Modular graph functions and asymptotic expansions of Poincaré series

Daniele Dorigoni

Joint work w/
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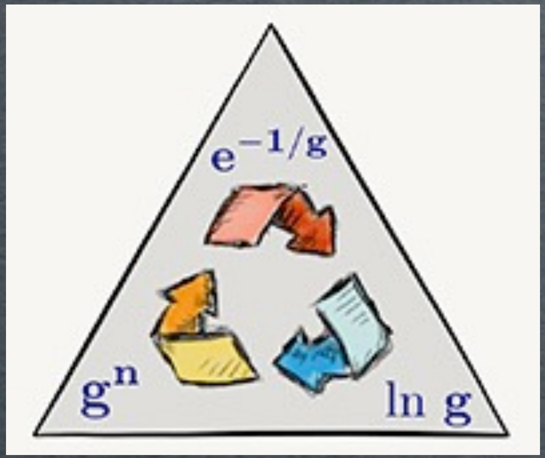
[1903.09250]
[2001.11035]



Resurgence @ KITP 2020 - Online Reunion

Modularity in String Theory

World-sheet:
Modular Graph Functions (MGF)

Space-time:
Coefficient functions for
low-energy expansion

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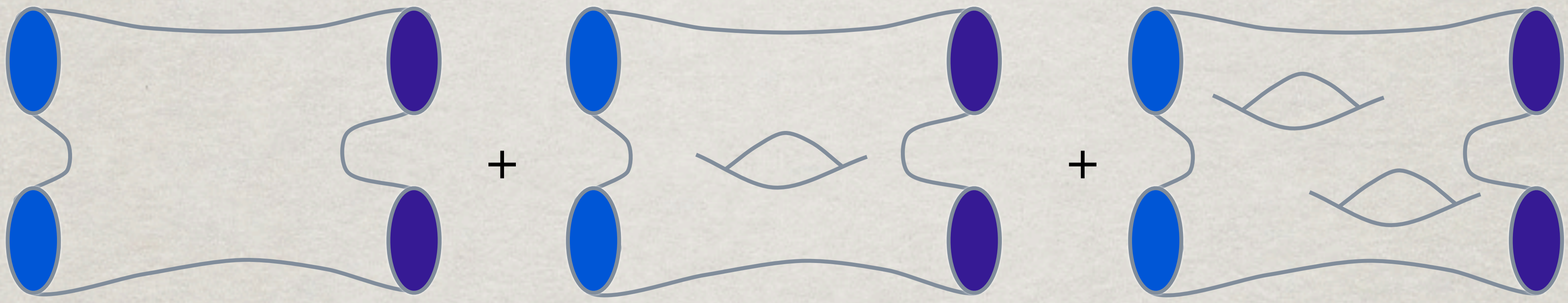
Modularity in String Theory

