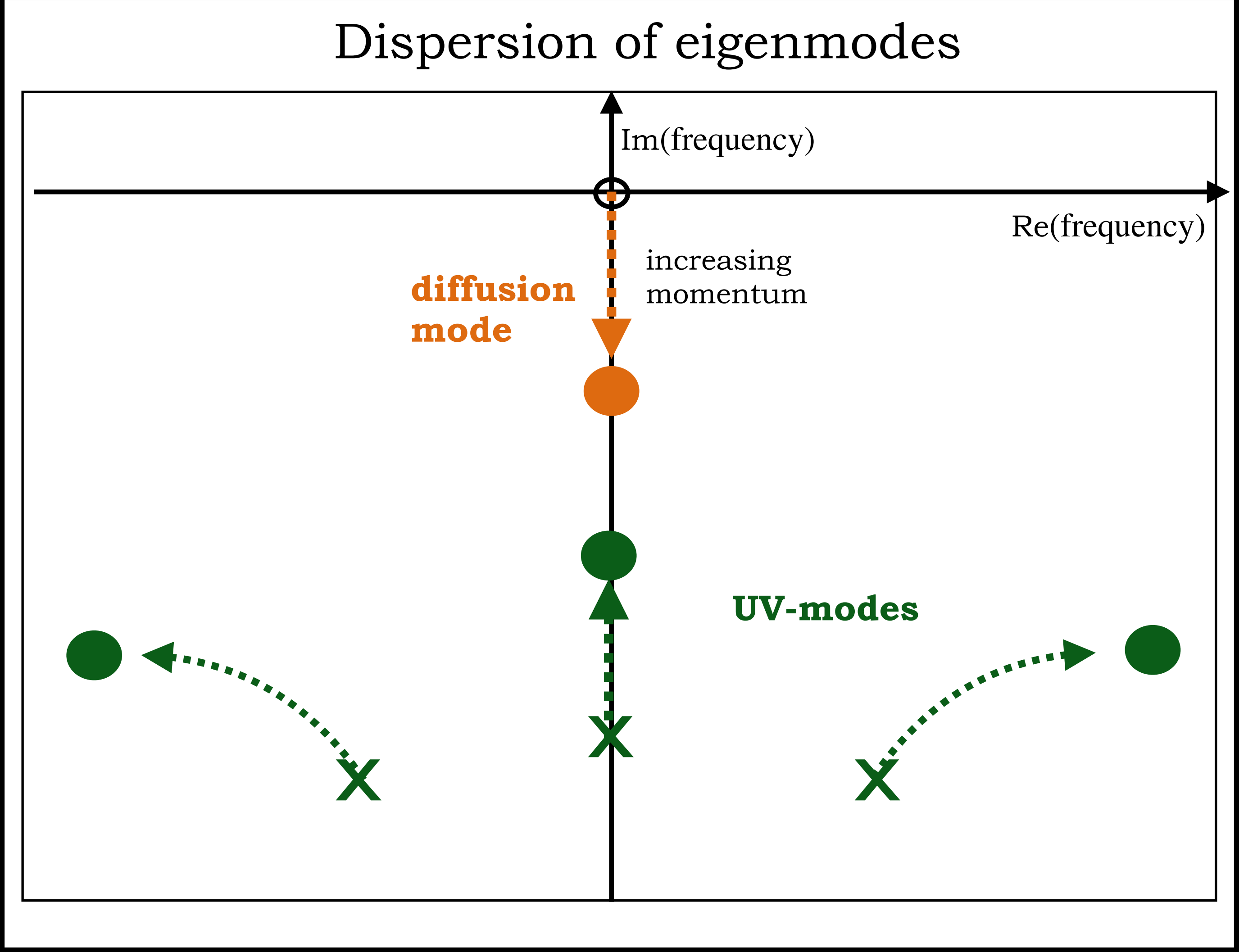
Ultraviolet-regulated theory of non-linear diffusion

KITP, Santa Barbara

July 5th, 2023



[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

Matthias Kaminski
University of Alabama

U.S. DEPARTMENT OF ENERGY

Objective

How do nonlinear quantum fluctuations and statistical fluctuations modify diffusion when taking into account the slowest UV-mode ($\tau \neq 0$)?

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

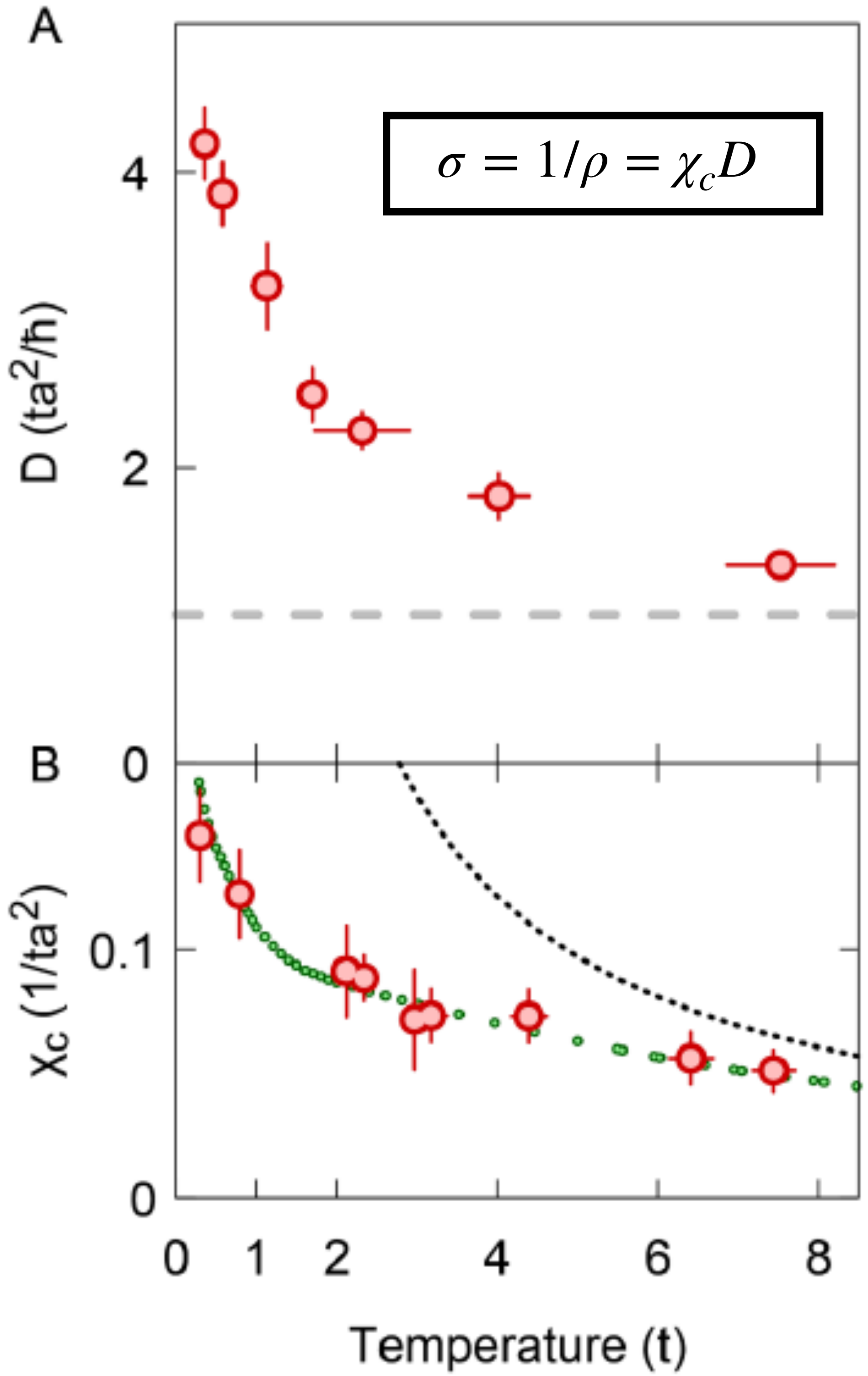
$\tau = 0$: Long time tails? *[Kovtun, Yaffe; PRD (2003)]*

Transport renormalized? *[Kovtun, Moore, Romatschke; PRD (2011)]*

Motivation

Ultracold atom measurements: *Bad metallic transport in a cold atom Fermi-Hubbard system*

[Brown et al.; Science (2018)]



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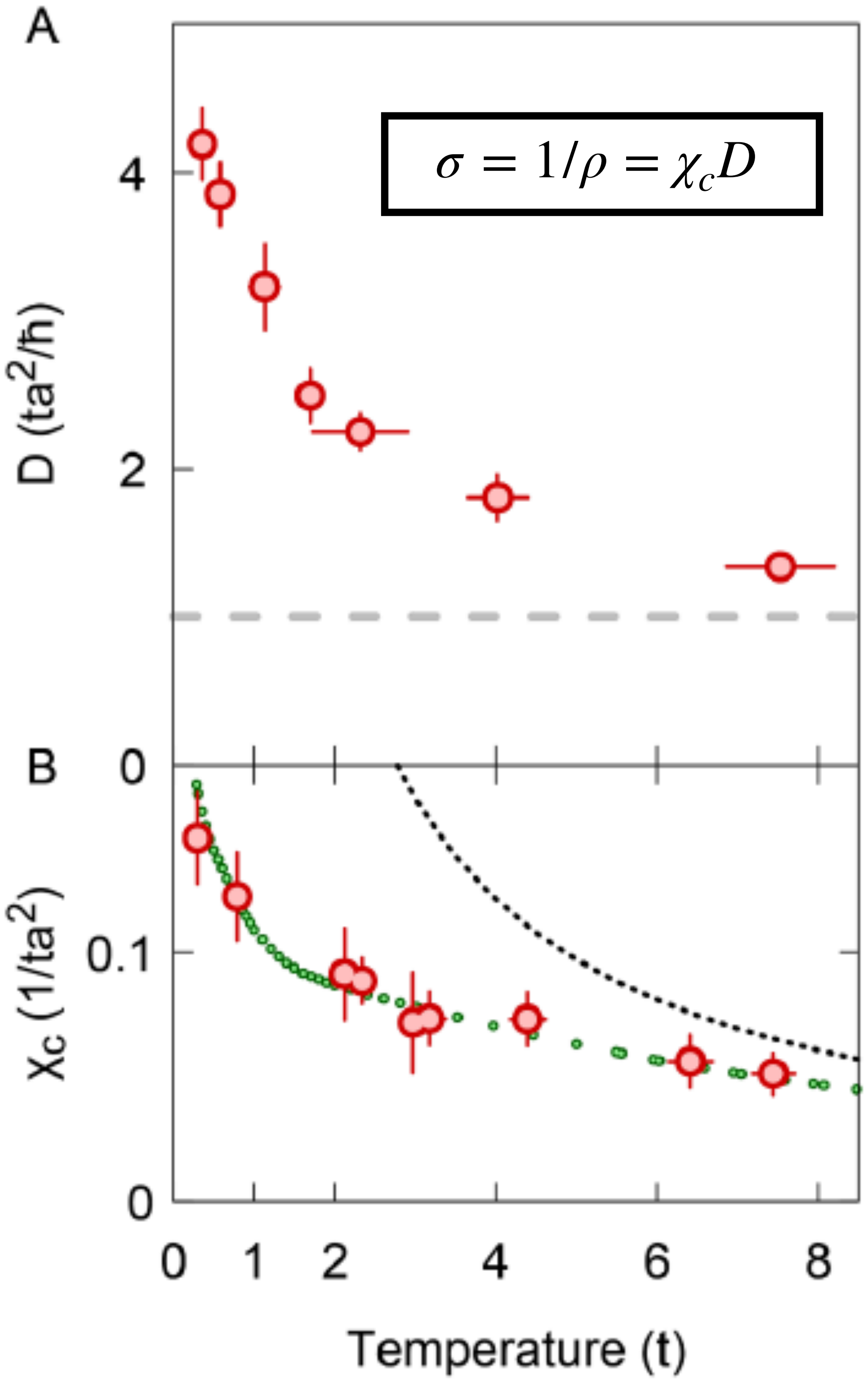

Diffusion coefficient modified by quantum-statistical fluctuations (e.g. near critical points)

[Chen-Lin, Delacrétaz, Hartnoll; PRL (2019)]

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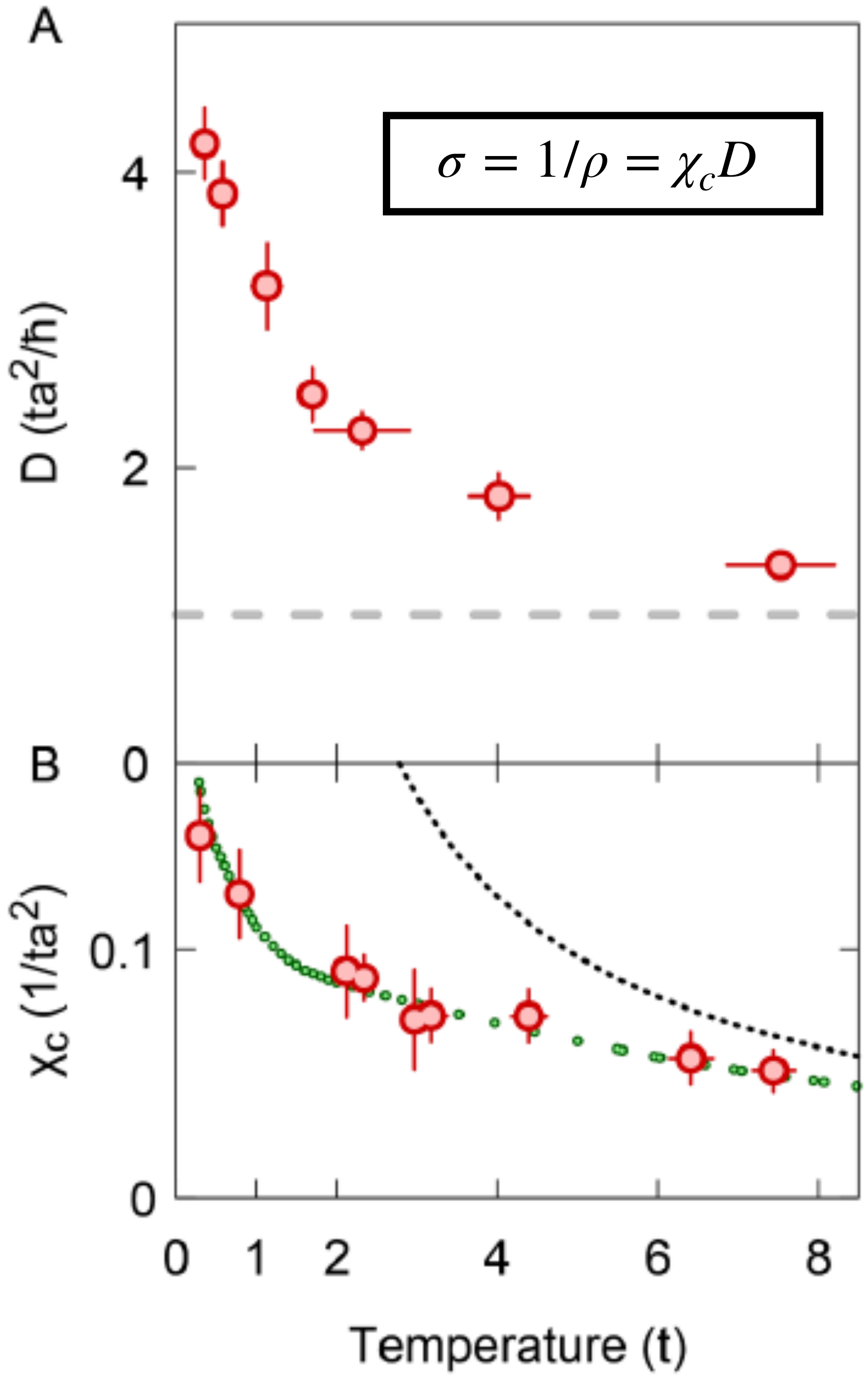

