On some integrable models of interacting particles.

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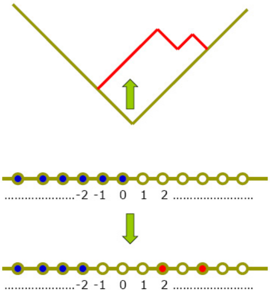
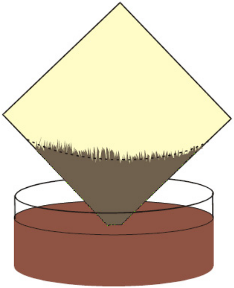
Outline

1. Introduction
2. Interacting particle models and integrability
   - Zero-range chipping models with factorized steady state
   - Integrability
3. TASEP with generalized update
   - Stationary state
   - Fluctuations of particle current

Interacting particle models





Interacting particle models gives a tool to study the universal scaling behaviour of a wide variety of large stochastic systems:

- Traffic flows
- Interface growth
- Polymers in random media
- Crystal shape

Integrability

Integrability is a special structure of the matrix of transition probabilities, which makes its complete diagonalization a solvable problem.

— The choice of dynamical rules is very restrictive.

+ The full exact analytic solution is possible.

Examples:

- SSEP $\longrightarrow$ EW
- ASEP $\longrightarrow$ KPZ

Can we extend the range of integrable models by including new interactions to see how the KPZ universality breaks down?

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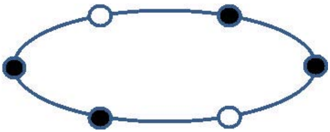
State space

M particles on the lattice



$\mathbf{n} = (0,1,1,0,1,1)$
$\mathbf{x} = (1,2,4,5)$,

- Particle configurations
    - Occupation numbers: $\mathbf{n} = \{n_i\}_{i\in\mathscr{L}}, \sum_{i\in\mathscr{L}} n_i = M$, $n_i \in \{0,1\}$-ASEP like, $n_i \in \mathbb{Z}_{\geq 0}$-ZRP like
    - Particle coordinates: $\mathbf{x} = (x_1, < \ldots, < x_M) \subset \mathscr{L}$-ASEP-like or $\mathbf{x} = (x_1, \leq \ldots, \leq x_M)$-ZRP like
- $\mathscr{L} = \mathbb{Z}$ - infinite lattice or $\mathscr{L} = \mathbb{Z}/L\mathbb{Z}$ - periodic lattice with L sites

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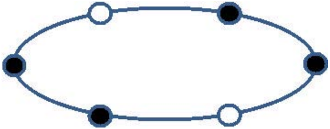

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Markov chain.

Chapman-Kolmogorov equation:

$$P_{t+1}(\mathbf{n}) = \sum_{\{\mathbf{n}'\}} M_{\mathbf{n},\mathbf{n}'} P_t(\mathbf{n}')$$

Stationary state

$$P_{st}(\mathbf{n}) = \sum_{\{\mathbf{n}'\}} M_{\mathbf{n},\mathbf{n}'} P_{st}(\mathbf{n}')$$

Factorized stationary measure:

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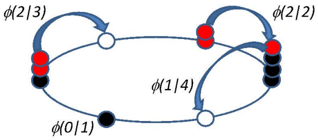

Dynamical rules:

- one-sided nearest neighbor hopping
- on-site (zero range) interaction
- $\varphi(m|n)$– probability for m particles to jump from a site with $n \geq m$ particles

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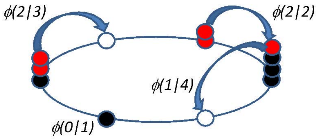

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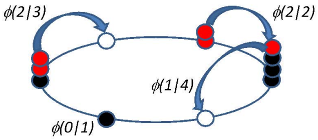

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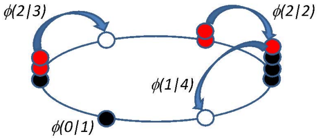

Markov matrix:

