KPZ Equation Limit of Interacting Particle Systems

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joint work with Ivan Corwin and with Amir Dembo

$(1+1)$-dim. KPZ Universality Class

For a class of random interface growth, $\varepsilon^1(h(\varepsilon^{-3}t, \varepsilon^{-2}x) - \mathbf{E}(h(\varepsilon^{-3}t, \varepsilon^{-2}x)))$ *converges to a universal process* $\mathscr{H}(t,x)$ *depending on the initial condition.*

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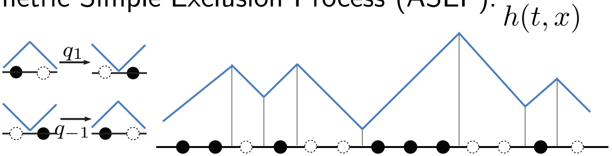

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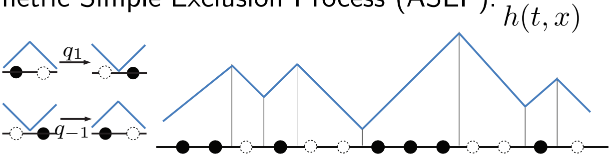

- Such a statement of currently out of reach without integrability.

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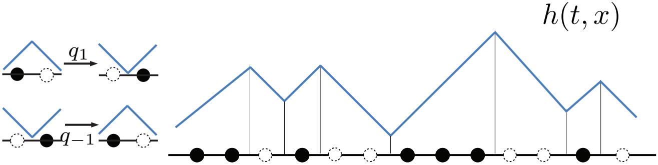

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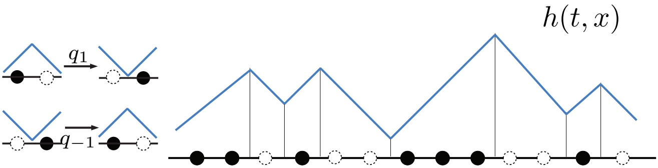

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$$b\varepsilon^1 (h(t\varepsilon^{-4}, x\varepsilon^{-2}) - at\varepsilon^{-3}) \Rightarrow \mathcal{H}(t, x),$$

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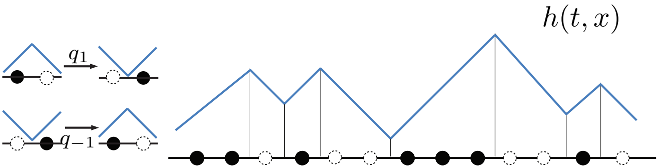

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discrete models converges to KPZ equation

if we tune parameters properly while doing scaling.

What scaling?

- Under $\mathcal{H}_\varepsilon(t,x) := \varepsilon^b \mathcal{H}(\varepsilon^{-z}t, \varepsilon^{-a}x)$, the KPZ equation scales as

$$\partial_t \mathcal{H}_\varepsilon = \varepsilon^{2a-z}\tfrac{1}{2}\partial_{xx}\mathcal{H} - \varepsilon^{2a-z-b}\tfrac{1}{2}(\partial_x \mathcal{H})^2 + \varepsilon^{b-\frac{z}{2}+\frac{a}{2}}\xi.$$

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- Other scalings are possible, e.g. the weak noise scaling $(z,a,b) = (4,2,0)$ and $\xi \to \varepsilon \xi$.

Strategy

- Apply the Hopf-Cole transformation:

$$\partial_t \mathcal{H} = \tfrac{1}{2}\partial_{xx}\mathcal{H} - \tfrac{1}{2}(\partial_x \mathcal{H})^2 + \xi \text{ (KPZ eq)},$$

