Coalescence of Geodesics in Last-Passage Percolation

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Last-Passage Percolation Model

Last-passage time

Let $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^2$, with $\mathbf{x} \leq \mathbf{y}$, and denote $\Gamma(\mathbf{x}, \mathbf{y})$ the set of all up-right oriented paths from $\mathbf{x}$ to $\mathbf{y}$. Consider a collection $\{W_{\mathbf{x}} \,:\, \mathbf{x} \in \mathbb{Z}^2\}$ of i.i.d. $\mathrm{Exp}(1)$ random variables and define

$$L(\mathbf{x}, \mathbf{y}) := \max_{\gamma \in \Gamma(\mathbf{x}, \mathbf{y})} \sum_{\mathbf{z} \in \gamma} W_{\mathbf{z}} \,.$$

Last-Passage Percolation Model

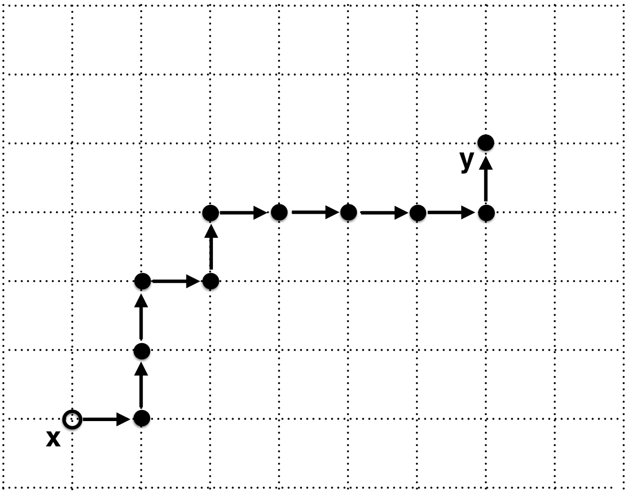

Figure: An up-right path.

Last-Passage Percolation Model

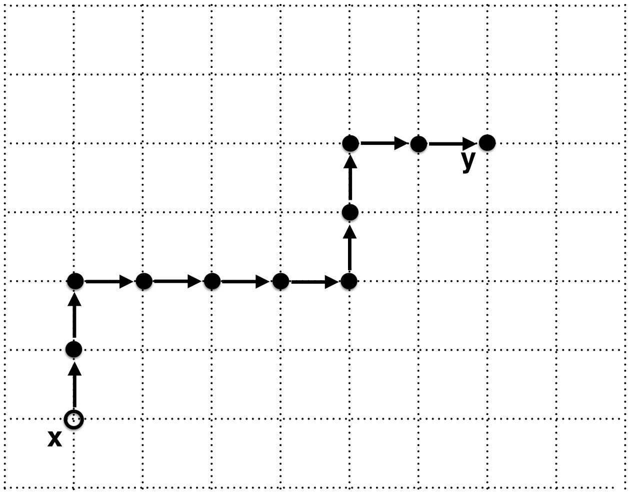

Figure: Another up-right path.

Last-Passage Percolation Model

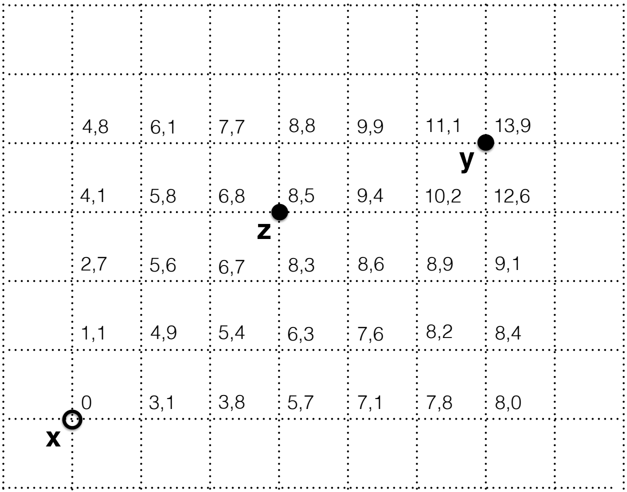

Figure: $L(\mathbf{x}, \mathbf{y}) = 13, 9$, $L(\mathbf{x}, \mathbf{z}) = 8, 5$.

Last-Passage Percolation Model

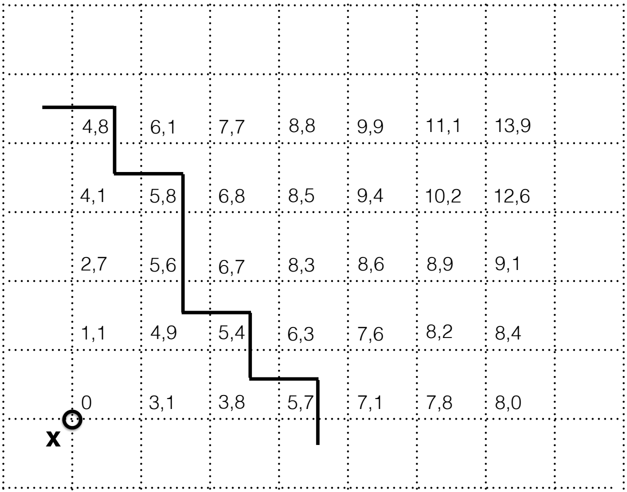

Figure: $\{y \ : \mathbf{y} \geq \mathbf{x}\,,\ L(\mathbf{x}, \mathbf{y}) \leq 6\}$.