- $\mathbf{M}_{\mathbf{n},\mathbf{n}'} = \sum_{\{m_k \in \mathbb{Z}_{\geq 0}\}_{k \in \mathscr{L}}} \prod_{i \in \mathscr{L}} T^{m_{i-1}, m_i}_{n_i, n'_i}$
- $T^{m_{i-1}, m_i}_{n_i, n'_i} = \delta_{(n_i - n'_i),(m_{i-1} - m_i)} \varphi(m_i | n'_i)$

Factorization of stationary measure

Theorem (Evans, Majumdar, Zia 2004)

The stationary measure of zero-range chipping models on a ring is the product measure iff the chipping probability is of the form

$$\varphi(m|n) = \frac{v(m)w(n-m)}{\sum_{i=0}^{n} v(i)w(n-i)},$$

where $w(k), v(m) \geq 0$*, in which case*

$$P_{st}(\mathbf{n}) = \frac{1}{Z(M,N)} \prod_{i=1}^{N} f(n_i) \quad \textit{with} \quad f(n) = \sum_{i=0}^{n} v(i)w(n-i)$$

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Diagonalize that

Eigenvalue problem

$$\mathbf{M}\Psi = \Lambda\Psi, \quad \overline{\Psi}\mathbf{M} = \Lambda\overline{\Psi}$$

Forward-backward symmetry:

$$\Pi\mathbf{M}^T\Pi = \mathbf{D}^{-1}\mathbf{M}\mathbf{D}$$

where $\mathbf{D}_{n,m} = P_{st}(n)$ and $\Pi(x_1,\dots,x_N) = (-x_N,\dots,-x_1)$.

Look for the eigenvector in the form:

$$\Psi_{\mathbf{n}} = \Psi^0_{\mathbf{n}} P_{st}(\mathbf{n}), \quad \overline{\Psi}_{\mathbf{n}} = \Pi\Psi^0_{\mathbf{n}}$$

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One and two particle problems

One particle problem

$$\Lambda\Psi^0(x) = p\Psi^0(x-1) + (1-p)\,\Psi^0(x);\ (p := \varphi(1|1))$$

Two particle problem (Free, $x_1 < x_2$)

$$\Lambda_2\Psi^0(x_1,x_2) = (1-p)[p\Psi^0(x_1-1,x_2) + (1-p)\Psi^0(x_1,x_2)] + p[p\Psi^0(x_1-1,x_2-1) + (1-p)\Psi^0(x_1,x_2-1)]$$

Two particle problem (Interacting, $x_1 = x_2 := x$)

$$\Lambda_2\Psi^0(x,x) = f(2)^{-1}[w(2)\Psi^0(x,x) + v(1)w(1)\Psi^0(x-1,x) + v(2)\Psi^0(x-1,x-1)]$$

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Boundary conditions

$$\Psi^0(x,x-1) = \alpha\Psi^0(x-1,x-1) + \beta\Psi^0(x-1,x) + \gamma\Psi^0(x,x)$$

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Two-particle reducibility

Many particle problem (Free)

$$\sum_{k_1=0}^{1}\cdots\sum_{k_n=0}^{1} p^{k_1+\cdots+k_n}(1-p)^{n-(k_1+\cdots+k_n)}\Psi^0(\dots,x-k_1,\dots,x-k_n,\dots)$$

Many particle problem (Interacting)

$$\sum_{k=0}^{n}\varphi(k|n)\Psi^0(\dots,(x-1)^k,x^{n-k},\dots)$$

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Generalized quantum binomial

