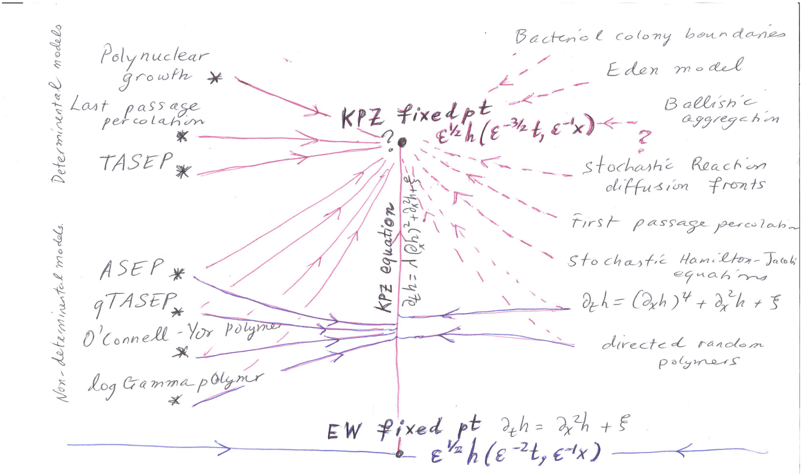
KPZ 1:2:3 scaling $h_\epsilon(t,x) = \epsilon^{1/2} h(\epsilon^{-3/2} t, \epsilon^{-1} x)$

$$\partial_t h_\epsilon = (\partial_x h_\epsilon)^2 + \epsilon^{1/2} \partial_x^2 h_\epsilon + \epsilon^{1/4} \xi$$

All KPZ class models have analogue of h and 1:2:3 scaling
Goal to understand the limit $\mathfrak{h}$ of h_ϵ as $\epsilon \to 0$ KPZ fixed point
Or $u = \partial_x h$, $\mathfrak{u} = \partial_x \mathfrak{h}$ stochastic Burgers
Philosophy: Understand KPZ universality by showing it is stable fixed point of 1:2:3 scaling.
But we don't even know what it is! What do we know?

- Depends on only one free parameter ($= 1$ by rescaling time)
- Depends on the initial data: Markov process
- Local dynamics
- KPZ 1:2:3 scaling invariance
- u weak solution of Burgers equation $\partial_t u = \partial_x u^2$. *Not* unique. *Dissipative* limit $\partial_t u = \partial_x u^2 + \epsilon \partial_x^2 u \quad \neq \quad$ *dispersive* limit $\partial_t u = \partial_x u^2 + \epsilon \partial_x^3 u$ (Lax, Levermore, Venakidis, Deift, Zhou, ...)

Conjectural KPZ Universality Class



Markov processes $\rightarrow$ fluctuations depend on initial data

Special initial data determined by $\epsilon^{1/2} h_0(\epsilon^{-1} x)$ invariance

Fractional Burgers equation

Goncalves-Jara conjecture the KPZ fixed point

$$\partial_t \mathfrak{u} = \partial_x(\mathfrak{u}^2) - (-\Delta)^{3/4}\mathfrak{u}dt + \sqrt{(-\Delta)^{3/4}}\xi$$

status of the equation is that Gubinelli-Jara can prove "energy solutions" exist, preserve white noise, but not that they are unique

- ✓ Invariant under 1:2:3 scaling
- X Nonlocal Ornstein-Uhlenbeck $-(-\Delta)^{3/4}\mathfrak{u}dt + \sqrt{(-\Delta)^{3/4}}\xi$ looks rather unnatural
- X more than one free parameter

$$\partial_t u = \partial_x(u^2) - \delta(-\Delta)^{3/4}udt + \sqrt{\delta(-\Delta)^{3/4}}\xi$$

Conclusion: one parameter family of non-local perturbations off KPZ fixed point

Conjecture: converges to KPZ fixed point as $\delta \searrow 0$

What we do know are Airy process limits from special scale invariant initial data