Last-Passage Percolation Model

Geodesics

There exists a.s. a unique $\gamma(\mathbf{x},\mathbf{y}) \in \Gamma(\mathbf{x},\mathbf{y})$ such that

$$\sum_{\mathbf{z}\in\gamma(\mathbf{x},\mathbf{y})} W_{\mathbf{z}} = L(\mathbf{x},\mathbf{y})\,.$$

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Backward Algorithm

If $\gamma(\mathbf{x},\mathbf{y}) = (\mathbf{x}_1,\cdots,\mathbf{x}_n)$, with $\mathbf{x}_n = \mathbf{y}$, then

$$\mathbf{x}_{j-1} = \arg\max\{L(\mathbf{x}_j - \mathbf{e}_1, \mathbf{x}_j), L(\mathbf{x}_j - \mathbf{e}_2, \mathbf{x}_j)\}\,.$$

Finite Geodesic

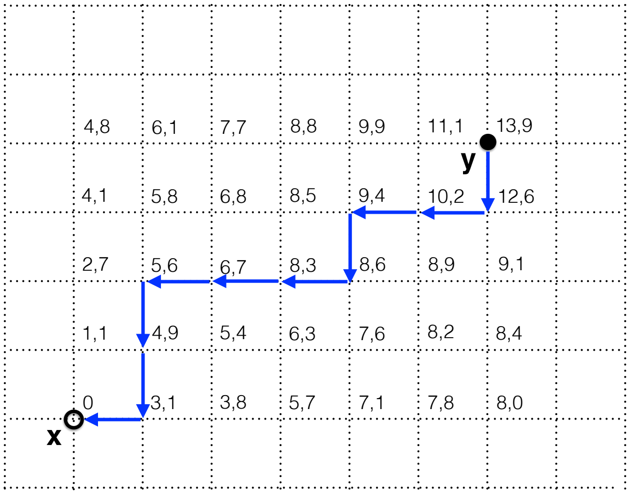

Figure: Backward Algorithm.

Existence and Coalescence of Geodesics

An up-right semi-infinite path $\gamma(\mathbf{x}_0) = (\mathbf{x}_0, \mathbf{x}_1, \cdots)$ is a geodesic

$$\text{if} \quad \gamma(\mathbf{x}_i, \mathbf{x}_j) = (\mathbf{x}_{i+1}, \cdots, \mathbf{x}_j)\,.$$

We say that it has direction $\mathbf{d} = (1, 1)$

$$\text{if} \quad \lim_{n\to\infty} \frac{\mathbf{x}_n}{|\mathbf{x}_n|} = \mathbf{d}\,.$$

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Existence and Coalescence

- a.s. $\exists^1$ semi-infinite geodesic $\gamma^{\uparrow}(\mathbf{x})$ with direction $\mathbf{d}$;
- a.s. $\exists\ \mathbf{c} \in \mathbb{Z}^2$ (random) such that

$$\gamma^{\uparrow}(\mathbf{x}) = \gamma(\mathbf{x}, \mathbf{c}) + \gamma^{\uparrow}(\mathbf{c}) \quad \text{and} \quad \gamma^{\uparrow}(\mathbf{y}) = \gamma(\mathbf{y}, \mathbf{c}) + \gamma^{\uparrow}(\mathbf{c})\,.$$

(Ferrari, P. '05, Coupier '11.)

Existence and Coalescence of Geodesics

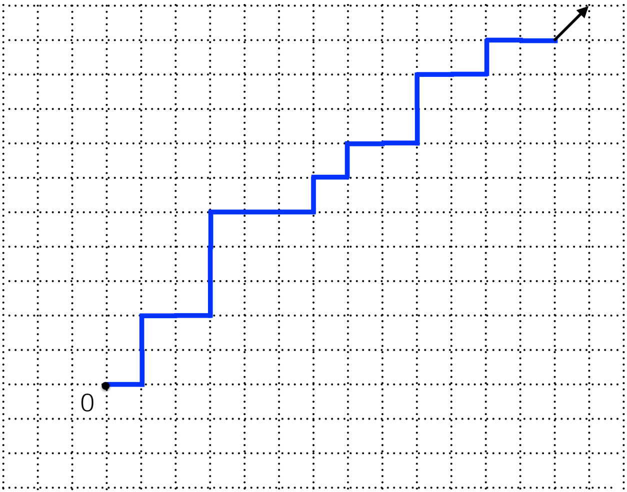

Figure: Directional Geodesic.