Problem reformulation

Consider an associative algebra with two generators A, B satisfying general homogeneous quadratic relation

$$BA = \alpha AA + \beta AB + \gamma BB,$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\alpha + \beta + \gamma = 1$. Find the coefficients for the generalized quantum binomial

$$(pA + (1-p)B)^n = \sum_{m=0}^{n} \varphi(m|n) A^m B^{n-m}.$$

Generalized quantum binomial

Theorem (Rosengren 2000, P. 2013)

$$\varphi(m|n) = \mu^m \frac{(\nu/\mu;q)_m(\mu;q)_{n-m}}{(\nu;q)_n} \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}},$$

where $\alpha = \frac{\nu(1-q)}{1-q\nu}, \beta = \frac{q-\nu}{1-q\nu}, \gamma = \frac{1-q}{1-q\nu}, \mu = p+\nu(1-p)$ *and* $\nu \neq q^{-k}$, *for* $k \in \mathbb{N}$. *(For* $\nu = q^{-k}$ *see Corwin, Petrov, 2015)*

In particular the functions $v(k), w(k)$ and $f(k)$ are

$$v(k) = \mu^k \frac{(\nu/\mu;q)_k}{(q;q)_k}, \;\; w(k) = \frac{(\mu;q)_k}{(q;q)_k}, \;\; f(n) = \frac{(\nu;q)_n}{(q,q)_n}$$

Bethe ansatz

Eigenvector:

$$\Psi^0(\mathbf{x}|\mathbf{u}) = \sum_{\sigma\in S_N} \mathrm{sgn}(\sigma) \prod_{i=1}^{M} \prod_{j>i} \frac{u_{\sigma_i} - q u_{\sigma_j}}{u_i - q u_j} \left(\frac{1-\nu u_{\sigma_i}}{1-u_{\sigma_i}} \right)^{x_i}$$

Eigenvalue:

$$\Lambda_N = \prod_{i=1}^{N} \left(\frac{1-\mu u_i}{1-\nu u_i} \right)$$

Periodic boundary conditions:

$$\Psi(x_1,\dots,x_N|\mathbf{u}) = \Psi(x_2,\dots,x_N,x_1+L|\mathbf{u})$$

Bethe equations:

$$\left(\frac{1-\nu u_i}{1-u_i} \right)^L = (-1)^{N-1} \prod_{j=1}^{N} \frac{u_i - q u_j}{u_j - q u_i}, \; i = 1,\dots,N.$$

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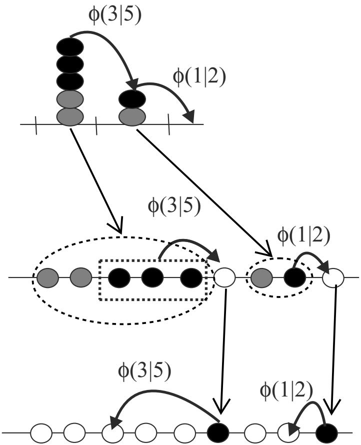
ZRP-ASEP mapping and particle hole transformation

ZRP-ASEP:

Particle-hole:



q-Hahn process. Particular cases.

- $q=1;\varphi(m|n)=p^m(1-p)^{n-m}C_n^m$: Independent particles
- $\mu=q\nu;\varphi(m|n)=p[n]_q$ q-boson (Sasamoto,Wadati 98) and q-TASEP (Borodin, Corwin, 2011)
- $\nu\to\mu=q,\ \varphi(m|n)\simeq dt/[n]_{1/q}$ MADM and and long range hopping models, (Sasamoto Wadati, 1998; Alimohammadi, Karimipour, Khorrami, 1998)
- $\nu=0$, Geometric q-TASEP (Borodin, Corwin, 2013)
- $q\to1,\mu=q^\alpha,\nu=q^{\alpha+\beta},\ \varphi(m|n)$ — Beta-Binomial distribution (Barraquand, Corwin, 2015)
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$$q=0,\varphi(m|n)=\begin{cases}(1-p), & m=0;\\ p\mu^{m-1}(1-\mu), & 0<m<n;\\ p\mu^{n-1} & m=n,\end{cases}$$

