Universal dynamics around nonthermal attractors

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KITP, ‘The Many Faces of Relativistic Fluid Dynamics’, 27.06.2023,
based on TP, Heller, Berges, PRL 130, 031602 (2023),
Heller, Mazeliauskas, TP, to appear soon

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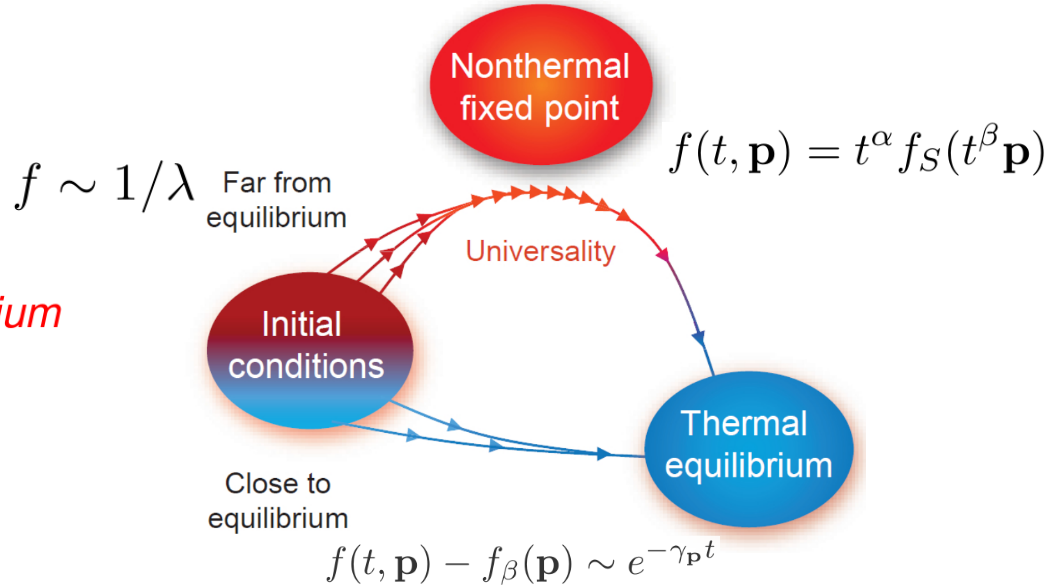
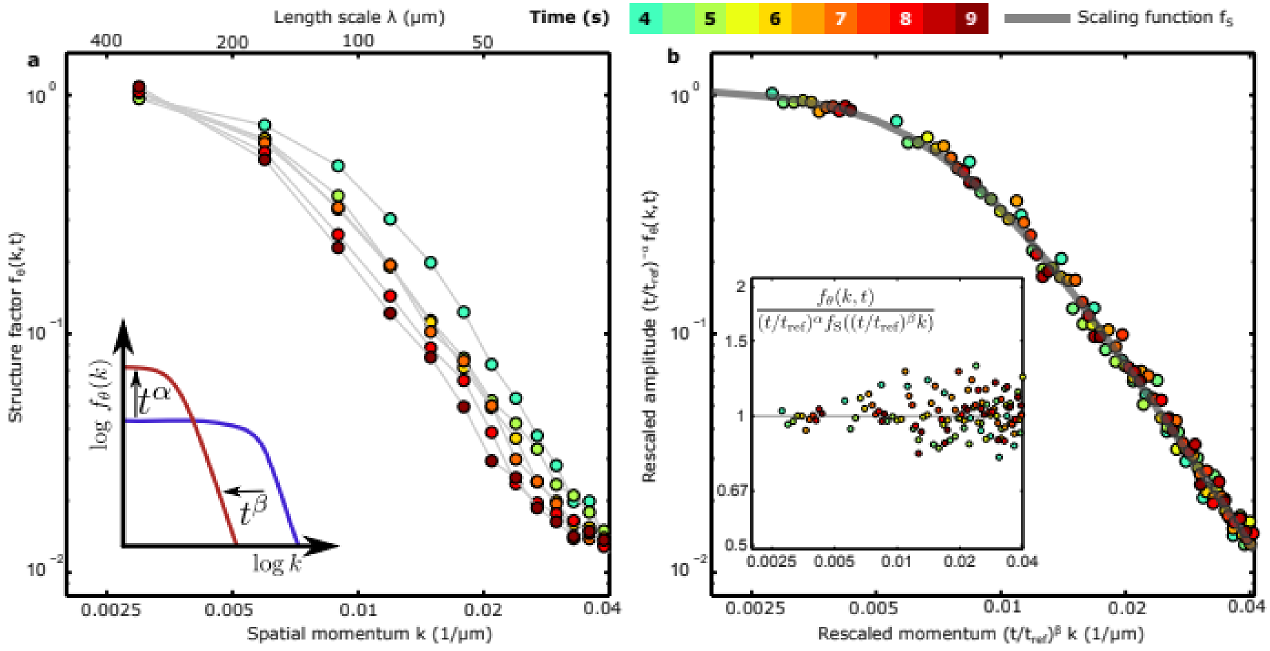
STRUCTURES CLUSTER OF EXCELLENCE

Nonthermal fixed points

- **Universal scaling in space and time**
  → *`attractor solutions´ / `self-organized scaling´ far from equilibrium*
- **New universality classes**
  → *unexpected links between different physical systems*

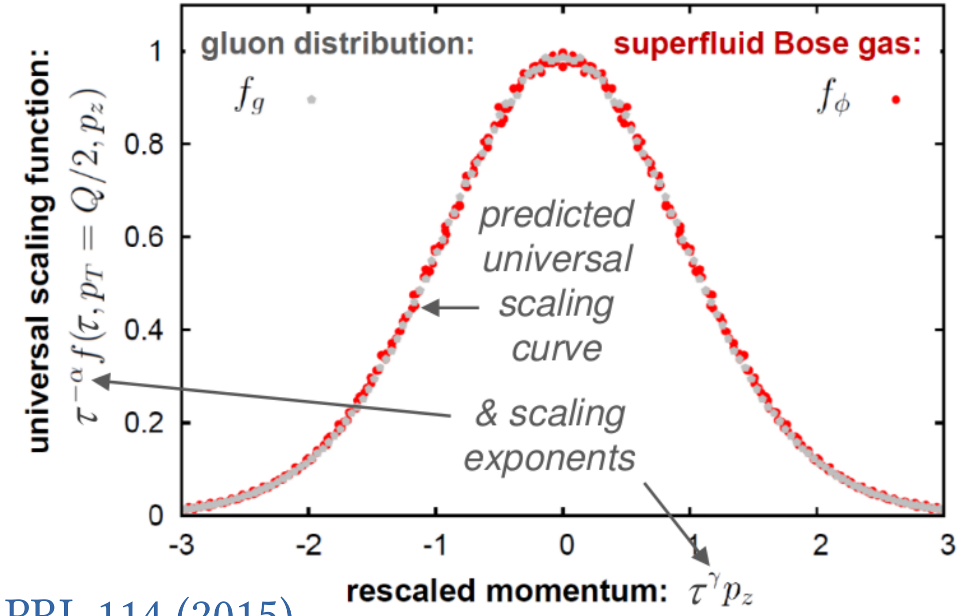


Spin-1 BEC quench in a quasi 1D optical trap:



'Bottom-up' thermalization process for heavy-ion collision

$$f(t, p_T, p_z) = t^\alpha f_S(t^\beta p_T, t^\gamma p_z)$$

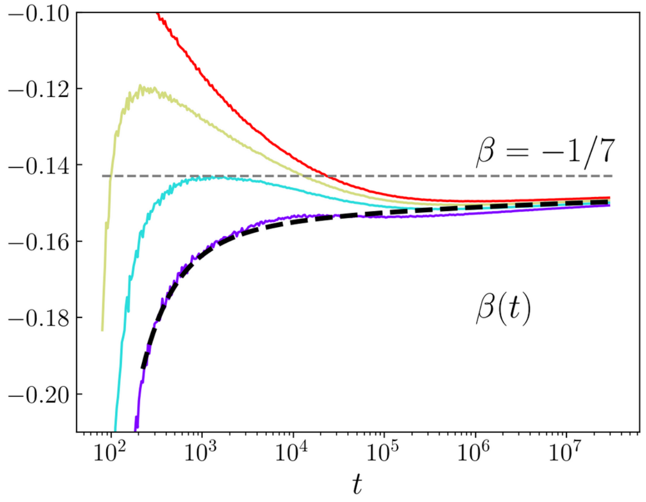


Prüfer et al., Nature 563, 217 (2018), Erne et al., Nature 563, 225 (2018), Glidden et al., Nature Phys. 17, 457 (2021).

