Birds, soap, and sandblasting: surprising connections in the theory of incompressible flocks

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Related: Order-disorder phase transition in incompressible flocks:
New J. Phys. **17,** 042002 (2015)

Includes a free short film of me
(only 20 takes for 2 minutes of footage!)

Outline

I) What's an incompressible flock? Are there any?

II) Hydrodynamic model (arbitrary d)

III) Linearized theory ⇒ Weirdness:
fluctuations in d=2 << fluc's in d=3 !

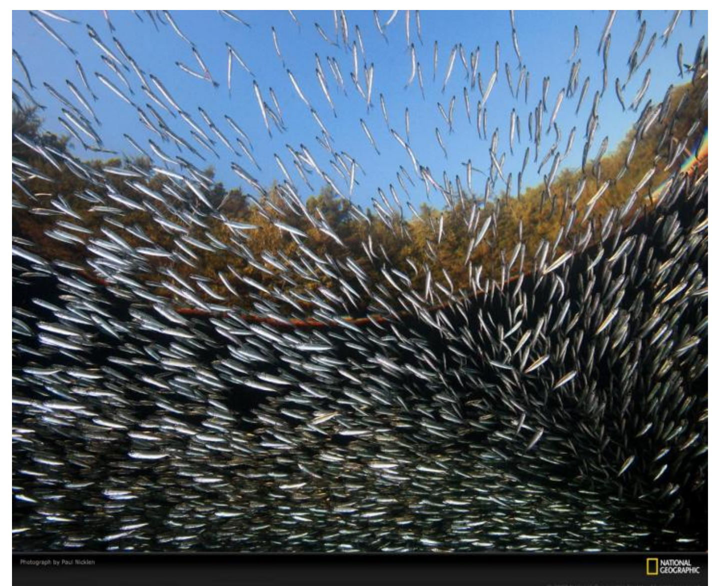
IV) Solving the non-linear theory:
Exact Mappings in d=2:

2d incompressible flock → 2d "incompressible" magnet
↓
1+1d KPZ equation ("sandblasting") ← 2d smectic ("soap")

I) What are incompressible flocks? Are there any?

First, what’s a flock?

Ordered Active fluids (aka “flocks”): large numbers of self-propelled “particles” which tend to align their velocities along the same direction



Flocks: Essential Features

- Only Local interactions: short ranged in space and time
- No external fields (no signs, no compasses):"Rotation invariance": order is spontaneous ("emergent"):
- i.e., Picked by the system, not a priori
- Ferromagnetic interactions (favor alignment)
- "Birds" keep moving $(\vec{v} \neq \vec{0})$ and making errors

I) What's an incompressible flock?

- As for fluids, "incompressible" means density can't change
- Very common in fluid mechanics to approximate fluids as incompressible
- (valid for speeds $v << c^{o}$ sound speed)

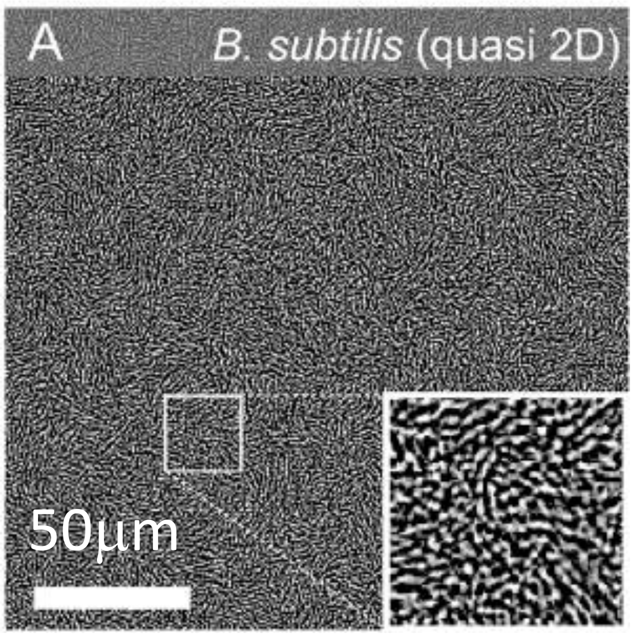
Can an incompressible flock exist in nature?

Hell, yes!

Examples of incompressible active fluids

Dense colony of *B. subtilis* bacteria

Picture from Wensink, Dunkel, Heidenreich, Drescher, Goldstein, Löwen and Yeomans (2012) PNAS



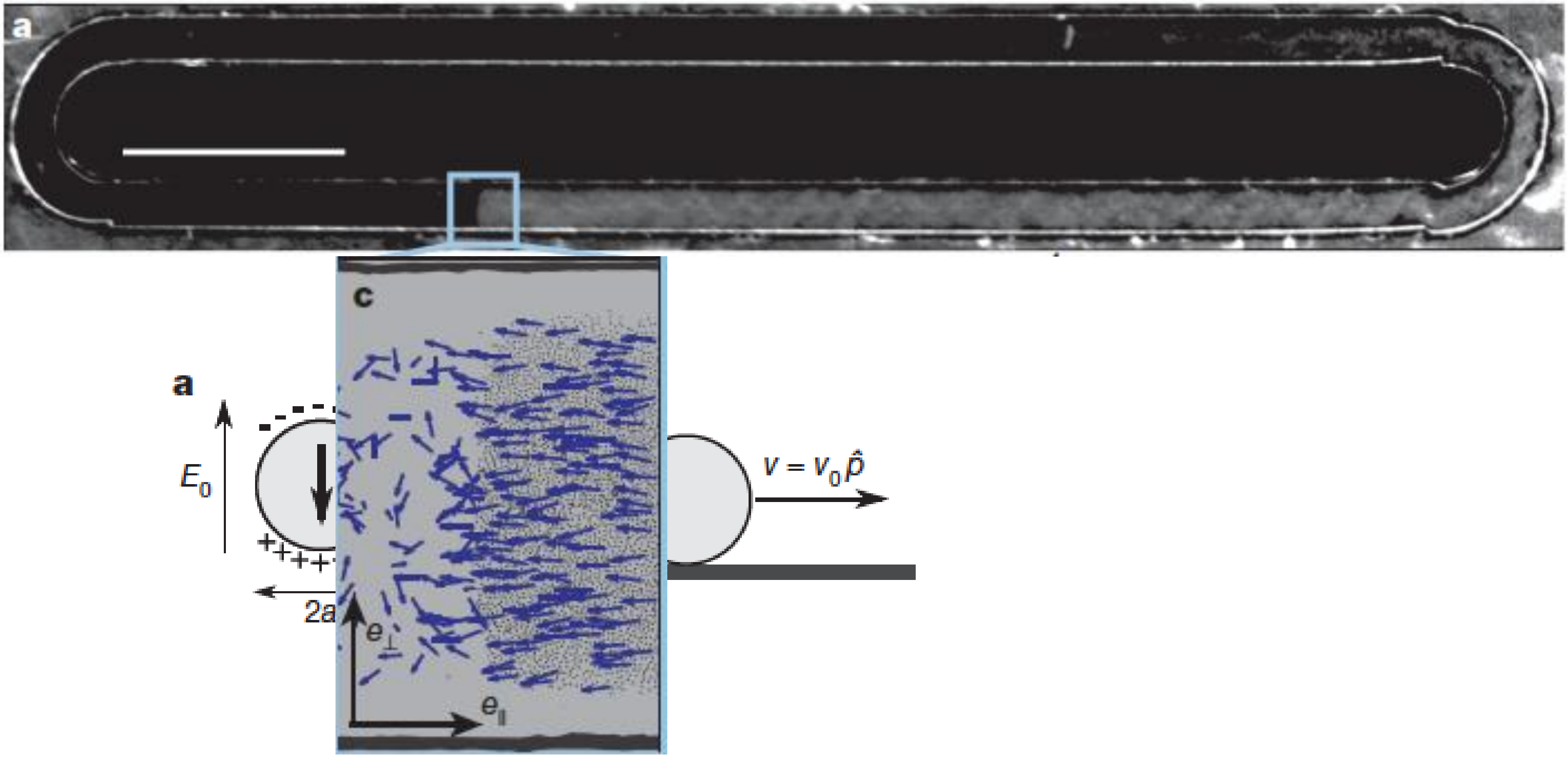
High density regime → compressibility ≈ 0

Bats in a cave

Picture from phys.org [Credit: Gerry Carter]

True incompressibility is possible via long-ranged repulsive interactions (a bat can "see" through the colony)

3) Systems with long-ranged hydrodynamic Interactions (e.g., “quinke rotators” (1))



II) Hydrodynamic Theory of Incompressible Flocks

- Hard (impossible) to solve microscopic model with ~10^5 birds
- Harder to figure out what happens if you change model (universal vs system-specific)
- Historical analog: Fluid mechanics (Navier, Stokes, 1822):

  No theory of atoms and molecules

  No statistical physics

  No computers, ipad, ipod, etc

- So, how'd they do it?