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- $q = 1; \varphi(m|n) = p^m(1-p)^{n-m}C_n^m$: Independent particles
- $\mu = q\nu; \varphi(m|n) = p[n]_q$ q-boson (Sasamoto,Wadati 98) and q-TASEP (Borodin, Corwin, 2011)
- $\nu \to \mu = q$, $\varphi(m|n) \simeq dt/[n]_{1/q}$ MADM and and long range hopping models, (Sasamoto Wadati, 1998; Alimohammadi, Karimipour, Khorrami, 1998)
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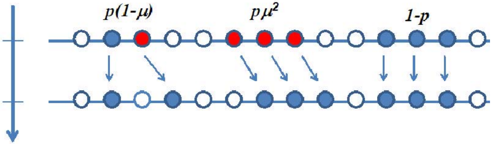
Discrete time dynamics



- Clusterwise update: *At every time step each cluster is updated independently.*
- First particle of a cluster jumps forward with probability p or stays with probability $(1-p)$.
- If the first particle decided to jump, the next particle follows it with probability μ and so do the second, third, e.t.c.
- Exclusion interaction (jumps to occupied sites are forbidden).

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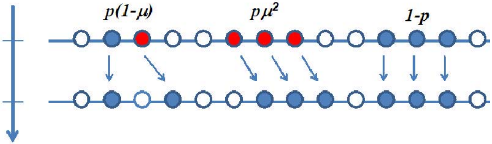

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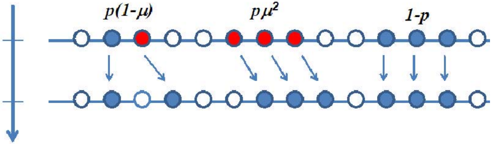

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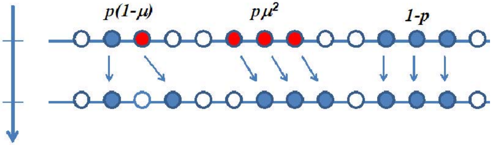

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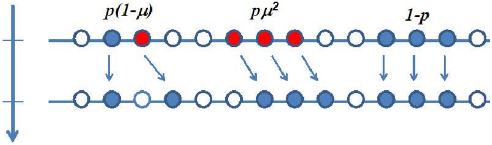

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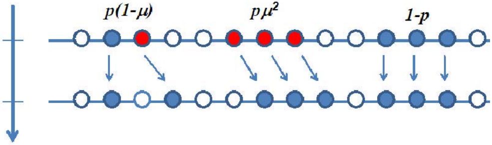

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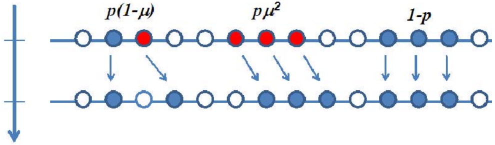

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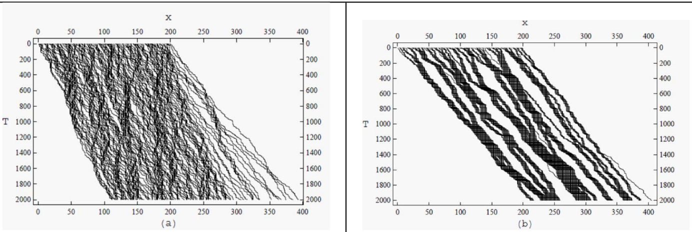
Two regimes



$p = 0.1, \mu = 0$ | $p = 0.1, \mu = 0,995$

- We expect change of behaviour the limit $\mu \to 1$:
  - $1 - \mu > 0$ - KPZ-like behaviour, $\Delta \sim L^{-1/2}$
  - $\mu \to 1$ - DA limit (all particles stick together into a single cluster, which moves diffusively)$\Delta = const$

What is in between?

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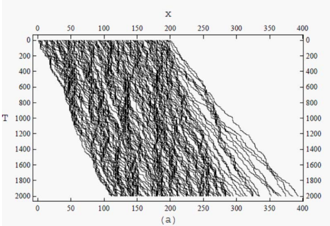

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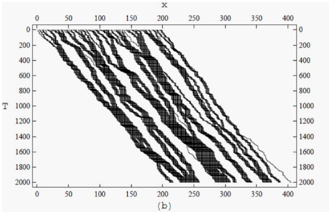

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What is in between?