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Hydrodynamics as an effective field theory

[Jensen, Kaminski, Kovtun,Meyer,Ritz,Yarom.; PRL (2012)]
[Banerjee et al. JHEP (2012)]
[Crossley, Glorioso, Liu; JHEP (2017)]
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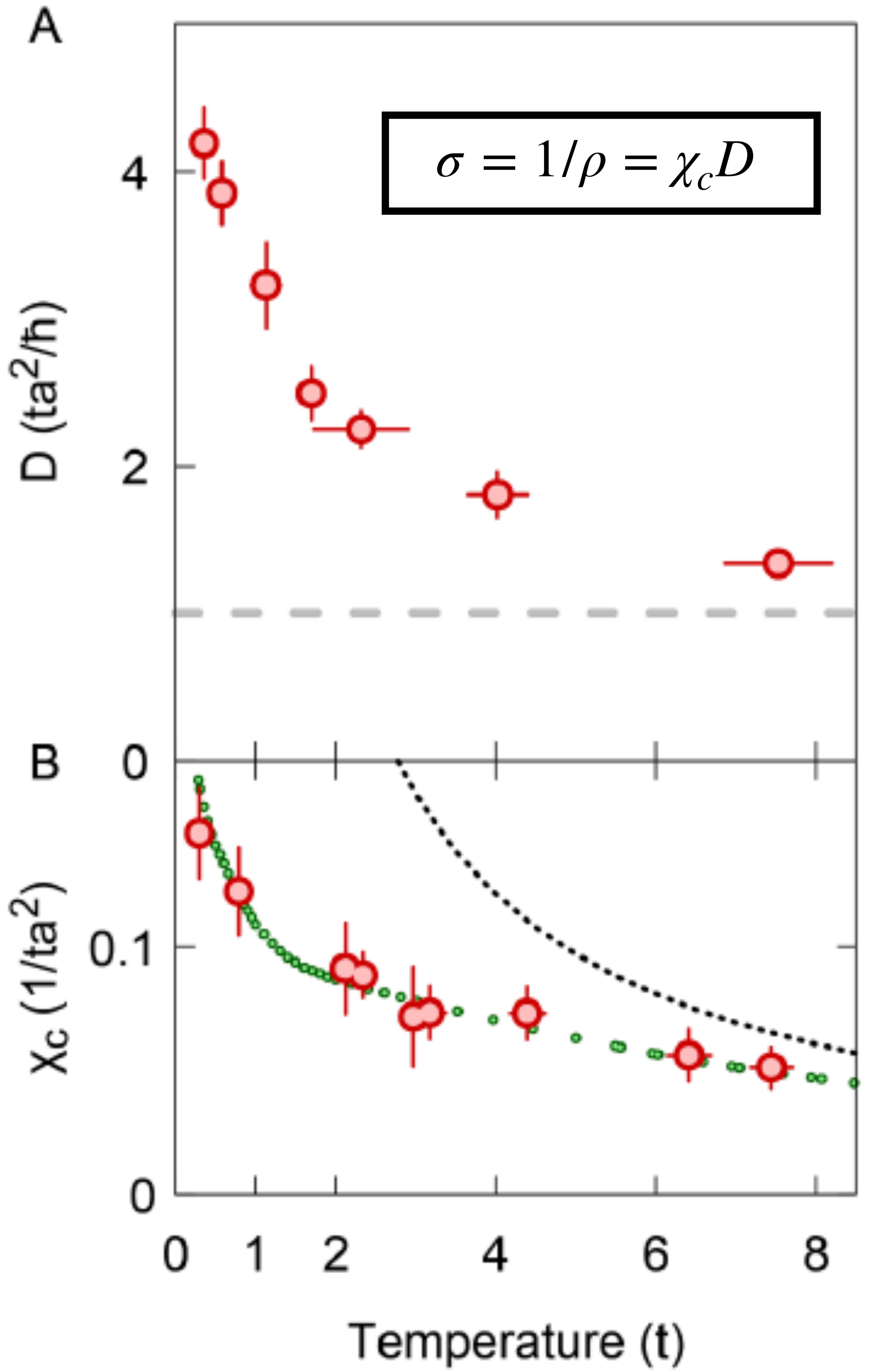

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BUT: hydrodynamics needs to be regulated by UV-mode(s) to be *causal and stable*
(e.g. Mueller-Israel-Stewart theory, BDNK, ...)

[Hiscock & Lindblom; PRD (1985)]
[Bemfica, Disconzi, Noronha; PRD (2018)] *[PRX (2022)]*
[Hoult,Kovtun; JHEP (2020)] *[Kovtun; JHEP (2019)]*
...
[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

Method

linear diffusion

linearized in hydrodynamic fields

(e.g., energy density ϵ ~ temperature T)

nonlinear diffusion via effective action from exponentiated e.o.m. Martin-Siggia-Rose formalism (MSR)

write stochastic differential equations as a field theory formulated using path integrals

[Martin, Siggia, Rose; PRA (1973)]

nonlinear diffusion via effective action via Schwinger-Keldysh formalism (SK)

effective field theory for dissipative hydrodynamics

[Crossley, Glorioso, Liu; JHEP (2017)]

MSR is used here.

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

adding one regulating UV-mode

charge diffusion

SK was used in

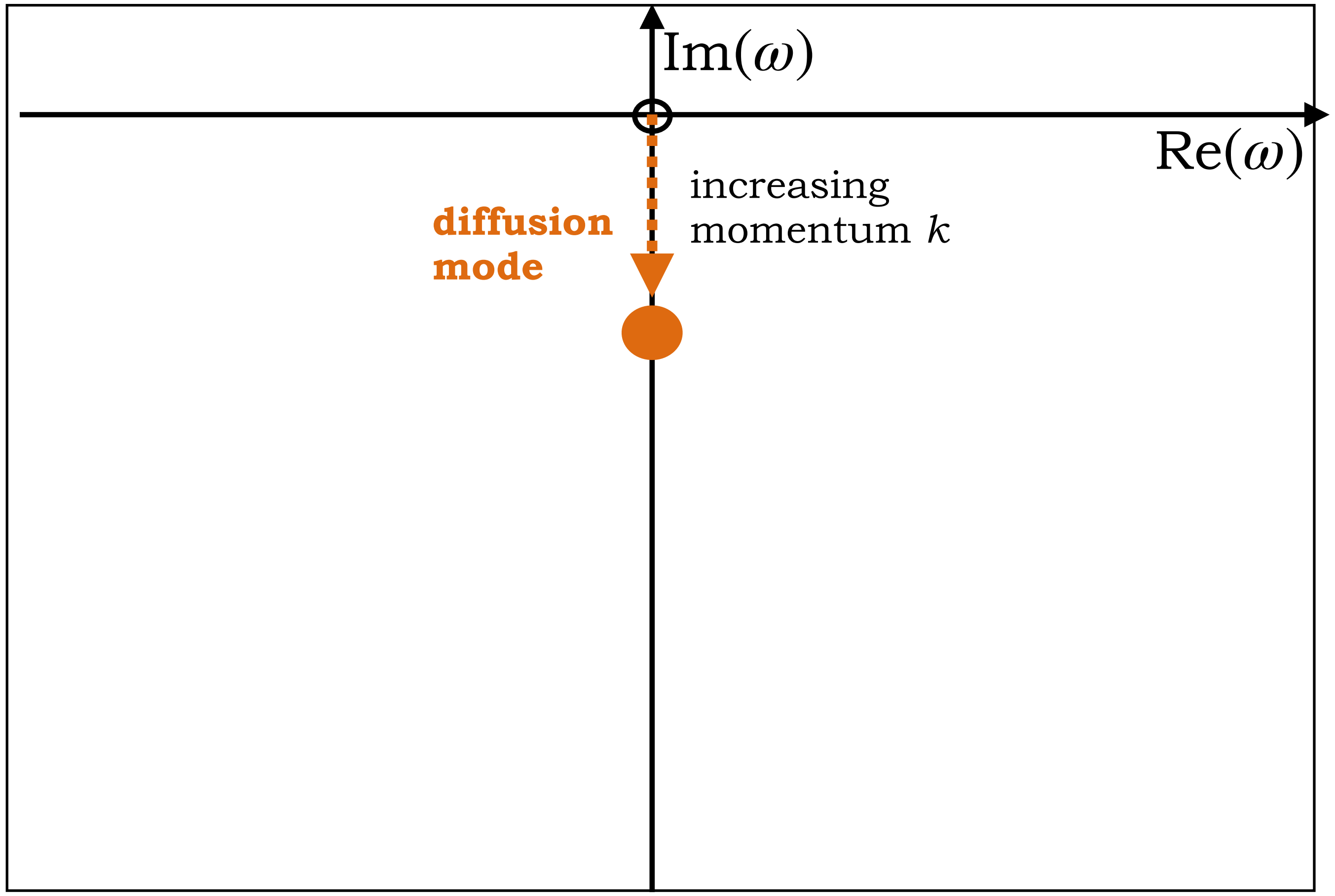
[Chen-Lin, Delacrétaz, Hartnoll; PRL (2019)]

no regulating UV-mode

heat diffusion

Method

Dispersion of eigenmodes in complex frequency plane



Consider one conserved charge n

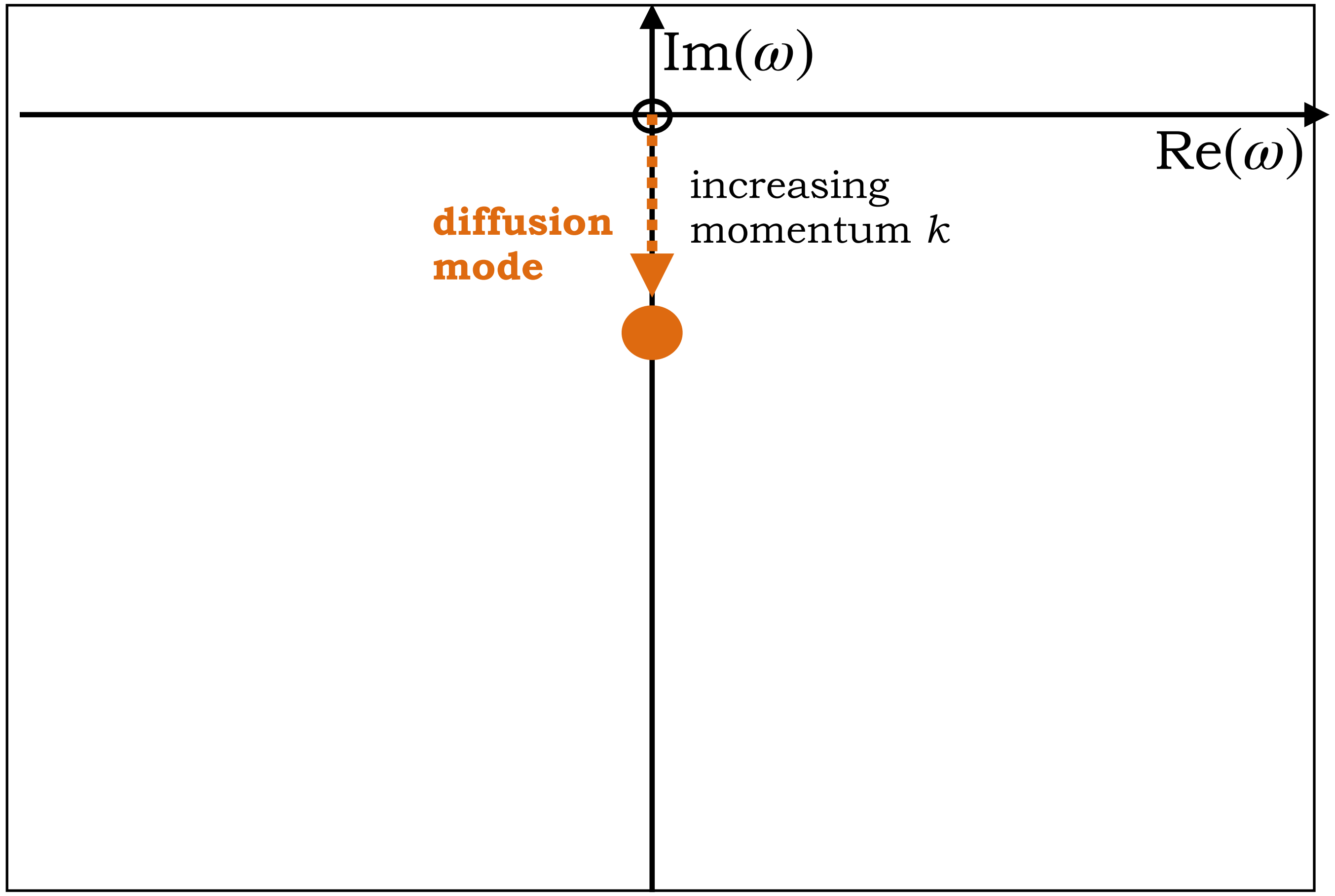
$$\partial_t n + \mathbf{\nabla} \cdot \mathbf{J} = 0\,, \qquad \mathbf{J} + D\,\mathbf{\nabla} n = 0$$