World-sheet: n-point amplitude, e.g. 4 gravitons

on genus h Riemann surface Σ_τ



Modularity in String Theory

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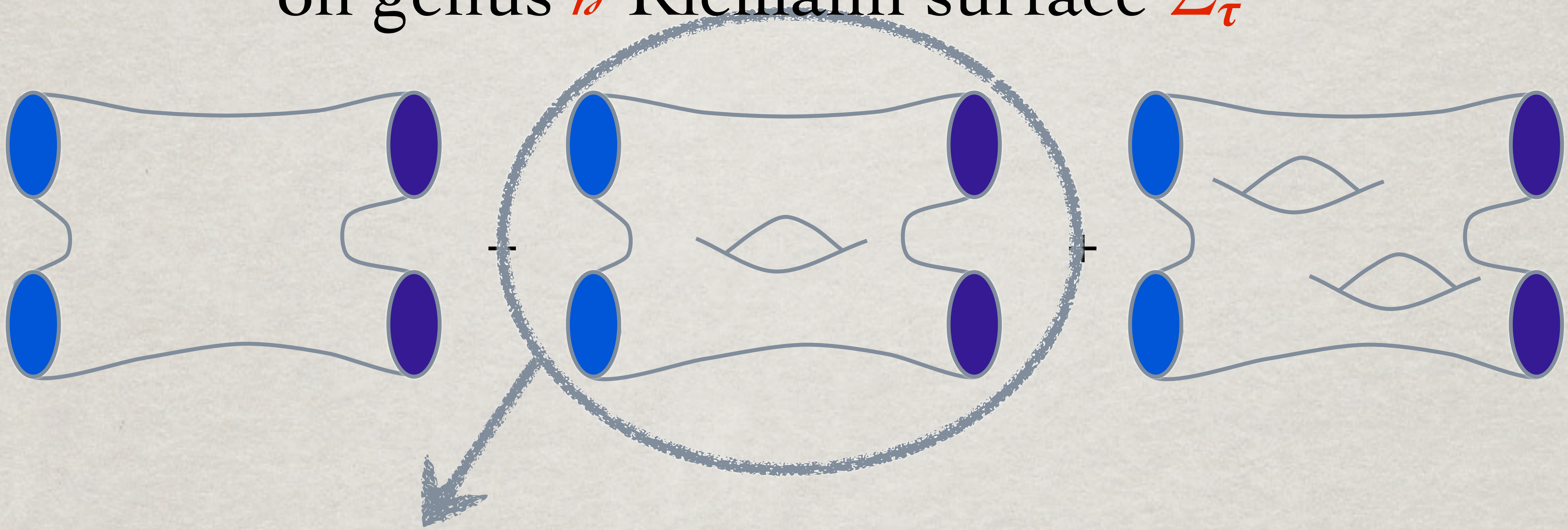
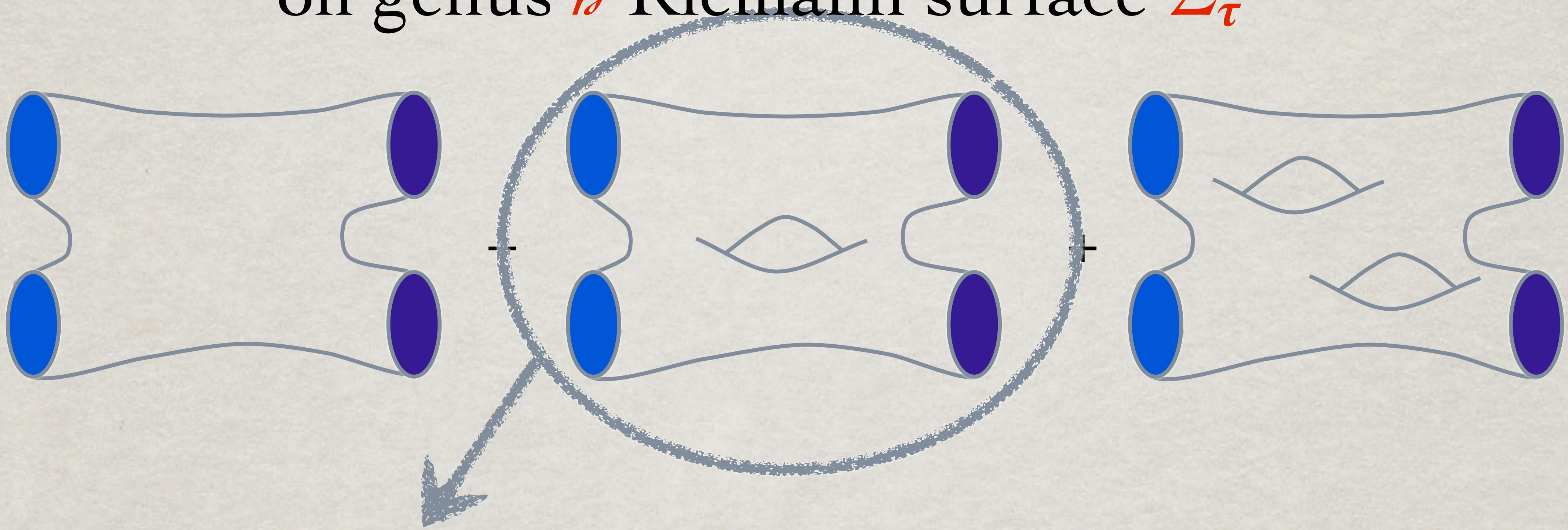


In particular genus 1 Case: torus amplitude

Modularity in String Theory

<u>World-sheet:</u> n-point amplitude, e.g. 4 gravitons on genus h Riemann surface Σ_τ



In particular genus 1 Case: torus amplitude

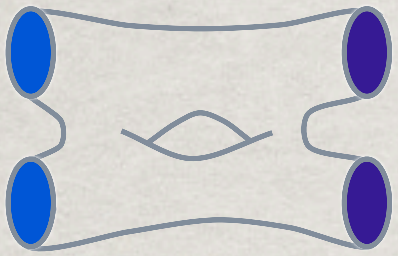
We need to integrate over complex structure $\boldsymbol{\tau}$

$\Sigma_\tau = \Sigma_{\tau'}$ if $\tau' = \gamma\tau$ with γ in $SL(2,\mathbb{Z})$

Modular Invariance

$SL(2,\mathbb{Z})$

Modularity in String Theory

World-sheet:  e.g. 4 pt amplitude

$$\mathcal{A}_1 = \int_{\mathcal{M}_1} d\mu(\tau) \int_{\Sigma_\tau} \prod_{i=1}^{4} \frac{d^2 z_i}{\text{“vol}\, SL(2,\mathbb{Z})\text{”}} \langle V_1(z_1) \cdots V_4(z_4) \rangle$$