$$\Big\downarrow \, \mathcal{Z}(t,x) := e^{-\mathcal{H}(t,x)}$$

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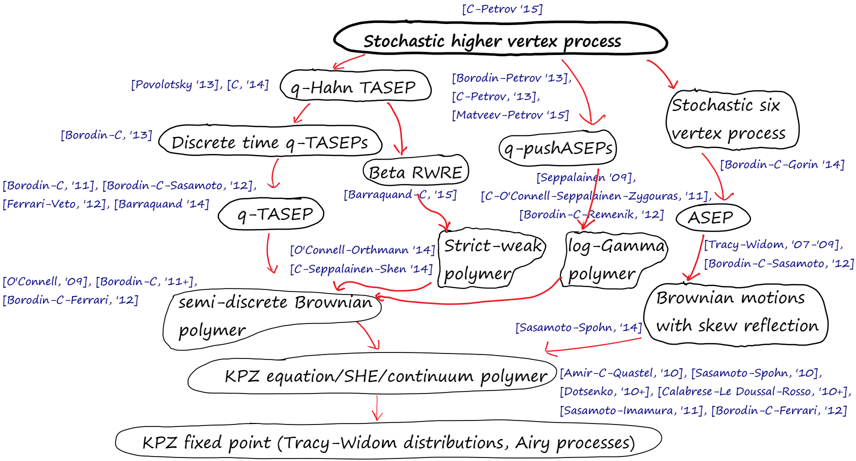
- For ASEP, Bertini and Giacomin utilizes the discrete HC transformation of Gärtner (1988).
- Q: when is a discrete HC transformation available?
- A: for *all* integrable models.

Higher Spin Exclusion Processes (Courtesy of Corwin)

Degenerations to known integrable stochastic systems in KPZ class



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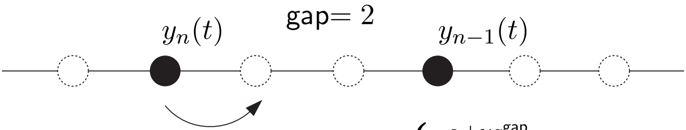
- Consider only $J = 1$ for simplicity.



with probability $\begin{cases} \frac{\alpha+\nu q^{\text{gap}}}{1+\alpha}, & \text{if } y_{n-1} \text{ moved}; \\ \frac{1+\alpha(1-q^{\text{gap}})}{1+\alpha}, & \text{otherwise}. \end{cases}$

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- Define $Z(t,x) := \lambda^t \rho^{x+\mu t} Q_{x+\mu t}(t)$, where $Q_n(t) := q^{y_n(t)+n}$,

$$\lambda := \frac{1+\alpha f}{1+\alpha q f} > 0,\ \mu := \frac{a-a'}{b-b'} > 0;$$

$$a := \frac{\alpha f}{1+\alpha f},\ a' := \frac{\alpha q f}{1+\alpha q f},\ b := \frac{f}{1-f},\ b' := \frac{\nu f}{1-\nu f},\ f := \frac{1-\rho}{1-\nu\rho}.$$

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Proposition (Corwin and Tsai, 2015)

$$\begin{aligned} &Z(t+1,x) - Z(t,x+\mu) \\ &\quad = \big[LZ(t,\bullet)\big](x) + Z(t,x+\mu)W(t,x+\mu), \end{aligned}$$

where $W(t,y)$ is an (explicit) $\mathscr{F}$-martingale.

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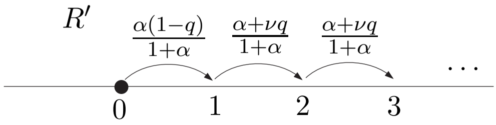
Here $p(y,x) := \mathbf{P}(R = x - y)$ and $L(y,x) := p(y,x) - \mathbf{1}_{y=x+\mu}$ be the semigroup and pseudo generator of the the following random walk.



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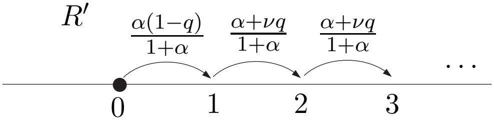

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$\sum_{n=0}^{\infty} \lambda\rho^n\,\mathbf{P}(R' = n) = 1.$

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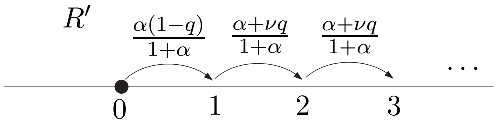

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$$Z(t+1,x) - Z(t,x+\mu)$$
$$= \big[LZ(t,\bullet)\big](x) + Z(t,x+\mu)W(t,x+\mu).$$

The cross-variance of $W(t,y)Z(t,y)$ is

$$Z(t,y)Z(t,y')\,\mathbf{E}(W(t,y)W(t,y)|\mathscr{F}(t))$$
$$= (\tfrac{\nu+\alpha}{1+\alpha})^{|y-y'|}\Theta_1(t,y\wedge y')\Theta_2(t,y\wedge y'),$$

$\Theta_1(t,y) := q\lambda Z(t,y) - [pZ(t,\bullet)](t)$ and
$\Theta_2(t,y) := -\lambda Z(t,y) + [pZ(t,\bullet)](t)$.

