# Resurgence and continuity with $Z_N$-twisted boundary condition


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Resurgence@KITP2020 07/22/20

# $Z_N$-twisted b.c. for compactified QFT

Adiabatic continuity conjecture: Vacuum structure & $Z_N$ symmetry persists during $Z_N$-twisted compactification

- Fractional instantons cause transition among classical $N$-minima
- makes $Z_N$ stable, leading to volume indep. of vacuum structure

't Hooft, Witten, Gonzales-arroyo, Okawa, Gross, Kitazawa….

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## Prospects and Known facts

- Resurgent structure on $R^{d-1} \times S^1$ may continue to $R^d$ Argyres, Dunne, Unsal (12) Fujimori, et.al (16-18)
- Weak-cplng confinement may be connected to strong-cplng one Unsal (07)
- Adiabatic continuity in 2D sigma model Sulejmanpasic (16) Tanizaki, TM, Sakai (17)

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I'll discuss the conjecture and the resurgent structure with $Z_N$-twist in 2D by using a couple of tools (1)semiclassics, (2)anomaly matching, (3)lattice.

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3. Adiabatic continuity in $\mathrm{CP}^{N\text{-}1}$ models
4. Other theories with $\mathrm{Z}_N$-twisted b.c.

# 1. Resurgence and bions in $\mathrm{CP}^{N\text{-}1}$ models

Dunne, Unsal (12), TM, Nitta, Sakai (14-16)
Fujimori, Kamata, TM, Nitta, Sakai(16-18)

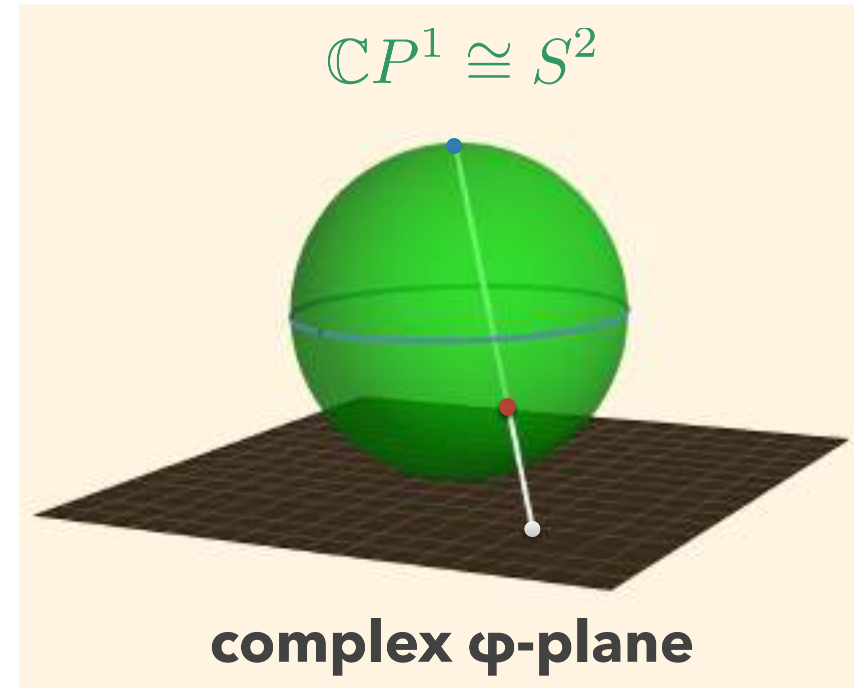
(For SUSY case, see also Dorigoni, Glass (17))

# $\mathrm{CP}^1$ sigma model on $\mathrm{R} \times \mathrm{S}^1$

- $\mathrm{CP}^1$ model on $\mathrm{R} \times \mathrm{S}^1$

$$\mathcal{L} = \frac{1}{g^2} \frac{|\partial_\mu \varphi|^2}{(1+|\varphi|^2)^2} \; + \; \mathcal{L}_F$$