Existence and Coalescence of Geodesics

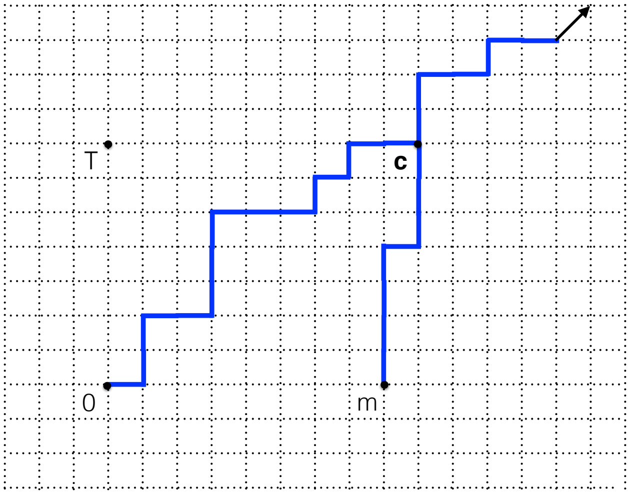

Figure: Coalescence.

Scaling Coalescence Times

Let $\mathbf{c}(\mathbf{x}, \mathbf{y})$ denote the first coalescence point (following the up-right orientation). Take $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = (m, 0)$, and denote T_m the second coordinate of $\mathbf{c}(\mathbf{x}, \mathbf{y})$.

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Questions

- Does $T_m \sim m^{\zeta}$?

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Conjectures

- $\zeta = 3/2$.

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Conjectures

- $\zeta = 3/2$.
- $\exists \ \lim_{m\to\infty} \frac{T_m}{\tau_1 m^{3/2}} \stackrel{dist.}{=} T$, for some $\tau_1 > 0$.

Scaling Coalescence Times

The Airy Process

Fluctuations of last-passage times are described by the Airy_2 process $(A(v)\,, v \in \mathbb{R})$. Denote

$$\mathbf{n} := (n, n) \text{ and } [v]_n := (2^{5/3} v n^{2/3}, 0)\,,$$

and define

$$A_n(v) := \frac{L(\mathbf{0}, \mathbf{n} + [v]_n) - \left(4n + 2^{8/3} v n^{2/3}\right)}{2^{4/3} n^{1/3}} + v^2\,.$$

Then

$$\lim_{n \to \infty} A_n(u) \stackrel{dist.}{=} A(v)$$

(Johansson '00, Corwin, Ferrari, Péché '10).

Scaling Coalescence Times

The Airy Sheet

It is conjectured that this convergence can be extend to a two dimensional setting: let

$$A_n(u,v) := \frac{L([u]_n, \mathbf{n} + [v]_n) - \left(4n + 2^{8/3}(v-u)n^{2/3}\right)}{2^{4/3}n^{1/3}} + (v-u)^2 .$$

Then (Corwin, Quastel, Remenik '15)

$$(?) \ \exists \lim_{n\to\infty} A_n(u,v) \stackrel{dist.}{=} A(u,v) ,$$

where $(A(u,v)\,,\ u,v \in \mathbb{R})$ is called the Airy Sheet.

Scaling Coalescence Times

Variational formula (partial result)

Suppose there exists a unique Airy Sheet (and is a nice sheet). Let B denote and independent standard two-sided Brownian Motion and define $U : \mathbb{R} \mapsto \mathbb{R}$ as

$$U(v) := \arg\max_{u \in \mathbb{R}} \left\{ \sqrt{2} B(u) + A(u, v) - (v - u)^2 \right\} .$$

Consider the associate counting measure $\mathcal{U}(s) := \# U((0, s])$.

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Consider the associate counting measure $\mathcal{U}(s) := \# U((0, s])$. Then (for $r > 0$)

$$\exists \lim_{m \to \infty} \mathbb{P} \left(\frac{T_m}{2^{-5/2} m^{3/2}} \leq r \right) = \mathbb{P} \left(\mathcal{U}(r^{-2/3}) = 0 \right) .$$

More on the Point Process $\mathcal{U}$

Power Law

Consider the distribution function

$$\mathbb{F}(r) := \begin{cases} 0\,, & \text{if } r \le 0\,; \\ \mathbb{P}\left(\mathcal{U}(r^{-2/3}) = 0\right) & \text{if } r > 0\,. \end{cases}$$

Then we have the power law behaviour

$$\exists \lim_{r \to \infty} r^{2/3}(1 - \mathbb{F}(r)) = \mathbb{E}\mathcal{U}(1)\,.$$

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Geodesic Forest

$\mathbb{F}$ also appears when one studies the height of a tree in the geodesic forest model.