Fick's law of diffusion:

$$\partial_t n - D\mathbf{\nabla}^2 n = 0$$

Method

Dispersion of eigenmodes in complex frequency plane



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Fick's law of diffusion:

$$\partial_t n - D\nabla^2 n = 0$$

Fourier transform $n(t,x) \propto e^{-i\omega t + ikx}\, n(\omega, k)$ to read off eigen-frequency:

$$\omega = -\,iDk^2$$

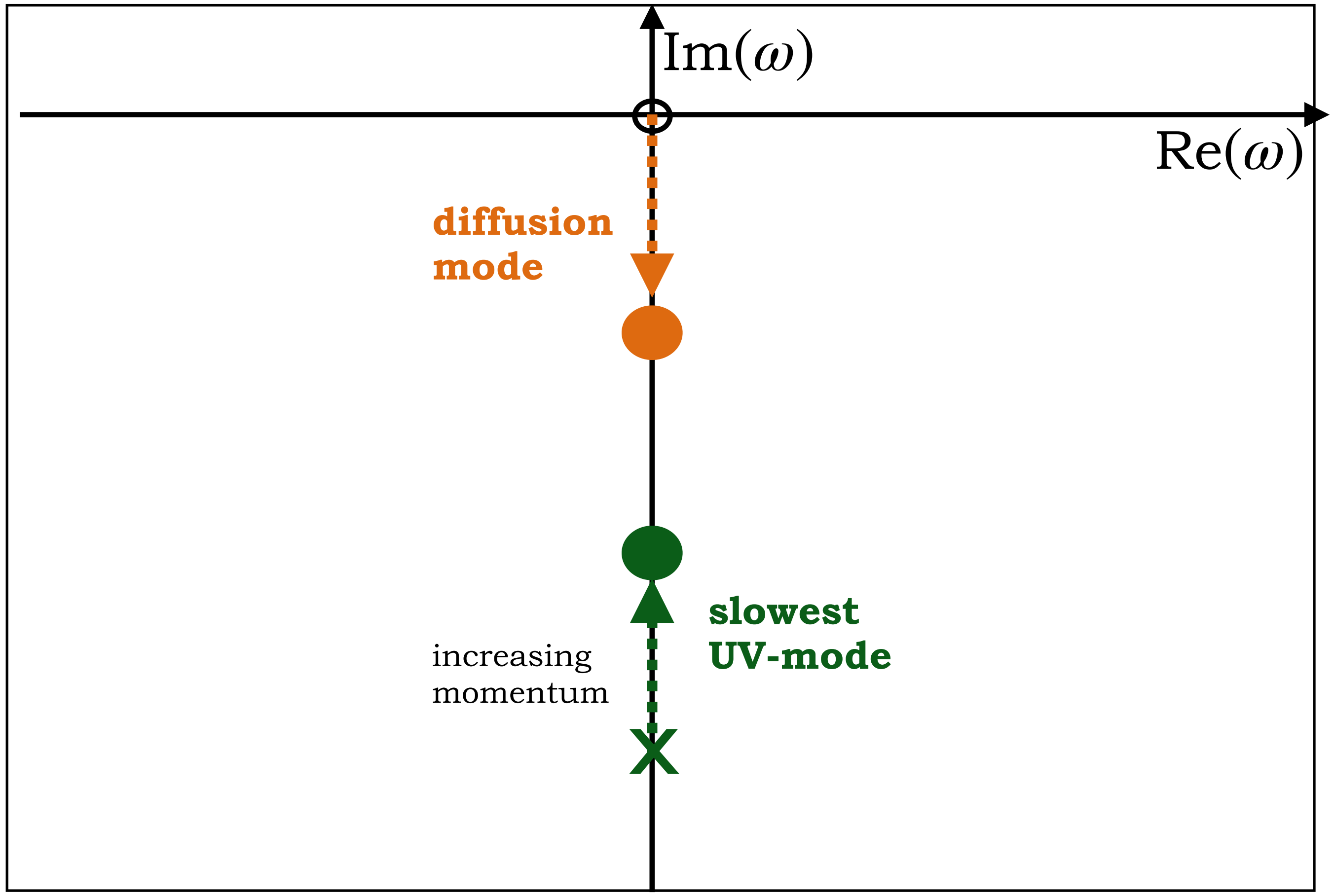
diffusion mode

Differential equation turned into algebraic equation by relations like $\partial_t\, e^{-i\omega t} = -i\omega\, e^{-i\omega t}$

Method

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

Dispersion of eigenmodes in complex frequency plane



Consider one conserved charge n and ***relaxation time*** τ:

$$\partial_t n + \nabla \cdot \mathbf{J} = 0\,, \quad \boxed{\tau\, \partial_t \mathbf{J}} + \mathbf{J} + D\,\nabla n = 0$$

Fick's law of diffusion ***(UV-regulated)***:

$$\boxed{\tau \partial_t^2 n} + \partial_t n - D\nabla^2 n = 0\,.$$

This was used to analyze experiment.

[Brown et al.; Science (2018)]

Method

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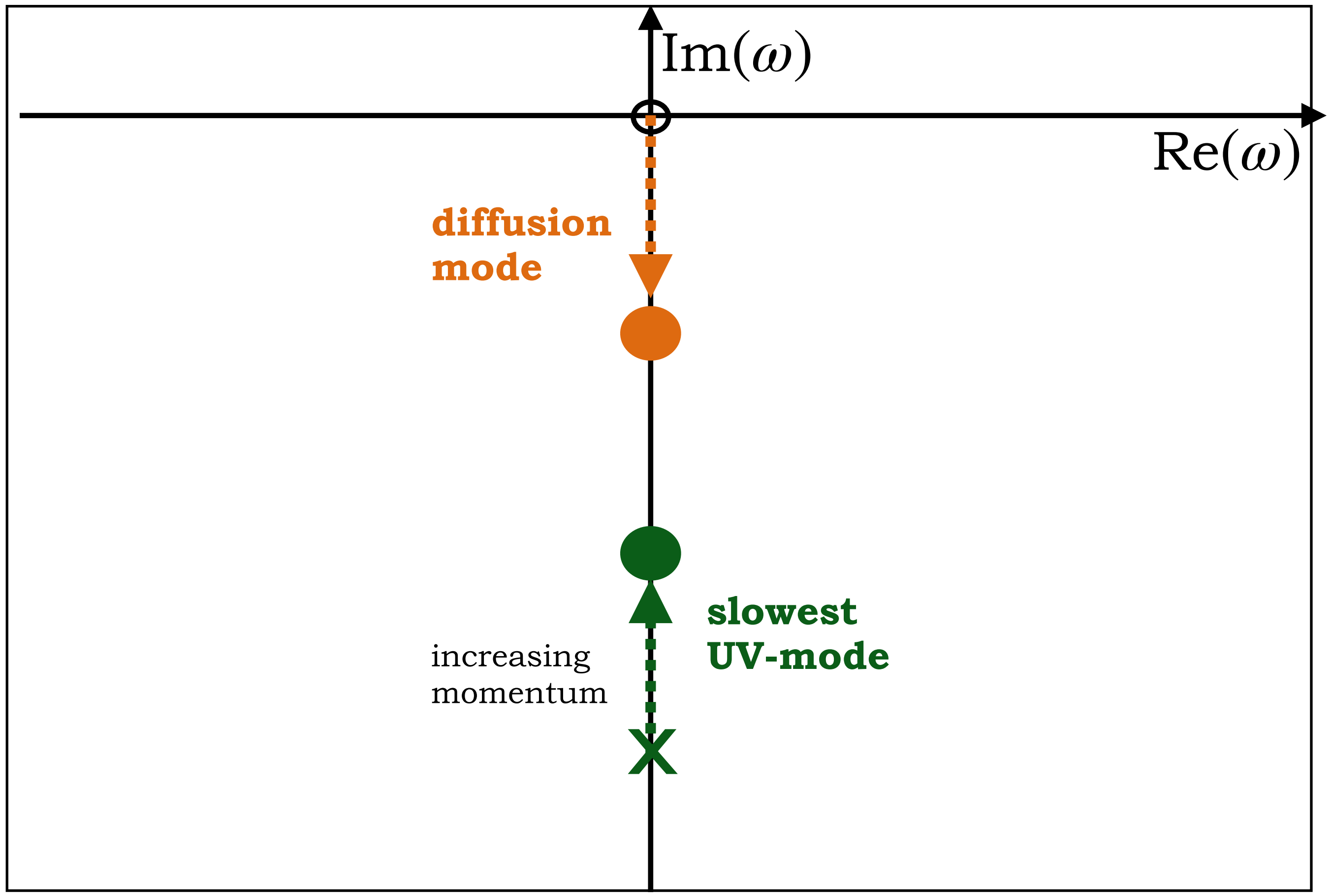
Dispersion of eigenmodes in complex frequency plane



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Fick's law of diffusion ***(UV-regulated)***:

$$\boxed{\tau \partial_t^2 n} + \partial_t n - D \nabla^2 n = 0\,.$$

Fourier transform to read off eigen-frequencies:

$$\boxed{\omega_{1,2} = -\frac{i}{2\tau}\left(1 \mp \sqrt{1 - 4\tau D \mathbf{k}^2}\right)}$$

diffusion mode and slowest UV-mode

This was used to analyze experiment.

[Brown et al.; Science (2018)]

e.g., Mueller-Israel-Stewart theory, one may also think of this as Hydro+

concise summary in my subsection V. B of white paper [Sorensen et al.; arXiv:2301.13253]

Method

Dense Nuclear Matter Equation of State from Heavy-Ion Collisions

Agnieszka Sorensen
Institute for Nuclear Theory, University of Washington, Seattle, WA 98195, USA

Kshitij Agarwal
Physikalisches Institut, Eberhard Karls Universität Tübingen, D-72076 Tübingen, Germany

Kyle W. Brown
Facility for Rare Isotope Beams, Michigan State University, East Lansing, MI 48824, USA and Department of Chemistry, Michigan State University, East Lansing, MI 48824, USA

Zbigniew Chajecki
Department of Physics, Western Michigan University, Kalamazoo, MI, 49008, USA

Paweł Danielewicz, William G. Lynch, Scott Pratt, and ManYee Betty Tsang
Department of Physics and Astronomy and Facility for Rare Isotope Beams Michigan State University, East Lansing, MI 48824 USA

Christian Drischler
Department of Physics and Astronomy and Institute of Nuclear and Particle Physics, Ohio University, Athens, OH 45701, USA

Stefano Gandolfi and Ingo Tews
Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

Jeremy W. Holt and Che-Ming Ko
Department of Physics and Astronomy and Cyclotron Institute, Texas A&M University, College Station, TX 77843, USA

Matthias Kaminski
Department of Physics and Astronomy, University of Alabama, Tuscaloosa, AL 35487, USA

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

Consider one conserved charge n and ***relaxation time*** τ:

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Fick's law of diffusion ***(UV-regulated)***:

$$\boxed{\tau \partial_t^2 n} + \partial_t n - D \nabla^2 n = 0$$

Fourier transform to read off eigen-frequencies:

$$\ldots \; i \; \left(1 \mp \sqrt{1 - 4\tau D k^2}\right)$$

e.g., Mueller-Israel-Stewart theory, one may also think of this as Hydro+

concise summary in my subsection V. B of white paper [Sorensen et al.; arXiv:2301.13253]

Method: Martin-Siggia-Rose

nonlinear diffusion via effective action from exponentiated e.o.m. Martin-Siggia-Rose formalism (MSR)

write stochastic differential equations as a field theory formulated using path integrals

[Martin, Siggia, Rose; PRA (1973)]

Idea:

$$\langle \mathcal{O} \rangle \sim e^{e.o.m.}$$

Stochastic differential equation (e.o.m.):

$$\partial_t x(t) = F(x(t),t) + \xi(x(t),t),$$

Noise correlation:

$$\langle \xi(x,t)\xi(x',t') \rangle = G(x,t,x',t').$$

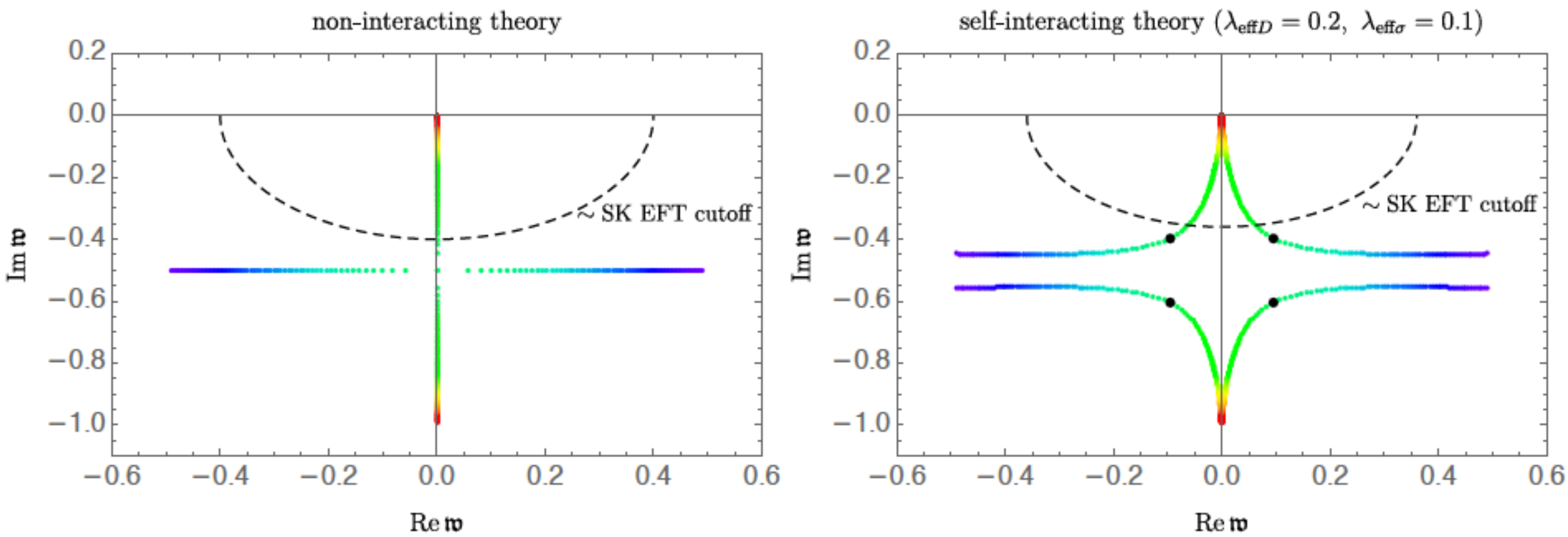
Observables averaged over solutions of this stochastic differential equation may be written:

$$\langle \mathcal{O}[x(t)] \rangle = \int \mathcal{D}[x,\tilde{x}]\, \mathcal{O}[x(t)]\, e^{-S[x,\tilde{x}]},$$

$$S[x,\tilde{x}] = \int_t i\tilde{x}(t)\,[\partial_t x(t) - F(x(t),t)] + \frac{1}{2}\int_{t,t'} G(x(t),t,x(t'),t')\tilde{x}(t)\tilde{x}(t').$$

Results: Spectrum of eigen-frequencies

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]



➡ larger range of applicability than SK without UV mode

Method

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

Self-interactions enter through dependence of diffusion coefficient on charge fluctuations:

$$D(n) = D + \lambda_D n + \frac{\lambda_D'}{2} n^2$$

Note: corrections to $\tau(n) = \tau + \lambda_{\tau,1} n + \lambda_{\tau,2} n^2 + \ldots$ **contribute to higher order only**

Method

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Leading to the nonlinear equation of motion:

$$\tau \partial_t^2 n + \partial_t n - \nabla^2 \left(D\, n + \frac{\lambda_D}{2} n^2 + \frac{\lambda'_D}{6} n^3 \right) = 0$$

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Exponentiate stochastic version of this equation to obtain path integral, from which the effective action can be read: *[Martin, Siggia, Rose; PRA (1973)]*

$$\mathcal{L} = i T \sigma\, \nabla n_a\, C\, \nabla n_a - n_a \left(\tau \partial_t^2 n + \partial_t n - D \nabla^2 n \right)$$

$$+ i T \chi \lambda_\sigma\, n \nabla n_a C \nabla n_a + \frac{\lambda_D}{2} \nabla^2 n_a\, n^2 + \frac{1}{2} i T \chi \lambda'_\sigma\, n^2 \nabla n_a C \nabla n_a + \frac{\lambda'_D}{6} \nabla^2 n_a\, n^3$$

with conductivity $\sigma(n) = \sigma + \chi \lambda_\sigma \delta n + \frac{1}{2} \chi \lambda'_\sigma \delta n^2$ and $C = \left(\frac{i\partial_t}{2T}\right) \coth\left(\frac{i\partial_t}{2T}\right)$

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Perform perturbation theory computation to one-loop order, like done in particle physics (e.g. QED).