Point-to-point directed random polymers, random growth on curved substrate, stochastic heat eq (KPZ) starting from delta function $Z(0,x) = \delta_0(x)$, etc

$$h(t,x) \sim c_1 t + c_2 \frac{x^2}{t} - c_3 t^{1/3} \mathcal{A}(c_4 t^{-2/3} x)$$

Airy process
stationary, marginals $\mathrm{F}_{\mathrm{GUE}}$

$t \to \infty$ c_i non-universal, in some cases computable

Point-to-line directed random polymers, random growth on flat substrate, stochastic heat eq (KPZ) starting from a constant $Z(0,x) \equiv 1$, etc

$$h(t,x) \sim c_5 t - c_6 t^{1/3} \mathcal{A}_1(c_7 t^{-2/3} x)$$

Airy_1 process
stationary, marginals $\mathrm{F}_{\mathrm{GOE}}$

"Stationary" case e.g. stochastic Burgers equation with invariant white noise, KPZ with Brownian motion $\mathcal{A}_{\mathrm{stat}}(x)$

Mixed cases start with one of the above to the right of the origin, and another to the left. $\mathcal{A}_{1\to 2}(x)$, $\mathcal{A}_{1\to\mathrm{stat}}(x)$, $\mathcal{A}_{2\to\mathrm{stat}}(x)$

6(+) universality subclasses $\Longleftrightarrow$ 6(+) Airy processes, determinental f.d.d.'s

Prahofer, Spohn, Sasamoto, Borodin, Baik, Johansson, . . . 00's

Airy process

$\mathcal{A}(\cdot)$ is the most basic. Here is a new formula for it (Q-Remenik, in prep) $g(t) \geq c - \kappa t^2$, $\kappa < 3/4$, $\int_{-L}^{L} |g'|^2 < \infty$ for all L

$$\mathbb{P}\left(\mathcal{A}(t) \leq g(t) + t^2,\ t \in \mathbb{R}\right) = \det\Big(I - K_{\mathrm{Ai}} + B_0 P_0 \mathcal{S} P_0 B_0\Big)$$

$$\mathcal{S}(x_1, x_2) = \int_{-\infty}^{g(0)} dy G_g(x_1, y, g) G_{g(-t)}(y, x_2)$$

$$G_g(x, y) = B_0(x_1, y) - \int_0^{\infty} \mathbb{P}_y(\tau_g \in dt)(e^{-t\Delta} B_0)(x, g(t))$$

$B_0(x, y) = \mathrm{Ai}(x + y)$, $\tau_g = \inf\{t \geq 0 \colon \sqrt{2} B_1(t) \geq g(t)\}$

Our old formula with Ivan was only for $\mathbb{P}\left(\mathcal{A}(t) \leq g(t) + t^2,\ t \in [-L, L]\right)$ and then one had to take a (highly singular) limit as $L \nearrow \infty$

KPZ with more general initial data $h_0(x)$

E.g. Initial data h_0 for KPZ By linearity of stochastic heat eqn

$$h(t,x) \sim \log \int e^{-\frac{(x-y)^2}{2t} + t^{1/3}\mathcal{A}(t^{-2/3}(x-y)) - h^0(y)} dy$$

$$h_\epsilon(t,x) \sim \epsilon^{1/2} \log \int e^{\epsilon^{-1/2}\{-\frac{(x-y)^2}{2t} + t^{1/3}\mathcal{A}(t^{-2/3}(x-y)) - \epsilon^{1/2} h^0(\epsilon^{-1} y)\}} dy$$

so if $\epsilon^{-1/2} h_0(\epsilon^{-1} y) \to \mathfrak{h}_0(y)$

$$\mathfrak{h}(t,x) = \sup_y \left\{ -\frac{(x-y)^2}{2t} + t^{1/3}\mathcal{A}(t^{-2/3}(x-y)) - \mathfrak{h}^0(y) \right\}$$

- only in sense of $1-d$ distributions
- limit class only depends on $\mathfrak{h}_0(x) = \lim_{\epsilon\to 0} \epsilon^{-1/2} h_0(\epsilon^{-1} x)$ (can prove this using Corwin-Hammond),
- eg. $h_0(x) = -|x|^{1/2+\delta}$ gives F_{GUE}, whereas $h_0(x) = c|x|^{1/2-\delta}$ gives flat. Boundary between curved and flat is $h_0(x) = c|x|^{1/2}$, and it has an exact formula.