Stationary state

Consider a limit $L \to \infty, M \to \infty, M/L = c$

Questions to answer

- Cluster distribution.
- Particle current.
- Correlation length.

When $\mu = 1$, there is a single cluster moving diffusively with the velocity p. How this regime is approached?

Partition function for ZRP-like model

- Partition function (M paricles, $N = L - M$ sites):

$$Z(M,N) = \sum_{n_1,\dots,n_N \geq 0} \delta_{\|n\|,M} \prod_{i=1}^{N} f(n_i) = \oint_{\Gamma_0} \frac{[F(z)]^N}{z^{M+1}},$$

- Occupation number distribution:

$$P(n) = f(n) \frac{Z(M-n, N-1)}{Z(M,N)}.$$

- Mean number of particles jumping per time step

$$J = \frac{N}{Z(M,N)} \oint_{\Gamma_0} \frac{[F(z)]^N}{z^M} \frac{V'(z)}{V(z)} \frac{dz}{2\pi i},$$

where $V(z) = \sum_{k \geq 0} v(k) z^k$ and $F(z) = \sum_{k \geq 0} v(k) z^k$

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Exact formuals

$$Z(M,N) = \binom{L-1}{M} {}_2F_1\left(-M,-N;1-L;\nu\right),$$

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Gauss hypergeometri function ${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n$;
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Saddle point approximation

- The local stationary state observable are reduced to evaluation of integrals of the form

$$\mathscr{I}_N(h(z),g(z)) = \oint_{\Gamma_0} e^{Nh(z)} g(z) \frac{dz}{2\pi i z},$$

where $h(z) = \ln(1-\nu z) - \ln(1-z) - \rho \ln z$.

- Critical point, $h'(z_c) = 0$:

$$z_c = 1 + \frac{(1-\nu)}{2c\nu}\left(1 - \sqrt{1 + \frac{4(1-c)c\nu}{1-\nu}}\right)$$

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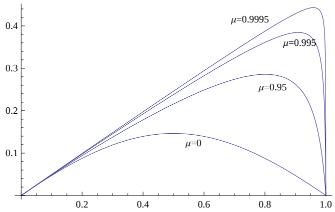
Particle current

$$j^{gTASEP} = \lim_{L\to\infty} J/L$$

$$= \frac{cp(1+(1-2c)\mu)}{2\mu+2c(p(1-\mu)-\mu)} - \frac{cp\sqrt{(1-\mu)(1-4(1-c)c(p-\mu)-\mu)}}{2\mu+2c(p(1-\mu)-\mu)}$$



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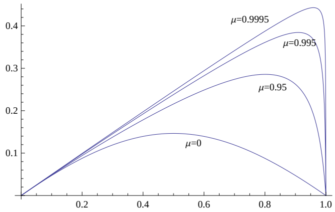

Cluster size distribution in gTASEP:

$$P(n) = z_c^n (1 - z_c)^{-1}$$

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Validity range of saddle point approximation

Consider a limit

$$\mu \to 1, \nu \to 1, p = \frac{\mu - \nu}{1 - \nu} = const.$$

Let

$$\lambda := (1 - \nu)^{-1} \to \infty.$$

How large can it be for the saddle point analysis to be valid, given $h_k \sim \lambda^{\frac{k-1}{2}}$:

$$\lim_{N \to \infty} \left| \frac{N^{1-k/2} h_k}{h_2^{k/2}} \right| = 0 \Rightarrow \lambda / N^2 \to \infty.$$

Transition regime, $\lambda N^{-2} = \mathrm{const}$

- Deform contour $Z(M,N) = -\oint_{\Gamma_1} e^{Nh(z)} \frac{dz}{2\pi i z}$.
- Choose the right integration scale $z = 1 + \frac{e^{i\varphi}}{\sqrt{\rho\lambda}}$ to get.