Modularity in String Theory

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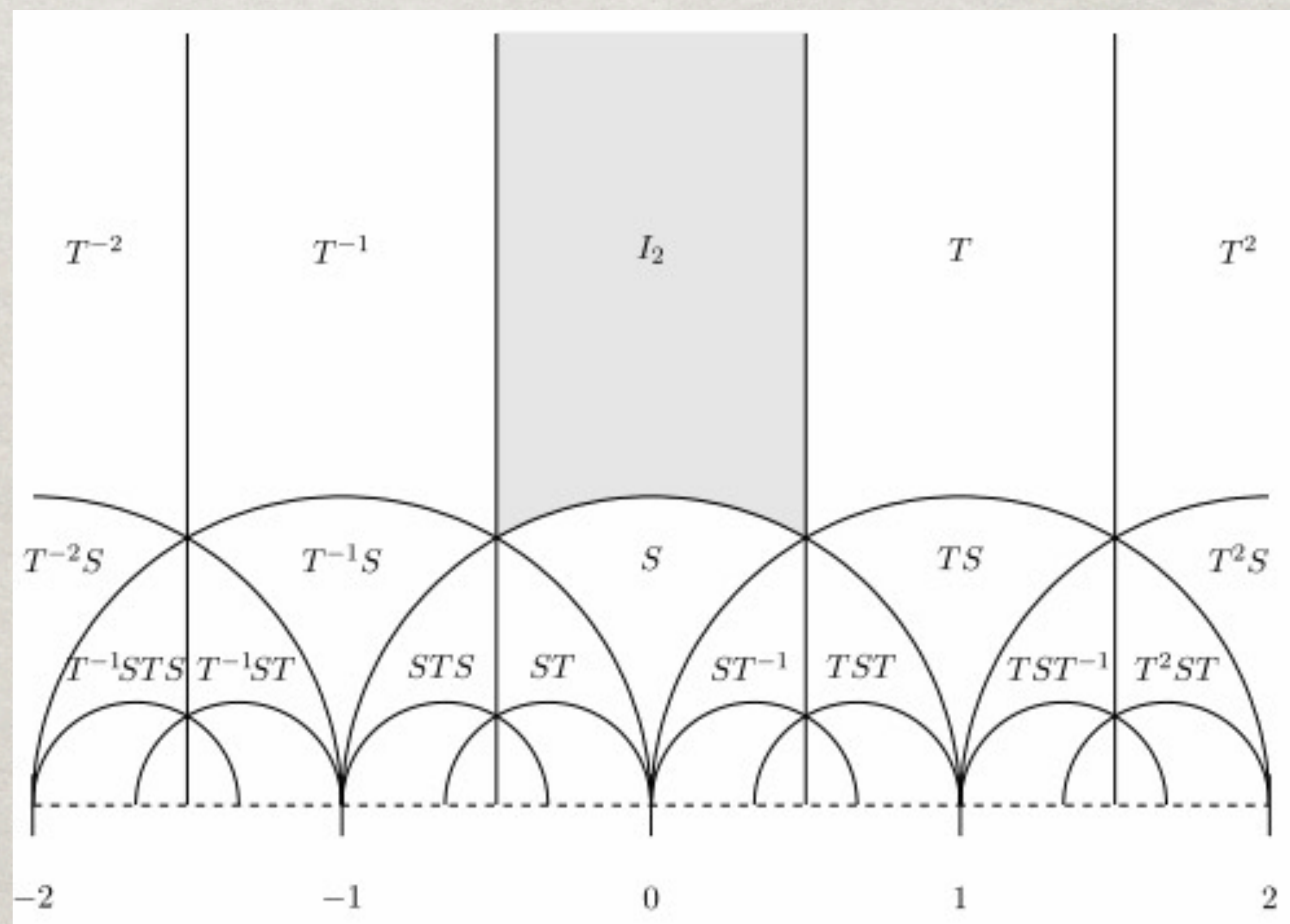
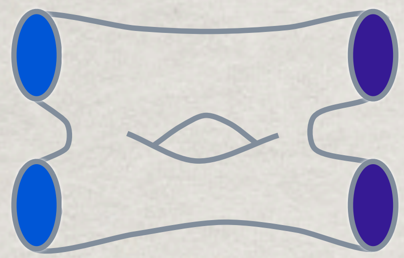
Integral over
complex struct
i.e. fundamental
domain of SL(2,Z)



Modularity in String Theory

World-sheet:  e.g. 4 pt amplitude

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Integral over complex struct.

Modular invariant integrand!
computable in $\alpha' = \ell_s^2$ expansion

invariant under SL(2,Z)

Modular Graph Functions

Modularity in String Theory

$$\int_{\Sigma_\tau} \prod_{i=1}^{4} \frac{d^2 z_i}{\text{“vol}\, SL(2,\mathbb{Z})\text{”}} \langle V_1(z_1) \cdots V_4(z_4) \rangle \sim \int_{\Sigma_\tau} \prod_{i=1}^{4} d^2 z_i \exp\Big[\sum_{i<j} s_{ij} G(z_i - z_j|\tau) \Big]$$

Modularity in String Theory

$$\int_{\Sigma_\tau} \prod_{i=1}^{4} \frac{d^2 z_i}{\text{“vol}\, SL(2,\mathbb{Z})\text{”}} \langle V_1(z_1) \cdots V_4(z_4) \rangle \sim \int_{\Sigma_\tau} \prod_{i=1}^{4} d^2 z_i \exp\left[\sum_{i<j} s_{ij} G(z_i - z_j|\tau) \right]$$