LPP Geodesic Forest

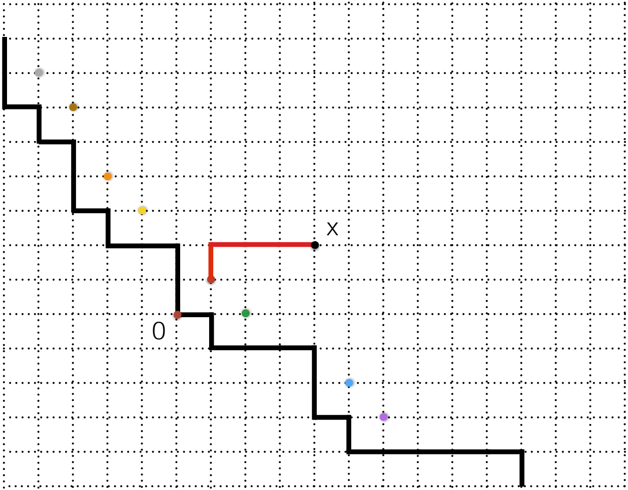
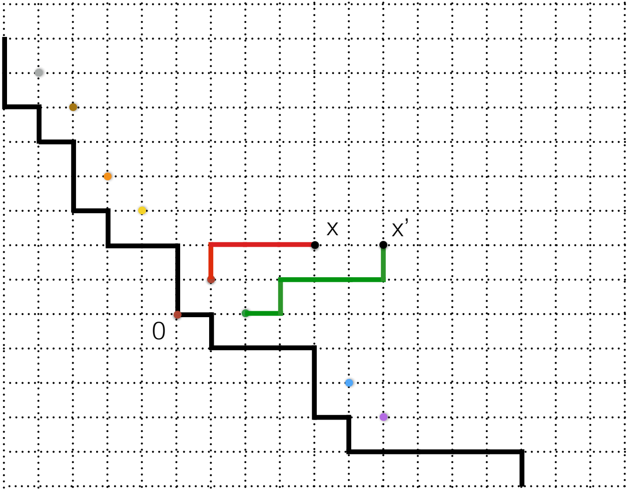


Figure: Point to substrate geodesic.

LPP Geodesic Forest



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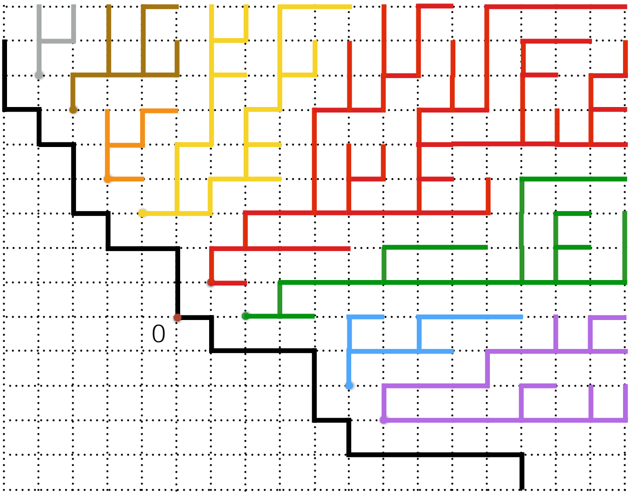

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Burges Equation with Random Forcing

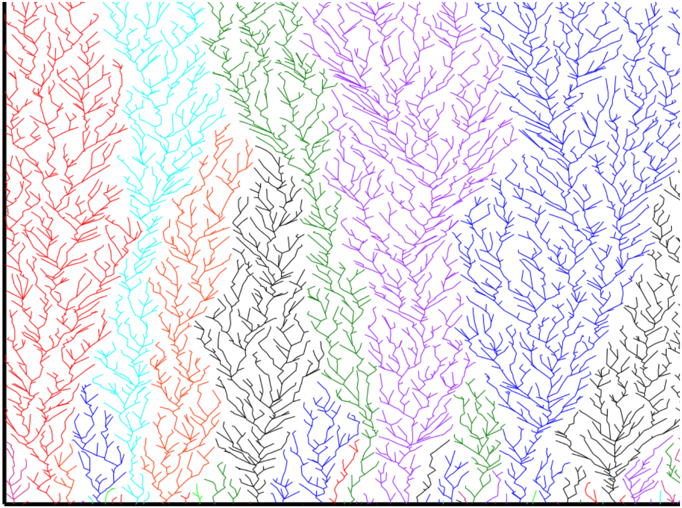

Figure: Bakhtin and Goel (authors).

Scaling Geodesic Forests

- Let h_z denote the height of the tree at z. For random walk type (no drift) of substrate, then we should have that

$$H_m := \max_{z=1,\cdots,m} h_z \,,$$

under $m^{3/2}$ rescaling, converges to $\mathbb{F}(r)$.

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- For flat type of substrate, one expects a similar result but the limit point process will be with respect to the Airy sheet minus a drifting parabola (Lopez, P. '15).