Continuum Approach

Replace $\vec{r}_i(t) \rightarrow$ Continuous fields:

$r(\vec{r},t)$: Coarse grained number density

$\vec{v}(\vec{r},t)$: Coarse grained velocity

Valid for: Length scales L >> interatomic distance

Time scales t >> collision time

Our (Yu-hai Tu and JT) idea: same approach, different symmetry

- No Galilean invariance (birds move through a Special “rest frame” (e.g., air, water, surface of Serengeti. Etc….))

Equations of motion for $\rho(\vec{r},t), \vec{v}(\vec{r},t)$

Make 'em up!

Rules: -Lowest order in space, time derivatives

-Lowest order in fluctuations

$$\delta\rho(\vec{r},t) \equiv \rho(\vec{r},t) - <\rho(\vec{r},t)>$$

$$\delta\vec{v}(\vec{r},t) \equiv \vec{v}(\vec{r},t) - <\vec{v}(\vec{r},t)>$$

Respect Symmetries (for flocks, Rotation invariance)

Worked for fluids, should work for flocks

Hydrodynamic equations for Incompressible active fluids

$$\rho = \rho_0$$

Density EOM: $\partial_t r + \vec{\nabla} \cdot (r \vec{v}) = 0$

$$\downarrow$$

$$\vec{\nabla} \cdot \vec{v} = 0$$

How do you determine P?

Result:

$$\partial_t \vec{v} + l_1 (\vec{v} \cdot \vec{\nabla}) \vec{v} = a \vec{v} - b |\vec{v}|^2 \vec{v} - \vec{\nabla} P + D_T \nabla^2 \vec{v} + D_2 (\vec{v} \cdot \vec{\nabla})^2 \vec{v} + \vec{f}$$

(1)

$$\vec{\nabla} \cdot \vec{v} = 0$$ (2)

$\vec{\nabla}$ ·(both sides of Eq. (1)); solve for P in terms of $\vec{v}$ in Fourier space.

Plugging P back into Eq. (1) , we obtain a closed non linear EOM of $\vec{v}$.

III) Linearized theory

- First, look for uniform, steady state solution for $\vec{f} = \vec{0}$

$$\vec{v}(\vec{r},t) = \vec{v}_o = \text{constant}$$

$$\cancel{\partial_t \vec{v}} + \cancel{\lambda_1 (\vec{v}\cdot\vec{\nabla})\vec{v}} = a\vec{v} - b|\vec{v}|^2\,\vec{v} - \cancel{\vec{\nabla}P} + \cancel{D_T \nabla^2 \vec{v}} + \cancel{D_2 (\vec{v}\cdot\vec{\nabla})^2 \vec{v}} + \cancel{\vec{f}}$$

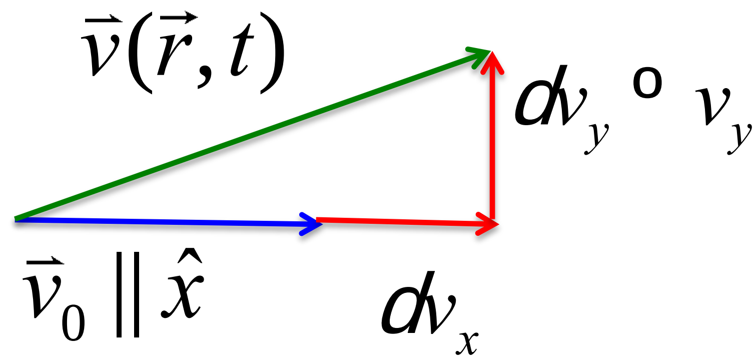
$$\Rightarrow \quad \vec{0} = a\vec{v}_0 - b|\vec{v}_0|^2\,\vec{v}_0 \quad \Rightarrow \quad v_0 \equiv |\vec{v}_0| = \sqrt{\frac{a}{b}}$$

Direction of $\vec{v}_0$ arbitrary

(consequence of rotation invariance)

Now, look for small fluctuations about this for $\vec{f} \neq \vec{0}$

$$\vec{v}(\vec{r},t) = \vec{v}_0 + \delta\vec{v}(\vec{r},t)$$



y : component perpendicular to $\vec{v}_0$

v_y: Goldstone mode in the compressible case

The Mexican hat, massive and massless modes

$$\partial_t \vec{v} + \lambda_1 (\vec{v} \cdot \vec{\nabla})\vec{v} = \boxed{a\vec{v} - b|\vec{v}|^2\,\vec{v}} - \vec{\nabla} P + D_T \nabla^2 \vec{v} + D_2 (\vec{v} \cdot \vec{\nabla})^2 \vec{v} + \vec{f}$$

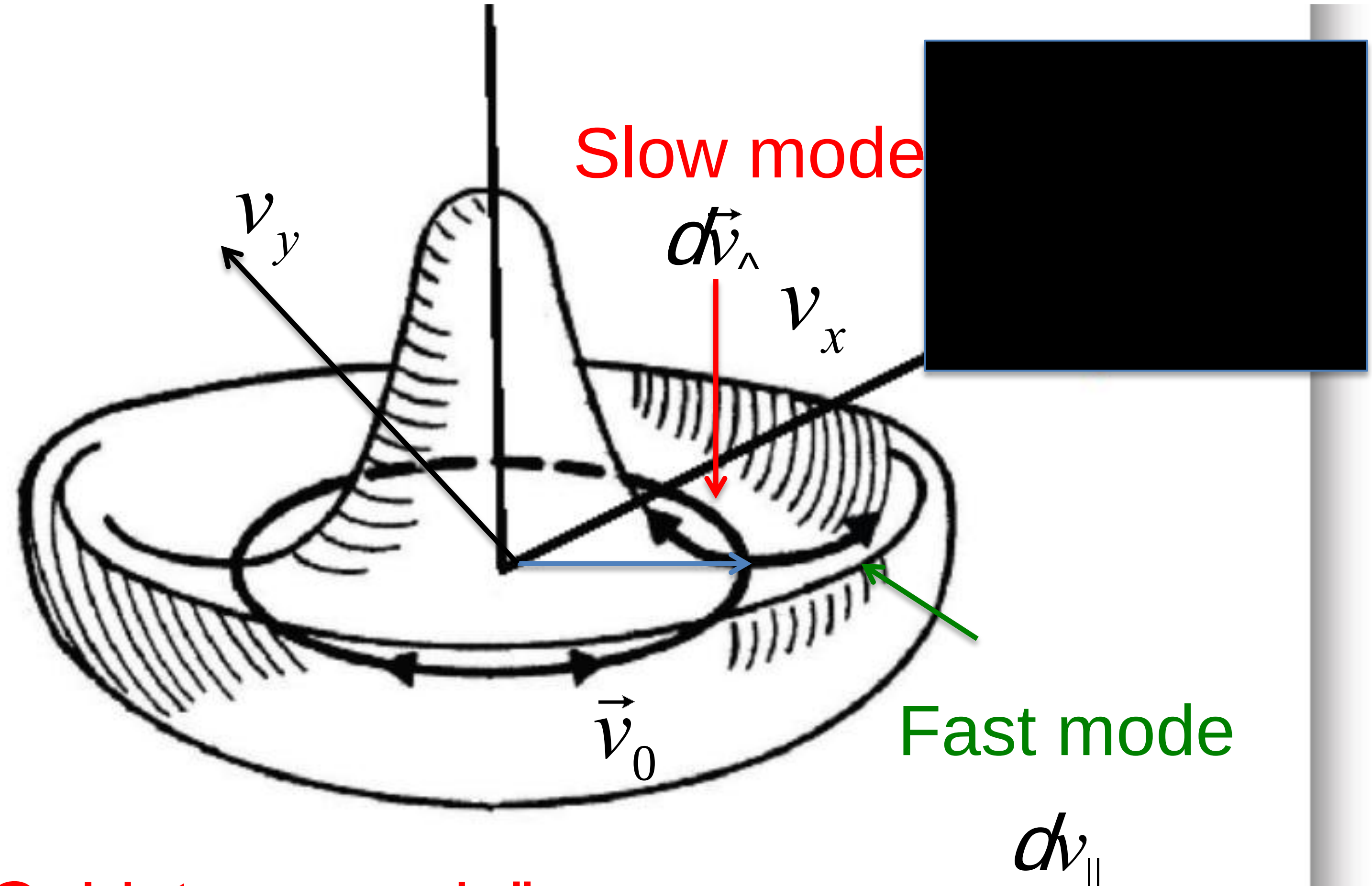
$$-\frac{\partial U_m}{\partial \vec{v}}$$