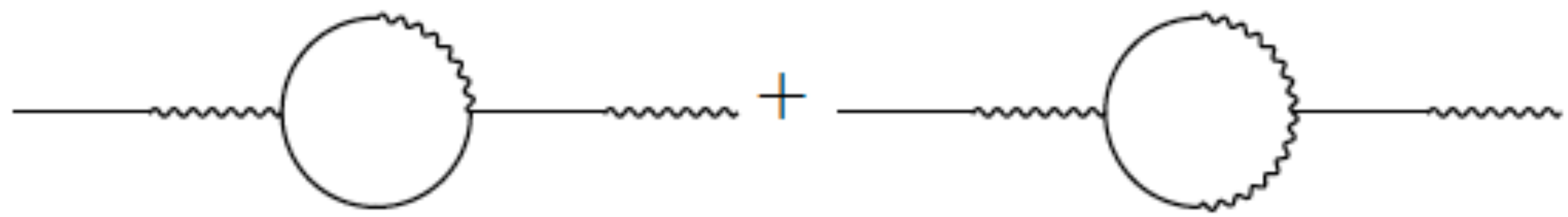
Method

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

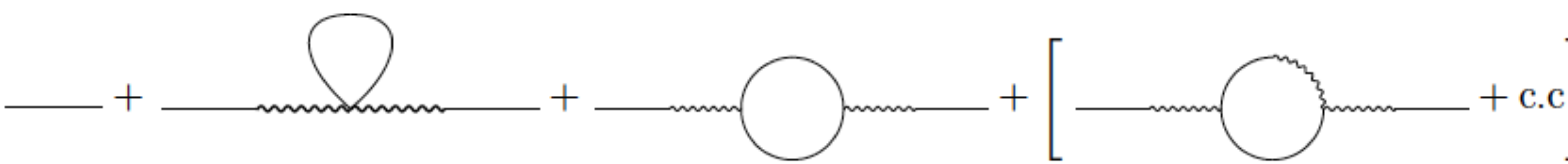
Perform perturbation theory computation to one-loop order, like done in particle physics (e.g. QED).

$$G_{nn_a}(p) = G^{(0)}_{nn_a}(p) + G^{(0)}_{nn_a}(p)(-\Sigma(p))G^{(0)}_{nn_a}(p) = \frac{1}{\omega + iD_0\mathbf{k}^2 - i\tau\,\omega^2 + \Sigma(\omega,\mathbf{k})}$$

$$G^{(0)}_{nn_a}\Sigma(p)G^{(0)}_{nn_a} =$$



$$G^{(0)}_{nn_a}(p)(-C(p))G^{(0)}_{n_a n}(p) =$$

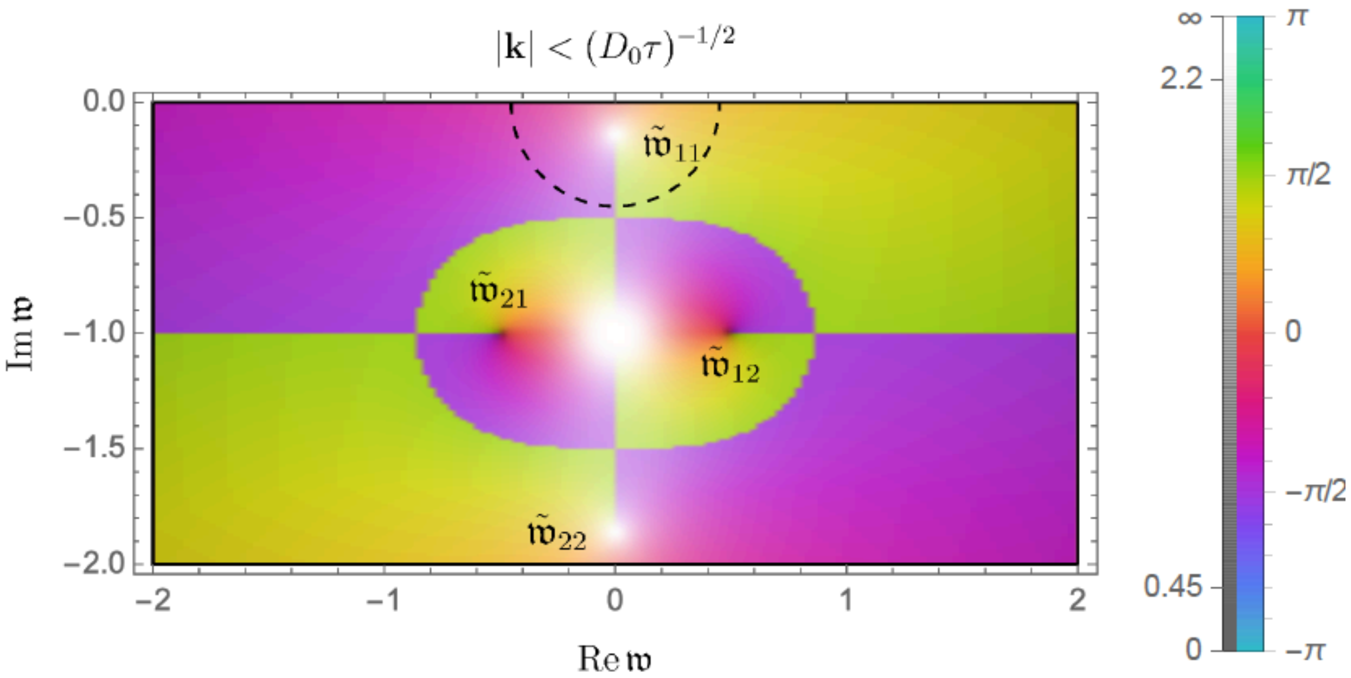


Results: charge correlation function

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

Charge correlator to one-loop order:

$$G^R_{nn}(\omega,\mathbf{k}) = \frac{i\left(\sigma+\delta\sigma(\omega,\mathbf{k})\right)\mathbf{k}^2}{-i\tau\omega^2+\omega+iD\mathbf{k}^2+\Sigma(\omega,\mathbf{k})}$$



$\alpha_1 = |\alpha_1| e^{i\varphi}$

Branch point singularities:

$$\tilde{\omega}_{11}\ \&\ \tilde{\omega}_{22} = -\frac{i}{\tau}\left(1\mp\sqrt{1-D\mathbf{k}^2\tau}\right), \qquad \tilde{\omega}_{12}\ \&\ \tilde{\omega}_{21} = -\frac{i}{\tau}\pm|\mathbf{k}|\sqrt{\frac{D}{\tau}}$$

$$\Sigma_d(\omega,\mathbf{k}) = \alpha_d(\omega,\mathbf{k})\,(\tau D)^{\frac{2-d}{2}}\frac{T\chi}{D^2}\mathbf{k}^2\left[f_{1d}(\omega,\mathbf{k})\lambda_D^2+f_{2d}(\omega,\mathbf{k})\lambda_D\lambda_\sigma\right]$$

Non-analyticities:

$$\alpha_1(\omega,\mathbf{k}) = \frac{1}{16}\left(\frac{(1-i\tau\omega)^2(D\mathbf{k}^2\tau-i\omega\tau(2-i\tau\omega))}{D\mathbf{k}^2\tau+(1-i\tau\omega)^2}\right)^{-1/2}, \qquad (d=1)$$

$$\alpha_2(\omega,\mathbf{k}) = -\frac{1}{64\pi}\log\left(\frac{(1-i\tau\omega)^2(D\mathbf{k}^2\tau-i\omega\tau(2-i\tau\omega))}{D\mathbf{k}^2\tau+(1-i\tau\omega)^2}\right), \qquad (d=2)$$

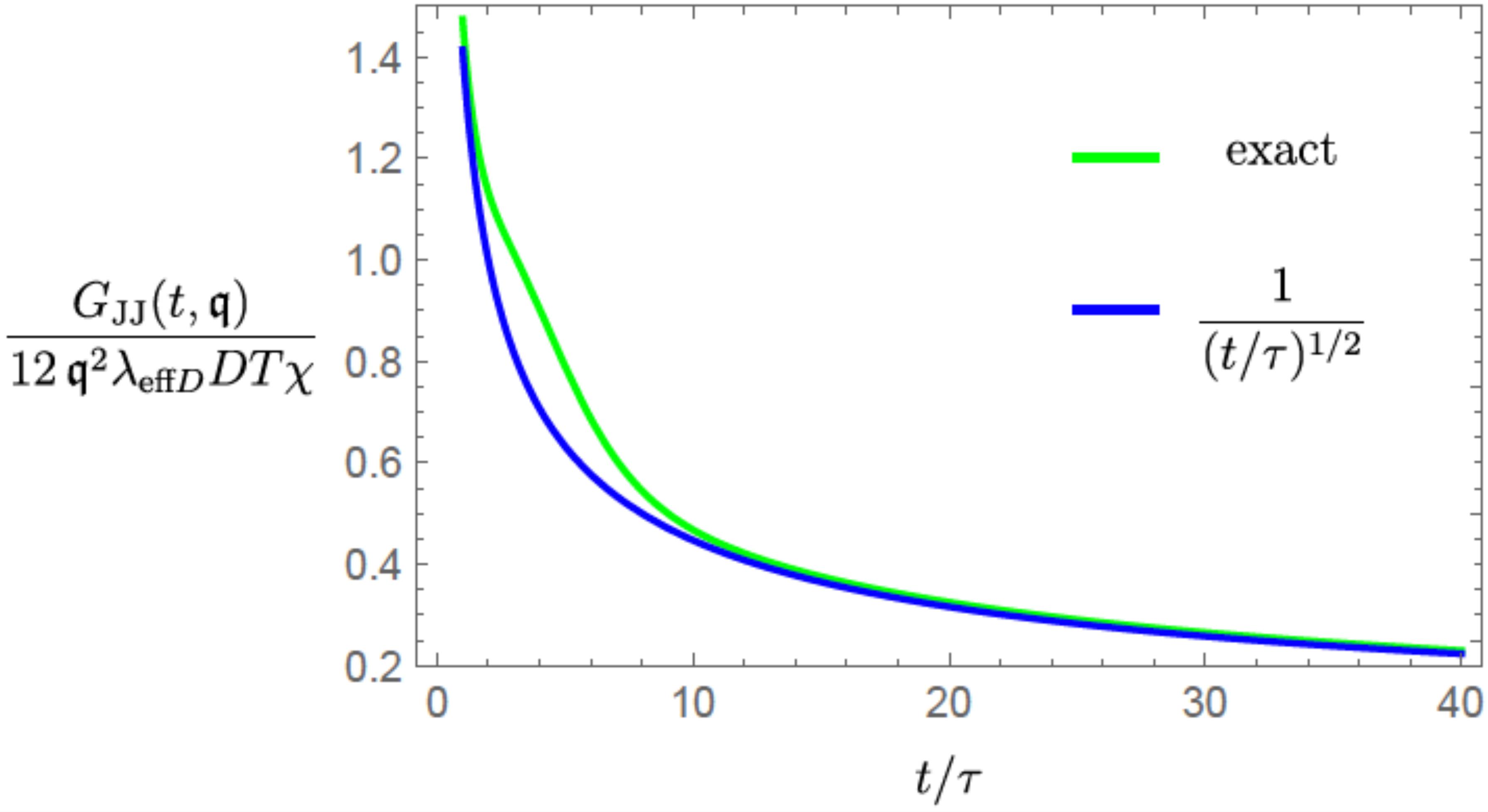
$$\alpha_3(\omega,\mathbf{k}) = -\frac{1}{128\pi}\left(\frac{(1-i\tau\omega)^2(D\mathbf{k}^2\tau-i\omega\tau(2-i\tau\omega))}{D\mathbf{k}^2\tau+(1-i\tau\omega)^2}\right)^{1/2}, \qquad (d=3)$$

Results: conductivity correction & current correlator

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

$$\frac{\delta\sigma_d(\omega,\mathbf{k})}{\sigma} = -\alpha_d(\omega,\mathbf{k})\,(\tau D)^{\frac{2-d}{2}}\,\frac{2T\chi}{D^2}\,\mathbf{k}^2\left[f_{3d}(\omega,\mathbf{k})\lambda_D^2 + f_{4d}(\omega,\mathbf{k})\lambda_D\lambda_\sigma\right]$$

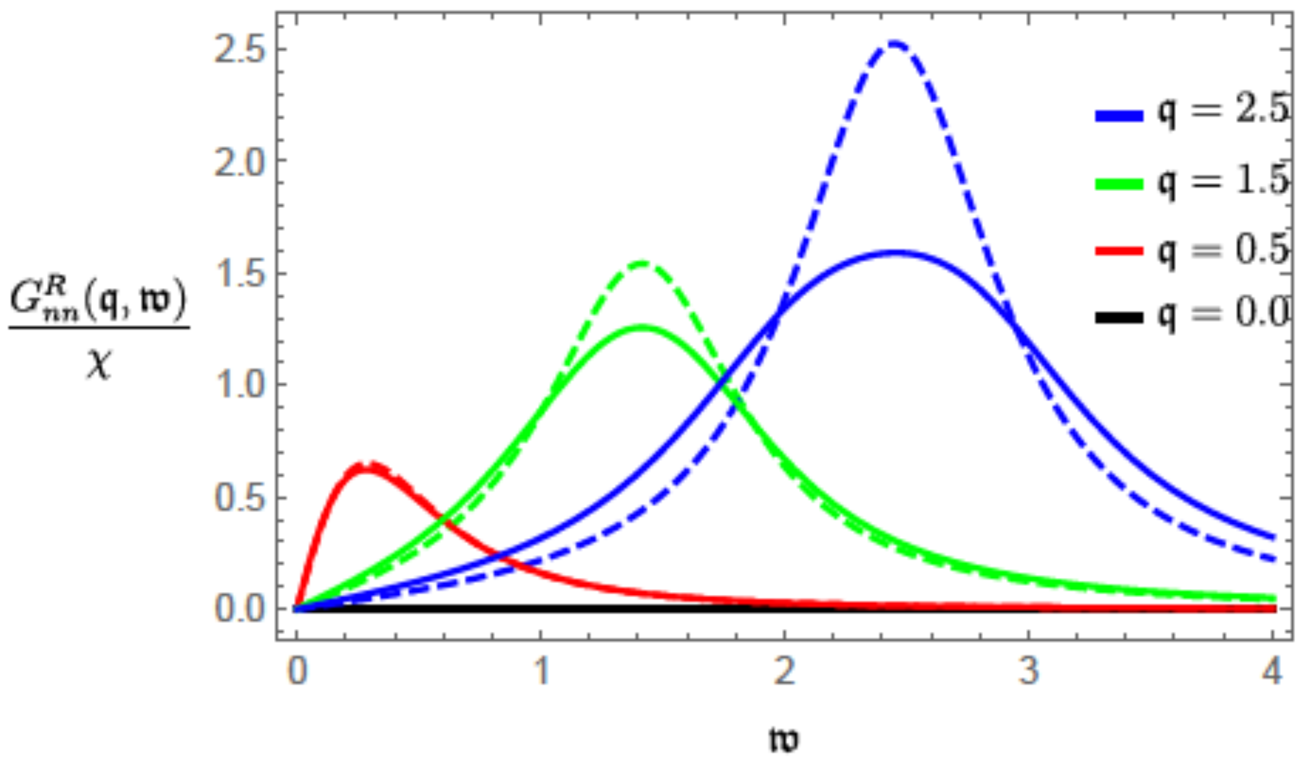
equivalent to result from [Chen-Lin, Delacrétaz, Hartnoll; PRL (2022)]