$s_{ij} = -\frac{\alpha'}{4}(p_i + p_j)^2$ Mandelstam variables

$G(z|\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2\, e^{2\pi i(nu - mv)}}{\pi |m\tau + n|^2}$ SL(2,Z) invariant Torus Green Function

$$z = u\tau + v$$

$$\tau = \tau_1 + i\tau_2$$

Modularity in String Theory

$$\int_{\Sigma_\tau} \prod_{i=1}^{4} \frac{d^2 z_i}{\text{“vol}\, SL(2,\mathbb{Z})\text{”}} \langle V_1(z_1)\cdots V_4(z_4)\rangle \sim \int_{\Sigma_\tau} \prod_{i=1}^{4} d^2 z_i \exp\Big[\sum_{i<j} s_{ij} G(z_i - z_j|\tau)\Big]$$

Expand integrand at low energies,
i.e. α' small, and integrate over insertion points

Obtain Feynman-like rules [D'Hoker, Green, Vanhove]

Modularity in String Theory

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Expand integrand at low energies,
i.e. $\boldsymbol{\alpha}$' small, and integrate over insertion points

Obtain Feynman-like rules [D'Hoker, Green, Vanhove]

$$C_{3,1,1}(\tau) = \sum_{(m_i,n_i)\neq(0,0)} \frac{(\tau_2/\pi)^5}{|m_1\tau+n_1|^6|m_2\tau+n_2|^2|(m_1+m_2)\tau+(n_1+n_2)|^2}$$

Modular Invariant (MGF)

Modularity in String Theory

Space-time: Type II(B) in 10-D has another SL(2,Z) action

Modular group simply acts on axio-dilaton

$$z = C_{(0)} + ie^{-\phi} \longrightarrow \gamma \cdot z = \frac{az+b}{cz+d}$$

$$\langle e^{\phi} \rangle = g_s \qquad \gamma \in SL(2,\mathbb{Z})$$

In general U-Duality Group via compactification on T^d
[Hull-Townsend]

Preserved at each order in low-energy expansion
i.e. α' expansion

Modularity in String Theory

Space-time:

SL(2,Z) Preserved at each order in α' expansion
i.e. low energy expansion of II(B) in 10-D

$$S = \frac{1}{(\alpha')^4}\int d^{10}x\sqrt{-g}\Big(R + (\alpha')^3 f_{R^4}(\tau)R^4 + (\alpha')^5 f_{D^4R^4}(\tau)D^4R^4 + (\alpha')^6 f_{D^6R^4}(\tau)D^6R^4 + \ldots\Big)$$

$$\tau = \langle C_{(0)} + ie^{-\phi}\rangle$$

Modularity in String Theory

Space-time:

SL(2,Z) Preserved at each order in α' expansion
i.e. low energy expansion of II(B) in 10-D

$$S = \frac{1}{(\alpha')^4}\int d^{10}x\sqrt{-g}\Big(R + (\alpha')^3 f_{R^4}(\tau)R^4 +$$
$$(\alpha')^5 f_{D^4R^4}(\tau)D^4R^4 + (\alpha')^6 f_{D^6R^4}(\tau)D^6R^4 + \ldots\Big)$$

Modular Invariant coefficient functions
(SUSY)

From N=4 Susy Loc. [Chester, Green, Pufu, Wang, Wen]

Modularity in String Theory

World-sheet:

Torus world-sheet, Modular Graph Functions (MGF)

τ really world-sheet torus

Space-time:

Coefficient functions for
low-energy expansion

τ really axio-dilaton, i.e. $1/g_s$

Modularity in String Theory

World-sheet:
Modular Graph Functions (MGF)

How do we study $f(\tau)$
invariant under SL(2,Z)?

Space-time:
Coefficient functions for
low-energy expansion

Modular Differential Eq:

Both cases have underlying (modular) differential eq.

$$(\Delta - s(s-1))f(\tau) = S(\tau)$$

$$\Delta = \tau_2^2(\partial_{\tau_1}^2 + \partial_{\tau_2}^2)$$
$$\tau = \tau_1 + i\tau_2$$

Modular Invariant Source term

<u>Origin of Diff. Eq:</u>

- ✳ World-Sheet: "just" act with Laplacian on multiple lattice-sums [D'Hoker, Green, Vanhove]
- ✳ Space-Time: Supersymmetry constrains coefficient functions [Green, Sethi, Vanhove; Bossard, Verschinin]

Modular Differential Eq:

E.g. Modular Graph Function

$$C_{3,1,1}(\tau) = \sum_{(m_i,n_i)\neq(0,0)} \frac{(\tau_2/\pi)^5}{|m_1\tau+n_1|^6|m_2\tau+n_2|^2|(m_1+m_2)\tau+(n_1+n_2)|^2}$$

$$(\Delta-6)C_{3,1,1}(\tau) = \frac{172}{5}\pi^{-5}\zeta_{10}\,\mathrm{E}_5(\tau) - 16\pi^{-5}\zeta_4\zeta_6\,\mathrm{E}_2(\tau)\mathrm{E}_3(\tau) + \frac{\zeta_5}{10}$$