$$U_m(\vec{v}) = U_m(|\vec{v}|^2) \equiv$$

"Mexican hat potential"

Due to rotation invariance

$$U_m(\vec{v}) = -\frac{a}{2}|\vec{v}|^2 + \frac{b}{4}|\vec{v}|^4$$



$\delta\vec{v}_\perp$ = "Goldstone mode"

Linear theory: Fourier space: mode with wavevector $\vec{q}$

To go to Fourier space, $\vec{\nabla} \rightarrow i\vec{q}$

$$\Rightarrow \vec{\nabla} \cdot \vec{v}(\vec{r},t) = 0 \Rightarrow \vec{q} \cdot \delta\vec{v}(\vec{q},t) = 0$$

Two constraints:

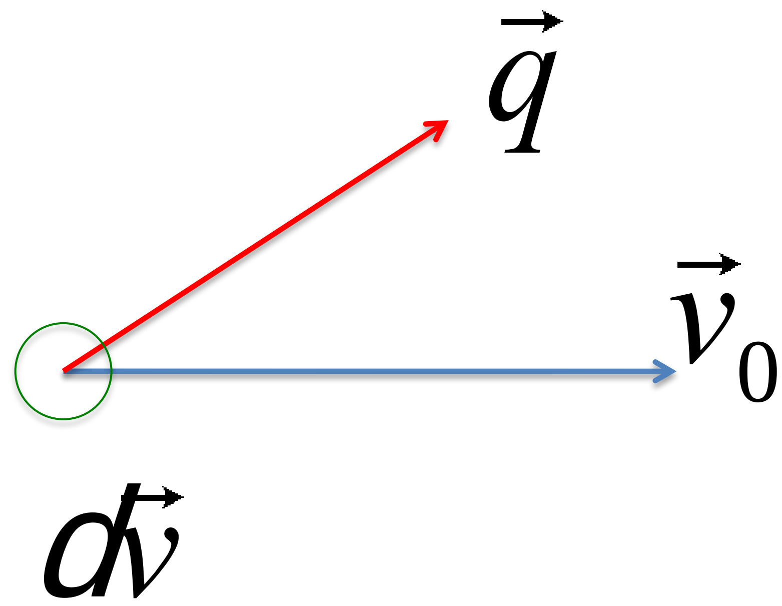
1) Incompressibility Þ $d\vec{v} \wedge \vec{q}$

2) “softness” (i.e., Goldstone mode)

Þ $d\vec{v} \wedge \vec{v}_0$

In d=3, no problem

But in d=2 ?



Þ No “soft” directions in d=2

Smaller fluctuations
In d=2 than in d=3!

IV) Solving the non-linear theory: Exact mappings in d=2:

Mapping # 1: Incompressible flock to “incompressible” magnet

Equation of motion written in terms of $\delta\vec{v}$

$$\partial_t \delta\vec{v} + \lambda_1 (\delta\vec{v}\cdot\vec{\nabla})\delta\vec{v} = -\frac{\partial U_m}{\partial \delta\vec{v}} - \vec{\nabla}P + D_T \nabla^2 \delta\vec{v} + \vec{f}$$

Irrelevant (due to smallness of $\delta\vec{v}$)

How to deal with them?

But still relevant non-linearities in here

Mapping to “incompressible magnet” (continued)

$$\partial_t \delta\vec{v} = -\frac{\partial U_m}{\partial \delta\vec{v}} - \vec{\nabla} P + D_T \nabla^2 \delta\vec{v} + \vec{f}$$

Remember constraint: $\vec{\nabla} \cdot \vec{v} = 0$

can write as TDGL model: $\partial_t \vec{v} = -\frac{\delta H}{\delta \vec{v}} + \vec{f}$

$$H = \int d^2 r [U_m(\vec{v}) + \frac{1}{2} D_T |\vec{\nabla}\vec{v}|^2]$$

Mapping to incompressible magnet (continued)

$$\partial_t d\vec{v} = -\frac{\partial U_m}{\partial d\vec{v}} - \vec{\nabla} P + D_T \nabla^2 d\vec{v} + \vec{f}$$

Enforces constraint: $\vec{\nabla} \cdot \vec{v} = 0$

can write as TDGL model: $\P_t \vec{v} = -\frac{dH}{d\vec{v}} + \vec{f}$

LaGrange Multiplier Enforcing constraint

$$H = \int d^2 r [U_m(\vec{v}) + \frac{1}{2} D_T |\vec{\nabla}\vec{v}|^2 + P(\vec{r}) \vec{\nabla} \cdot \vec{v}]$$

This is just the simplest (i.e., purely relational) dynamical model for an equilibrium XY model (i.e., a 2-component ferromagnet with magnetization $\vec{M}$ $(\vec{v} \rightarrow \vec{M})$

with constraint $\vec{\nabla} \cdot \vec{v} = 0$

Can get equal-time statistics from Equilibrium statistical mechanics with Boltzmann weight:

N.B. Here, $P(\{\vec{v}(\vec{r},t)\}) =$ Probability, Not Pressure

$$P(\{\vec{v}(\vec{r},t)\}) = e^{-bH} / Z$$

Sounds good, but H is non-trivial (non-quadratic)

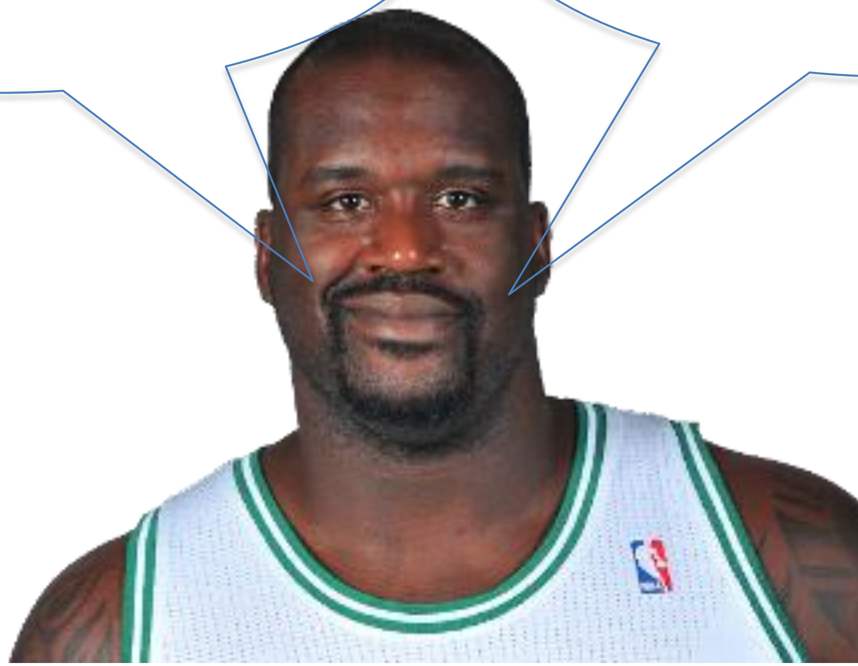
How to deal with this?