Exact formula for the square root class [Q-Remenik, in prep]

$$\mathbb{P}\Big(\sup_{x\in\mathbb{R}}\Big\{\mathcal{A}(x)-x^2-\alpha_1|x-a|^{1/2}\mathbf{1}_{x<a}-\alpha_2|x-a|^{1/2}\mathbf{1}_{x\geq a})\Big\}\leq r\Big)$$
$$=\det\Big(I-K_{\mathrm{Ai}}+B_0P_0\mathcal{S}_\infty^{\mathrm{sq},a,\alpha_1,\alpha_2}P_0B_0\Big)$$

$$\mathcal{S}_\infty^{\mathrm{sq},a,\alpha_1,\alpha_2}(x_1,x_2)$$
$$=\int_{-\infty}^{r}dy\,\Big[(e^{a\Delta}B_0)(x_1,y)-\int_0^\infty dt_1\,\rho_{r-y,\alpha_1}(t_1)(e^{(a-t_1)\Delta}B_0)(x_1,\alpha_1\sqrt{t_1}$$
$$\times\Big[(e^{-a\Delta}B_0)(y,x_2)-\int_0^\infty dt_2\,\rho_{r-y,\alpha_2}(t_2)(e^{-(a+t_2)\Delta}B_0)(\alpha_2\sqrt{t_2}+r,x_2)\Big].$$

$$\rho_{r-y,\alpha}(t)=\sum_{n=1}^\infty\frac{\big(\frac{1}{2}(y-r)\big)^{2\nu_n(\alpha)}\,t^{\nu_n(\alpha)-1}}{\partial_\nu\Upsilon(\nu,\frac{\alpha^2}{4})\big|_{\nu=\nu_n(\alpha)}},$$

where $\big(\nu_n(\alpha)\big)_{n\geq1}$ are the negative zeros of the function $\Upsilon(\cdot,\frac{\alpha^2}{4})$.

$$\Upsilon(\nu,\tfrac{z^2}{2})=2^\nu e^{z^2/4}D_{-2\nu}(z)$$

$D_n(z)$ is the parabolic cylinder function, $B_0(x,y)=\mathrm{Ai}(x+y)$.

Airy sheet

For fixed point at time t as a process in x need extra parameter

$$Z(0,y,t,x) \sim c_t e^{-\frac{(x-y)^2}{2t}+t^{1/3}\mathcal{A}(t^{-2/3}x,t^{-2/3}y)}$$

$$\mathfrak{h}(t,x) = \sup_y \left\{-\frac{(x-y)^2}{2t} + t^{1/3}\mathcal{A}(t^{-2/3}x, t^{-2/3}y) - \mathfrak{h}^0(y)\right\}$$

Lax-Oleinik formula for Burgers eq driven by Airy noise

Space-time Airy sheet

$$\mathcal{A}(s,y;t,x) = \lim_{\epsilon\to 0} \epsilon^{1/2} \log Z(\epsilon^{-3/2}s, \epsilon^{-1}y, \epsilon^{-3/2}t, \epsilon^{-1}x)$$