E.g. Low-energy expansion coefficient

$$(\Delta-12)f_{D^6R^4}(\tau) = -4\zeta_3^2\,\mathrm{E}_{3/2}(\tau)^2$$

Standard
modular invariant
non-holo
Eisenstein series

Modular Differential Eq:

Start from:

$$\boxed{(\Delta - s(s-1))f(\tau) = S(\tau)}$$

as common starting point to learn about f($\boldsymbol{\tau}$)

Methods:

- ✶ Fourier Decomposition [Green, Miller, Vanhove]
- ✶ Spectral decomposition [Green, Miller, Vanhove; Klinger-Logan]
- ✶ Lattice sums [Bossard, Kleinschmidt]
- ✶ Poincaré Series (+ Resurgence) [DD, Kleinschmidt]

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Solution by Poincaré Series:

[Ahlén, Kleinschmidt]

$$(\Delta - s(s-1))f(\tau) = S(\tau)$$

Assume modular Invariant Source term has Poincaré series representation

$$S(\tau) = \sum_{\gamma \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \rho(\gamma\tau)$$

$$B(\mathbb{Z}) = \left\{ \begin{pmatrix} \pm 1 & n \\ 0 & \pm 1 \end{pmatrix} \right\}$$

$$\rho(\tau + 1) = \rho(\tau)$$

Make ansatz for solution:

$$f(\tau) = \sum_{\gamma \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \sigma(\gamma\tau)$$

Solve for "Seed" Function

Solution by Poincaré Series:

[Ahlén, Kleinschmidt]

$$S(\tau) = \sum_{\gamma \in B(\mathbb{Z}) \backslash SL(2,\mathbb{Z})} \rho(\gamma\tau)$$

$$(\Delta - s(s-1))f(\tau) = S(\tau)$$

$$f(\tau) = \sum_{\gamma \in B(\mathbb{Z}) \backslash SL(2,\mathbb{Z})} \sigma(\gamma\tau)$$

$$\Big\downarrow$$

$$(\Delta - s(s-1))\sigma(\tau) = \rho(\tau)$$

<u>Key Pt:</u>

* Generically easier problem!
* We can go back, i.e. reconstruct $f(\tau)$ from seed!

e.g.:

$$(\Delta - 6)C_{3,1,1}(\tau) = \frac{172}{5}\pi^{-5}\zeta_{10}\,\mathrm{E}_5(\tau) - 16\pi^{-5}\zeta_4\zeta_6\,\mathrm{E}_2(\tau)\mathrm{E}_3(\tau) + \frac{\zeta_5}{10}$$

$$\Big\downarrow$$

$$C_{3,1,1}(\tau) = \sum_{\gamma\in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \sigma(\gamma\tau)$$

$$(\Delta - 6)\sigma(\tau) = \frac{172}{5}\pi^{-5}\zeta_{10}\tau_2^5 - 16\pi^{-5}\zeta_4\zeta_6\mathrm{E}_2(\tau)\tau_2^3 + \frac{\zeta_5}{10}\tau_2^{\epsilon}$$

Used Poincaré series rep. for non-Holo E_s:

$$\mathrm{E}_s(\tau) = \frac{1}{2\zeta_{2s}} \sum_{(c,d)\neq(0,0)} \frac{\tau_2^s}{|c\tau + d|^{2s}} = \sum_{\gamma\in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \Big[\mathrm{Im}(\gamma\tau)\Big]^s$$

$$(\Delta - 6)\sigma(\tau) = \frac{172}{5}\pi^{-5}\zeta_{10}\tau_2^5 - 16\pi^{-5}\zeta_4\zeta_6 \mathrm{E}_2(\tau)\tau_2^3 + \frac{\zeta_5}{10}\tau_2^\epsilon$$

Decompose in Fourier and solve mode by mode

$$\sigma(\tau) = \sum_{n\in\mathbb{Z}} c_n(\tau_2) e^{2\pi i n \tau_1}$$

$$c_0(\tau_2) = Pol(\tau_2)$$

$$c_n(\tau_2) = \frac{2\pi^2}{945}\sigma_{-3}(|n|)\tau_2^2 e^{-2\pi|n|\tau_2}$$

Divisors Sum

$$\sigma_s(n) = \sum_{d|n} d^s$$

$$(\Delta - 6)\sigma(\tau) = \frac{172}{5}\pi^{-5}\zeta_{10}\tau_2^5 - 16\pi^{-5}\zeta_4\zeta_6 \mathrm{E}_2(\tau)\tau_2^3 + \frac{\zeta_5}{10}\tau_2^{\epsilon}$$

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Divisors Sum

$$\sigma_s(n) = \sum_{d|n} d^s$$

Ok, so what does this tell us about $C_{3,1,1}$?