Berges et al., PRL 114 (2015)

Roadmap

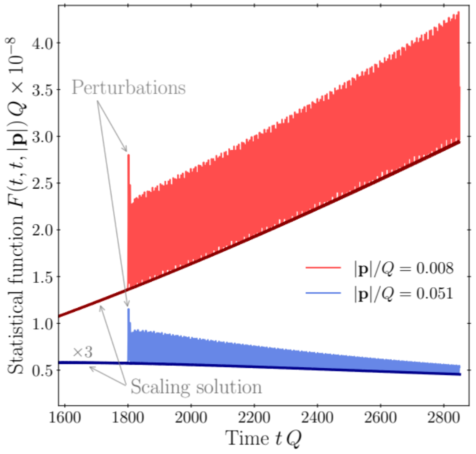
Discuss two aspects of universal dynamics in the approach to nonthermal attractors:



- Fast and slow time evolution of scaling exponents

  based on Heller, Mazeliauskas, TP, to appear soon

- Stability properties of nonthermal attractors via linear response



based on TP, Heller, Berges, PRL 130, 031602 (2023)

The derivation of scaling

Orioli, Boguslavski, Berges, PRD 92 (2015)

$$\partial_t f(t,\mathbf{p}) = C[f](t,\mathbf{p})$$

Make scaling ansatz:

$$f(t,\mathbf{p}) = (t/t_{\mathrm{ref}})^{\alpha} f_S((t/t_{\mathrm{ref}})^{\beta}\mathbf{p})$$

Determine scaling exponents via scaling relations from Boltzmann equation:

$$(t/t_{\mathrm{ref}})^{\alpha-1}[\alpha + \beta\bar{\mathbf{p}}\cdot\partial_{\bar{\mathbf{p}}}]f_S(\bar{\mathbf{p}}) = (t/t_{\mathrm{ref}})^{\mu}C[f_S](\bar{\mathbf{p}} = (t/t_{\mathrm{ref}})^{\beta}\mathbf{p})$$

and relevant conservation laws such as energy or particle number conservation:

$$\omega_{\mathbf{p}} \sim |\mathbf{p}|^{z}$$

$$\mathcal{E}(t) = \int \frac{d^d\mathbf{p}}{(2\pi)^d}\omega_{\mathbf{p}} f(t,\mathbf{p}) = (t/t_{\mathrm{ref}})^{\alpha-(d+z)\beta}\bar{\mathcal{E}}$$

$$n(t) = \int \frac{d^d\mathbf{p}}{(2\pi)^d} f(t,\mathbf{p}) = (t/t_{\mathrm{ref}})^{\alpha-d\beta}\bar{n}$$

$$\mu = \alpha - 1$$

$$\alpha = (d+z)\beta$$

$$\alpha = d\beta$$

1st Example: Direct energy cascade

$f(t,\mathbf{p}) = (t/t_{\text{ref}})^{\alpha} f_S((t/t_{\text{ref}})^{\beta}\mathbf{p})$

$$C^{2\leftrightarrow 2}[f](t,\mathbf{p}) = \frac{\lambda^2}{2}\int_{\mathbf{lqr}} \frac{(2\pi)^{d+1}\delta^{(3)}(\mathbf{p}+\mathbf{l}-\mathbf{q}-\mathbf{r})}{2\omega_{\mathbf{p}}2\omega_{\mathbf{l}}2\omega_{\mathbf{q}}2\omega_{\mathbf{r}}}\delta^{(1)}(\omega_{\mathbf{p}}+\omega_{\mathbf{l}}-\omega_{\mathbf{q}}-\omega_{\mathbf{r}})$$

$$\times\left[(f_{\mathbf{p}}+1)(f_{\mathbf{l}}+1)f_{\mathbf{q}}f_{\mathbf{r}} - f_{\mathbf{p}}f_{\mathbf{l}}(f_{\mathbf{q}}+1)(f_{\mathbf{r}}+1)\right]$$