1. *Independent increments.* $\mathcal{A}(u,y;t,x)$ is independent of $\mathcal{A}(u',y;t',x)$ if $(u,t)\cap(u',t')=\emptyset$;
2. *Space and time stationarity.* $\mathcal{A}(u,y;t,x) \overset{\text{dist}}{=} \mathcal{A}(u+h,y;t+h,x) \overset{\text{dist}}{=} \mathcal{A}(u,y+z;t,x+z)$;
3. *Scaling.* $\mathcal{A}(0,y;t,x) \overset{\text{dist}}{=} t^{1/3}\mathcal{A}(0,t^{-2/3}y;1,t^{-2/3}x)$;
4. *Semi-group property.* $u<s<t$, $\hat{\mathcal{A}}(u,y;t,x) := \mathcal{A}(u,y;t,x) - \frac{(x-y)^2}{2(t-u)}$,

$$\hat{\mathcal{A}}(u,y;t,x) = \sup_{z\in\mathbb{R}} \left\{\hat{\mathcal{A}}(u,y;s,z) + \hat{\mathcal{A}}(s,z;t,x)\right\}$$

Prelimiting Airy sheets are tight Corwin-Pimentel-Q, in prep in

1. Hammersley LPP;
2. Exponential and geometric (lattice) LPP;
3. Brownian semi-discrete LPP;
4. Log-gamma polymer;
5. Brownian semi-discrete polymer;
6. Continuum polymer (Hopf-Cole solutions of the KPZ).

Unfortunately, we have no uniqueness, so cannot even prove, eg. scaling property of the limit

Polymer fixed point

From x at time s to y at time t travels through $z_1, \ldots, z_n$ at times $s < s_1 < \cdots < s_n < t$ which optimize

$$\hat{\mathcal{A}}(s, x; s_1, z_1) + \hat{\mathcal{A}}(s_1, z_1; s_2, z_2) + \cdots + \hat{\mathcal{A}}(s_n, z_n; t, y)$$

Conjectured universal limit of directed polymers in $1 + 1$ dimensions

KPZ fixed pt via replicas+factorization ansatz (Corwin-Q-Remenik)

Enough to study $t \to \infty$ and then fine mesh limit $\mu, \nu \to \infty$ of the generating function

$$G(s, x; c, y) = \left\langle \exp\left(- e^{\frac{t}{24}} \left[\sum_{k,l=1}^{\mu,\nu} e^{-s_k - c_l} Z(0, y_l, t, x_k) \right] \right) \right\rangle.$$

Expand exp, write it using replicas as

$$\sum_{N=0}^{\infty} \frac{(-1)^N e^{tN/24}}{N!} \sum_r e^{-tE_r} |\psi_r(\mathbf{0})|^2 \Phi_r(\mathbf{x}, \mathbf{s}) \Phi_r^*(\mathbf{y}, \mathbf{c})$$

$$\Phi_r(\mathbf{x}, \mathbf{s}) := \sum_{k_1, \ldots, k_N = 1}^{\mu} e^{-(s_{k_1} + \cdots + s_{k_N})} \frac{\psi_r(x_{k_1}, \ldots, x_{k_N})}{\psi_r(\mathbf{0})} \qquad (1)$$

E_r, ψ_r are the eigenvalues, eigenfunctions of the attractive (symmetric) δ-Bose gas $-\frac{1}{2}\sum_{i=1}^{N} \partial_{x_i}^2 - \frac{1}{2}\sum_{i \neq j = 1}^{N} \delta(x_i - x_j)$

Problem: Summation (1) only explicit in very special cases.

Eigenfunctions

$$\psi_{\boldsymbol{q},\boldsymbol{n}}(x_1, ..., x_N) =$$
$$C \sum_{p\in\mathcal{P}} \mathrm{sgn}(p) \prod_{1\leq a<b\leq N} \big(\xi_{p(a)} - \xi_{p(b)} + i\,\mathrm{sgn}(x_a - x_b)\big)\, e^{i\sum_{c=1}^{N} \xi_{p(c)} x_c}$$

$$\xi_a = q_\alpha - \frac{i}{2}(n_\alpha + 1 - 2r_\alpha) \quad \text{for} \quad a = \sum_{\beta=1}^{\alpha-1} n_\beta + r_\alpha\ , \quad \alpha = 1, \ldots, M$$

n_α =number of particles in the α-th cluster
$N = \sum_{\alpha=1}^{M} n_\alpha$ =total number of particles