Last-Passage-Percolation with Boundary

For $n \geq 1$ and $x \in \mathbb{Z}$, let

$$L_M(x,n) := \max_{z \leq x} \{M(z) + L_z(x,n)\} \ ,$$

where $L_z(x,n) := L\left((z,1),(x,n)\right) + W_{(z,1)}$ and

$$M(z) := \begin{cases} 0\,, & \text{if } \ z = 0\,; \\ \sum_{k=1}^{z} \mathrm{Exp}_k(1/2)\,, & \text{if } \ z > 0\,; \\ -\sum_{k=z}^{-1} \mathrm{Exp}_k(1/2)\,, & \text{if } \,, z < 0\,, \end{cases}$$

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LPP-invariance

For all $n \geq 1$

$$L_M(y,n) - L_M(x,n) \stackrel{dist.}{=} M(y) - M(x)\,.$$

Last-Passage-Percolation with Boundary

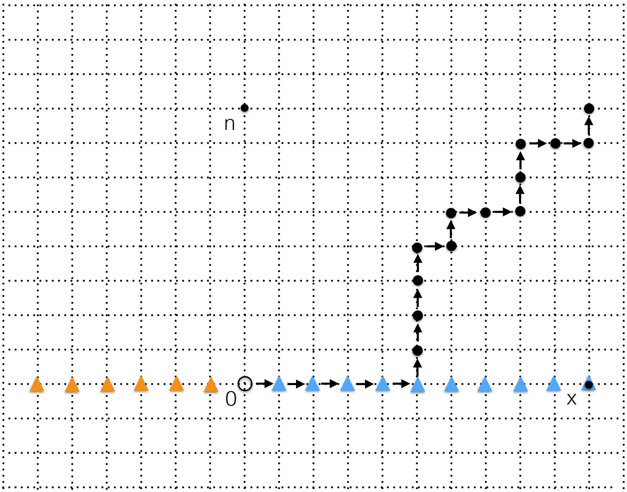

Figure: Signed $\mathrm{Exp}(1/2)$ boundary.

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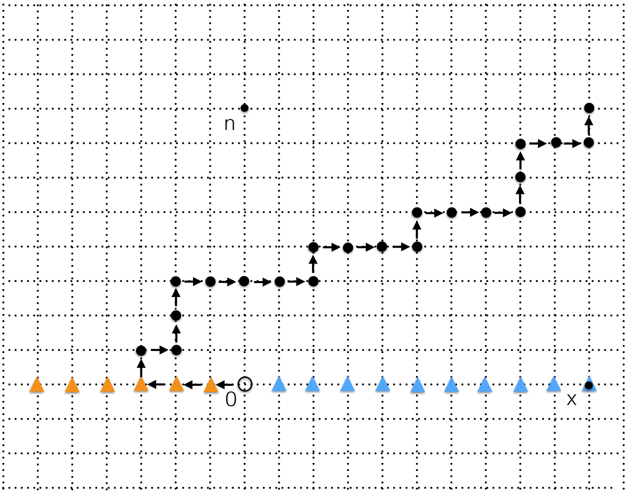

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The Exit-Point Process

For fixed $n \geq 1$ define the exit-point process $(Z_n(x)\,, x \in \mathbb{Z})$ as

$$Z_n(x) \overset{a.s.}{:=} \arg\max_{z \leq x} \{M(z) + L_z(x,n)\}\ ,\ \text{ for } x \in \mathbb{Z}$$

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Let

$$\zeta_n(z) := \mathbb{1}\{z = Z_n(x) \text{ for some } x \in \mathbb{Z}\}\ ,$$

and define the counting measure

$$\mathcal{Z}_n(m) := \sum_{z \in (0,m]} \zeta_n(z)\,.$$

The Exit-Point Process

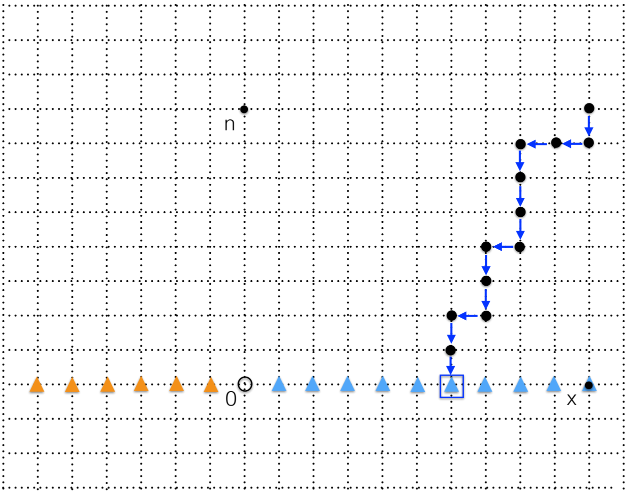

Figure: Backward algorithm for exit points.