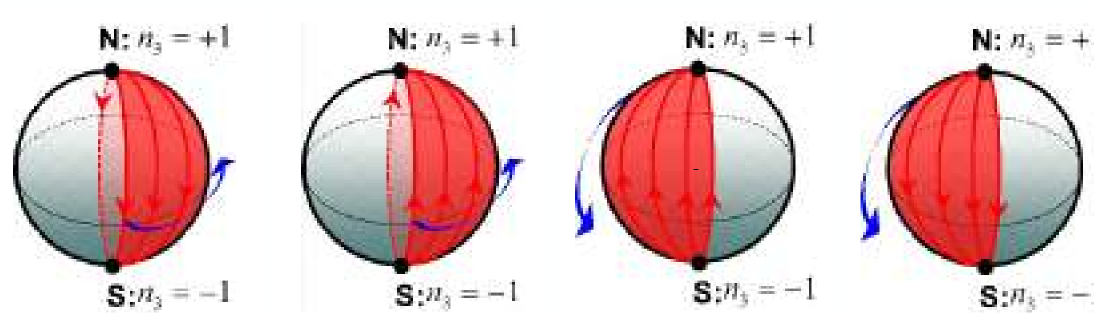
asymptotically-free theory




- $Z_2$ twisted boundary condition

$$\varphi(y+L) = e^{imL} \varphi(y) \qquad (m=\pi/L) \;\rightarrow\; \text{exact } Z_2 \text{ symmetry}$$

→ **Fractional instantons ($Q=1/2$, $S=S_I/2$)**




Lee, Yi(97)
Lee, Lu(97)
Kraan, van Baal(97)
Eto, et.al. (04)(06)
Bruckmann (05)

# $\mathrm{CP}^1$ sigma model on $\mathrm{R} \times \mathrm{S}^1$

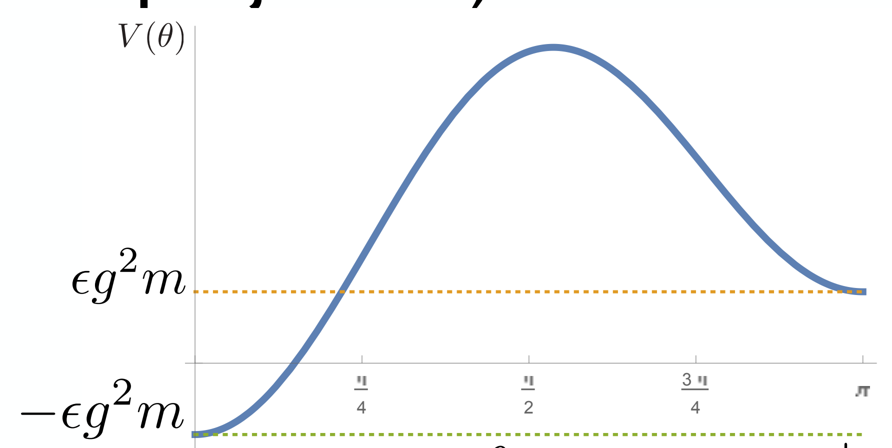
- $\mathrm{CP}^1$ quantum mechanics ($\varepsilon$=1: SUSY)

$$L \;=\; \frac{1}{g^2} G\Big[\partial_t\varphi\partial_t\bar{\varphi} - m^2\varphi\bar{\varphi} + i\bar{\psi}\mathcal{D}_t\psi + \epsilon m(1+\varphi\,\partial_\varphi \log G)\bar{\psi}\psi\Big] \qquad G = \frac{1}{(1+|\varphi|^2)^2}$$

- Ground-state effective bosonic theory (fermion # projection)

$$[H, \psi\bar{\psi}] = 0 \;\Rightarrow\; \bar{\psi}|\Psi\rangle = 0$$

$$\Rightarrow\; V \;=\; \frac{m^2}{4}\sin^2\theta \;-\; \epsilon m g^2 \cos\theta$$