$$= (t/t_{\text{ref}})^{\mu} C[f_S](\bar{\mathbf{p}})$$

$\omega_{\mathbf{p}} \sim |\mathbf{p}|$

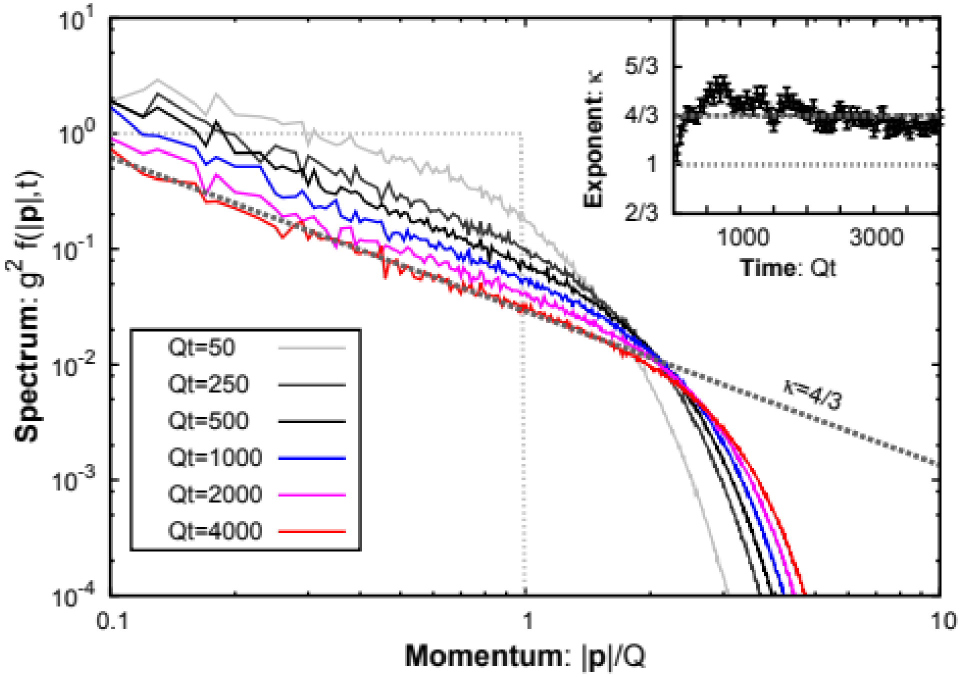
$1/\lambda \gg f_{\mathbf{p}} \gg 1$

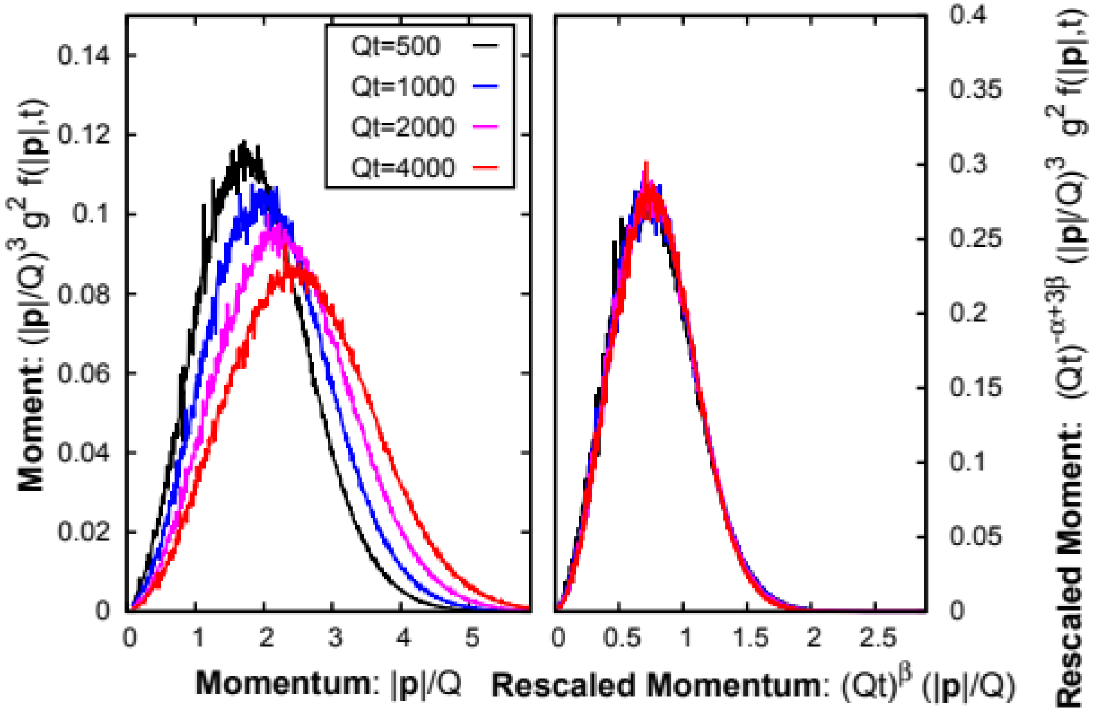
$\alpha = (d+z)\beta$

$\mu = 3\alpha - \beta = \alpha - 1$

$(\alpha,\beta) = (-4/7, -1/7)$

Classical-Statistical SU(2):





Also, Inflationary Cosmology: Micha, Tkachev, PRL 90, 121301 (2003), …

Berges, Boguslavski, Schlichting, Venugopalan, PRD 89 (2014)

New derivation of prescaling

To appear: Heller, Mazeliauskas, TP

$$\alpha(t) \to \alpha$$

Make prescaling ansatz

$$f(t, \mathbf{p}) = A(t) f_S(B(t)\mathbf{p})$$

with $A(t) = \exp\left[\int_{t_{\text{ref}}}^{t} dt' \frac{\alpha(t')}{t'}\right] \to (t/t_{\text{ref}})^{\alpha}$

Determine prescaling exponents via scaling relations from Boltzmann equation:

$$\partial_t f(t, \mathbf{p}) = \frac{A(t)}{t} \left[\alpha(t) + \beta(t)\bar{\mathbf{p}} \cdot \partial_{\bar{\mathbf{p}}}\right] f_S(\bar{\mathbf{p}})$$

$$= \exp\left[\int_{t_{\text{ref}}}^{t} \frac{dt'}{t'} \mu[\alpha, \beta](t')\right] C_S[f_S](\bar{\mathbf{p}})$$

and relevant conservation laws

$$\alpha(t) = (d + z)\beta(t)$$
$$\alpha(t) = d\beta(t)$$
$$\alpha(t) = \sigma\beta(t)$$

via separation of variables

$$\frac{1}{D_1} = \frac{\left[\sigma + \bar{\mathbf{p}} \cdot \partial_{\bar{\mathbf{p}}}\right] f_S(\bar{\mathbf{p}})}{C_S(\bar{\mathbf{p}})}$$

$$= \exp\left[\int_{t_{\text{ref}}}^{t} \frac{dt'}{t'} \left(\mu[\alpha, \beta](t') - \alpha(t')\right)\right] \frac{t}{\beta(t)}$$

We obtain the time evolution of prescaling exponents !

$$t\partial_t \beta(t) = \left(\mu[\alpha, \beta](t) - \alpha(t) + 1\right) \beta(t)$$

Prescaling coined: Berges, Mazeliauskas, PRL 122, 122301 (2019)

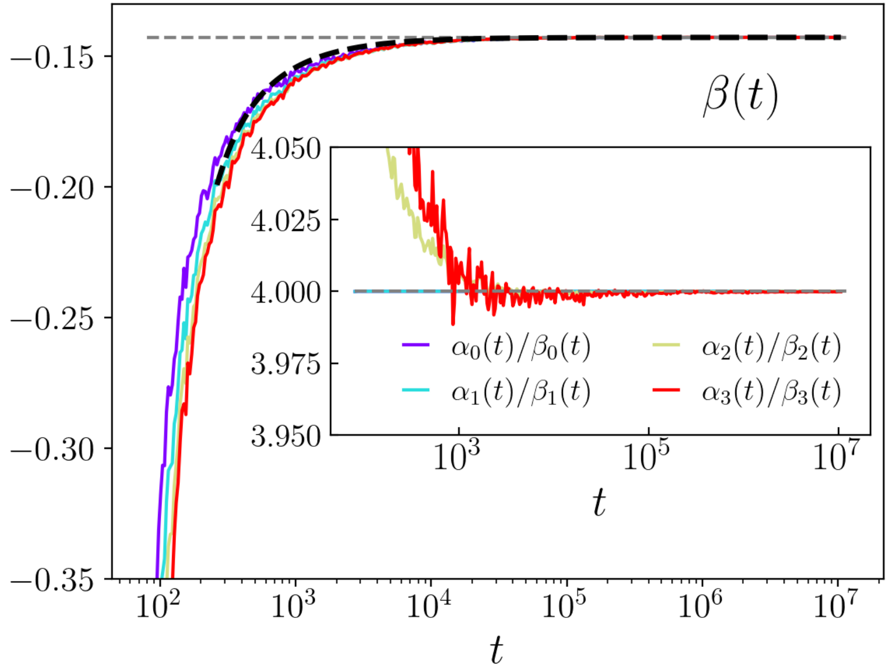
Prescaling in direct energy cascade

$$\alpha(t) = (d+z)\beta(t)$$
$$1/\lambda \gg f_{\mathbf{p}} \gg 1$$

Study prescaling behavior of perturbative collision kernel

$$C^{2\leftrightarrow 2}[f](t,\mathbf{p}) = \frac{\lambda^2}{2}\int_{\mathbf{lqr}} \frac{(2\pi)^{d+1}\delta^{(3)}(\mathbf{p}+\mathbf{l}-\mathbf{q}-\mathbf{r})}{2\omega_{\mathbf{p}}2\omega_{\mathbf{l}}2\omega_{\mathbf{q}}2\omega_{\mathbf{r}}}\delta^{(1)}(\omega_{\mathbf{p}}+\omega_{\mathbf{l}}-\omega_{\mathbf{q}}-\omega_{\mathbf{r}})$$
$$\times\left[(f_{\mathbf{p}}+1)(f_{\mathbf{l}}+1)f_{\mathbf{q}}f_{\mathbf{r}} - f_{\mathbf{p}}f_{\mathbf{l}}(f_{\mathbf{q}}+1)(f_{\mathbf{r}}+1)\right]$$
$$= A^3(t)B^{-1}(t)C[f_S](\bar{\mathbf{p}})$$

To appear: Heller, Mazeliauskas, TP



Simple power-law prescaling dynamics

$$t\partial_t\beta(t) = (1+7\beta(t))\beta(t)$$

$$\beta(t) = \frac{\beta(t_{\mathrm{ref}})\,t/t_{\mathrm{ref}}}{1-7\beta(t_{\mathrm{ref}})[t/t_{\mathrm{ref}}-1]}$$
$$\alpha(t) = \frac{\alpha(t_{\mathrm{ref}})\,t/t_{\mathrm{ref}}}{1-\frac{7}{4}\alpha(t_{\mathrm{ref}})[t/t_{\mathrm{ref}}-1]}$$

$$\xrightarrow{t\to\infty} \quad (\alpha,\beta) = (-4/7, -1/7)$$

$$n_m(t) = \int_{\mathbf{p}} |\mathbf{p}|^m f(t,\mathbf{p}) = A(t)B^{-(3+m)}(t)n_m(t_{\mathrm{ref}})$$