Prolhac-Spohn factorization assumption

$$\Phi_r(\mathbf{x}, \mathbf{s}) \approx \prod_{\alpha=1}^{M} e^{-\frac{1}{4}\sum_{a,b=1}^{\mu} |x_a - x_b| \partial_{s_a} \partial_{s_b}} \left(\sum_{k=1}^{\mu} e^{-s_k + i q_\alpha x_k}\right)^{n_\alpha}$$

Factorization assumption

- Allows one to put generating function in form so that one can take $t \to \infty$
- For narrow wedge initial data get full Airy process (Prolhac-Spohn)
- For Brownian initial data get correct two point function of $\mathcal{A}_{\mathrm{stat}}$ (Imamura-Sasamoto)
- For narrow wedge two points Dotsenko's summation formula,

$$\frac{\langle \overbrace{x_1,\ldots,x_1}^{L} \overbrace{x_2,\ldots,x_2}^{R} |\Psi_z\rangle}{\langle 0|\Psi_z\rangle} \delta_{\sum_{\alpha=1}^{M} n_\alpha, L+R} =$$

$$\binom{L+R}{L}^{-1} \sum_{m_\alpha+l_\alpha>0} \prod_{\alpha=1}^{M} \binom{n_\alpha}{m_\alpha} \mathrm{e}^{\mathrm{i}x_1 m_\alpha q_\alpha + \mathrm{i}x_2 l_\alpha q_\alpha - m_\alpha l_\alpha x/2} \delta_{m_\alpha+l_\alpha, n_\alpha} \delta_{\sum_{\alpha=1}^{M} m_\alpha, L} \delta_{\sum_{\alpha=1}^{M} l_\alpha, R}$$

$$\times \prod_{\alpha\neq\beta} \frac{\Gamma[1+\frac{1}{2}(m_\alpha+m_\beta+l_\alpha+l_\beta)+\mathrm{i}(q_\alpha-q_\beta)]\Gamma[1+\frac{1}{2}(-m_\alpha+m_\beta+l_\alpha-l_\beta)+\mathrm{i}(q_\alpha-q_\beta)]}{\Gamma[1+\frac{1}{2}(-m_\alpha+m_\beta+l_\alpha+l_\beta)+\mathrm{i}(q_\alpha-q_\beta)]\Gamma[1+\frac{1}{2}(m_\alpha+m_\beta+l_\alpha-l_\beta)+\mathrm{i}(q_\alpha-q_\beta)]}$$

Setting double product $= 1$ (value at steepest descent point) equivalent to factorization assumption (Imamura-Sasamoto-Spohn)

On this evidence we proceed and obtain transition probabilities.

KPZ fixed pt transition probs from replica + factorization assumption (Corwin-Q-Remenik)

$$P(\mathfrak{h}(1,x) \le g(x) \text{ on } [a,b] \mid \mathfrak{h}(0,x) = f(x) \text{ on } [c,d]) = \det(I - K + L)$$

where $L(z,z')$ is given by

$$\int\int dmdu\; e^{(d-c)H} K(\partial_m Y_{[c,d]}^{\tilde{f}(\cdot)+m})(z',u) Y_{[a,b]}^{\tilde{g}(\cdot)+u-m} e^{(b-a)H} K(u,z)$$

$Y_{[a,b]}^{f}$ is solution operator of $\left\{ \begin{array}{ll} \partial_t u = -Hu & a < t < b \\ u(t,x) = 0 & x \ge f(t) \end{array} \right\}$

$H = -\partial_x^2 + x$ is the Airy operator

Hard to verify.

Test: Flat $\mapsto$multipoint should give $\mathcal{A}_1$. One pt ok but 2 point function unclear. Passes some non-trivial tests, but even numerics is inconclusive