KPZ equation limit for HSEP, scaling

Scaling

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$$t_\varepsilon(t) := t^*_\varepsilon \varepsilon^{-3} t, \quad x_\varepsilon(t, x) := r_* \varepsilon^{-1} x + \tfrac{\mu_\varepsilon}{\varepsilon} \varepsilon^{-2} t.$$

Here $r_* := (b - b')^{-1} > 0$, $\frac{\mu_\varepsilon}{\varepsilon} = \mu + O(\varepsilon)$, and
$t^*_\varepsilon := \varepsilon^{-1}(a^2 - {a'_\varepsilon}^2 - (a - a'_\varepsilon)(b + b'))^{-1} = t^* + O(\varepsilon)$.

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- Equivalently, $Z_\varepsilon(\tau, x) := \exp(-H_\varepsilon(\tau, x))$, where $H_\varepsilon(\tau, x) := \varepsilon^1 y_{n_\varepsilon(\tau,x)}(t_\varepsilon(\tau)) + C_{1,\varepsilon} t + C_{2,\varepsilon} x$,

KPZ equation limit for HSEP

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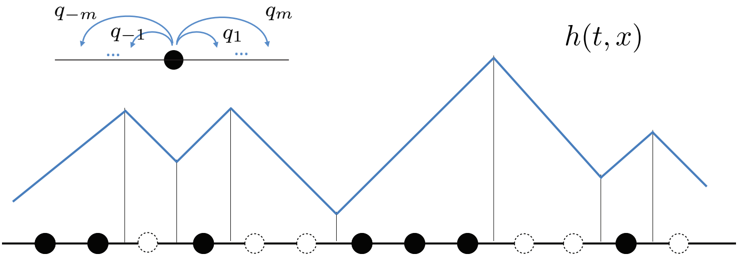
Theorem (Corwin and Tsai, 2015)

Let $\mathcal{Z}(t,x)$ be the $C(\mathbb{R}_+,\mathbb{R})$-valued solution of the SHE starting from $\mathcal{Z}_{ic}(x) \in C(\mathbb{R})$. If $Z_\varepsilon(0,\bullet)$ satisfies certain moment conditions and $Z_\varepsilon(0,\bullet) \Rightarrow \mathcal{Z}_{ic}(\bullet)$, then

$$Z_\varepsilon(\bullet,\bullet) \Rightarrow \mathcal{Z}(\bullet,\bullet).$$

Non-nearest Neighbor Exclusion Processes



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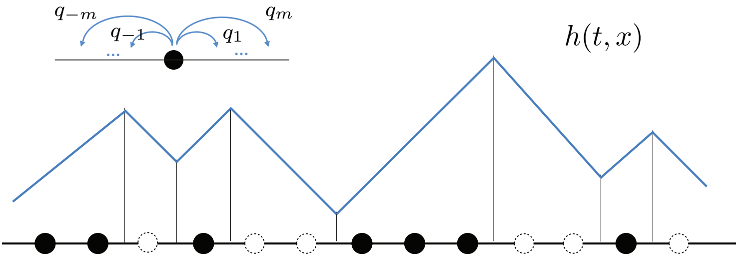

- Height function $h(t,x)$ defined for $x \in \mathbb{Z}$;
- Occupation variable $\eta(y)$ defined for $y \in \mathbb{Z} + \frac{1}{2}$, as

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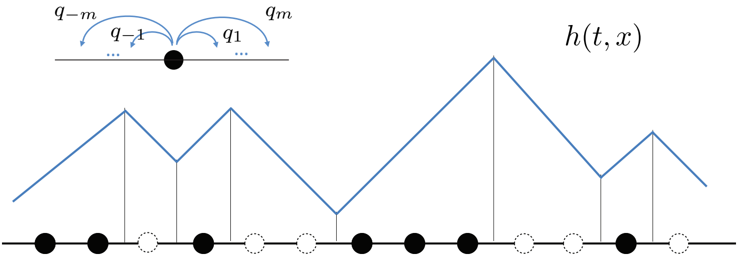

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- Weak asymmetry:

$$q_i := \tfrac{1}{2}(\kappa_i - \varepsilon^1 \gamma_i), \;\; q_{-i} := \tfrac{1}{2}(\kappa_i + \varepsilon^1 \gamma_i), \quad i = 1, \dots, m.$$

Non-nearest Neighbor Exclusion Processes

Assume $m = 3$.