From Seed to Function:

$$f(\tau) = \sum_{\gamma \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \sigma(\gamma\tau)$$

$$\sigma(\tau) = \sum_{n\in\mathbb{Z}} c_n(\tau_2) e^{2\pi i n \tau_1}$$

$$? \downarrow ?$$

$$f(\tau) = \sum_{n\in\mathbb{Z}} a_n(\tau_2) e^{2\pi i n \tau_1}$$

Focus on Zero-Mode (i.e. "Topologically trivial sector" $\tau_1 \text{``}=\text{''} \theta$)

Really thinking of different topological sectors as different Instantons sectors!
Topologically trivial sector should contain
Perturbation theory + IIbar sector

From Seed to Function:

$$f(\tau) = \sum_{\gamma \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \sigma(\gamma\tau)$$

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$$\downarrow$$

$$f(\tau) = \sum_{n\in\mathbb{Z}} a_n(\tau_2) e^{2\pi i n \tau_1}$$

Focus on Zero-Mode (i.e. "Topologically trivial sector" τ_1 " $=$ " θ)

$$a_0(\tau_2) = c_0(\tau_2) + \tau_2 \sum_{c>0} \sum_{m\in\mathbb{Z}} \sum_{q\in(\mathbb{Z}/c\mathbb{Z})^\times} e^{2\pi i m q/c} \int_{\mathbb{R}} e^{-2\pi m \frac{it}{c^2(1+t^2)}} c_m\left(\frac{\tau_2^{-1}}{c^2(1+t^2)}\right) dt$$

Non-Zero-Modes more complicated (Kloosterman Sums) but can still be reconstructed

Key Pt: we can reconstruct all Fourier coefficients from seed function!

$\downarrow$

Can we obtain the "Weak" coupling $\tau_2 \to \infty$ expansion for MGF and low-energy coefficient functions?

- τ world-sheet torus
- τ axio-dilaton, i.e. $1/g_s$

DIFFERENT nature of NP corrections!
Abuse of notation: I will refer to NP physics as Instantons/anti-Instantons in both cases

Key Pt: we can reconstruct all Fourier coefficients from seed function!

$\downarrow$

Can we obtain the "Weak" coupling $\tau_2 \to \infty$ expansion for MGF and low-energy coefficient functions?

Focus on 0-mode, i.e. "topologically" trivial sector, i.e. independent from $\theta \sim \mathrm{Re}\,\tau = \tau_1$

We studied the weak coupling expansion for the zero Fourier mode obtained from the general seed

$$\boxed{c_n(\tau) = \sigma_a(|n|)(4\pi|n|)^b \tau_2^r e^{-2\pi|n|\tau_2}}$$

Weak Coupling Expansion:

Using

$$a_0(\tau_2) = c_0(\tau_2) + \tau_2 \sum_{c>0} \sum_{m\in\mathbb{Z}} \sum_{q\in(\mathbb{Z}/c\mathbb{Z})^\times} e^{2\pi i m q/c} \int_{\mathbb{R}} e^{-2\pi m \frac{it}{c^2(1+t^2)}} c_m\left(\frac{\tau_2^{-1}}{c^2(1+t^2)}\right) dt$$

We have to compute expansion for $\tau_2 \to \infty$ of:

$$\sum_{c>0} \sum_{m>0} \sum_{q\in(\mathbb{Z}\backslash c\mathbb{Z})^\times} e^{2\pi i m q/c} \sigma_a(m) \left(\frac{4\pi m}{c^2\tau_2}\right)^b \int_{\mathbb{R}} e^{-2\pi m \frac{1+it}{c^2\tau_2(1+t^2)}} \frac{dt}{(1+t^2)^r}$$

Note that τ_2 only appears in the combination m/τ_2

So after some manipulations we are asking about the asymptotic expansion of

$$\sum_{m\geq 0} \varphi((m+h)t) \sim ? \qquad \text{as } t = 1/\tau_2 \to 0$$

Zagier's Trick:

$$\boxed{\sum_{m\geq 0}\varphi((m+h)t)\sim\ ? \qquad \text{as } t=1/\tau_2\to 0}$$

If the function φ has expansion around t~0:

$$\boxed{\varphi(t)\sim\sum_{n\geq 0}b_n t^n}$$

Zagier's Trick:

$$\sum_{m\geq 0} \varphi((m+h)t) \sim \,? \qquad \text{as } t = 1/\tau_2 \to 0 \quad 0 < h \leq 1$$

If the function φ has expansion around t~0:

$$\varphi(t) \sim \sum_{n\geq 0} b_n t^n \quad \Longrightarrow \quad \sum_{m\geq 0} \varphi((m+h)t) \sim \frac{I_\varphi}{t} + \sum_{n\geq 0} b_n \zeta(-n,h) t^n$$

Zagier's Trick:

$$\boxed{\sum_{m\geq 0} \varphi((m+h)t) \sim \,? \qquad \text{as } \underset{0<h\leq 1}{t = 1/\tau_2 \to 0}}$$

If the function φ has expansion around t~0:

$$\boxed{\varphi(t) \sim \sum_{n\geq 0} b_n t^n} \longrightarrow \boxed{\sum_{m\geq 0} \varphi((m+h)t) \sim \frac{I_\varphi}{t} + \sum_{n\geq 0} b_n \zeta(-n,h) t^n}$$