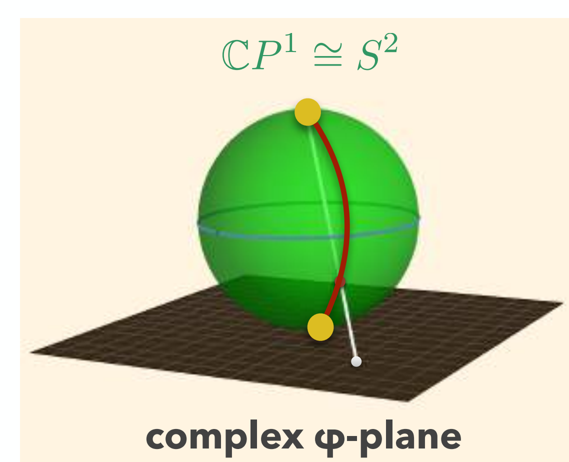

$$\theta = 2\arctan|\varphi|$$

- Two local minima

North and south poles

- Instanton solution for $\varepsilon$=0 $\qquad S_I = \dfrac{m}{g^2}$

Tunneling effect between two minima



# Real bion solutions

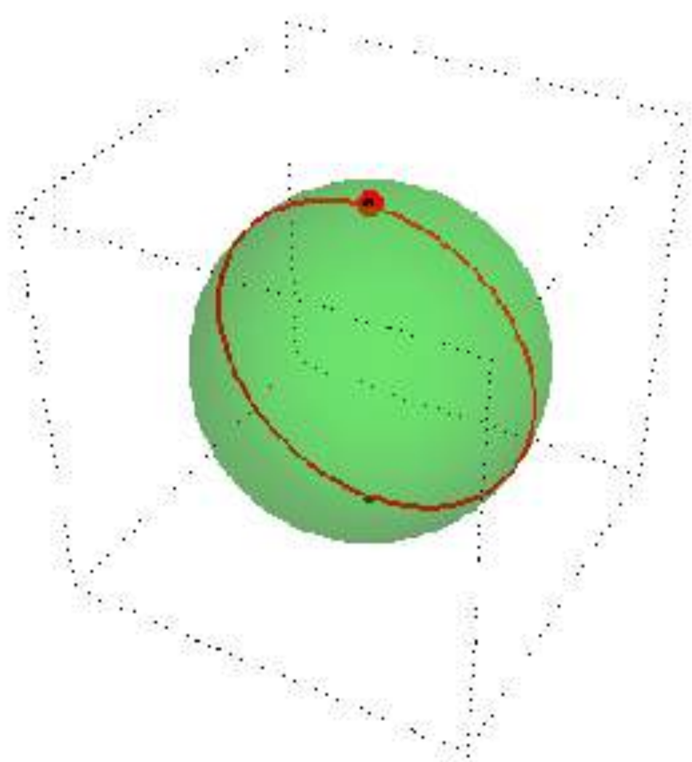
Fujimori, Kamata, TM, Nitta, Sakai(16)
See Behtash, et.al. (15) for other QM

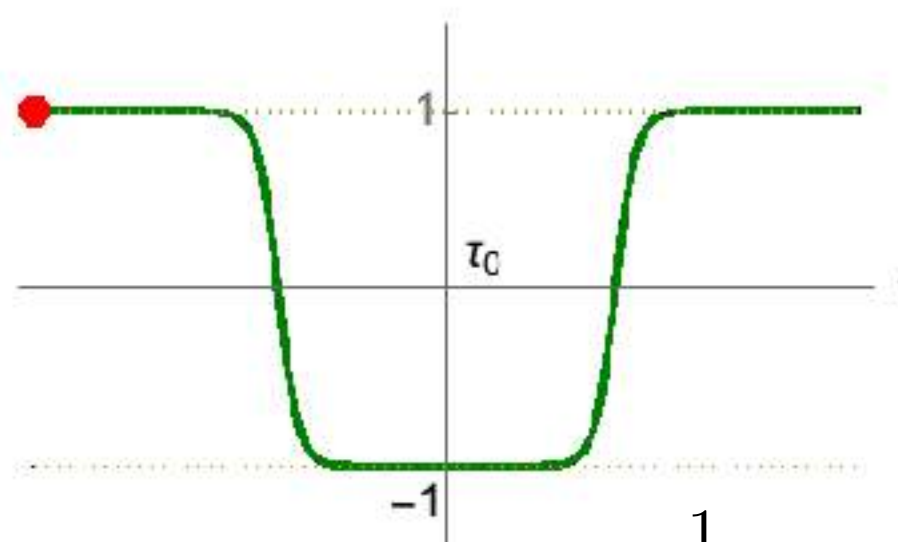
**real solution**

$$\varphi = \sqrt{\frac{\omega^2}{\omega^2 - m^2}} \frac{e^{i\phi_0}}{i \sinh \omega(\tau - \tau_0)}$$