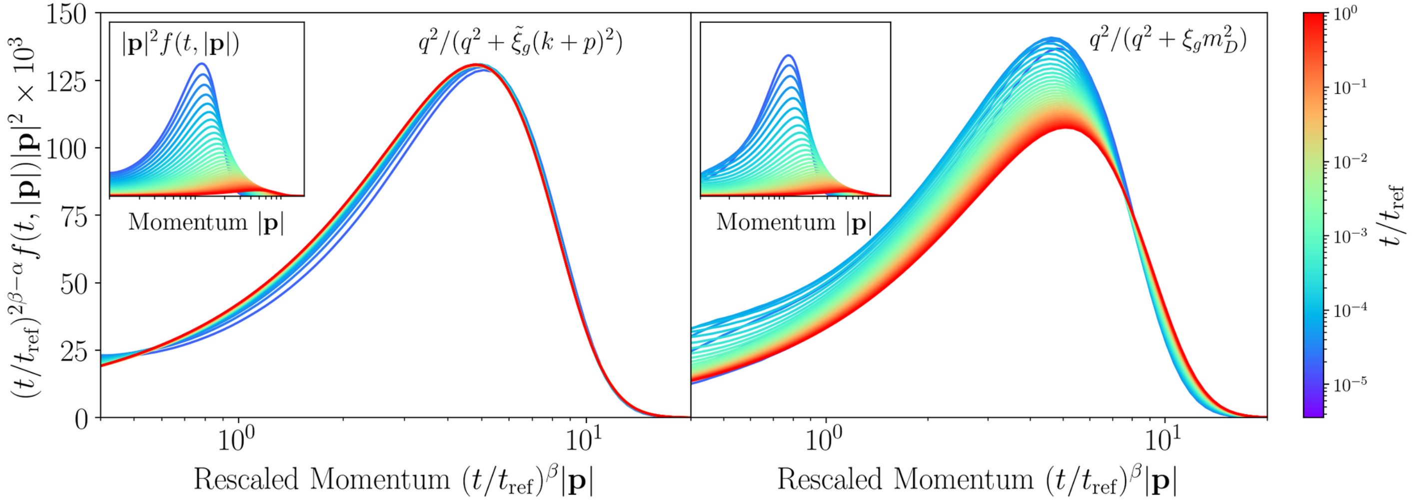
Breaking of scaling in QCD KT

Debye mass in QCD kinetic theory breaks scaling $|\mathcal{M}|^2[B(t)\bar{m}_D]$

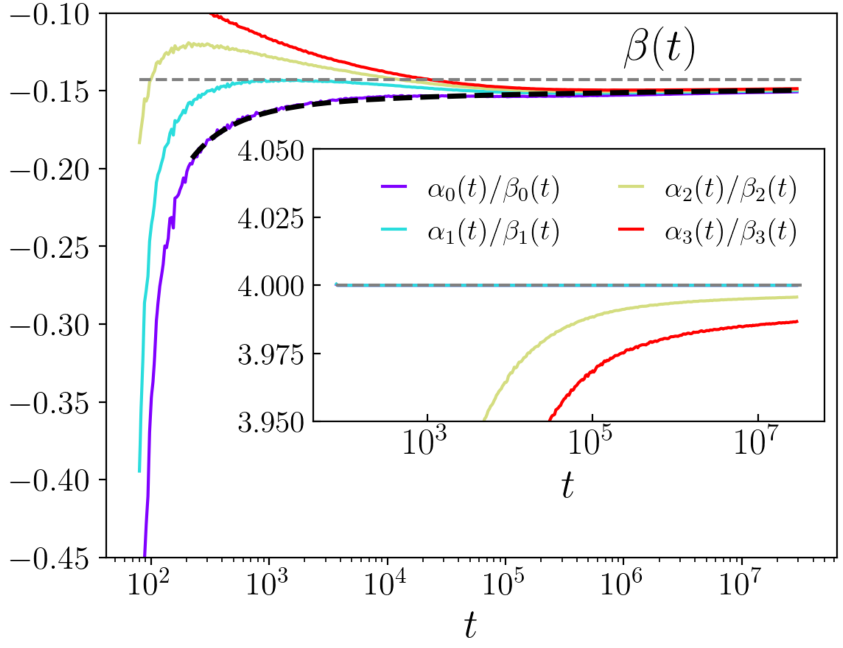
$m_D(t) \sim t^{-1/7}$

$$\alpha(t) = (d+z)\beta(t)$$
$$(\alpha, \beta) = (-4/7, -1/7)$$
$$1/\lambda \gg f_{\mathbf{p}} \gg 1$$

Brewer et. al JHEP 145 (2022)



To appear: Heller, Mazeliauskas, TP



$$C^{\rm FP}[f](t,\mathbf{p})$$
$$= \ln\left[\frac{\langle p\rangle(t)}{m_D(t)}\right]\tilde{C}^{\rm FP}[f](t,\mathbf{p})$$
$$= A^3(t)B^{-1}(t)\ln\left[B^{-2}(t)\frac{\langle\bar{p}\rangle}{\bar{m}_D}\right]$$
$$\times\, \tilde{C}_S^{\rm FP}[f_S](\bar{\mathbf{p}})$$

Modifies Scaling !

$$\frac{\ddot{\beta}(t)}{\beta(t)} = \frac{\dot{\beta}(t)^2}{\beta(t)^2} + \frac{14\dot{\beta}}{t} - \frac{(7\beta(t)+1)^2}{t^2}$$

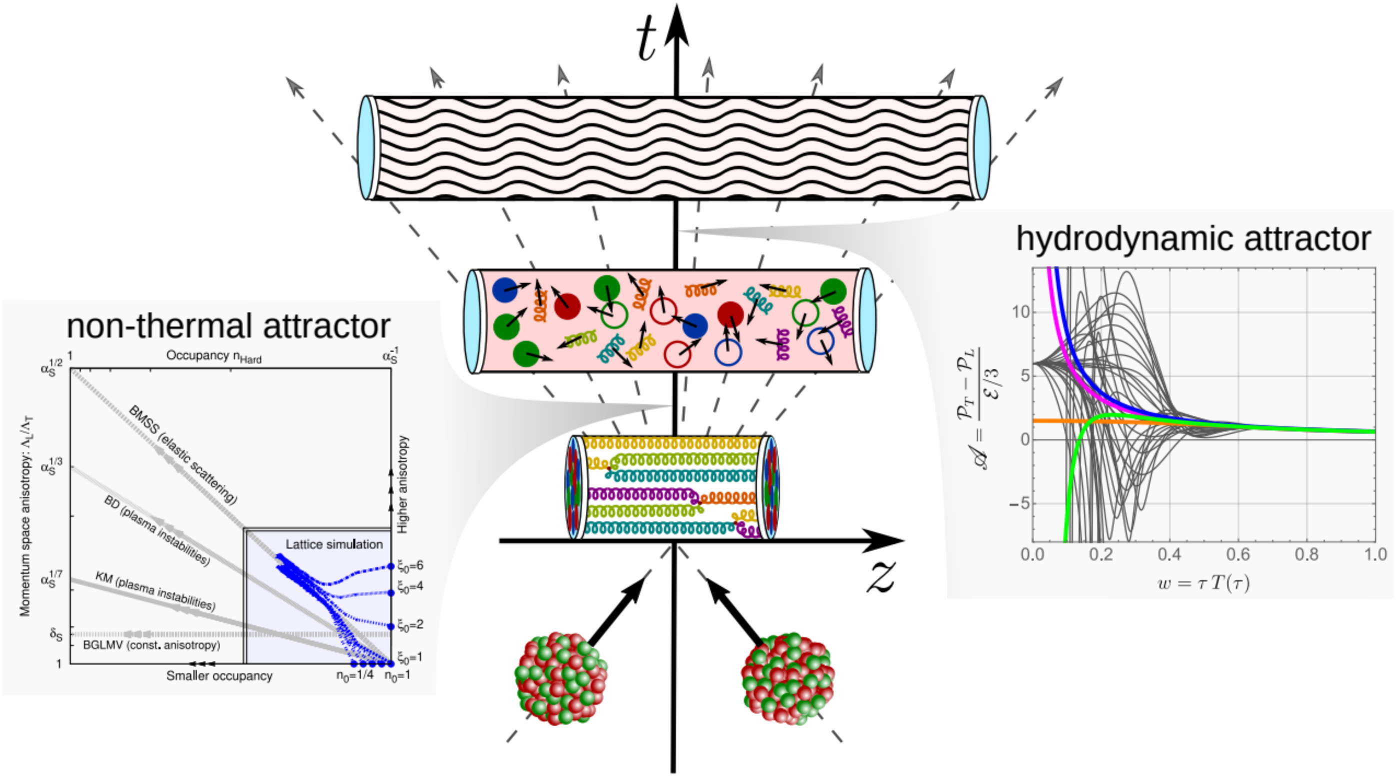
$$\beta(t) = \beta = -1/7$$

At late times:

$$\beta(t) \overset{t\gtrsim 1}{=} -\frac{1}{7}\left[1+\frac{2}{\log(t/t_{\rm ref})}\right]$$