"Riemann" Term

$$I_\varphi = \int_0^\infty \varphi(t)dt$$

Zagier's Trick:

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"Riemann" Term

$$I_\varphi = \int_0^\infty \varphi(t)dt$$

"Euler" Term (exchange sums)

$$\zeta(-n,h) \text{``}=\text{''} \sum_{m\geq 0} (m+h)^n$$

Hurwitz zeta and 0<h≤1

Weak Coupling Expansion:

Apply this method to $C_{3,1,1}$

$y = \pi\tau_2$

$$a_0 \sim \frac{2}{155925}y^5 + \frac{2\zeta_3}{945}y^2 - \frac{\zeta_5}{180} + \frac{7\zeta_7}{16y^2} - \frac{\zeta_3\zeta_5}{2y^3} + \frac{43\zeta_9}{64y^4}$$

[D'Hoker, Green, Vanhove]

and f_{D6R4}

$$a_0 \sim \frac{2}{3}\zeta_3^2 g_s^{-3} + \frac{4}{3}\zeta_2\zeta_3 g_s^{-1} + 4\zeta_4 g_s + \frac{4\zeta_6}{27}g_s^3$$

[Green, Miller, Vanhove]

In all these cases we find a truncating perturbative expansions BUT we still expect NP physics!

Cheshire Cat Resurgence:

[Dunne, Ünsal] - [Kozçaz, Sulejmanpasic, Tanizaki, Ünsal]

Consider deformation for seed of $C_{3,1,1}$

$$\sigma(\tau) = \sum_{n\in\mathbb{Z}} c_n(\tau_2) e^{2\pi i n \tau_1}$$

$$c_0(\tau_2) = Pol(\tau_2)$$

$$c_n(\tau_2) = \frac{2\pi^2}{945}\sigma_{-3}(|n|)\tau_2^2 e^{-2\pi|n|\tau_2}\,(4\pi|n|)^b$$

Only at the end consider $b \to 0$

Cheshire Cat Resurgence:

$y = \tau_2$

Consider deformation for seed of $C_{3,1,1}$

Repeating Zagier's trick the contribution to the zero-mode sector now becomes

$$I(b) = I_{\rm pert}(b) + \frac{16\pi^{5-b}}{945\,\zeta(6-2b)}\sin(\pi b)\sum_{n\geq 0}(4\pi y)^{-n-5}\frac{(n+6)\Gamma(n+2+b)\Gamma(n+5+b)}{(n+4)!}$$
$$\times\frac{\zeta(5+n+b)\zeta(2+n+b)\zeta(7+n-b)\zeta(10+n-b)}{\zeta(2n+12)}\,.$$

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$$\times\frac{\zeta(5+n+b)\zeta(2+n+b)\zeta(7+n-b)\zeta(10+n-b)}{\zeta(2n+12)}\,.$$

When $b \to 0$ it reduces to truncating perturbative expansion

Cheshire Cat Resurgence:

Consider deformation for seed of $C_{3,1,1}$

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Factorially divergent asymptotic Cheshire tail

Cheshire Cat Resurgence:

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$$\times\frac{\zeta(5+n+b)\zeta(2+n+b)\zeta(7+n-b)\zeta(10+n-b)}{\zeta(2n+12)}.$$

Borel transforming factorially divergent asymptotic Cheshire tail seems hopeless however we can use Dirichlet series:

$$\frac{\zeta(2+n)\zeta(5+n)\zeta(7+n)\zeta(10+n)}{\zeta(2n+12)} = \sum_{m>0}\sigma_{-3}(m)\sigma_{-5}(m)m^{-n-2},$$

Cheshire Cat Resurgence:

$$I(b) = I_{\text{pert}}(b) + \frac{16\pi^{5-b}}{945\,\zeta(6-2b)}\sin(\pi b)\sum_{n\geq 0}(4\pi y)^{-n-5}\frac{(n+6)\Gamma(n+2+b)\Gamma(n+5+b)}{(n+4)!}$$
$$\times\frac{\zeta(5+n+b)\zeta(2+n+b)\zeta(7+n-b)\zeta(10+n-b)}{\zeta(2n+12)}\,.$$

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$$I_{\text{asy}}(b) = \frac{16}{\pi}(4\pi y)^{-3}\sin(\pi b)\sum_{m>0}\sigma_{-3}(m)\sigma_{-5}(m)\sum_{n\geq 0}(4\pi m y)^{-n-2}(6+n)\Gamma(n+2)$$

Cheshire Cat Resurgence:

$$I_{\rm asy}(b) = \frac{16}{\pi}(4\pi y)^{-3}\sin(\pi b)\sum_{m>0}\sigma_{-3}(m)\sigma_{-5}(m)\sum_{n\geq 0}(4\pi m y)^{-n-2}(6+n)\Gamma(n+2)$$