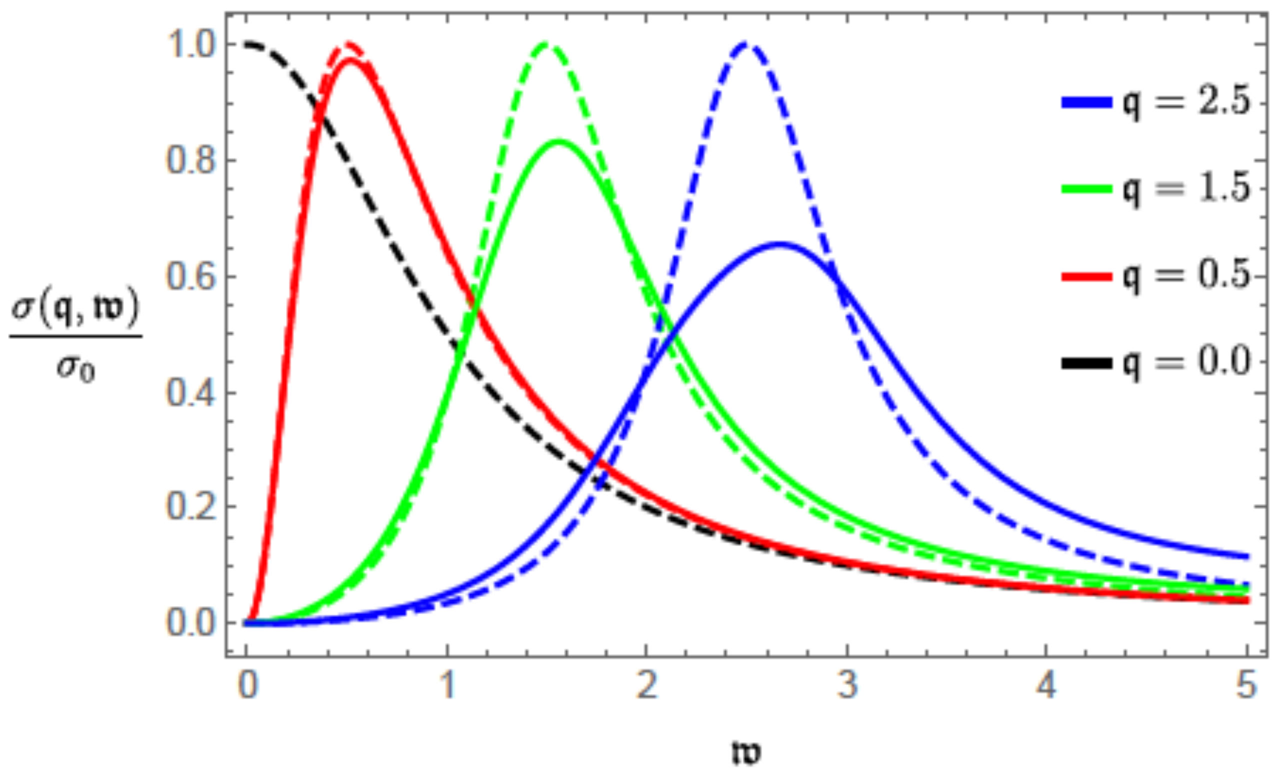


➡ long time tail: power law,
➡UV mode inconsequential for conductivity correction

Reduction of susceptiblity & conductivity peaks

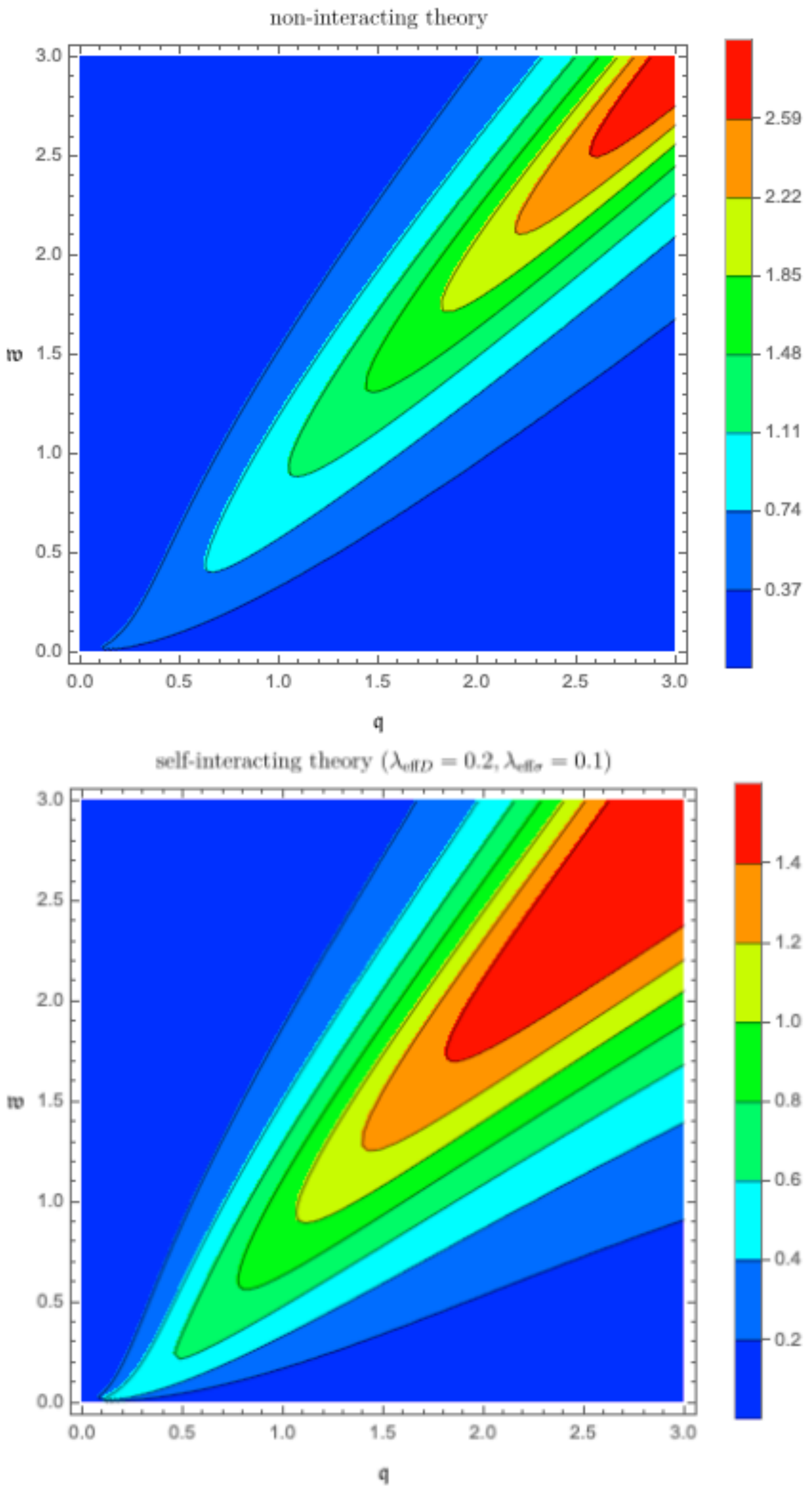
[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]





Dynamic susceptiblity

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]



dynamic susceptibility G^R_{nn}/χ

Method: Effective formalism for hydrodynamic fluctuations

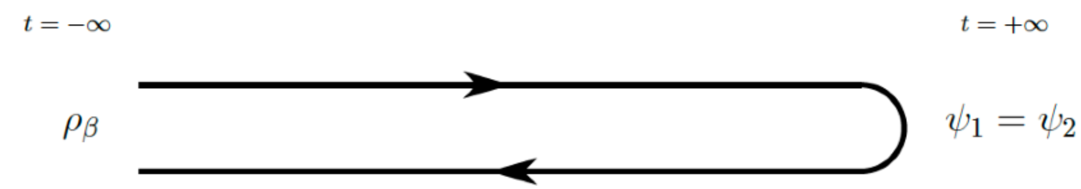
Supplemental Material of [Chen-Lin, Delacrétaz, Hartnoll; PRL (2019)]

Goal is to compute correlator:

$$\langle \varepsilon(t,x)\varepsilon(t',x')\cdots\rangle_\beta \equiv \mathrm{Tr}\Big(\rho_\beta\,\varepsilon(t,x)\varepsilon(t',x')\cdots\Big) \qquad \rho_\beta = e^{-\beta H}/\,\mathrm{Tr}\,e^{-\beta H}$$

Generating functional:

$$Z[A^1_\mu, A^2_\mu,] \equiv \mathrm{Tr}\left(U[A_1]\rho_\beta U[A_2]^\dagger\right) \qquad Z[A^1_\mu, A^2_\mu] = \int D\psi_1 D\psi_2\, e^{iS[\psi_1,A_1]-iS[\psi_2,A_2]}$$



Constraints on effective action:

$$Z[A_\mu, A_\mu] = 1\,,$$

reflection: $\quad Z[A^1_\mu, A^2_\mu] = Z^*[A^2_\mu, A^1_\mu]\,,$

gauge invariance: $\quad Z[A^1_\mu, A^2_\mu] = Z[A^1_\mu + \partial_\mu\lambda^1, A^2_\mu + \partial_\mu\lambda^2]\,,$

KMS condition: $\quad Z[A^1_\mu, A^2_\mu] = Z[A^1_\mu(-t, x_{PT}), A^2_\mu(-t-i\beta, x_{PT})]\,,$

Local effective action I: $\quad Z[A^1_\mu, A^2_\mu] = \int D\varphi_1 D\varphi_2\, e^{iI[B^1_\mu, B^2_\mu]} \qquad B_\mu = A_\mu + \partial_\mu\varphi$

Auxiliary fields: $\quad \varphi_r = \frac{1}{2}(\varphi_1 + \varphi_2)\,, \qquad \varphi_a = \varphi_1 - \varphi_2$

Most general isotropic quadratic action:

$$\beta\mathcal{L}_2 = c\dot\varphi_r\dot\varphi_a + \kappa\dot\varphi_r\nabla^2\varphi_a + iT\tilde\kappa(\nabla\varphi_a)^2 + \cdots$$

$$\mathcal{L} = iT^2\kappa(\nabla\varphi_a)^2 - \varphi_a(\dot\varepsilon - D\nabla^2\varepsilon) + \nabla^2\varphi_a\left(\frac{\lambda}{2}\varepsilon^2 + \frac{\lambda'}{3}\varepsilon^3\right) + icT^2(\nabla\varphi_a)^2(\tilde\lambda\varepsilon + \tilde\lambda'\varepsilon^2) + \cdots,$$

$$\varepsilon = cT\dot\varphi_r$$

quartic action (constraints imposed)

Results: Effective formalism for hydrodynamic fluctuations

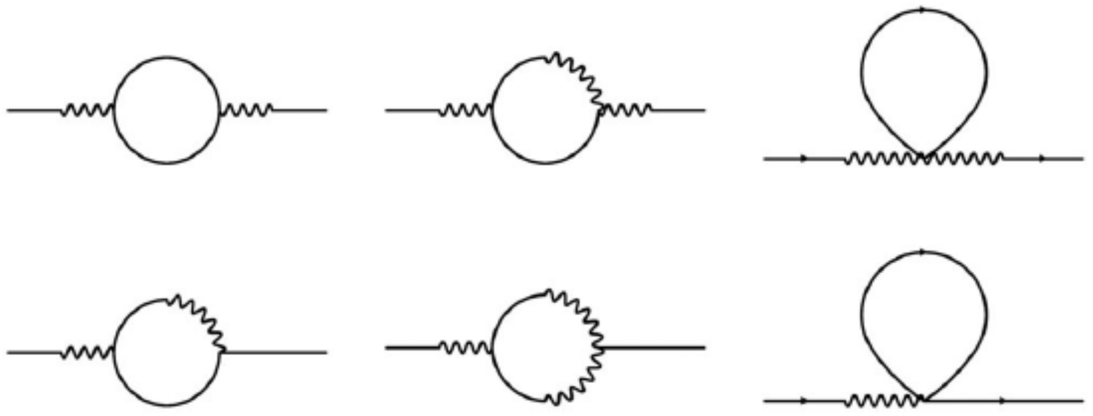
[Chen-Lin, Delacrétaz, Hartnoll; PRL (2019)]

Energy correlator:

$$G^R_{\varepsilon\varepsilon}(\omega, k) = \frac{i[\kappa + \delta\kappa(\omega, k)]Tk^2}{\omega + iDk^2 + \Sigma(\omega, k)}$$

$$\delta\kappa(\omega, k) = \delta\kappa + \kappa_\star(\omega, k),$$
$$\Sigma(\omega, k) = i\delta Dk^2 + \Sigma_\star(\omega, k),$$

$$\mathcal{L} = iT^2\kappa(\nabla\varphi_a)^2 - \varphi_a(\dot{\varepsilon} - D\nabla^2\varepsilon) + \nabla^2\varphi_a\left(\frac{\lambda}{2}\varepsilon^2 + \frac{\lambda'}{3}\varepsilon^3\right) + icT^2(\nabla\varphi_a)^2(\tilde{\lambda}\varepsilon + \tilde{\lambda}'\varepsilon^2) + \cdots,$$

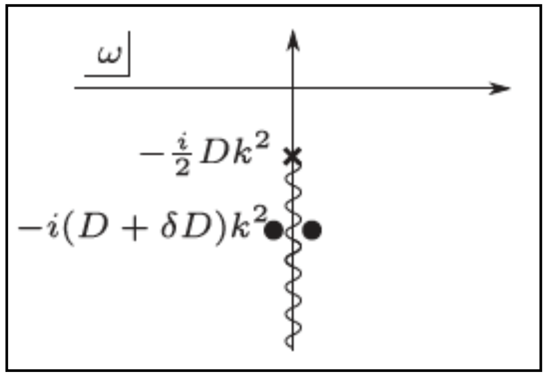


Analytic corrections to transport:

$$\frac{\delta\kappa}{\kappa} = \frac{f_d}{c\ell^d_{\text{th}}}\lambda_\kappa, \qquad \frac{\delta D}{D} = \frac{f_d}{c\ell^d_{\text{th}}}\lambda_D$$

