Skewness): $\langle A_T^3 \rangle_c = T^2 \left[\frac{6\sqrt{3}}{\pi} - 2 \right]$
- Forth cumulant (Flatness):

$$\langle A_T^4 \rangle_c = T^{5/2} \frac{32}{5\sqrt{\pi}} \left[5\sqrt{2} - 4 + \frac{3}{\pi} \left\{ 4 - 4\sqrt{2} \arccos\left(\frac{5}{3\sqrt{3}} \right) - 3\sqrt{2} \arccos\left(\frac{1}{3} \right) \right\} \right]$$

Scaling of the n-th cumulant: $\langle A_T^n \rangle_c \sim T^{(n+1)/2}$

These results have been obtained by a perturbative expansion of the MFT equations. Although the Bethe Ansatz can be applied to this system, the MFT approach seems more efficient when comparing the complexity of the calculations.

Generalizations

The aim would be to derive the full cumulant generating function of the Area. Are the MFT equations integrable?

Consider the same Ising ferromagnet with nearest-neighbor interactions, but in the presence of *a magnetic field* favoring the majority phase. The corresponding particle system is the *totally asymmetric simple exclusion process (TASEP).*

For the TASEP case, the average area is $\langle A_T \rangle = T^2/6$. It is known that the limiting shape is the parabola $\sqrt{x} + \sqrt{y} = \sqrt{T}$.

The variance scales as $T^{7/3}$. A precise calculation remains an open problem. Besides, the MFT scheme can not be applied per se to the TASEP, which is a non-diffusive system.

Conclusions

The asymmetric exclusion process is a paradigm for the behaviour of systems far from equilibrium in low dimensions. The ASEP is important for theory but also for its multiple applications. The tagged particle plays the role of a probe for the dynamics. Single-file in 1d is one of the simplest example of anomalous diffusion.

The Macroscopic Fluctuation Theory is a versatile tool to understand non-equilibrium properties of interacting particle systems. It generalizes the Onsager-Machlup theory of fluctuations close to equilibrium. In particular, it provides us with a physical picture of how a non-reversible fluctuation can be generated.

The calculation of the full statistics of a tracer in SEP is a difficult and unsolved problem.

The asymmetric case (with the anomalous scaling $t^{2/3}$ in the quenched case) is an open problem.