Apply standard Borel Ecalle median resummation

$$I_{\rm asy}(b) = \frac{16(4\pi y)^{-3}}{\pi}\sum_{m>0}\sigma_{-3}(m)\sigma_{-5}(m)\left[\sin(\pi b)\mathcal{S}_{\pm}[F](4\pi m y) - \pi(\pm i\sin(\pi b))e^{-4\pi m y}(4+4\pi m y)\right]$$

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$$\mathcal{S}_{\pm}[F](4\pi m y) = \lim_{\theta\to 0^{+}}\int_{0}^{\infty e^{\mp i\theta}} e^{-4\pi m y t}\frac{t(6-5t)}{(t-1)^2}dt$$

Usual lateral Borel resummation of asymptotic series

Cheshire Cat Resurgence:

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Very simple Stokes automorphism,
Infinitely many Instantons/anti-Instantons BUT
untangled by Dirichlet series

[Arutyunov,DD, Savin] - [DD, Kleinschmidt]

Cheshire Cat Resurgence:

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Hypothesis: Transseries parameter exponentiates

$\pm i\sin(\pi b) \mapsto e^{\pm i\pi b}$

(Numerically and analytically confirmed in the context of Lambert series and in known cases)

Cheshire Cat Resurgence:

$$I_{\text{asy}}(b) = \frac{16(4\pi y)^{-3}}{\pi} \sum_{m>0} \sigma_{-3}(m)\sigma_{-5}(m)\left[\sin(\pi b)\mathcal{S}_{\pm}[F](4\pi m y) - \pi(\pm i \sin(\pi b))e^{-4\pi m y}(4 + 4\pi m y)\right]$$

b→0

Hypothesis: Transseries parameter exponentiates

$\pm i \sin(\pi b) \mapsto e^{\pm i\pi b}$

$$\boxed{a_0^{NP} \sim \sum_{m>0} \sigma_{-3}(m)\sigma_{-5}(m)(\pi y)^{-2} m e^{-4\pi m y}\left(1 + \frac{1}{\pi m y}\right)}$$

Reproduce [D'Hoker, Duke]

NP corrections (IIbar) entirely captured by asymptotic (truncating) perturbative expansion

These are really D-instanton/anti-D-instanton pairs for D^6R^4 [Green, Miller, Vanhove]

Lambert Series & Iterated integrals

[DD, Kleinschmidt]

$$\mathcal{L}_s(q) = \sum_{k=1}^{\infty} k^{-s} \frac{q^k}{1-q^k} = \sum_{m=1}^{\infty} \sigma_{-s}(m) q^m \sim \int_{\tau}^{i\infty} d\tau_1 \cdots \int_{\tau_{\ell-1}}^{i\infty} d\tau_\ell G_{s+1}(\tau_\ell)$$

Connection with theory of Iterated integrals and one loop string amplitudes [Brown, Brödel, Schlotterer, Zerbini,...]

Lambert Series & Iterated integrals

[DD, Kleinschmidt]

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Asymptotic Expansion for $q = e^{2\pi i\tau} \to 1$

$\mathrm{Im}\tau = \tau_2 > 0$

Lambert Series & Iterated integrals

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$$\mathrm{Li}_s(q^n) = \mathrm{Li}_s(e^{2\pi i n\tau})$$

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For parameter s=m <u>odd integer</u>: Cheshire resurgence

$$\mathcal{L}_m(\tau) = Pert(\tau) + \tau^{m-1} \sum_{n=1}^{\infty} \sigma_{-m}(n) e^{-\frac{2\pi i n}{\tau}}$$

Lambert Series & Iterated integrals

[DD, Kleinschmidt]

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Truncating Taylor-Laurent
Perturbative Expansion

Lambert Series & Iterated integrals

[DD, Kleinschmidt]

$$\boxed{\mathcal{L}_s(q) = \sum_{k=1}^{\infty} k^{-s}\frac{q^k}{1-q^k} = \sum_{m=1}^{\infty}\sigma_{-s}(m)q^m = \sum_{n=1}^{\infty}\mathrm{Li}_s(q^n)}$$

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For parameter s=m odd integer: Cheshire resurgence

$$\boxed{\mathcal{L}_m(\tau) = Pert(\tau) + \tau^{m-1}\sum_{n=1}^{\infty}\sigma_{-m}(n)e^{-\frac{2\pi i n}{\tau}}}$$

NP terms just
S-Dual transformation

$\tau^{m-1}\mathcal{L}_m(-1/\tau)$

Conclusions:

- Poincaré Series representation combined with modular Laplace equations to understand Taylor-Laurent "perturbative" expansion;

- Very interesting numerology (Open/Closed strings, multiple-zeta values and sv prescription) [Brown, Brödel, Schlotterer, Zerbini,...]

- Use of Dirichlet series to disentangle single IIBar contribution from infinitely many

- Cheshire cat resurgence to read NP IIBar sector from deformed asymptotic tail (TS parameter exponentiation?)

- NP corrections are crucial to reproduce correct modularity property!
  Perturbative expansion = period polynomials
  NP contributions = S-Dual transformation
  Connections with iterated integrals [DD, Kleinschmidt]

See Sergei's talk for CS story(maybe)

Thanks!