Consult the wisdom of the ancients

The Big Aristotle: (aka Shaquille O’ Neal)

“My game’s a mystery;
no-one understands it;

It’s like the
Pythagorean theorem”



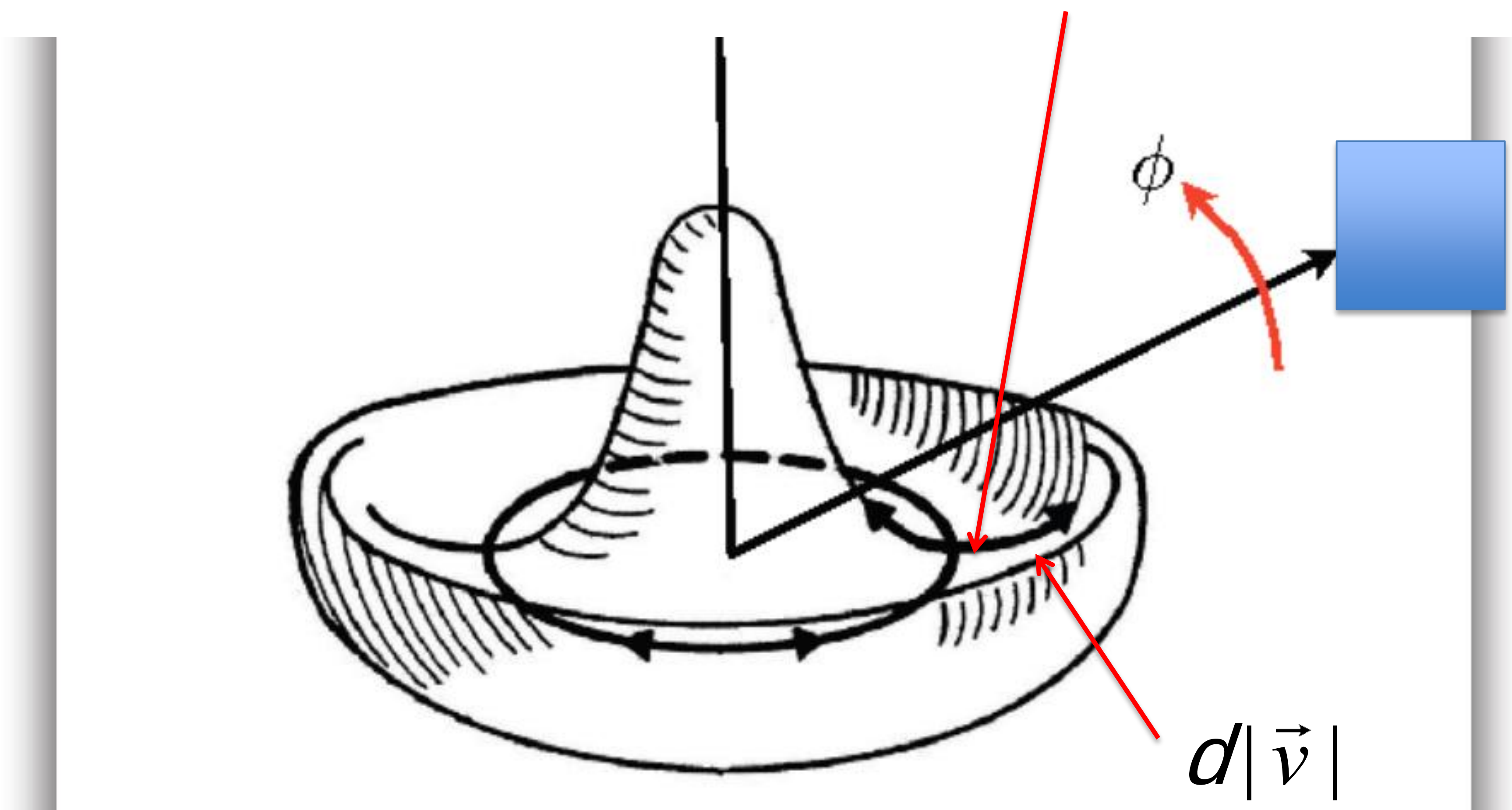
How does Pythagorean theorem help?

Anharmonic terms come from Mexican hat potential

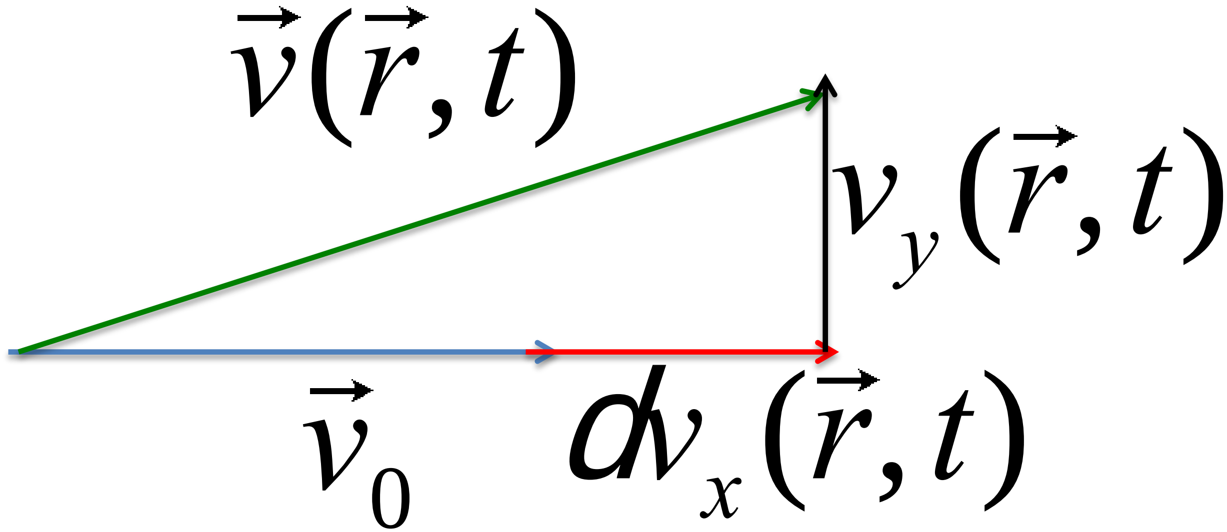
$$H = \int d^2 r [U_m(\vec{v}) + \frac{1}{2} D_T |\vec{\nabla}\vec{v}|^2 + P(\vec{r})\vec{\nabla}\cdot\vec{v}]$$

Mexican hat potential: only depends on $|\vec{v}|$
need to know change $d|\vec{v}|$ in $|\vec{v}|$
due to fluctuations

Curvature o m “mass”



Pythagorean theorem:



$$\delta|\vec{v}(\vec{r},t)| \equiv |\vec{v}(\vec{r},t)| - v_0 = \sqrt{(v_0+\delta v_x)^2 + v_y^2} - v_0 \approx \delta v_x + \frac{1}{2}v_y^2$$

Equilibrium XY model with constraints

Hamiltonian:

$$H = \frac{1}{2}\int d^2r \left[m\left(\delta v_x + \frac{v_y^2}{2v_0} \right)^2 + D_T \left| \vec{\nabla} v_y \right|^2 \right]$$

Constraint: $\partial_x \delta v_x + \partial_y v_y = 0$

Mapping #2: 2d magnet + constraint to 2d smectic

Dealing with constraint

Old 2D fluid mechanic's trick: Streaming function

$$v_x = -v_0 \partial_y f, \quad v_y = v_0 \partial_x f$$

Contours of constant f are flow lines

$$\partial_x v_x + \partial_y v_y = v_0[-\partial_x \partial_y f + \partial_y \partial_x f] = 0$$

Automatically satisfies constraint:

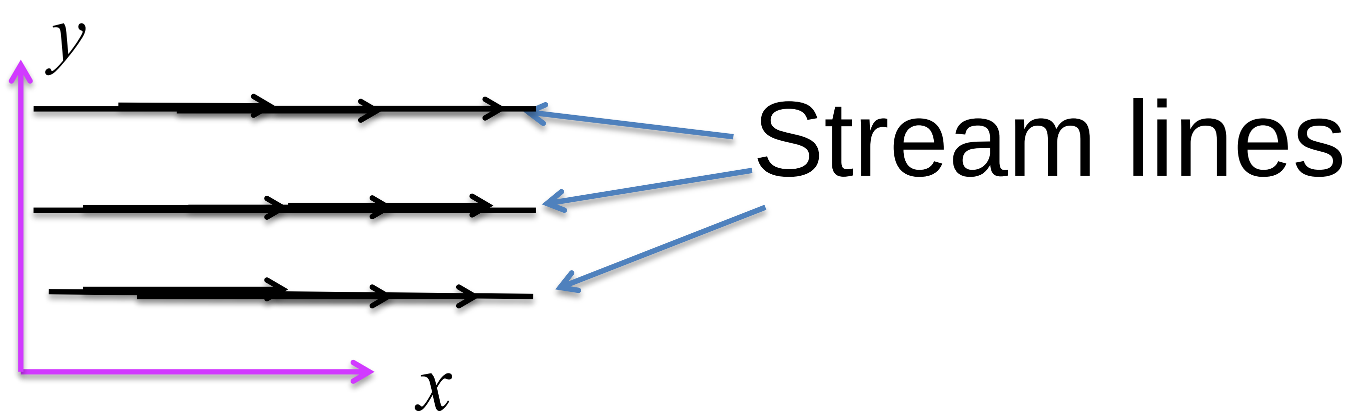
$$\vec{v} \cdot \nabla f = v_x \partial_x f + v_y \partial_y f$$

$$= v_0[-(\partial_x f)\partial_y f + (\partial_y f)\partial_x f] = 0$$

What do these contours look like in absence of fluctuations?

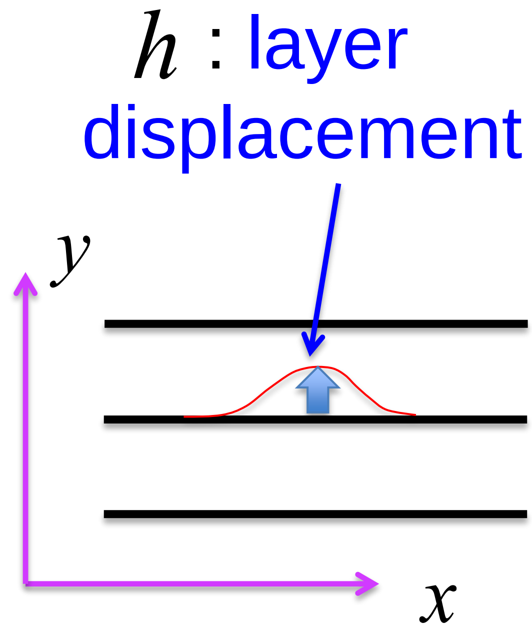
Parallel, uniformly spaced "layers" (or stripes):

$$v_x = v_0 = -v_0 \partial_y \phi, \quad v_y = 0 = v_0 \partial_x \phi \Rightarrow \phi = -y$$



What do fluctuations do?

Displace the layers!



$$f = -y + h(x, y)$$ Þ

This looks like an equilibrium 2D “smectic” (2d stack of 1d fluids; “liquid crystal”)!

This is not a superficial similarity

In fact, it's an exact mapping:

$$\phi = -y + h(x,y) \Rightarrow \delta v_x = -v_0 \partial_y h, \quad v_y = v_0 \partial_x h$$

Plug into: $H = \frac{1}{2}\int d^2 r \left[m\left(\delta v_x + \frac{v_y^2}{2v_0} \right)^2 + D_T \left| \vec{\nabla} v_y \right|^2 \right]$

Get: $H = \frac{1}{2}\int d^2 r \left[B\left(\partial_y h - \frac{(\partial_x h)^2}{2} \right)^2 + K\left(\partial_x^2 h\right)^2 \right]$

$B \equiv m v_0^2$: compression modulus

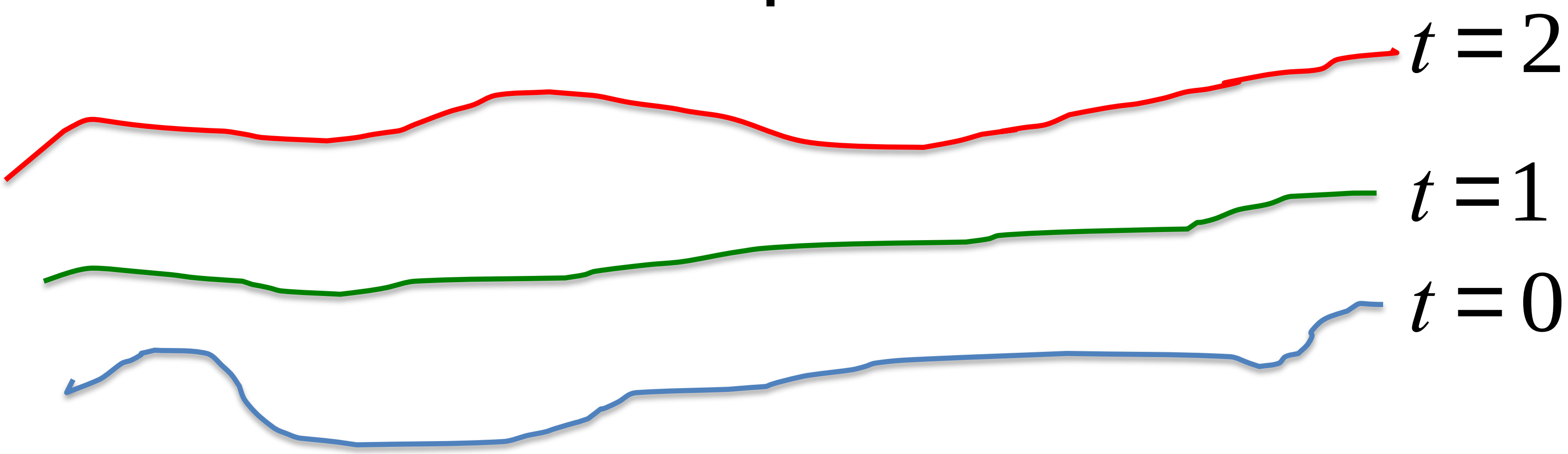
Elastic model for equilibrium 2D smectic!

$K \equiv D_T v_0^2$: bend modulus

Mapping #3 (last one):

2d smectic->1 (space) +1 (time) dimensional KPZ equation (surface growth)

Vapor



Crystal

Described by Kardar, Parisi, and Zhang equation:

$$\partial_t h = D\partial_x^2 h + \lambda(\partial_x h)^2 + f(x,t)$$

Golubovic and Wang showed that this exactly maps onto a 2d smectic:

$$y = 2\lambda t \qquad \frac{K}{kT} = \frac{D^2}{2\lambda\Delta} \qquad \frac{B}{kT} = \frac{2\lambda}{\Delta}$$

2) L. Golubović and Z.-G. Wang, Phys. Rev. Lett. **69**, 2535 (1992);
L. Golubović and Z.-G. Wang, Phys. Rev. E **49**, 2567 (1994).