$$h(z) = -2\sqrt{\frac{\rho}{\lambda}}\cos\varphi + O(1/\lambda)$$

- Then we obtain

$$Z(M,N) = \frac{-1}{\sqrt{\rho\lambda}}\int_0^{2\pi} e^{-2N\sqrt{\rho/\lambda}\cos\varphi + i\varphi}\frac{d\varphi}{2\pi} \simeq \frac{\theta}{2M} I_1(\theta),$$

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Transitional distribution over the macroscopic scale

Cluster fraction distribution ($\chi = n/M,\ M \to \infty$):

$$\begin{aligned} \mathrm{Prob}(\chi = 1) &= \frac{1}{I_0(\theta)}, \\ \mathrm{Prob}(\chi < x) &= \frac{\theta}{2I_0(\theta)} \int_0^x \frac{I_1\left(\theta\sqrt{1-y}\right)}{\sqrt{1-y}} dy. \end{aligned}$$

Correlation function

$$C(k) \equiv \langle \tau_1 \tau_{1+k} \rangle - \langle \tau_1 \rangle \langle \tau_{1+k} \rangle \simeq c(1-c)e^{-k/\xi}, \;\; \mu < 1$$

$$\xi \simeq \sqrt{\lambda c(1-c)}, \;\; \lambda \to \infty.$$

$$C(Lr) = \frac{(1-2c)e^{-1/\widetilde{\xi}} + c(1-c)(e^{-r/\widetilde{\xi}} + e^{-(1-r)/\widetilde{\xi}})}{1+e^{-1/\widetilde{\xi}}},$$

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Large deviation function

- Introduce deformed Markov matrix

$$\mathbf{M}^{\gamma}_{\mathbf{n},\mathbf{n}'} = \mathbf{M}_{\mathbf{n},\mathbf{n}'} \exp\left(\gamma \mathscr{N}(\mathbf{n},\mathbf{n}')\right),$$

where $\mathscr{N}(\mathbf{n},\mathbf{n}')$ is the number of particle jumps in the one-step transition from $\mathbf{n}'$ to $\mathbf{n}$.

- The log of its largest eigenvalue $\Lambda_0(\gamma)$ is the rescaled cumulant generating function of total number of particle jumps

$$\ln \Lambda_0(\gamma) = \lim_{t\to\infty} \frac{\ln\left\langle e^{\gamma Y_t}\right\rangle}{t}.$$

- Its Legendre transform is the large deviation function:

$$\lim_{t\to\infty} t^{-1} \ln \mathbb{P}(Y_t/t > y) = \sup_{\gamma}(y\gamma - \ln \Lambda_0(\gamma))$$

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What would we expect in scaling limit?

- Large deviation hypothesis:

$$\mathbb{P}(Y_t/t > y) \simeq \exp\left(\frac{t}{aL^z}\hat{G}\left(\frac{y-\overline{y}}{ab}\right)\right)$$

$$\lim_{t\to\infty} t^{-1}\ln\left\langle e^{\gamma Y_t}\right\rangle = \gamma\overline{y} + aL^{-z}G(\gamma bL^z), \quad (L\to\infty, \gamma bL^z = const)$$

- KPZ, $z = 3/2$, (Derrida-Lebowitz, 1998):

$$G_{DL}(\gamma) = -Li_{5/2}(B)$$
$$\gamma = -Li_{3/2}(B)$$

- DA limit, $z = 2$, CLT for random walk of particle of mass M,:

Exact results.

- Integral expressions

$$\ln \Lambda_0(\gamma) = (\mu - \nu) \oint_{\Gamma_0} \frac{\ln\left[1 - \frac{B(1-\nu u)^N}{(1-u)^N u^M}\right]}{(1-\mu u)(1-\nu u)} \frac{du}{2\pi i}.$$

$$\gamma = \frac{1-\nu}{M} \oint_{\Gamma_0} \frac{\ln\left[1 - \frac{B(1-\nu u)^N}{(1-u)^N u^M}\right]}{(1-u)(1-\nu u)} \frac{du}{2\pi i}.$$