Theorem (Dembo and Tsai, 2013)

Given any $\kappa_1, \kappa_2, \kappa_3 \in (0,1)$ with $\kappa_1 + \kappa_2 + \kappa_3 = 1$ and $\gamma > 0$, for the following choice of asymmetry

$$\gamma_j = \gamma\Big(\sum_{i=j}^{3} \tfrac{2(i-j)}{j}\kappa_i + \kappa_j\Big) + O(\varepsilon), \quad j = 1, 2, 3. \qquad (\star)$$

Let $Z(t,x) := \exp(-\gamma\varepsilon^1 h(t,x) + c_\varepsilon t)$ and $Z_\varepsilon(t,x) := Z(\varepsilon^{-4}t, \varepsilon^{-2}x)$. If $Z_\varepsilon(0,\cdot)$ satisfies the preceding moment conditions and if $Z_\varepsilon(0,\cdot) \Rightarrow \mathcal{Z}_{ic}(\cdot)$, then

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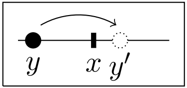

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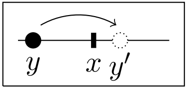

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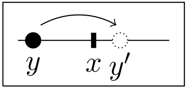

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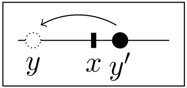

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where $\mathcal{L} := \sum_{i=1}^{3}(\frac{\gamma\kappa_i}{2} + O(\varepsilon))\mathcal{L}_i$, $\mathcal{Q} := \sum_{k=1}^{3} b_k \sum_{(y,y')\ni x}\Big(\eta(y)\eta(y')Z(t,x)\Big)$.

$\mathcal{L}_i := (-\eta(x+\frac{1}{2}-i) - \ldots - \eta(x-\frac{1}{2}) + \eta(x+\frac{1}{2}) + \ldots + \eta(x-\frac{1}{2}+i))Z(t,x).$

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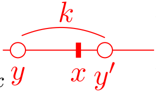
where $\mathcal{L} := \sum_{i=1}^{3}\big(\frac{\gamma\kappa_i}{2} + O(\varepsilon)\big)\mathcal{L}_i$, $\mathcal{Q} := \sum_{k=1}^{3} b_k \sum_{(y,y')\ni x}\Big($ 

$\Big)$.

$\mathcal{L}_i := (-\eta(x+\frac{1}{2}-i) - \ldots - \eta(x-\frac{1}{2}) + \eta(x+\frac{1}{2}) + \ldots + \eta(x-\frac{1}{2}+i))Z(t,x).$

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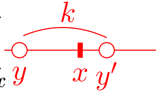
where $\mathcal{L} := \sum_{i=1}^{3}\big(\frac{\gamma\kappa_i}{2} + O(\varepsilon)\big)\mathcal{L}_i$, $\mathcal{Q} := \sum_{k=1}^{3} b_k \sum_{(y,y')\ni x}\Big($ 

$\Big)$.

- Wish to match $(\varepsilon\mathcal{L} + \varepsilon^2\mathcal{Q})$ to

$$\text{Laplacian} = \qquad \big(Z(t,x+i) + Z(t,x-i) - 2Z(t,x)\big)$$

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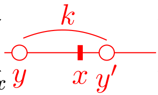
where $\mathcal{L} := \sum_{i=1}^{3}(\frac{\gamma\kappa_i}{2} + O(\varepsilon))\mathcal{L}_i,\ \mathcal{Q} := \sum_{k=1}^{3} b_k \sum_{(y,y')\ni x}\Big($ 

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- Wish to match $(\varepsilon\mathcal{L} + \varepsilon^2\mathcal{Q})$ to

$$\text{Laplacian} = \tfrac{1}{2}\sum_{i=1}^{3}\widetilde{\kappa}_i(Z(t,x+i) + Z(t,x-i) - 2Z(t,x))$$

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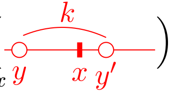
where $\mathcal{L} := \sum_{i=1}^{3}\big(\frac{\gamma\kappa_i}{2} + O(\varepsilon)\big)\mathcal{L}_i$, $\mathcal{Q} := \sum_{k=1}^{3} b_k \sum_{(y,y')\ni x}\Big($  $\Big)$.