Gaussian noise=> $P(\{f(\vec{r},t)\}) = \frac{1}{2D}\int dx\,dt[f(\vec{r},t)]^2$

KPZ eqn=> $f(\vec{r},t) = \partial_t h - \lambda(\partial_x h)^2 - D\partial_x^2 h$

$$P(\{h(x,t)\}) =$$

$$\exp(-\frac{1}{2D}\int dx\,dt[\partial_t h - \lambda(\partial_x h)^2 - D\partial_x^2 h]^2)$$

$$P(\{h(x,t)\}) =$$

$$\exp(- \frac{1}{2\mathrm{D}} \int dx\, dt [\partial_t h - / (\partial_x h)^2 - D\partial_x^2 h]^2)$$

$$[\partial_t h - / (\partial_x h)^2 - D\partial_x^2 h \ ]^2 =$$

$$[\partial_t h - / (\partial_x h)^2]^2 + D^2 (\partial_x^2 h)^2$$

Smectic terms (t->y)

$$- \frac{2D}{\mathrm{D}} (\partial_t h - / (\partial_x h)^2) \partial_x^2 h$$

Boundary terms

$$(\partial_x h)^2 \partial_x^2 h = \frac{1}{3} \partial_x ((\partial_x h)^3)$$

$$(\partial_t h)(\partial_x^2 h) = \frac{1}{2} \{ \partial_x [(\partial_t h)(\partial_x h) - h \partial_t \partial_x h] + \partial_t (h \partial_x^2 h) \}$$

We can finally stop all this mapping,
Because scaling laws for 1+1 dim
KPZ are known exactly (that’s why
KP and Z are famous)

They solve it using one more mapping:
KPZ->1d infinitely compressible
equilibrium fluid

Using their results (and some from
Rudy Hwa’s nephew Terry), we get:

Real-space fluctuations:

$$\left\langle \left| \delta\vec{v}(\vec{r},t) \right|^2 \right\rangle = \text{finite}$$

This implies long-ranged orientational order

Real space two-point correlation function:

$$\left\langle \delta\vec{v}\left(\vec{0},t\right)\cdot\delta\vec{v}\left(\vec{r},t\right) \right\rangle \propto \begin{cases} \exp(-x/y^{2/3})(1+4x^2/(9y^2)), & |x| << |y|^{3/2} \\ |y|^{-2/3}, & |x| >> |y|^{3/2} \end{cases}$$

Exponents (and expressions (and 4/9) are exact!

Summary:

I) Incompressible flocks exist

II) Interesting Hydrodynamic model (arbitrary d)

III) Linearized theory ⟹ Weirdness:
fluctuations in d=2 << fluc's in d=3 !

IV) Solving the non-linear theory:
Exact Mappings in d=2:

2d incompressible flock → 2d "incompressible" magnet

↓

1+1d KPZ equation ("sandblasting") ← 2d smectic ("soap")

Thank you for your attention!

Summary

- The order-disorder transition in “incompressible” active fluids can be continuous.
- The critical exponents are calculated to order $\varepsilon=4-d$, and two exact relations between these exponents are found.
- We reveal a theoretical connection between the ordered phase in 2D and 2D equilibrium smectics. Through this connection we find the exact scaling exponents.

Results from equilibrium 2D smectics

Elastic model for 2D smectic can be mapped to KPZ model in 1+1 dimensions (ref. 9)

Anomalous elasticity:

$$K \propto \begin{cases} q_x^{-\frac{1}{2}}, & q_y << q_x^{3/2} \\ q_y^{-\frac{1}{3}}, & q_y >> q_x^{3/2} \end{cases}$$

$$B \propto \begin{cases} q_x^{\frac{1}{2}}, & q_y << q_x^{3/2} \\ q_y^{\frac{1}{3}}, & q_y >> q_x^{3/2} \end{cases}$$

Fluctuations:

$$\left\langle \left| h\left(\vec{q},t\right)\right|^2 \right\rangle \propto \begin{cases} q_x^{-\frac{7}{2}}, & q_y << q_x^{3/2} \\ q_y^{-\frac{7}{3}}, & q_y >> q_x^{3/2} \end{cases}$$

Exponents are exact!

9) L. Golubović and Z.-G. Wang, Phys. Rev. Lett. **69**, 2535 (1992);
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Results from equilibrium 2D smectics

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Fluctuations:

Exponents are exact!

$$\left\langle \left| h(\vec{q},t) \right|^2 \right\rangle \propto \begin{cases} q_{\parallel}^{-\frac{7}{2}}, & q_{\perp} << q_{\parallel}^{3/2} \\ q_{\perp}^{-\frac{7}{3}}, & q_{\perp} >> q_{\parallel}^{3/2} \end{cases}$$

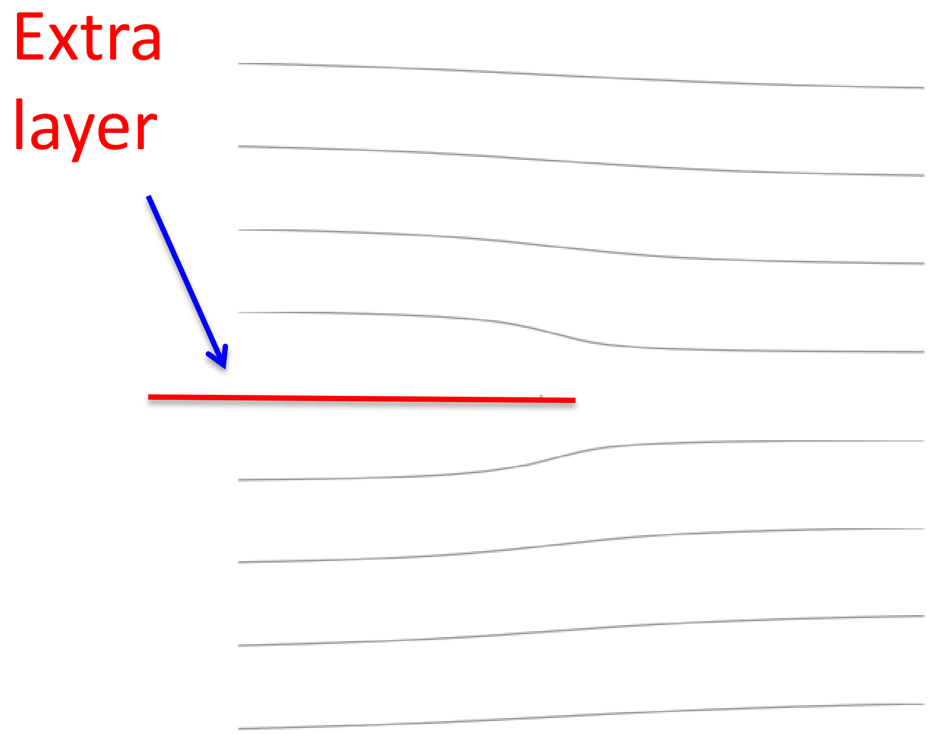
9) L. Golubović and Z.-G. Wang, Phys. Rev. Lett. **69**, 2535 (1992);
L. Golubović and Z.-G. Wang, Phys. Rev. E **49**, 2567 (1994).

DIslocations in 2D smectics always proliferate at any finite temperature T.

Smectic order only persists to length scales $\mu \exp\{-E_D/T\}$

E_D: energy of a single dislocation

Dislocation:



Results for 2D incompressible active fluid

Physically, variable *h* in our model is not the layer displacement of smectics.

Recall $u_x = -v_0 \partial_y h, \quad u_y = v_0 \partial_x h$

Smectic-type dislocation does not exist in this problem

Golubović-Wang scaling behavior holds up to arbitrary long length scale.

$$v_l(\vec{k},\omega) = G_0(\vec{k},\omega)\left[f_l(\vec{k},\omega) - \frac{i\lambda}{2} P_{lmn}(\vec{k}) \int_{\vec{q},\Omega} v_m(\vec{q},\Omega) v_n(\vec{k}-\vec{q},\omega-\Omega) \right]$$

$$- \frac{b}{3} G_0(\vec{k},\omega) Q_{lmno}(\vec{k}) \int_{\vec{q},\Omega,\vec{p},\nu} v_m(\vec{k}-\vec{q}-\vec{p},\omega-\Omega-\nu) v_n(\vec{q},\Omega) v_o(\vec{p},\nu)$$

where

$$G_0(\vec{k},\omega) = \left(-i\omega + a + D_T k^2\right)^{-1}$$

$$P_{lmn}(\vec{k}) = P_{lm}(\vec{k}) k_n + P_{\ln}(\vec{k}) k_m$$

$$Q_{lmno}(\vec{k}) = P_{lm}(\vec{k}) \delta_{no} + P_{\ln}(\vec{k}) \delta_{mo} + P_{lo}(\vec{k}) \delta_{mn}$$

$$P_{lm}(\vec{k}) = \delta_{lm} - \frac{k_l k_m}{k^2},$$ projection operator

Linearized theory

$$\vec{u}_\perp = \vec{u}_L + \vec{u}_T \qquad \vec{u}_L = \frac{\vec{q}_\perp(\vec{q}_\perp \cdot \vec{u}_\perp)}{q_\perp^2}$$

$$\partial_t \vec{u}_\perp = -\frac{2aq_\perp^2 + \Gamma(\vec{q})q_x^2}{q^2}\vec{u}_L - \Gamma(\vec{q})\vec{u}_T + \vec{f}_\perp$$

$$C_{LL}(\vec{q}) = \langle \vec{u}_L(-\vec{q}) \cdot \vec{u}_L(\vec{q}) \rangle = \frac{D}{2a\left(\frac{q_\perp}{q_x}\right)^2 + \Gamma(\vec{q})}$$

$$\propto \frac{D}{2a} \; for\ most\ q \to 0$$

Massive

This is because of the incompressibility constraint:

$$q_\wedge u_L + q_x u_x = 0$$

a picture might be more useful here; illustrate how,
if v is perpendicular to q, it must have a component along the ordering direction, which is massive. Then you can give the formula for <v^(q)>, and say that this is why there’s a mass for all directions of q except q_y<~q_x^2