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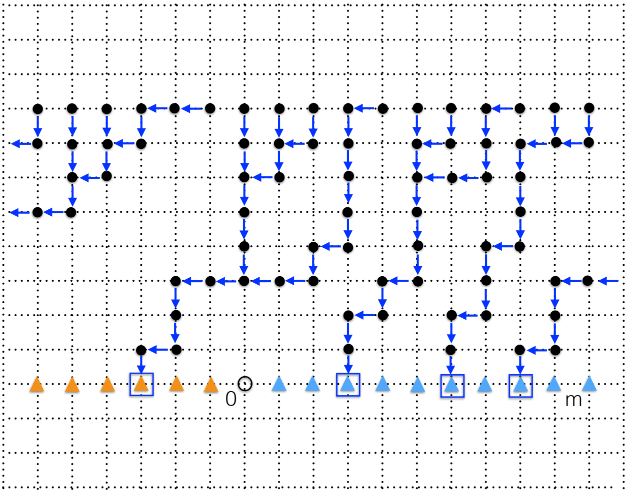

Figure: $\mathcal{Z}_n(m) = 3$

Duality: Coalescence Times and Exit-Points

Theorem

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- Busemann field and LPP-Reversibility;

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Proof

- Busemann field and LPP-Reversibility;
- Self-duality of the geodesic tree;
- Exit points are crossing points of directional geodesics.

Proof of Duality

Define the directional geodesic trees

$$\mathcal{L}^{\uparrow} := \left\{ \gamma^{\uparrow}(\mathbf{x}) \; : \; \mathbf{x} \in \mathbb{Z}^2 \right\} \text{ and } \mathcal{L}^{\downarrow} := \left\{ \gamma^{\downarrow}(\mathbf{x}) \; : \; \mathbf{x} \in \mathbb{Z}^2 \right\}$$

($\gamma^{\downarrow}(\mathbf{x})$ folows direction $-\mathbf{d}$), and let $\mathcal{L}^{\downarrow *}$ denote the dual of $\mathcal{L}^{\downarrow}$.

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$$\mathcal{L}^{\downarrow *} \stackrel{dist.}{:=} \mathcal{L}^{\uparrow} .$$

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$$\mathcal{L}^{\downarrow *} \overset{dist.}{:=} \mathcal{L}^{\uparrow} .$$

Consider the Busemann field

$$B^{\downarrow} := \left\{ B^{\downarrow}(\mathbf{x}) \,:\, \mathbf{x} \in \mathbb{Z}^2 \right\} , \quad \text{where} \quad B^{\downarrow}(\mathbf{x}) := L(\mathbf{c}, \mathbf{x}) - L(\mathbf{c}, \mathbf{0}) .$$

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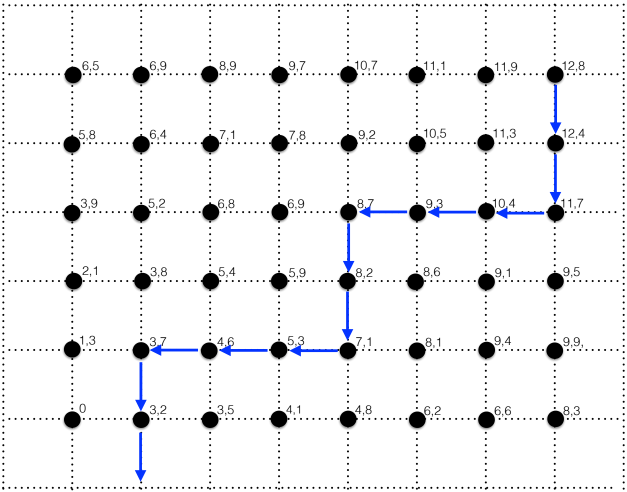

Figure: Busemann Field and Directional Geodesics $(-\mathbf{d})$.

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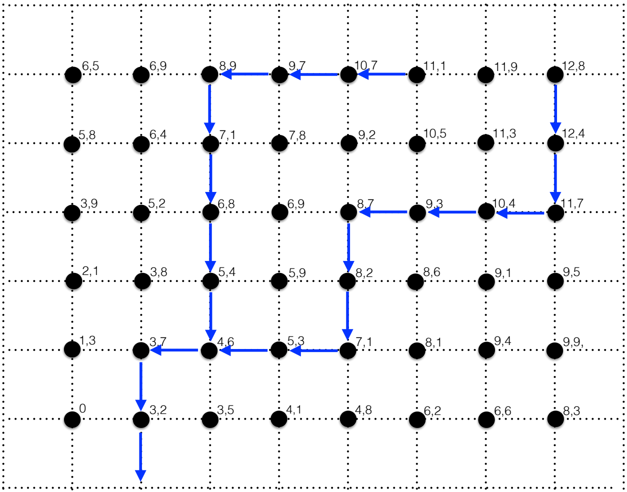

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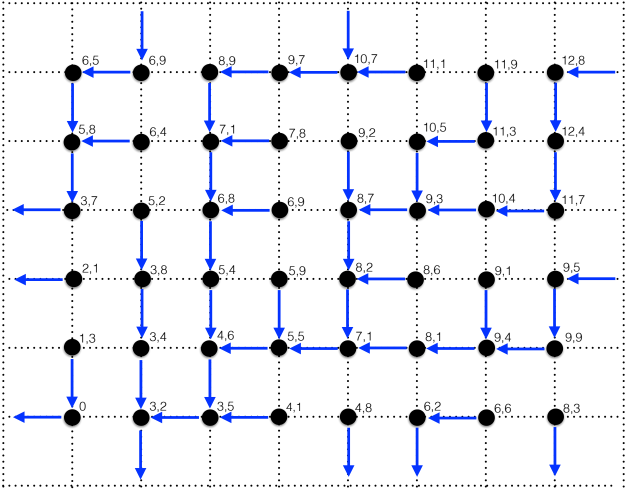

Figure: Directional Geodesic Tree $\mathcal{L}^{\downarrow}$.