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$$\text{Laplacian} = \frac{1}{2}\sum_{i=1}^{3}\widetilde{\kappa}_i\Big(\frac{Z(t,x+i)}{Z(t,x)} + \frac{Z(t,x-i)}{Z(t,x)} - 2\Big)Z(t,x)$$

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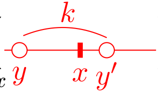
where $\mathcal{L} := \sum_{i=1}^{3}(\frac{\gamma\kappa_i}{2} + O(\varepsilon))\mathcal{L}_i$, $\mathcal{Q} := \sum_{k=1}^{3} b_k \sum_{(y,y')\ni x}\Big($ 

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$$\frac{Z(t,x+i)}{Z(t,x)} = \exp\Big(-\varepsilon\sum_{x<y<x+i}\eta(t,y)\Big)$$

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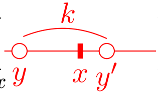
where $\mathcal{L} := \sum_{i=1}^{3}\big(\frac{\gamma\kappa_i}{2} + O(\varepsilon)\big)\mathcal{L}_i$, $\mathcal{Q} := \sum_{k=1}^{3} b_k \sum_{(y,y')\ni x}\Big($ 

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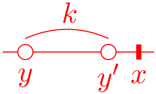
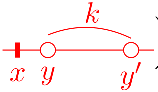
$$\text{Laplacian} = \sum_{i=1}^{3}\varepsilon\frac{\gamma\widetilde{\kappa}_i}{2}\mathcal{L}_i + \sum_{k=1}^{2}\varepsilon^2\widetilde{b}_k\Big(\sum_{y,y'\in(x-3,x)}$$ 

$$+ \sum_{y,y'\in(x,x+3)}$$ 

$$\Big) + \varepsilon^3\text{cubic} + \ldots$$

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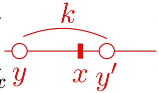
where $\mathcal{L} := \sum_{i=1}^{3}\big(\frac{\gamma\kappa_i}{2} + O(\varepsilon)\big)\mathcal{L}_i$, $\mathcal{Q} := \sum_{k=1}^{3} b_k \sum_{(y,y')\ni x}\Big($ 

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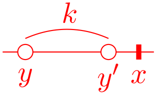

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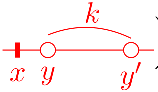

$$\Big) + \varepsilon^3\text{cubic} + \ldots$$

- Match linear terms by choosing $\widetilde{\kappa}_i = \kappa_i + O(\varepsilon)$.

Gradient Condition

- How to match quadratic terms?

Gradient Condition

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Definition

We call $f(x)$ a **gradient term** if

$$f(x) = \sum_{|j| \leq 3} c_j (g_j(x + j) - g_j(x)),$$

Gradient Condition

- How to match quadratic terms?

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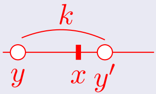
$$f(x) = \sum_{|j| \leq 3} c_j (g_j(x+j) - g_j(x)),$$

for some $g_j(x)$ of the form $g_j(x) = \eta(x+\ell_1) \cdots \eta(x+\ell_n) Z(t,x)$.

Gradient Condition

- How to match quadratic terms?

Proposition



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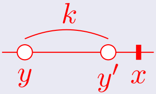
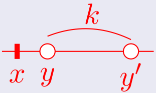


+ *grad. term*

=



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Gradient Condition

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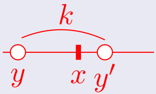
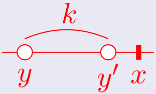


$=$


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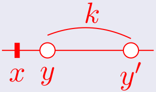


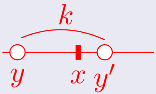
+ grad. term.

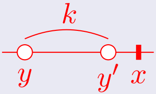
- Matching the quadratic terms amounts to matching the coefficients: $kb_k = 2(3-k)\widetilde{b}_k$, which gives the condition $(\star)$.

Gradient Condition

- How to match quadratic terms?

Proposition



$=$ 

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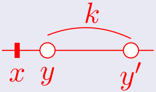
$=$ 

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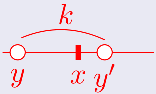
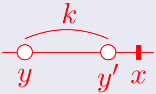
- Cubic terms: use $m \leq 3$.

Gradient Condition

- How to match quadratic terms?

Proposition



$=$ 

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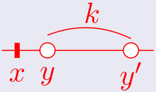
$=$ 

+ grad. term.

- Matching the quadratic terms amounts to matching the coefficients: $kb_k = 2(3-k)\widetilde{b}_k$, which gives the condition $(\star)$.
- Cubic terms: use $m \leq 3$.
- higher order terms: hydrodynamic limit argument.