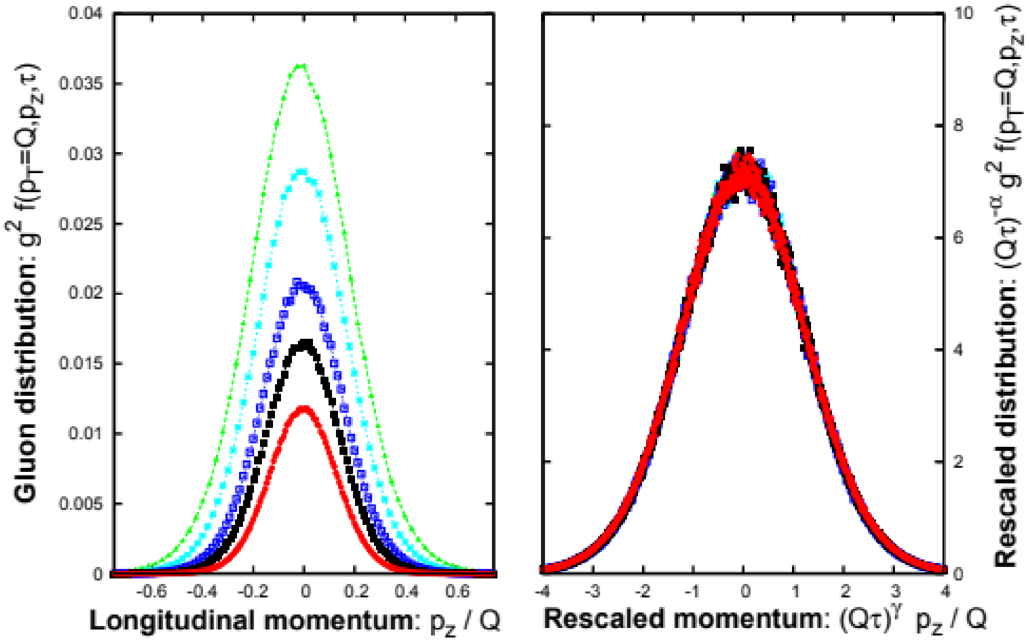
2nd Example: Nonthermal attractor in nuclear collisions

$$f(t, p_T, p_z) = t^\alpha f_S(t^\beta p_T, t^\gamma p_z)$$
$$\alpha = -2/3,\ \beta = 0,\ \gamma = 1/3$$



$$f(p_T, p_z, \tau_{\text{occ}}) = \frac{n_0}{2g^2}\Theta\left(Q - \sqrt{p_T^2 + (\xi_0 p_z)^2}\right)$$

3+1D Classical-Statistical Yang-Mills



Berges et al., 93 RMP (2020)
See also: Kurkela, Zhu, PRL 115 (2015),
Baier et al., PLB 502 (2001).

From/to

$$\tau Q_S \geq \log^2 \alpha_S^{-1}$$
$$\tau Q_S \geq \alpha_S^{-3/2}$$

Berges et al., PRD 89 (2014)

Prescaling with Bjorken expansion

$$\alpha(t) = (d+z)\beta(t)$$
$$(\alpha, \beta, \gamma)_{\mathrm{BMSS}} = (-2/3, 0, 1/3)$$

Prescaling ansatz $\quad 1/\lambda \gg f_{\mathbf{p}} \gg 1$

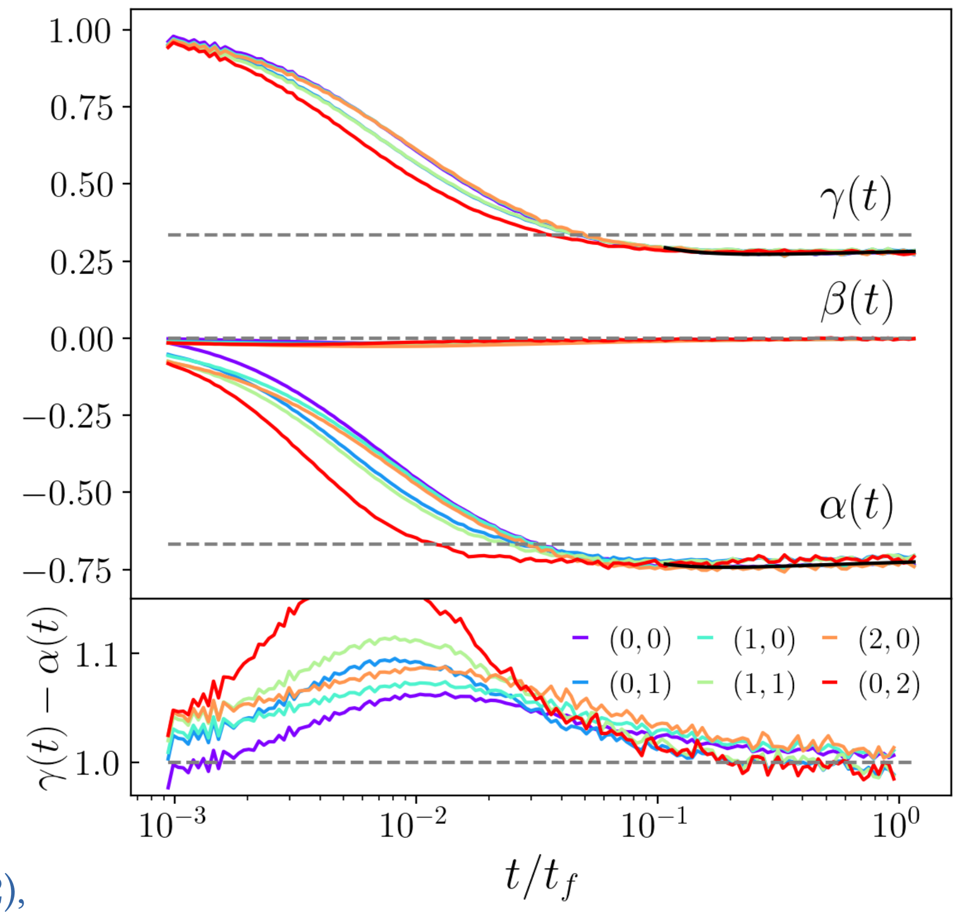
$$f(\tau, \mathbf{p}_\perp, p_z) = A(\tau) f_S(B(\tau)\mathbf{p}_\perp, G(\tau)p_z)$$

To appear: Heller, Mazeliauskas, TP

in Boltzmann gives $\quad G(\tau) = \exp[\int_{\tau_{\mathrm{ref}}}^{\tau} d\tau' \gamma(\tau')/\tau']$

$$\tau^2 \ddot{\alpha}(\tau) = 3\tau\dot{\alpha}(\tau)(1 + 3\alpha(\tau)) - \alpha(\tau)(3\alpha(\tau) + 2)^2$$

$$\alpha(\tau) = -2/3$$



via energy and particle number conservation $\quad \mathcal{P}_L \ll \mathcal{P}_\perp$

$$0 = \alpha(\tau) - 3\beta(\tau) - \gamma(\tau) + 1 \quad \text{energy}$$
$$0 = \alpha(\tau) - 2\beta(\tau) - \gamma(\tau) + 1 \quad \text{particle}$$