Nonanalytic corrections:

$$\kappa_\star(\omega, k) = f_\kappa(\omega, k)\alpha_d(\omega, k),$$
$$\Sigma_\star(\omega, k) = k^2 f_\Sigma(\omega, k)\alpha_d(\omega, k),$$



$$\alpha_1(\omega, k) = \frac{1}{4}\left(k^2 - \frac{2i\omega}{D}\right)^{-1/2}, \qquad (d = 1)$$
$$\alpha_2(\omega, k) = -\frac{1}{16\pi}\log\left(k^2 - \frac{2i\omega}{D}\right), \qquad (d = 2)$$
$$\alpha_3(\omega, k) = -\frac{1}{32\pi}\left(k^2 - \frac{2i\omega}{D}\right)^{1/2}. \qquad (d = 3)$$

$$f_\kappa(\omega, k) = \frac{cT^2}{D^2}k^2\lambda\tilde{\lambda},$$
$$f_\Sigma(\omega, k) = \frac{cT^2}{D^2}[\omega\lambda(\lambda + \tilde{\lambda}) + iDk^2\lambda\tilde{\lambda}]$$

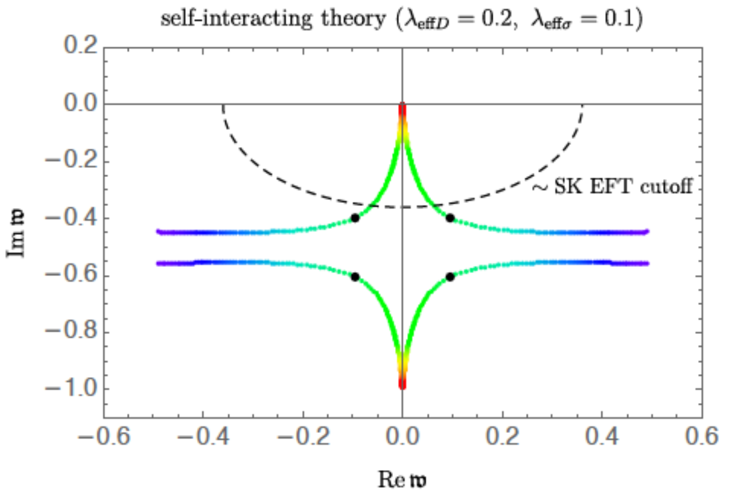
nonanaliticities in energy correlator introduce <u>branch point</u> half-way to <u>splitted</u> diffusion pole

Discussion

Summary

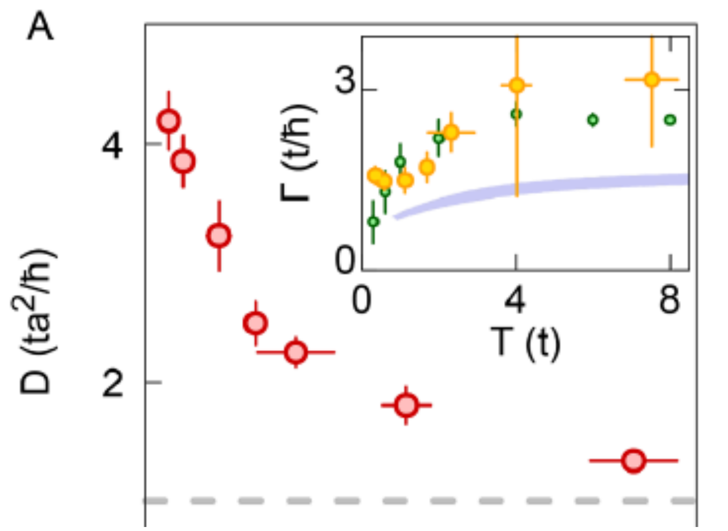
[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

- UV mode *does not affect renormalization* of diffusion coefficient D and conductivity σ
- bound on local thermalization time τ is *protected* from renormalization
- charge current relaxes with *power law*
- proposed *measurement in ultracold atoms*
- relevant for QCD near critical point & Hubbard



Outlook

- extracting the running of τ from data will be an important check for the theory constructed in this work
- repeat our computation in Schwinger-Keldysh (generally equivalent?)



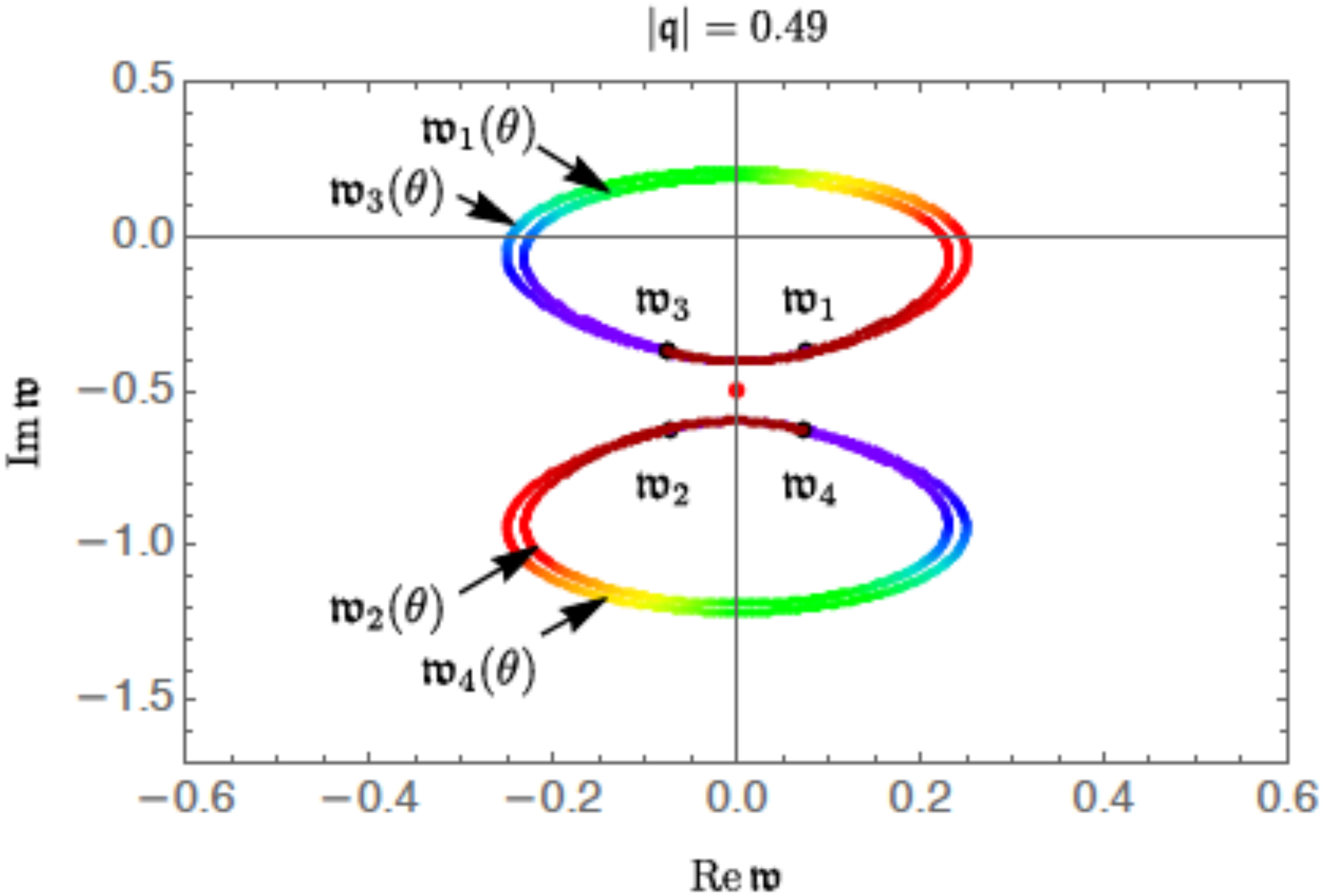
[Brown et al.; Science (2018)]

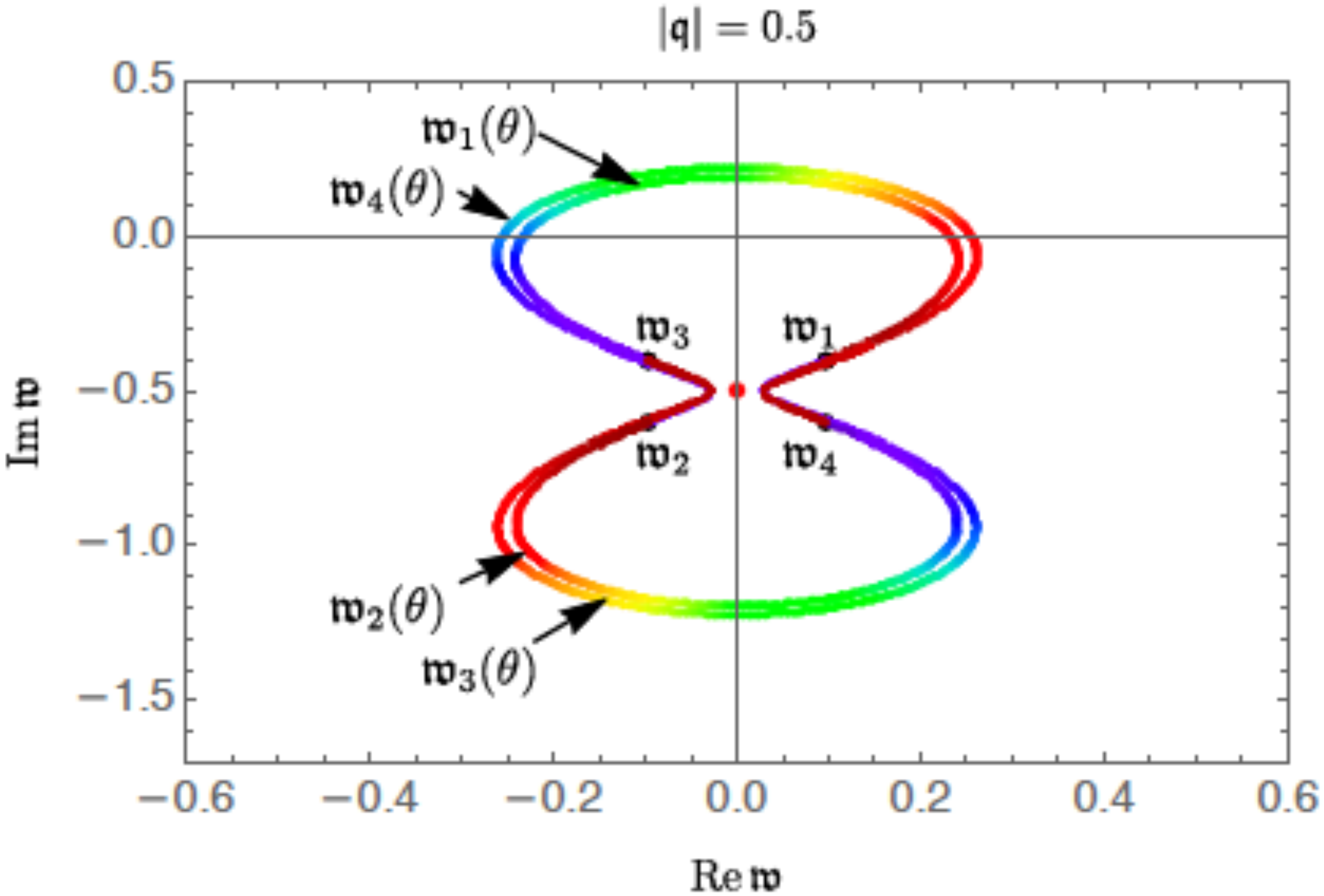
for stability and causality, see [Mullins, Hippert, Noronha; arXiv:2306.08635]

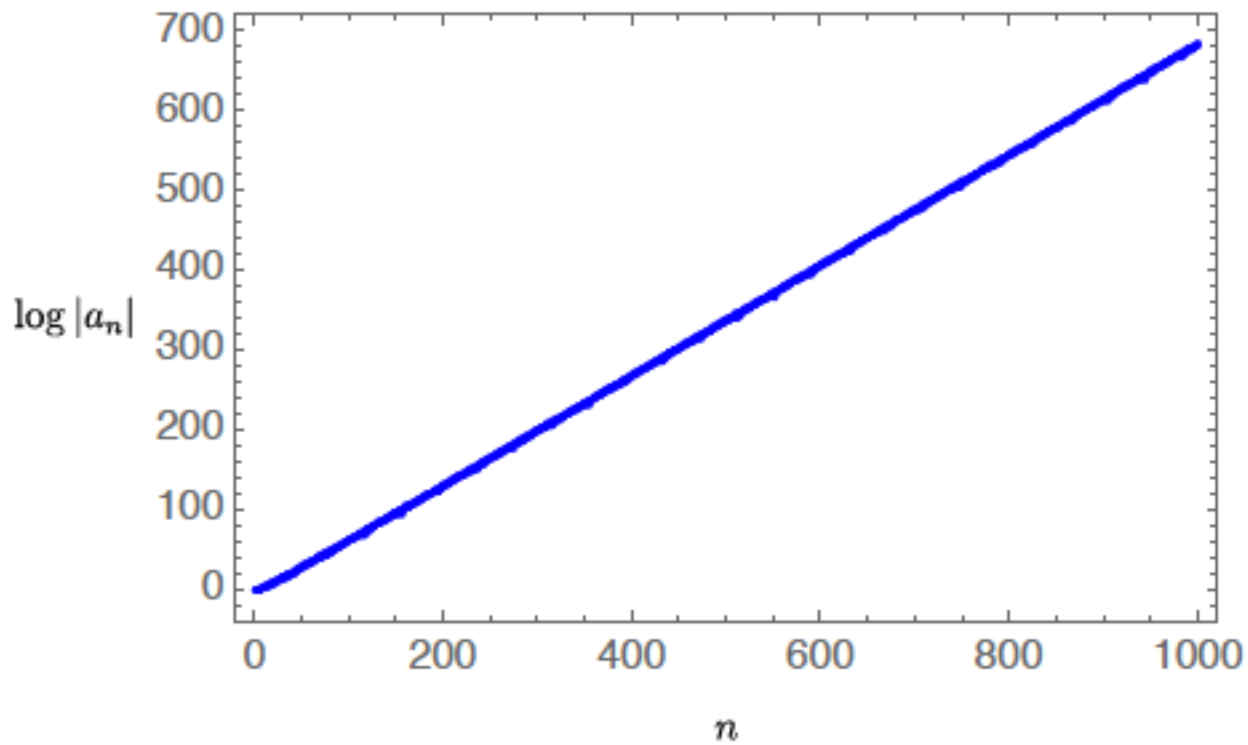
APPENDIX

Complex momentum spectra

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]