Proof of Duality

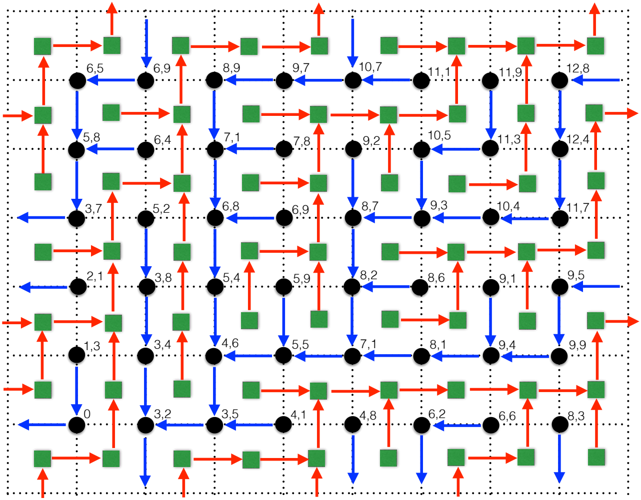

Figure: $\mathcal{L}^{\downarrow}$ (blue) and $\mathcal{L}^{\downarrow *}$ (red).

Proof of Duality

Backward algorithm Ψ

For $\gamma^{\downarrow}(\mathbf{x}) = (\mathbf{x}_n)_{n\geq 0}$ then $\mathbf{x}_0 = \mathbf{x}$ and

$$\mathbf{x}_{n+1} = \arg\max\left\{B^{\downarrow}(\mathbf{x}_n - \mathbf{e}_1)\,,\; B^{\downarrow}(\mathbf{x}_n - \mathbf{e}_2)\right\}\,,$$

and so $\mathcal{L}^{\downarrow}$ can be seen as the set composed of down-left oriented edges $(\mathbf{x}, \mathbf{e}_{\mathbf{x}})$ such that $\mathbf{x} \in \mathbb{Z}^2$ and

$$\mathbf{e}_{\mathbf{x}} = \begin{cases} \mathbf{x} - \mathbf{e}_1 & \text{if } B^{\downarrow}(\mathbf{x} - \mathbf{e}_1) > B^{\downarrow}(\mathbf{x} - \mathbf{e}_2)\,, \\ \mathbf{x} - \mathbf{e}_2 & \text{if } B^{\downarrow}(\mathbf{x} - \mathbf{e}_2) > B^{\downarrow}(\mathbf{x} - \mathbf{e}_1)\,. \end{cases}$$

Thus

$$\mathcal{L}^{\downarrow} = \Psi(B^{\downarrow})\,.$$

Proof of Duality

$\mathcal{L}^{\downarrow *}$ can be seen as the set composed of up-right oriented edges $(\mathbf{x}^*, \mathbf{e}_{\mathbf{x}^*})$ such that

$$\mathbf{e}_{\mathbf{x}^*} = \begin{cases} \mathbf{x}^* + \mathbf{e}_1 & \text{if } \mathbf{e}_{\mathbf{x}+\mathbf{d}} = (\mathbf{x}+\mathbf{d}) - \mathbf{e}_1 \,, \\ \mathbf{x}^* + \mathbf{e}_2 & \text{if } \mathbf{e}_{\mathbf{x}+\mathbf{d}} = (\mathbf{x}+\mathbf{d}) - \mathbf{e}_2 \,. \end{cases}$$

It can be rewritten as:

$$\mathbf{e}_{\mathbf{x}^*} = \begin{cases} \mathbf{x}^* + \mathbf{e}_1 & \text{if } B^{\downarrow *}(\mathbf{x}^* + \mathbf{e}_1) < B^{\downarrow *}(\mathbf{x}^* + \mathbf{e}_2) \,, \\ \mathbf{x}^* + \mathbf{e}_2 & \text{if } B^{\downarrow *}(\mathbf{x}^* + \mathbf{e}_2) < B^{\downarrow *}(\mathbf{x}^* + \mathbf{e}_1) \,, \end{cases}$$

where $B^{\downarrow *}(\mathbf{x}^*) := B^{\downarrow}(\mathbf{x})$.