- Series representations (term by term integration)

$$\ln \Lambda_0(\gamma) = -(\mu - \nu) \sum_{n=1}^{\infty} \frac{B^n}{n} \binom{Ln-2}{Mn-1} F_1\left(1-nM; 1-nN, 1; 2-nL; \nu, \mu\right),$$

$$\gamma = -\frac{1-\nu}{M} \sum_{n=1}^{\infty} \frac{B^n}{n} \binom{Ln-1}{Mn-1} {}_2F_1\left(1-Mn, 1-Nn; 1-nL; \nu\right).$$

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Asymptotic forms

- KPZ regime, $\lambda/N^2 \to 0, \lambda^{1/4} N^{3/2}\gamma = const$

$$\ln \Lambda(\gamma) = \gamma J_\infty + aL^{-z} G_{DL}(\gamma b L^z),$$

$a \sim \lambda^{1/4}$ and $b \sim \lambda^{-1/4}$as $\lambda \to \infty$.

- Transition regime

$$\ln \Lambda(\gamma) = \gamma p M + N^{-2} p(1-p) \mathcal{G}_\theta(N^2 \rho \gamma),$$

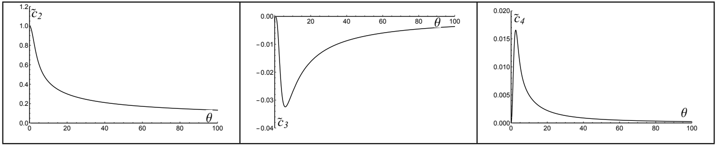
$$\mathcal{G}_\theta(t) = \frac{\theta^2}{4} \sum_{k=1}^{\infty} I_2(k\theta) \frac{B^k}{k}, \; t = -\frac{\theta}{2} \sum_{k=1}^{\infty} I_1(k\theta) \frac{B^k}{k}$$

Cumulants

- Mean current: $J \simeq Mp - p(1-p)\rho \frac{\theta}{2} \frac{I_2(\theta)}{I_1(\theta)}$,
- Diffusion coefficient: $\Delta = p(1-p)\left[\frac{I_1(2\theta)}{I_1^2(\theta)}\left(\frac{I_2(2\theta)}{I_1(2\theta)} - \frac{I_2(\theta)}{I_1(\theta)}\right)\right]$

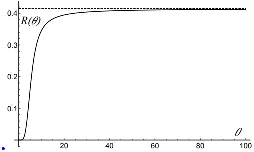
Cumulants

- Cumulants in the transition regime scale as $c_n \sim N^{2(n-1)}$ unlike $c_n \sim N^{3/2(n-1)}$ in the KPZ regime and $c_n \sim N^n$.



- Universal cumulant ratio:

$$R(\theta) = \frac{c_3^2}{c_2 c_4} = \frac{\left(\mathscr{G}_\theta^{(3)}(0)\right)^2}{\mathscr{G}_\theta''(0)\mathscr{G}_\theta^{(4)}(0)} \to \frac{2\left(3/2-8/3^{3/2}\right)^2}{15/2-24/\sqrt{3}+9/\sqrt{2}} \simeq 0.41517$$



Limiting forms of transitional LDF

$$\mathcal{G}_\theta(t) \simeq -\frac{\theta t}{2} + \frac{3}{8}\sqrt{\frac{\theta}{2\pi}} G_{DL}\left(t\sqrt{\frac{8\pi}{\theta}}\right), \ \theta \to \infty$$

$$\mathcal{G}_\theta(t) \simeq \frac{t^2}{2} - \frac{\theta^2 t}{8}, \ \theta \to 0$$