$$\begin{cases} 0 = \beta(\tau) \\ 1 = \gamma(\tau) - \alpha(\tau) \end{cases}$$

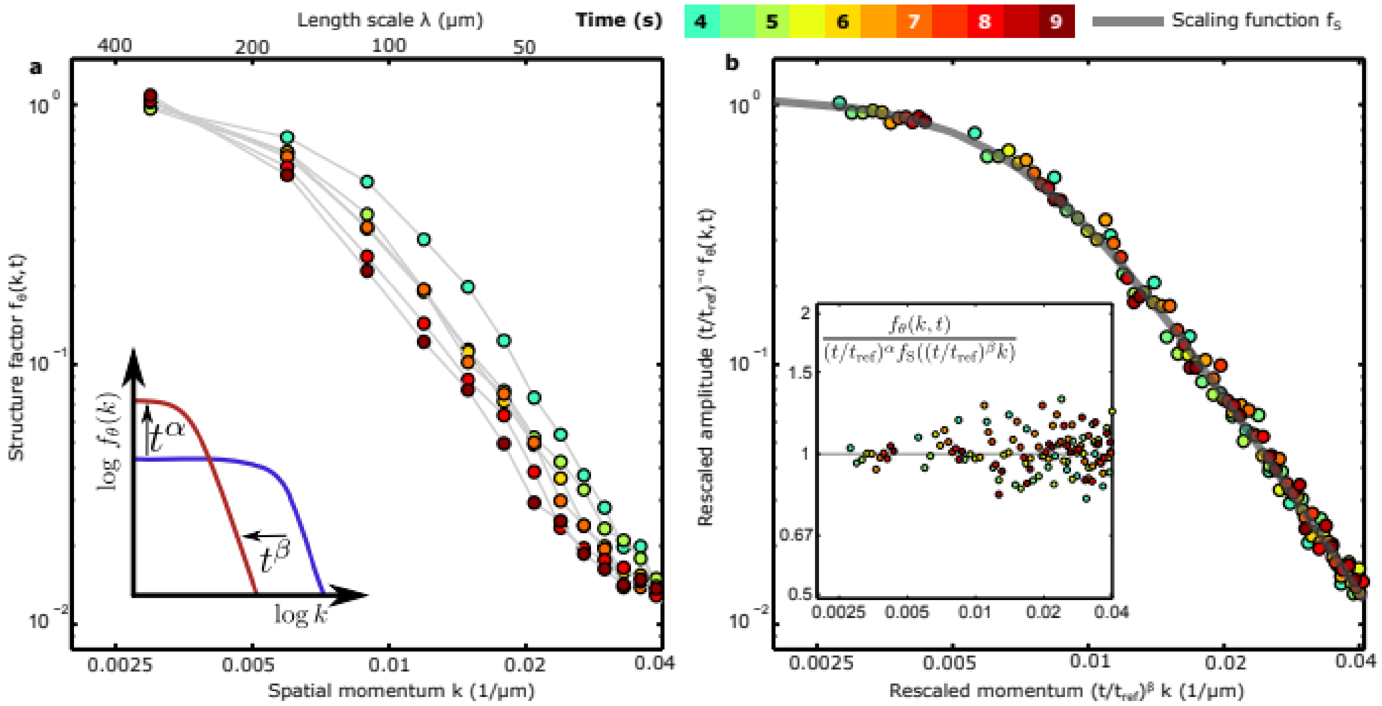
Earlier works: Berges, Mazeliauskas, PRL 122 (2019); Brewer et. al JHEP 145 (2022), Mikheev et. al, PRD 105 (2022)

3rd Example: Inverse particle cascade

$f \sim 1/\lambda$

$$\alpha = d\beta$$
$$(\alpha, \beta) = (d/2, 1/2)$$

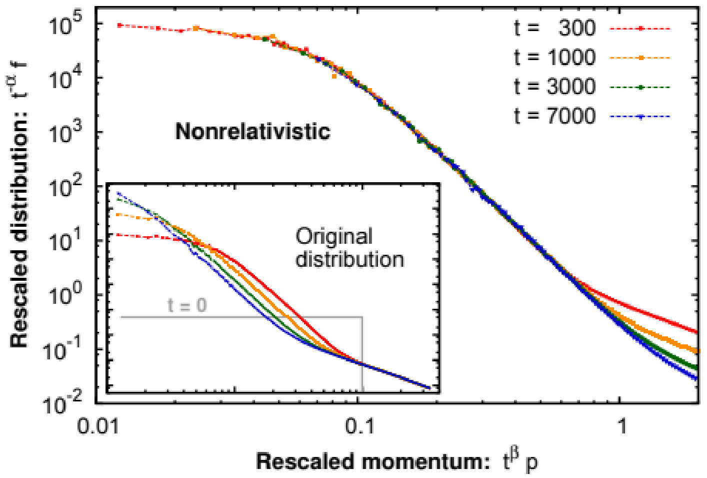
Ultra-cold quantum gases   Prüfer et al., Nature 563, 217 (2018)



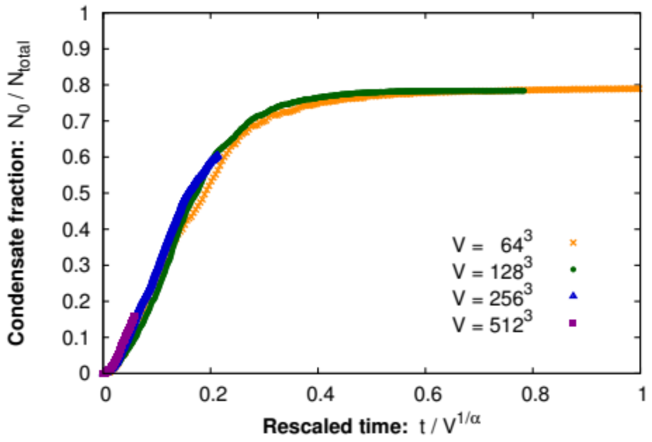
$\alpha = 0.33 \pm 0.08, \quad \beta = 0.54 \pm 0.06$

See also: Erne et al., Nature 563, 225 (2018),
Glidden et al., Nature Phys. 17, 457 (2021).

$\alpha = 1.66 \pm 0.12\,, \qquad \beta = \ 0.55 \pm 0.03$



Gross–Pitaevskii

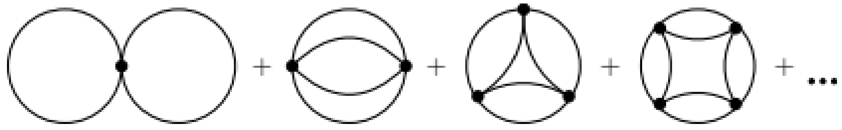


Orioli, Boguslavski, Berges, PRD 92 (2015)

Inverse cascade via far-from-equilibrium quantum fields

TP, Heller, Berges, PRL 130 (2023)

$$\hat{H}(t) = \int d^3x \left[\frac{1}{2} \left(\partial_t \hat{\Phi}_a(t,\mathbf{x}) \right)^2 + \frac{1}{2} \left(\nabla_{\mathbf{x}} \hat{\Phi}_a(t,\mathbf{x}) \right)^2 + \frac{\lambda}{4!N} \left(\hat{\Phi}_a(t,\mathbf{x}) \hat{\Phi}_a(t,\mathbf{x}) \right)^2 \right]$$