Proof of Duality

Let $\phi : \mathbf{x} \in \mathbb{Z}^2 \mapsto \phi(\mathbf{x}) := (-\mathbf{x})^* \in \mathbb{Z}^{2*}$ and set

$$\tilde{B}(\mathbf{x}) := -B^{\downarrow *}(\phi(\mathbf{x}))\,.$$

Thus $\phi^{-1}(\mathcal{L}^{\downarrow *})$ can be represented as the set composed of down-left oriented edges $(\mathbf{x}, \mathbf{e}_{\mathbf{x}})$ such that

$$\mathbf{e}_{\mathbf{x}} = \begin{cases} \mathbf{x} - \mathbf{e}_1 & \text{if } \tilde{B}(\mathbf{x} - \mathbf{e}_1) > \tilde{B}(\mathbf{x} - \mathbf{e}_2)\,, \\ \mathbf{x} - \mathbf{e}_2 & \text{if } \tilde{B}(\mathbf{x} - \mathbf{e}_2) > \tilde{B}(\mathbf{x} - \mathbf{e}_1)\,. \end{cases}$$

Or, equivalently,

$$\phi^{-1}(\mathcal{L}^{\downarrow *}) = \Psi(\tilde{B})\,.$$

Proof of Duality

- The Busemann field is the stationary TASEP ($p = 1/2$) conditioned to have a particle jumping from site 0 to site 1 at time zero (Cator, P. '12). $B(i,j)$ is the time the ith hole and the jth particle interchange positions;

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- Hence $\phi^{-1}(\mathcal{L}^{\downarrow *}) = \Psi(\tilde{B}) \stackrel{dist.}{=} \Psi(B) = \mathcal{L}^{\downarrow}$;
- In particular, $\mathcal{L}^{\downarrow *} \stackrel{dist.}{=} \mathcal{L}^{\uparrow} \;\Rightarrow\; T_m(\mathcal{L}^{\uparrow}) \stackrel{dist.}{=} T_m(\mathcal{L}^{\downarrow *})$.

Proof of Duality

Fix $n \geq 0$ and for $x \in \mathbb{Z}$ denote $Z_n^{\downarrow}(x)$ the first point in $\gamma^{\downarrow}((x, n))$, following the down-left orientation, that intersects transversally the horizontal axis $\mathbb{Z} \times \{0\}$ (crossing point).

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$$\mathcal{Z}_n^{\downarrow}(m) := \sum_{z \in (0,m]} \zeta_n^{\downarrow}(z)\,, \ \text{ for } m \geq 0\,,$$

where

$$\zeta_n^{\downarrow}(z) = \begin{cases} 1 & \text{if } z = Z_n^{\downarrow}(x) \text{ for some } x \in \mathbb{Z}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

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$\mathcal{L}^{\downarrow}$ and its dual $\mathcal{L}^{\downarrow *}$ satisfy

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Therefore

$$\begin{aligned}
\mathbb{P}\,(T_m < n) &= \mathbb{P}\left(T_m(\mathcal{L}^{\downarrow *}) < n\right) \\
&= \mathbb{P}\left(\mathcal{Z}_n^{\downarrow}(m) = 0\right) \\
&= \mathbb{P}\,(\mathcal{Z}_n(m) = 0)\;.
\end{aligned}$$

Scaling the Exit-Point Process

Theorem

$\mathcal{Z}_n$ is translation invariant and ergodic. Furthermore, there exists $\epsilon_0 > 0$ such that

$$\liminf_{n\to\infty} n^{2/3} p_n > \epsilon_0 \,,$$

where $p_n := \mathbb{P}(\zeta_n(0) = 1)$.

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- (Balázs, Cator, Seppälaïnen '06)

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Power Law Behaviour

Corollary

There exists $\epsilon_0 > 0$ *such that*

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Proof

$$n^{2/3}\mathbb{P}\left(T_m \geq n\right) = n^{2/3}\mathbb{P}\left(\mathcal{Z}_n(m) \geq 1\right) \geq n^{2/3}p_n\,.$$

Scaling the Exit Point

Let

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Theorem

$$\lim_{n\to\infty} \frac{Z_n(n)}{2^{5/3} n^{2/3}} \stackrel{dist.}{=} U.$$

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Denote

$$\mathbb{G}(r) := \liminf_{m\to\infty} \mathbb{P}\left(\frac{T_m}{2^{-5/2}m^{3/2}} > r\right) \text{ and } \mathbb{F}(s) := \mathbb{P}\left(U \le s\right) .$$

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Conjectural Picture

Let

$$U_n(v) := \frac{Z_n(n + 2^{5/3} n^{2/3} v)}{2^{5/3} n^{2/3}} .$$

We expect that

$$\exists \lim_{n \to \infty} U_n(v) \stackrel{dist.}{=} U(v) ,$$

and hence

$$\exists \lim_{r \to \infty} r^{2/3} \mathbb{G}(r) = \lim_{\delta \to 0^+} \delta^{-1} \mathbb{P} \left(\mathcal{U}(\delta) \geq 1 \right) = \mathbb{E} \mathcal{U}(1) ,$$

and

$$\exists \lim_{n \to \infty} \mathbb{E} \mathcal{Z}_n \left(\lfloor 2^{5/3} n^{2/3} \rfloor \right) = 2^{5/3} \lim_{n \to \infty} n^{2/3} p_n = \mathbb{E} \mathcal{U}(1) .$$

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Thank you for your attention.