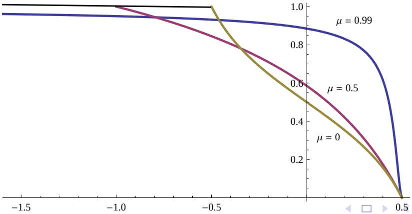
Nonstationary case

Hydrodynamics: $\partial_t \rho + \partial_x j(\rho) = 0$

Particle number vs coordinate (step initial conditions, $\chi = x/t, \theta = n/t$):

$$\chi = -\frac{(\mu - \nu)(-1 + u(2 - u\mu + u(-1 + (2 + (-2 + u)u)\mu)\nu))}{(-1 + u\mu)^2(-1 + \nu)(-1 + u^2\nu)}$$

$$\theta = \frac{u^2(-1 + \mu)(\mu - \nu)}{(-1 + u\mu)^2(-1 + u^2\nu)}; \quad u \in [0,1]$$



Nonstationary case

Exact distribution:

$$P(x_{n_1} > a_1, \ldots, x_{n_k} > a_k) = \det(\mathbb{1} - \chi_a K \chi_a), \quad \chi_a = \prod_i \mathbb{1}(x_i < a_i)$$

$$K(n_1, x_1; n_2, x_2) = (1-\nu)\left[\oint_{\Gamma_1} \frac{dv}{2\pi i} \frac{(1-v)^{n_2+x_2-n_1-x_1-1}}{v^{n_2-n_1}(1-\nu v)^{n_2+x_2-n_1-x_1+1}} \right.$$
$$+ \oint_{\Gamma_1} \frac{du}{2\pi i} \oint_{\Gamma_0} \frac{dv}{2\pi i} \frac{u^{n_1}(1-\mu u)^t (1-\nu u)^{n_1+x_1-t-1}}{(1-u)^{x_1+n_1+1}}$$
$$\left. \times \frac{(1-v)^{x_2+n_2}}{v^{n_2}(1-\mu v)^t (1-\nu v)^{n_2+x_2-t}} \frac{1}{(v-u)} \right]$$

KPZ regime

Exact distribution:

$$\chi_i = \chi_0 + s_i t^{-1/3} \kappa_f^{-1}, \; \theta = \theta_i + u_i t^{-2/3} \kappa_c^{-1}$$

$$\kappa_f^{-1} t^{1/3} K(n_1, x_1; n_2, x_2) \sim K_{Airy_2}(u_1, s_1, u_2, s_2)$$

Transition regime

$$\lambda \to \infty, t \to \infty, t/\lambda^{2/3} = \sigma$$

Scaling: $n_i/t = \theta_i, x_i/t = p - \theta - s_i/\lambda^{1/3}$

$$\lambda^{1/3}\sigma^{-1}K(n_1,x_1;n_2,x_2) \to K_{tr}(\theta_1,s_1;\theta_2,s_2) :=$$
$$\mathbb{1}(s_1 > s_2)\int_{\Gamma_0}\frac{du}{2\pi i}\exp\left(\sigma\left((s_1-s_2)u + \frac{\theta_1-\theta_2}{u}\right)\right) +$$
$$\int_{\Gamma_-}\frac{du}{2\pi i}\int_{\Gamma_0}\frac{dv}{2\pi i}\frac{u}{v}\frac{\exp\left(\sigma\left(\frac{p(1-p)}{2}(u^2-v^2)+s_1u+\frac{\theta_1}{u}-s_2v-\frac{\theta_2}{v}\right)\right)}{u-v}$$

Limiting behaviour:

$K_{tr} \to K_{Airy_2}, \sigma \to \infty,\ s_i \sim \sigma^{-2/3}$

$K_{tr}(\theta,s_1,\theta,s_2) \to \frac{e^{-y_1^2/2p(1-p)}}{\sqrt{2\pi p(1-p))}}$, as $\sigma \to 0$ and $y_1 = s_1\sqrt{\sigma} \sim x_1/\sqrt{t}$

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Summary

- The KPZ universality breaks down when the saddle point method fails.
- The KPZ scaling function keeps its form up to the diffusive scale, all the change being in model dependent constants.
- We obtained the LDF and the kernel interpolating between Gaussian and KPZ regimes

- Outlook
  - Search for new integrable particle models.
  - Combinatorial structure of gTASEP. (Should the RSK be modified?)