Modified dispersion relation (in case d=1, convergent)

$$m = 1,3: \quad \mathfrak{w}_m = \sum_{n=2} a_n \, (-1)^{n\left[\frac{m-1}{2}\right]} \, (\mathfrak{q}^2)^{n/2} \quad = a_2 \, \mathfrak{q}^2 \pm a_3 \, (\mathfrak{q}^2)^{3/2} + \cdots \,,$$

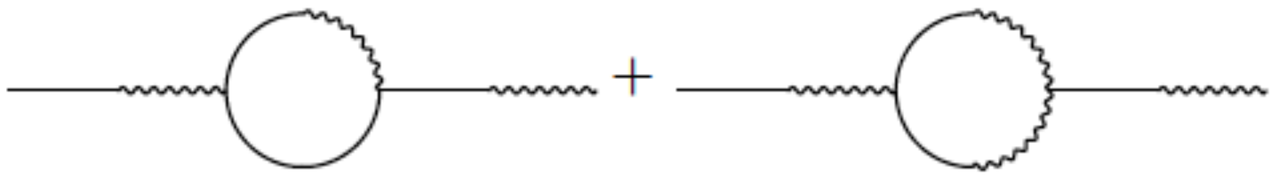
$$m = 2,4: \quad \mathfrak{w}_m = -i + \sum_{n=2} c_n \, (-1)^{n\left[\frac{m}{2}\right]} \, (\mathfrak{q}^2)^{n/2} = -i + c_2 \, \mathfrak{q}^2 \mp c_3 \, (\mathfrak{q}^2)^{3/2} + \cdots \,.$$

Loop calculations

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

$$G_{nn_a}(p) = G^{(0)}_{nn_a}(p) + G^{(0)}_{nn_a}(p)(-\Sigma(p))G^{(0)}_{nn_a}(p) = \frac{1}{\omega + iD_0\mathbf{k}^2 - i\tau\,\omega^2 + \Sigma(\omega,\mathbf{k})}$$

$$G^{(0)}_{nn_a}\Sigma(p)G^{(0)}_{nn_a} =$$



$$\Sigma(p) = \lambda_D^2\,\mathbf{k}^2\int_{p'}\mathbf{k}'^2 G^{(0)}_{n_a n}(p')G^{(0)}_{nn}(p'+p)$$
$$-\frac{i}{2}\chi T\lambda_D\lambda_\sigma\mathbf{k}^2\int_{p'}(\mathbf{k}'^2+\mathbf{k}\cdot\mathbf{k}')\Big[Q(\omega')+Q(\omega+\omega')\Big]G^{(0)}_{n_a n}(p')G^{(0)}_{nn_a}(p+p')$$

$$f_1(\omega,\mathbf{k}) = -\frac{\omega(1-i\tau\omega)(D\mathbf{k}^2\tau-\tau^2\omega^2-3i\tau\omega+2)^2}{\left(D\mathbf{k}^2\tau+(1-i\tau\omega)^2\right)^2}\left[1+\frac{1}{(T\tau)^2}Q_1^{(2)}+\frac{1}{(T\tau)^4}Q_1^{(4)}\right],$$
$$Q_1^{(2)} = -\frac{(D\mathbf{k}^2\tau-i\tau\omega(1-i\tau\omega))^2}{48(D\mathbf{k}^2\tau+(1-i\tau\omega)^2)},$$
$$Q_1^{(4)} = \frac{(D\mathbf{k}^2\tau-i\tau\omega(1-i\tau\omega))^2\left(3(D\mathbf{k}^2\tau)^2-2(i\tau\omega)(D\mathbf{k}^2\tau)(3-i\tau\omega)-(i\tau\omega)^2(1-i\tau\omega)^2\right)}{11520(D\mathbf{k}^2\tau+(1-i\tau\omega)^2)^2}$$

$$f_2(\omega,\mathbf{k}) = \frac{2i(D\mathbf{k}^2-i\omega-i\tau\omega^2)(1-i\tau\omega)(D\mathbf{k}^2\tau-\tau^2\omega^2-3i\tau\omega+2)}{\left(D\mathbf{k}^2\tau+(1-i\tau\omega)^2\right)^2}\left[1+\frac{1}{(T\tau)^2}Q_2^{(2)}+\frac{1}{(T\tau)^4}Q_2^{(4)}\right],$$
$$Q_2^{(2)} = \frac{(D\mathbf{k}^2\tau)^2-2(i\tau\omega)(D\mathbf{k}^2\tau)-(i\tau\omega)^2(1-i\tau\omega)^2}{48(D\mathbf{k}^2\tau+(1-i\tau\omega)^2)},$$
$$Q_2^{(4)} = -\frac{1}{11520(D\mathbf{k}^2\tau+(1-i\tau\omega)^2)^2}\Big[(D\mathbf{k}^2\tau)^4-4(D\mathbf{k}^2\tau)^3(i\tau\omega)(1+i\tau\omega),$$
$$-2(D\mathbf{k}^2\tau)^2(i\tau\omega)^2(1-10i\tau\omega-5\tau^2\omega^2)+4(D\mathbf{k}^2\tau)(i\tau\omega)^3(1-i\tau\omega)^2(3-i\tau\omega)+(i\tau\omega)^4(1-i\tau\omega)^4\Big]$$

Loop calculations - continued

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

$$\lim_{D\mathbf{k}^2 \ll \omega} f_{1d}(\omega, \mathbf{k}) = -\omega(2 - 3i\tau\omega + \tau^2\omega^2)\left[1 + \frac{\omega^2}{48T^2} - \frac{\omega^4}{11520T^4}\right]$$

$$\lim_{D\mathbf{k}^2 \ll \omega} f_{2d}(\omega, \mathbf{k}) = 2\omega(1 - i\tau\omega)\left[1 + \frac{\omega^2}{48T^2} - \frac{\omega^4}{11520T^4}\right].$$

$$f_3(\omega, \mathbf{k}) = -\frac{i\omega(1 - i\tau\omega)(D\mathbf{k}^2\tau - \tau^2\omega^2 - 3i\tau\omega + 2)^2}{\left(D\mathbf{k}^2\tau + (1 - i\tau\omega)^2\right)^2}\left[\frac{1}{(T\tau)^2}Q_3^{(2)} + \frac{1}{(T\tau)^4}Q_3^{(4)}\right],$$

$$Q_3^{(2)} = \frac{3(D\mathbf{k}^2\tau)^2 - 2(D\mathbf{k}^2\tau)(i\tau\omega)(3 - i\tau\omega) - (i\tau\omega)^2(1 - i\tau\omega)^2}{96\,(D\mathbf{k}^2 + i\omega(1 + i\tau\omega))\,(D\mathbf{k}^2\tau + (1 - i\tau\omega)^2)},$$

$$Q_3^{(4)} = \frac{(i\tau\omega)^2\left(5(D\mathbf{k}^2\tau)^2 - 2(D\mathbf{k}^2\tau)(i\tau\omega)(5 - 3i\tau\omega) + (i\tau\omega)^2(1 - i\tau\omega)^2\right)}{5760\,(D\mathbf{k}^2 + i\omega(1 + i\tau\omega))\,(D\mathbf{k}^2\tau + (1 - i\tau\omega)^2)},$$

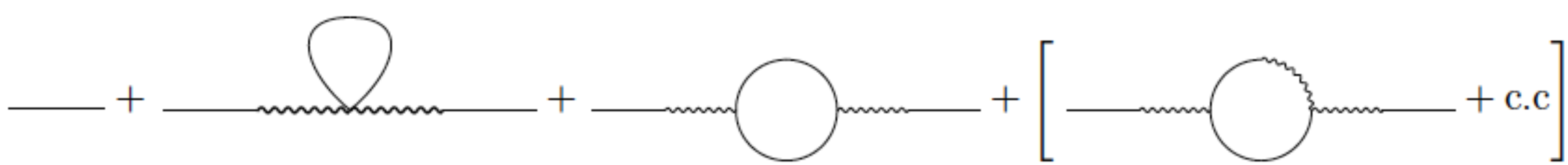
$$(\omega + iD\mathbf{k}^2 - i\tau\omega^2)\left(1 - \frac{\delta\sigma(\omega, \mathbf{k})}{\sigma}\right) + \Sigma(\omega, \mathbf{k}) = 0$$

$$C(p) = 2T\chi D\mathbf{k}^2 Q(\omega) + 2\chi T\lambda'_\sigma Q(\omega)\int_{p'} G_{nn}^{(0)}(p') + \frac{1}{2}\lambda_D^2\mathbf{k}^4\int_{p'} G_{nn}^{(0)}(p')G_{nn}^{(0)}(p - p')$$
$$+ i\chi T\lambda_\sigma\lambda_D\mathbf{k}^2\int_{p'}\mathbf{k}\cdot\mathbf{k}' G_{n_a n}^{(0)}(p')G_{nn}^{(0)}(p + p')(Q(\omega) + Q(\omega')).$$

$$G_{nn}(\omega, \mathbf{k}) = \frac{C(\omega, \mathbf{k})}{\omega^2 + D^2\mathbf{k}^4 + 2\,\omega\,\mathrm{Re}\,\Sigma(\omega, \mathbf{k}) + 2\,(D\,\mathbf{k}^2 - \tau\omega^2)\,\mathrm{Im}\,\Sigma(\omega, \mathbf{k})}$$

$$C(p) = 2T\chi D\mathbf{k}^2 Q(\omega)\left[1 + \frac{\mathrm{Re}\delta\sigma(p)}{\sigma} + \frac{D\mathbf{k}^2 - \tau\omega^2}{\omega}\frac{\mathrm{Im}\delta\sigma(p)}{\sigma} + \frac{\mathrm{Re}\Sigma(p)}{\omega}\right]$$

$$G_{nn_a}^{(0)}(p)(-C(p))G_{n_a n}^{(0)}(p) =$$



Singular points of plane curves

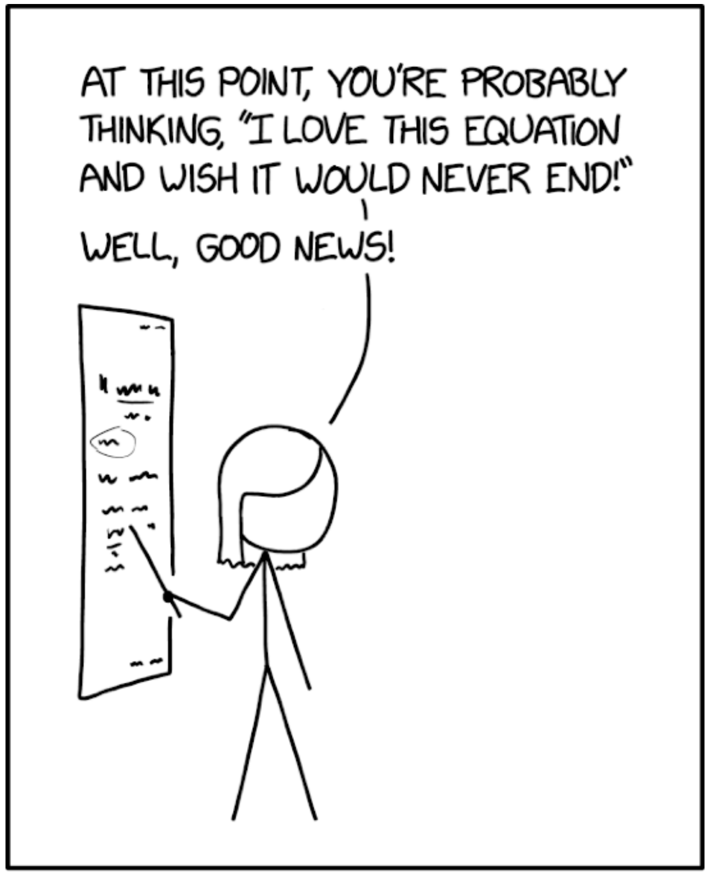
[C.T.C. Wall (2004)]

Puiseux theorem:

Any equation f(x, y) = 0, where f is a polynomial with f(O) = 0 or more generally f ∈ C[[x, y]] with zero constant term, admits at least one solution in formal power series of the form

$$x = t^n, \quad y = \sum_{1}^{\infty} a_r t^r$$

(some n ∈ N).
Thus, y can be expressed as power series in fractional powers of x.



TAYLOR SERIES EXPANSION IS THE WORST.

[https://xkcd.com/2605/]

Example: hydrodynamics

$$x = k, \quad y = \omega, \quad f(x,y) = \mathscr{P}(\omega, k)$$

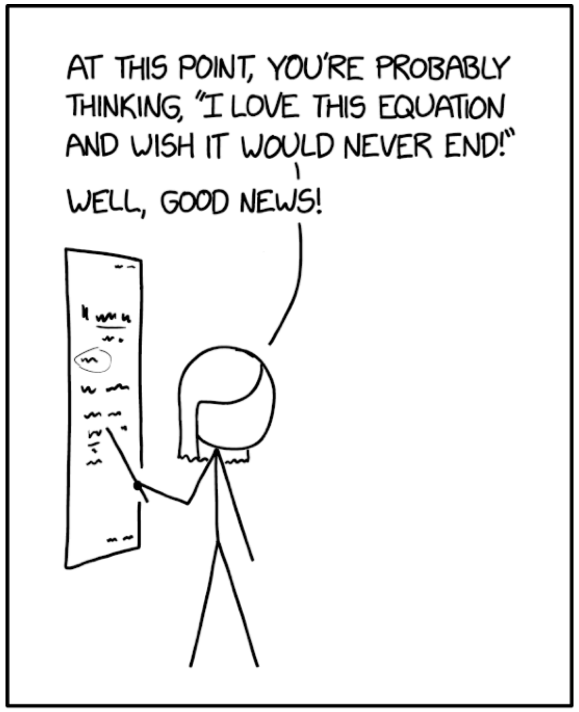
$$\mathscr{P}\phi = 0 \Rightarrow \mathscr{P} = \omega + iDk^2 + \mathscr{O} = 0$$

➡There exists convergent hydrodynamic expansion. Critical points limit the radius of convergence in complex *k*.