$\langle \hat{H}(t) \rangle \longrightarrow$ 

Solve quantum evolution equations of

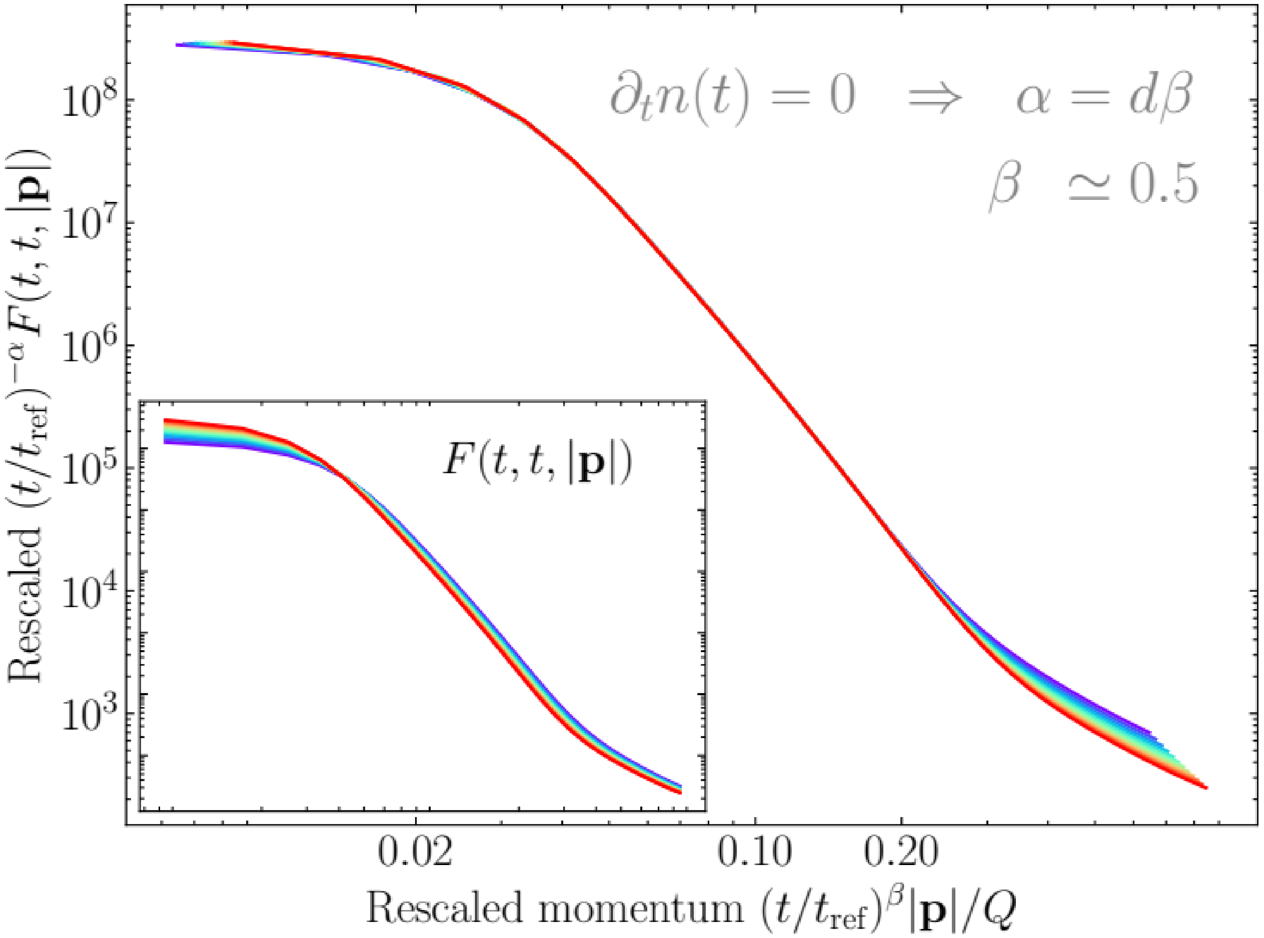
$$F_{ab}(t,t',\mathbf{x}-\mathbf{x}') = \frac{1}{2} \langle \{ \hat{\Phi}_a(t,\mathbf{x}), \hat{\Phi}_b(t',\mathbf{x}') \} \rangle - \langle \hat{\Phi}_a(t,\mathbf{x}) \rangle \langle \hat{\Phi}_b(t',\mathbf{x}') \rangle$$

$$\rho_{ab}(t,t',\mathbf{x}-\mathbf{x}') = i \langle [\hat{\Phi}_a(t,\mathbf{x}), \hat{\Phi}_b(t',\mathbf{x}')] \rangle$$

$$F(t_0,t_0,\mathbf{p}) = \frac{1}{\sqrt{\mathbf{p}^2+m^2}} \left[\left(\frac{N n_0}{\lambda} \right) \theta(Q - |\mathbf{p}|) + \frac{1}{2} \right]$$

We consider $\lambda = 0.01, N = 4$ and $n_0 = 25$ in this work.

$$F(t,t,|\mathbf{p}|) = (t/t_{\text{ref}})^{\alpha} F_S \left((t/t_{\text{ref}})^{\beta} |\mathbf{p}| \right)$$



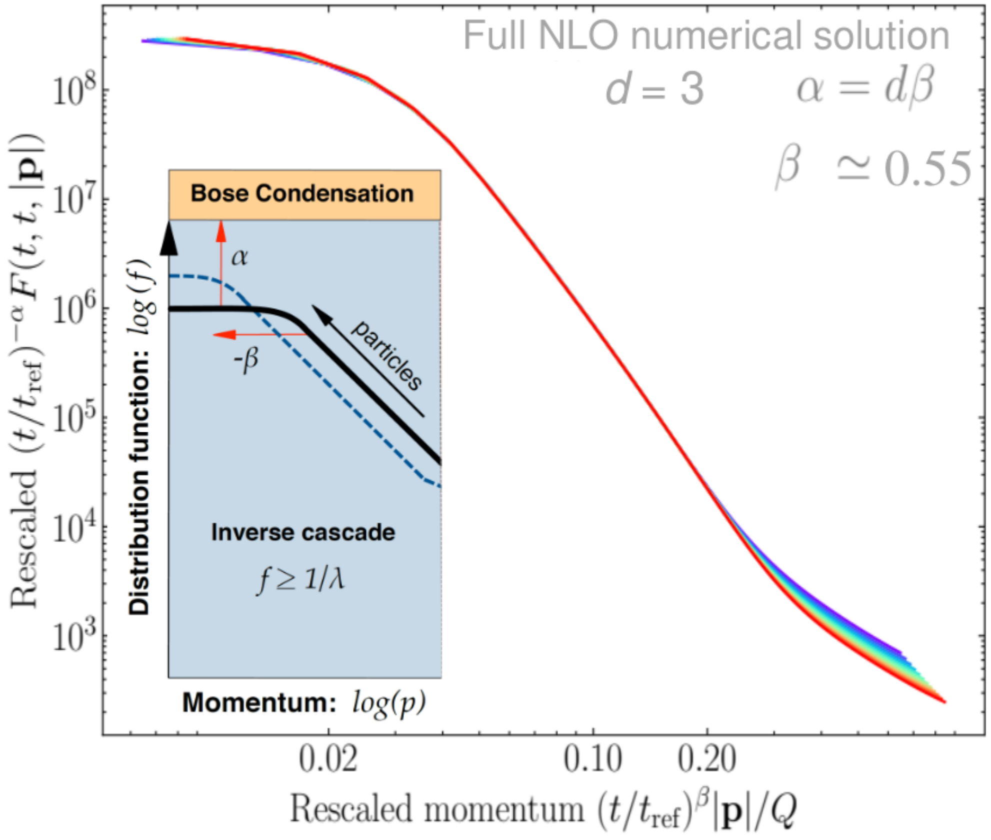
See also: Berges, Rothkopf, Schmidt, PRL 101, 041603 (2008)

Linear response far from equilibrium analysis

TP, Heller, Berges, PRL 130 (2023)