Critical exponents

Order parameter:

$$\left|\left\langle \vec{v} \right\rangle\right| \propto \left| x - x_c \right|^{\beta}$$

x: control parameter

$$\beta = \frac{1}{2} - \frac{6}{127}\varepsilon + O\left(\varepsilon^2\right)$$

The equal-time correlation function:

$$\left\langle \vec{v}\left(\vec{r},t\right)\cdot\vec{v}\left(\vec{r}\,',t\right)\right\rangle = \left|\vec{r}-\vec{r}\,'\right|^{2-d-\eta} Y\left(\frac{\left|\vec{r}-\vec{r}\,'\right|}{\xi}\right)$$

$$e = 4 - d$$

Correlation length:

$$x \, \mu \left| x - x_c \right|^{-n}$$

$$\nu = \frac{1}{2} + \frac{65}{508}\varepsilon + O\left(\varepsilon^2\right)$$

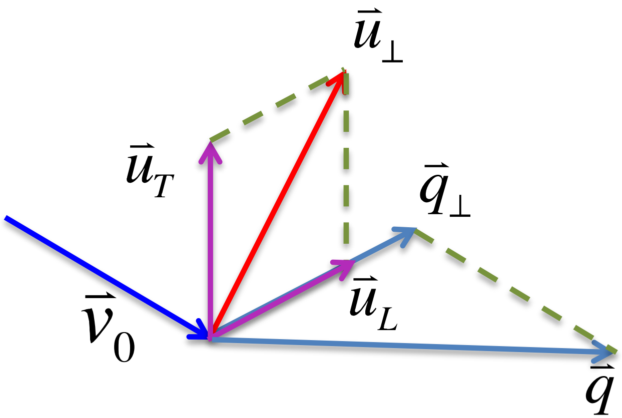
Susceptibility:

$$\chi \propto \left| x - x_c \right|^{-\gamma} \qquad \gamma = 1 + \frac{27}{254}\varepsilon + O\left(\varepsilon^2\right)$$

The field(h)-dependence of the order parameter right at the critical point:

$$\left| \left\langle \vec{v} \right\rangle \right| \propto h^{\frac{1}{\delta}} \qquad \delta = 3 + \frac{51}{127}\varepsilon + O\left(\varepsilon^2\right)$$

Linear theory



$$\vec{u} = \vec{u}_{||} + \vec{u}_\perp = \vec{u}_{||} + \vec{u}_T + \vec{u}_L$$

The incompressibility constraint ($\vec{q} \cdot \vec{u} = 0$) gives

$$q_\wedge u_L + q_{||} u_{||} = 0$$

$\vec{u}_L$ is massive

$$\langle \vec{u}_L(-\vec{q}) \cdot \vec{u}_L(\vec{q}) \rangle = \frac{D}{2a\left(\frac{q_\perp}{q_{||}}\right)^2 + \Gamma(\vec{q})} \qquad \propto \frac{D}{2a} \; for\ most\ q \to 0$$

$\Gamma(\vec{q}) \to \mu q^2$

Phase transitions in active fluids were first studied by Viscek et al in 1995 (Ref.1).

The analogy between active fluids and magnets.

1) T. Vicsek, A. CzirÓk, E. Ben-Jacob, I. Cohen, and O. Schochet, 1995, Phys. Rev. Letter. **75**, 1226 (1995)

Hydrodynamic equations for Flocks:

(“convective Derivative”:also in Navier-Stokes eqn)

New terms (forbidden in NS equations due to Galilean invariance)

Velocity EOM:

$$\partial_t \vec{v} + l_1(\vec{v}\cdot\vec{\nabla})\vec{v} + l_2\vec{v}(\vec{\nabla}\cdot\vec{v}) + l_3(\vec{\nabla}|\vec{v}|^2) = a\vec{v} - b|\vec{v}|^2\vec{v}$$

$$-\vec{\nabla}P(r) - \vec{v}(\vec{v}\cdot\vec{\nabla}P_2(r)) + D_B\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) + D_T\nabla^2\vec{v} + D_2(\vec{v}\cdot\vec{\nabla})^2\vec{v} + \vec{f}$$

Noise

Density EOM:

$$\partial_t r + \vec{\nabla}\cdot(r\vec{v}) = 0$$

Number conservation (“immortal” flock)

Fluctuations for 2D incompressible active fluid

Exponents are exact!

$$u_{\parallel} = -v_0 \partial_{\perp} h, \quad u_{\perp} = v_0 \partial_{\parallel} h$$

$$\left\langle \left| u_{\perp}(\vec{q},t) \right|^2 \right\rangle = q_{\parallel}^2 \left\langle \left| h(\vec{q},t) \right|^2 \right\rangle \propto \begin{cases} q_{\parallel}^{-\frac{3}{2}}, & q_{\perp} \ll q_{\parallel}^{3/2} \\ q_{\parallel}^2 q_{\perp}^{-\frac{7}{3}}, & q_{\perp} \gg q_{\parallel}^{3/2} \end{cases}$$

$$\left\langle \left| u_{\parallel}(\vec{q},t) \right|^2 \right\rangle = q_{\perp}^2 \left\langle \left| h(\vec{q},t) \right|^2 \right\rangle \propto \begin{cases} q_{\perp}^2 q_{\parallel}^{-\frac{7}{2}}, & q_{\perp} \ll q_{\parallel}^{3/2} \\ q_{\perp}^{-\frac{1}{3}}, & q_{\perp} \gg q_{\parallel}^{3/2} \end{cases}$$

Analog between active fluids and magnets

Arrows: active particles, Arrow heads: directions of velocities

Local alignment interactions

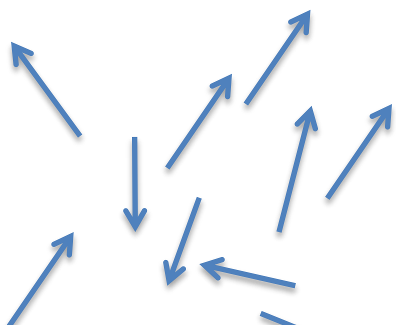
Errors (noises)

$< \vec{v}(\vec{r},t) >= \vec{0}$

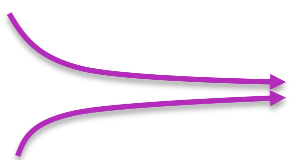
Disordered

$< \vec{v}(\vec{r},t) > \neq \vec{0}$

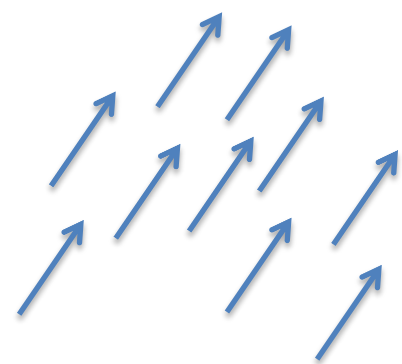
Ordered



noise



temperature



Paramagnetic phase

Ferromagnetic phase

non equilibrium

Hydrodynamic equations for compressible active fluids (ref. 6)

Velocity EOM:

Rotation invariance

$$\partial_t \vec{v} + \lambda_1 (\vec{v}\cdot\vec{\nabla})\vec{v} + \lambda_2 \vec{v}(\vec{\nabla}\cdot\vec{v}) + \lambda_3 (\vec{\nabla}|\vec{v}|^2) = -a\vec{v} - b|\vec{v}|^2\,\vec{v}$$

$$-\vec{\nabla}P(\rho,|\vec{v}|) + D_T \nabla^2 \vec{v} + D_B \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) + \vec{f}$$

← Noise (errors)

Density EOM: $\partial_t r + \vec{\nabla}\cdot(r\vec{v}) = 0$ Number conservation

6) J. Toner and Y. Tu, Phys. Rev. Lett. 92, 025702 (2004)

Hydrodynamic equations for Flocks:

Connection to Equilibrium Ferromagnet (Pointers) (and Mermin-Wagner Theorem):

New terms (forbidden in NS equations due to Galilean invariance)

Velocity EOM:

("convective Derivative")

Other "2 V, and grad" terms

"move faster," Slowpoke

"But, not Too Fast!"

$$\partial_t \vec{v} + \lambda_1 (\vec{v}\cdot\vec{\nabla})\vec{v} + \lambda_2 \vec{v}(\vec{\nabla}\cdot\vec{v}) + \lambda_3(\vec{\nabla}|\vec{v}|^2) = \alpha\vec{v} - \beta|\vec{v}|^2\vec{v}$$

$$-\vec{\nabla}P(\rho) - \vec{v}(\vec{v}\cdot\vec{\nabla}P_2(\rho)) + D_B\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) + D_T\nabla^2\vec{v} + D_2(\vec{v}\cdot\vec{\nabla})^2\vec{v} + \vec{f}$$

Anisotropic pressure

Anisotropic viscosity

Noise (errors)

Density EOM:

$$\partial_t \rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

Number conservation ("immortal" flock)

Hydrodynamic equations for Compressible Flocks:

$$\partial_t \vec{v} + \lambda_1 (\vec{v}\cdot\vec{\nabla})\vec{v} + \lambda_2 \vec{v}(\vec{\nabla}\cdot\vec{v}) + \lambda_3 (\vec{\nabla}|\vec{v}|^2) = \alpha\vec{v} - \beta|\vec{v}|^2\,\vec{v}$$

$$-\vec{\nabla}P(\rho) - \vec{v}(\vec{v}\cdot\vec{\nabla}P_2(\rho)) + D_B\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) + D_T\nabla^2\vec{v} + D_2(\vec{v}\cdot\vec{\nabla})^2\vec{v} + \vec{f}$$

Anisotropic pressure

Anisotropic viscosity

Noise (errors)

Density EOM:

$$\partial_t \rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

Number conservation
("immortal" flock)

Connection to other models

$$\partial_t \vec{v} + \lambda_1 (\vec{v} \cdot \vec{\nabla}) \vec{v} = -a\vec{v} - b|\vec{v}|^2 \vec{v} - \vec{\nabla} P + D_T \nabla^2 \vec{v} + \vec{f}$$

$$\vec{\nabla} \cdot \vec{v} = 0$$

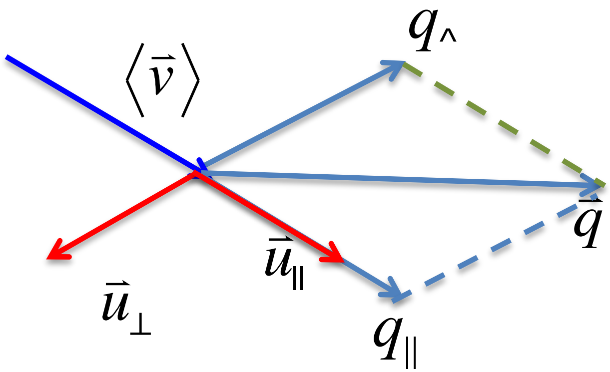
Time-Dependent randomly forced incompressible fluid considered in (Ref. 7)

The model for magnets with dipolar interaction (due to $\nabla \cdot \vec{v} = 0$ constraint) (Ref. 8)

7) D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A, **16**, 732 (1977)

8) M. E. Fisher and A. Aharony, Phys. Rev. Lett. **30**, 559 (1973)
A. Aharony and M. E. Fisher, Phys. Rev. B, **8**, 3323 (1973)

The incompressibility constraint in Fourier space:



$$q_\wedge u_\wedge + q_\parallel u_\parallel = 0$$

$u_\wedge$ becomes massive too

$$\langle \vec{u}_\perp(-\vec{q}) \cdot \vec{u}_\perp(\vec{q}) \rangle = \frac{D}{2a\left(\frac{q_\perp}{q_\parallel}\right)^2 + \Gamma(\vec{q})} \quad \propto \frac{D}{2a} \; \textit{for most } q \to 0$$

$\Gamma(\vec{q})$ → μ q^2

No Goldstone modes

The nonlinear effects are interesting.

The full effective model (in Fourier space) is given by:

$$\partial_t u_\perp = \frac{q_\parallel^2}{q^2} F_{\vec{q}} \left[-\frac{2a}{v_0} \left(u_\parallel + \frac{u_\perp^2}{2v_0} \right) u_\perp + D_T \nabla^2 u_\perp \right] + \frac{q_\parallel q_\perp}{q^2} F_{\vec{q}} \left[2a \left(u_\parallel + \frac{u_\perp^2}{2v_0} \right) \right] + f_\perp$$

The λ_1-term (i.e., the convective term) is irrelevant!

This model can be mapped to an equilibrium model.

Mapping #2: 2d magnet + constraint to 2d smectic

Dealing with constraint $\partial_x v_x + \partial_y v_y = 0$

Old 2D fluid mechanic's trick: Streaming function ϕ

$$v_x = -v_0 \partial_y \phi, \quad v_y = v_0 \partial_x \phi$$

Contours of constant ϕ are flow lines

Automatically satisfies constraint:

$$\partial_x v_x + \partial_y v_y = v_0 [-\partial_x \partial_y \phi + \partial_y \partial_x \phi] = 0$$

Happy Guy Fawkes Day!

Mastermind of the “gunpowder plot”; Foiled on Nov 5, 1605

“The last man to enter Parliament with honest intentions”

Hydrodynamic equations for Compressible Flocks:

Velocity EOM:

$$\partial_t \vec{v} + l_1(\vec{v}\cdot\vec{\nabla})\vec{v} + l_2\vec{v}(\vec{\nabla}\cdot\vec{v}) + l_3(\vec{\nabla}|\vec{v}|^2) = a\vec{v} - b|\vec{v}|^2\,\vec{v}$$
$$-\vec{\nabla}P(r) - \vec{v}(\vec{v}\cdot\vec{\nabla}P_2(r)) + D_B\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) + D_T\nabla^2\vec{v} + D_2(\vec{v}\cdot\vec{\nabla})^2\vec{v} + \vec{f}$$

Noise

Density EOM:

$$\partial_t r + \vec{\nabla}\cdot(r\vec{v}) = 0$$

Number conservation
("immortal" flock)

V equation of motion, incompressible case

$$\vec{\nabla}\cdot\vec{v}=0 \qquad r=r_0=\text{constant} \qquad P_2(r)=P_2(r_0)=\text{constant}$$

$$\partial_t\vec{v}+/_1(\vec{v}\cdot\vec{\nabla})\vec{v}+/_2\vec{v}(\vec{\nabla}\cdot\vec{v})+/_3(\vec{\nabla}|\vec{v}|^2)=a\vec{v}-b|\vec{v}|^2\,\vec{v}$$
$$-\vec{\nabla}P(r)-\vec{v}(\vec{v}\cdot\vec{\nabla}P_2(r))+D_B\vec{\nabla}(\vec{\nabla}\cdot\vec{v})+D_T\nabla^2\vec{v}+D_2(\vec{v}\cdot\vec{\nabla})^2\vec{v}+\vec{f}$$

$$\partial_t\vec{v}+/_1(\vec{v}\cdot\vec{\nabla})\vec{v}=a\vec{v}-b|\vec{v}|^2\,\vec{v}-\vec{\nabla}P+D_T\nabla^2\vec{v}+D_2(\vec{v}\cdot\vec{\nabla})^2\vec{v}+\vec{f}$$