Invitation: Hydrodynamic expansion is asymptotic

Hydrodynamic expansion of dispersion relations
around far-from-equilibrium state

- asymptotic expansion: *coefficients* $\sim n!$
- attractors *[Heller, Spalinski; PRL (2015)]* *[Heller et al; PRL (2021)]*
- resurgence
- far-from-equilibrium holography *[Kurkela et al; PRL (2019)]* *[Janik, Jankowski, Soltanpanahi; PRL (2017)]*
- far-from-equilibrium fluid dynamics *[Romatschke; PRL (2017)]*

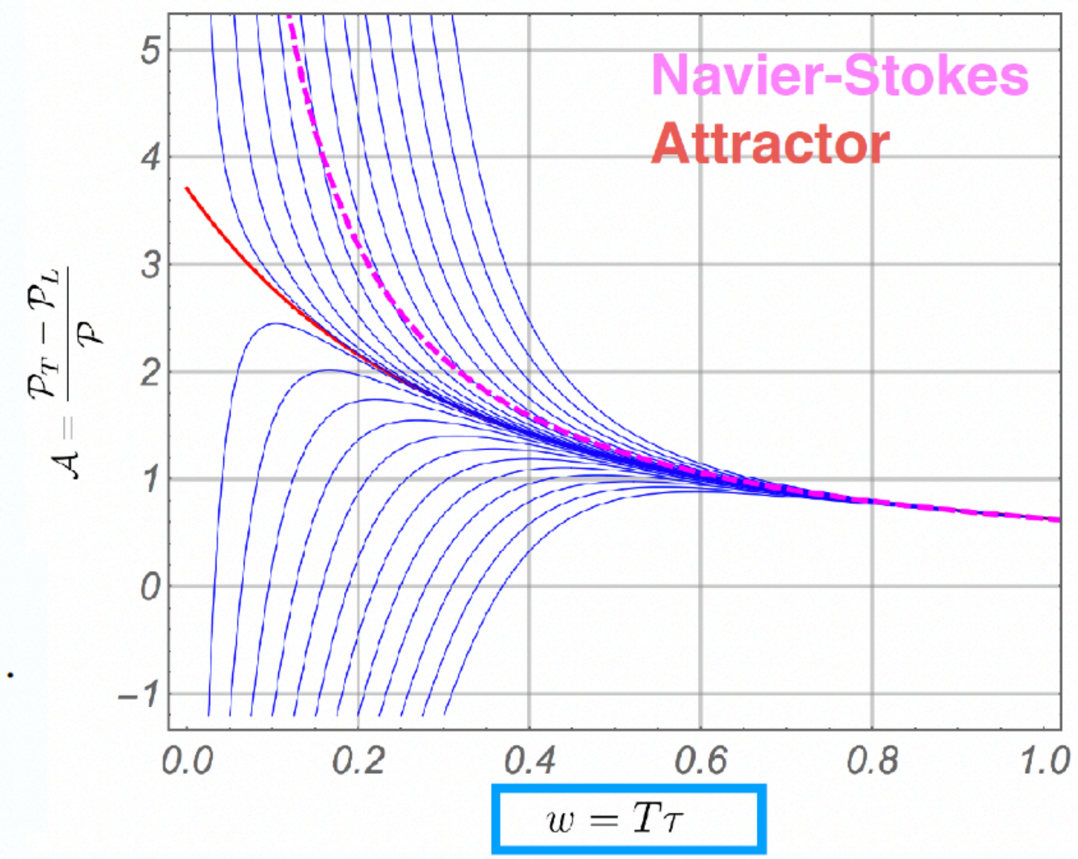


TAYLOR SERIES EXPANSION IS THE WORST.

➡asymptotic is worse

Pressure anisotropy in *N*=4 SYM:

$$\mathcal{A} = \underbrace{\frac{8C_\eta}{w}}_{\text{Navier–Stokes}} + \underbrace{\frac{16 C_\eta C_\tau}{3w^2}}_{\text{2nd order}} + \ldots = \underbrace{\sum_{n>0} \frac{a_n^{(0)}}{w^n}}_{\text{gradient expansion}} + \underbrace{\left(\sigma\, w^{\frac{C_\eta}{C_\tau}} e^{-\frac{3}{2C_\tau} w}\right) \sum_{n\geq 0} \frac{a_n^{(1)}}{w^n}}_{\text{transseries sectors}} + \ldots$$



[from Talk by Spalinski at QuarkMatter22]

AdS5 Schwarzschild pole-skipping:

$$\mathfrak{w} = i, \quad \mathfrak{q} = \pm\sqrt{\frac{3}{2}}i\,.$$

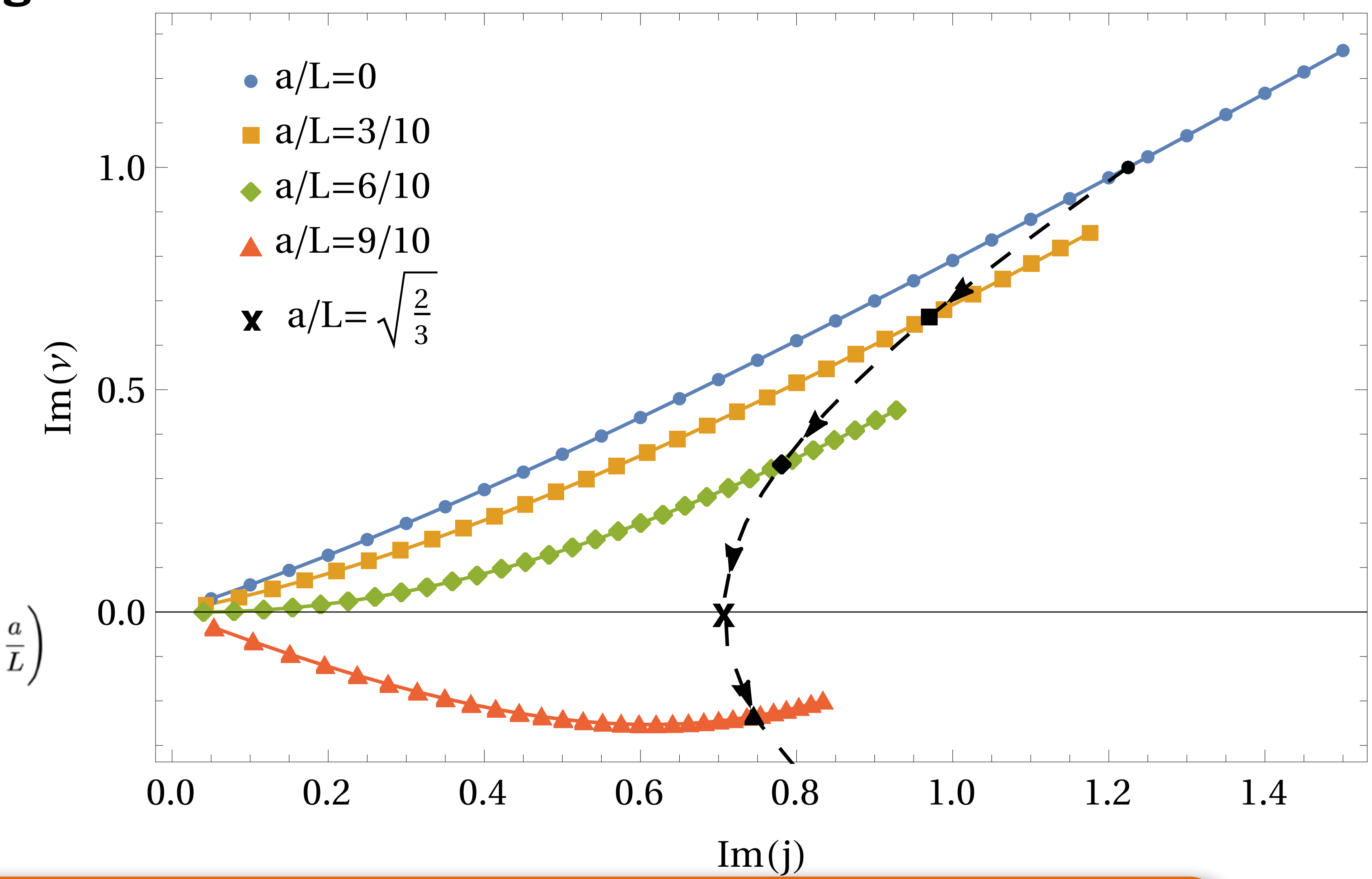
Apply transformation:

$$\mathfrak{q} = \frac{a\nu + j}{\sqrt{1-a^2}}, \quad \mathfrak{w} = \frac{aj+\nu}{\sqrt{1-a^2}}$$

Shifted pole-skipping points:

$$\nu_{scalar} = \frac{i}{\sqrt{1-a^2/L^2}}\left(1 \mp \frac{\sqrt{3}}{\sqrt{2}}\frac{a}{L}\right), \quad j_{scalar} = \frac{i}{\sqrt{1-a^2/L^2}}\left(\pm\frac{\sqrt{3}}{\sqrt{2}} - \frac{a}{L}\right)$$

[Amano(Garbiso),Blake,Cartwright,Kaminski,Thompson; (2022)]



$$\lambda_L = 2\pi T\left(1 - \sqrt{\frac{3}{2}}\frac{|a|}{L}\right) = 2\pi T\left(1 - |v|/v_B^{(0)}\right)$$

quantum Lyapunov exponent

$$v_B^{\pm} = \frac{\sqrt{\frac{2}{3}} \mp \frac{a}{L}}{1 \mp \sqrt{\frac{2}{3}}\frac{a}{L}}$$

butterfly velocity

Agrees with shock-wave computation and with near-horizon expansion method.

Pole-skipping points in rotating black holes in AdS4: *[Blake, Davison; JHEP (2021)]*

More topics in hydrodynamics

- Spin hydrodynamics

  [Hongo,Huang,Kaminski,Stephanov,Yee; JHEP (2021)]

- magnetohydrodynamics

  [Ammon, Grieninger, Hernandez, Kaminski, Koirala, Leiber, Wu; JHEP (2021)]

- non-relativistic

  [Kaminski, Moroz; PRB (2014)] *[Garbiso, Kaminski; JHEP (2019)]*

  [Davison, Grozdanov, Janiszewski, Kaminski; JHEP (2016)]

- far from equilibrium fluid dynamics

  [Cartwright, Kaminski; JHEP (2019)] *[Cartwright, Kaminski, Knipfer; arXiv:2207.02875]*

  [Wondrak, Kaminski, Bleicher; PRB (2020)] *[Cartwright, Kaminski, Schenke; PRC (2022)]*

- quantum chaos

  [Blake, Lee, Liu; JHEP (2018)] *[Amano(Garbiso),Blake,Cartwright, Kaminski,Thompson; JHEP (2022)]*

- convergence under rotation

  [Cartwright,Garbiso-Amano;Kaminski,Noronha,Speranza;arXiv:2112.10781]

Thank you for listening!

Vision: Quantum fluids far from equilibrium

Hydrodynamics

- far from equilibrium

[Romatschke; PRL (2018)]

[Glorioso,Liu]
[Haehl,Loganayagam,Rangamani]

- quantum chaos

[Blake, Lee, Liu; JHEP (2018)]
[Grozdanov et al. (2019)]

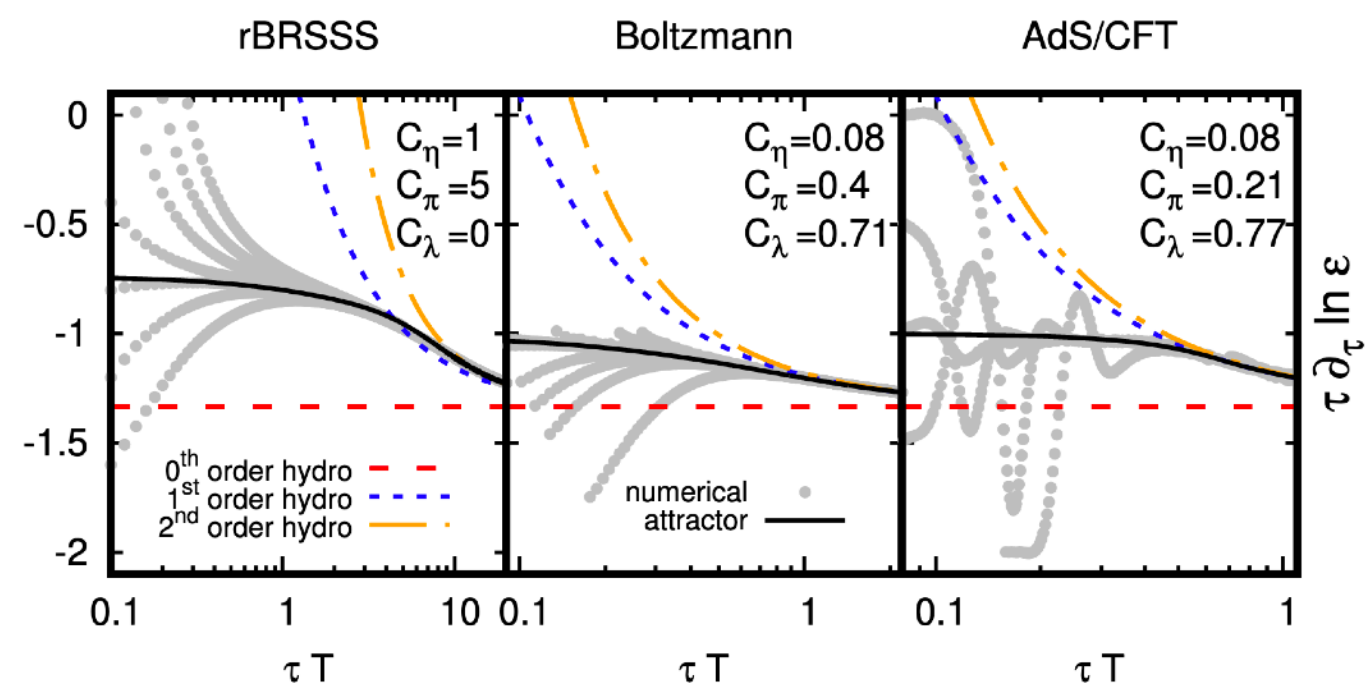
- convergence & stability

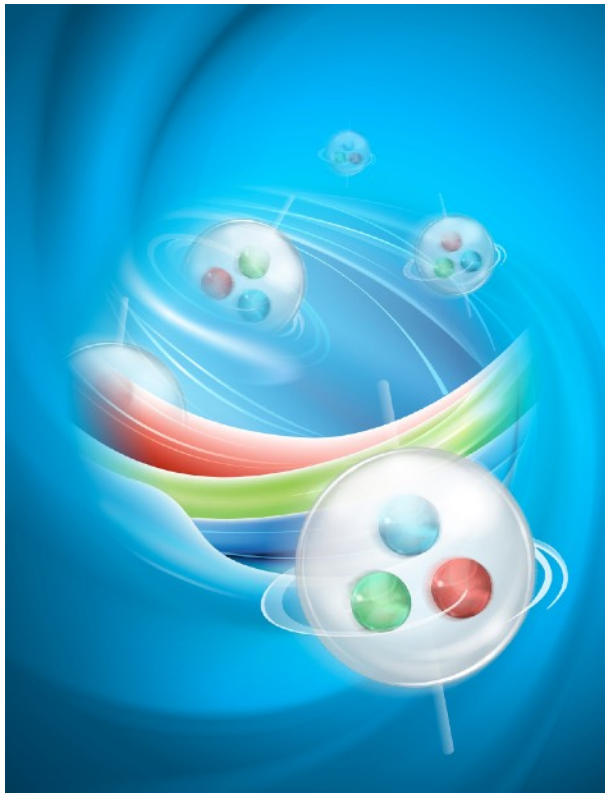
[Kovtun; JHEP (2019)]
[Grozdanov, Kovtun, Starinets, Tadic; PRL (2019)]
[Withers; JHEP (2018)]
[Heller, Janik, Witaszczyk; PRL (2013)]
[Heller, Spalinski; PRL (2018)]

- most vortical fluid

[Garbiso, Kaminski; JHEP (2019)]

[Cartwright, Garbiso-Amano; Kaminski, Noronha, Speranza;arXiv:2112.10781]





[STAR; Nature (2017)]