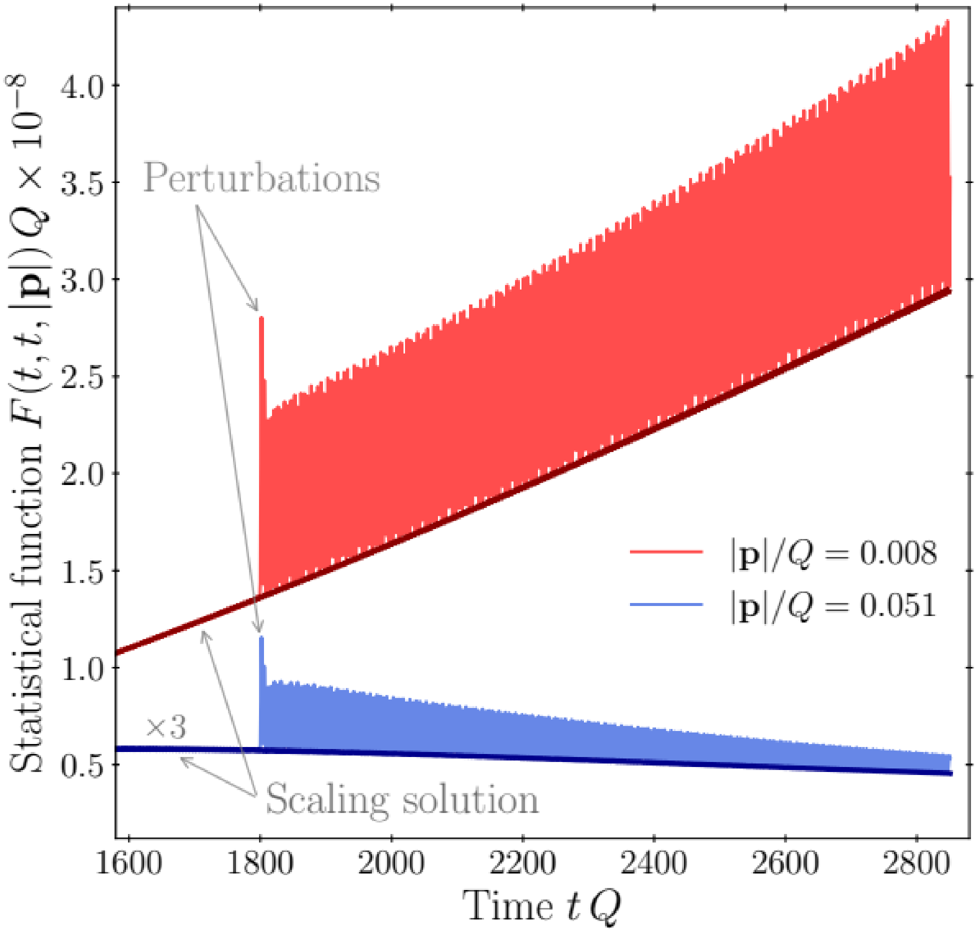
Unperturbed scaling solution

$$F(t,t,|\mathbf{p}|) = (t/t_{\mathrm{ref}})^{\alpha} F_S\left((t/t_{\mathrm{ref}})^{\beta}|\mathbf{p}|\right)$$



Response after a perturbation

$$F(t,t,|\mathbf{p}|) = (t/t_{\mathrm{ref}})^{\alpha} F_S((t/t_{\mathrm{ref}})^{\beta}|\mathbf{p}|) + \delta F(t,t,|\mathbf{p}|)$$



perturbations grow at lower momenta !

Attractor property ?

... and decay at higher momenta

Universal scaling of perturbations

TP, Heller, Berges, PRL 130 (2023)

Perturbations are described by time- and momentum-dependent rate integral $\delta F(t, t, \mathbf{p}) \sim e^{-\Gamma(t,\mathbf{p})}$

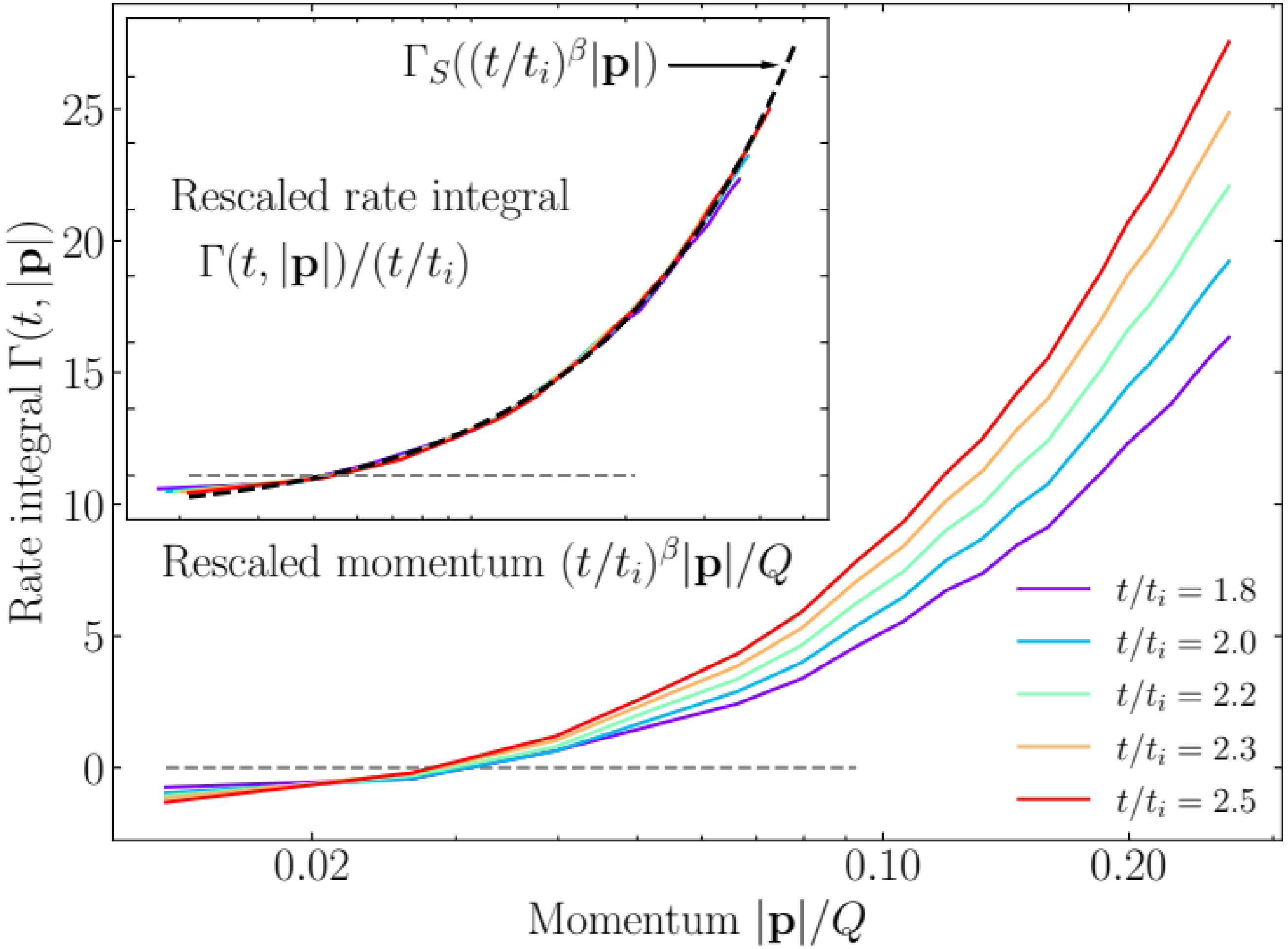
$$\Gamma(t, |\mathbf{p}|) = (t/t_{\text{ref}})\Gamma_S((t/t_{\text{ref}})^\beta |\mathbf{p}|)$$

where the scaling function can be captured by

$$\Gamma_S((t/t_{\text{ref}})^\beta |\mathbf{p}|) = A\ (t/t_{\text{ref}})^\beta |\mathbf{p}|/Q - B$$

Universal amplitude ratio: $A/B^{\beta+1} = 20.8 \pm 1.2$

For a given momentum $|\mathbf{p}|$, the positive (stable) contribution $\sim A\, t^{\beta+1}|\mathbf{p}|$ will eventually outgrow the negative (unstable) $\sim B\, t$ term for all non-zero momenta in the scaling regime.



Scaling instability

Finite systems ($|\mathbf{p}|_{\text{low}} \sim 1/L$) will always detect attractor properties at late enough times ('self-organized' scaling)

Summary

based on TP, Heller, Berges, PRL 130, 031602 (2023), Heller, Mazeliauskas, TP, to appear soon

Two aspects of universal dynamics around nonthermal attractors:
1) Dynamics of slow degrees of freedom (prescaling exponents) in the basin of attraction of a nonthermal attractor. In QCD kinetic theory, scaling breaking terms lead to an approach characterized by fast (power-law) and slow (log) modes.

2) Stability properties: System shows attractor behavior after $t \sim 1/|\mathbf{p}|^{1/\beta}$ for non-vanishing $|\mathbf{p}|$ due to a scaling instability in the presence of unstable directions.
Self-organized scaling (no fine-tuning) can be realized in the presence of both stable and unstable directions for the dynamics.

Open Questions

1) Relation between prescaling and stability analysis ?

2) Utilize prescaling equations to study thermalization dynamics for ultra-relativistic heavy ion collisions at realistic coupling strengths ?

Thank you for your attention !