$$\omega^2 = m^2 + 2\epsilon m g^2$$

**Moduli parameters are** $\tau_0$ **: position** $\phi_0$ **: phase**






$$\frac{1 - |\varphi|^2}{1 + |\varphi|^2}$$ **: kink profile**

$$\tau_+ - \tau_- = \frac{1}{\omega} \log \frac{4\omega^2}{\omega^2 - m^2}$$

# Complex bion solution

Fujimori, Kamata, TM, Nitta, Sakai(16)
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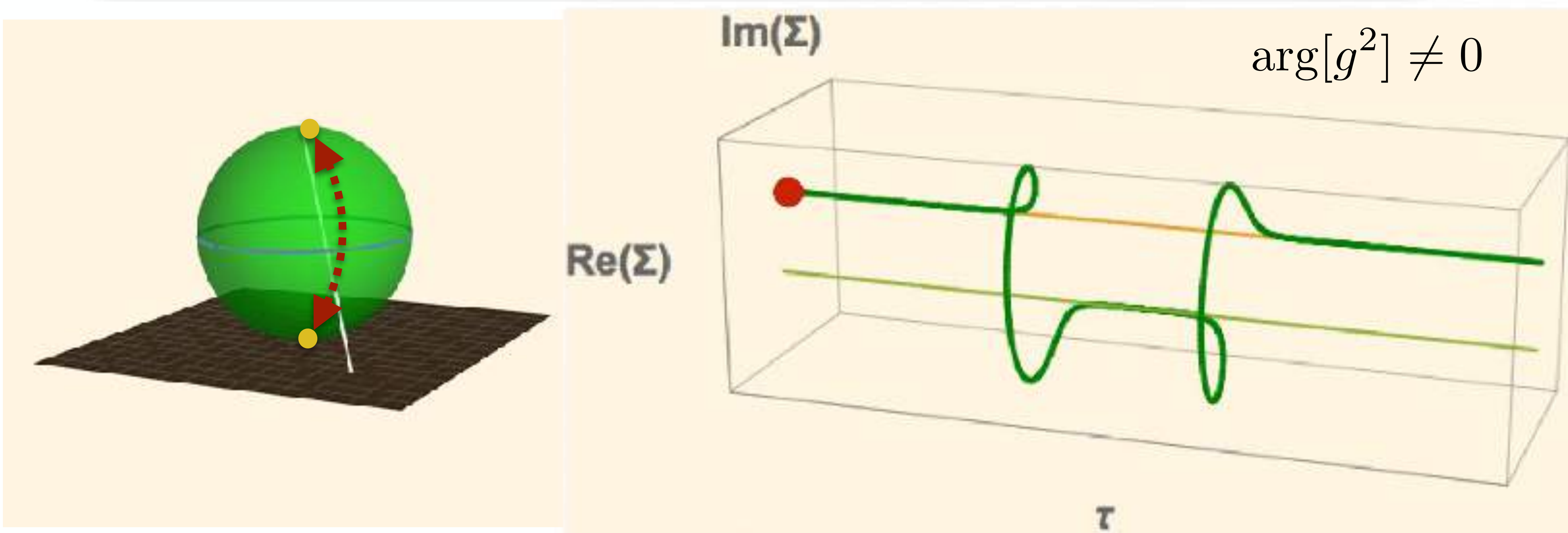
**Complex solution**

$$\varphi = \sqrt{\frac{\omega^2}{\omega^2 - m^2}} \frac{e^{i\phi_0}}{\cosh \omega(\tau - \tau_0)} \qquad \tilde{\varphi} = -\varphi^*$$

$$T^*CP^1 = \frac{SU(2)^C}{U(1)^C}$$




$$\tau_+ - \tau_- = \frac{1}{\omega}\left(\log \frac{4\omega^2}{\omega^2 - m^2} + \pi i\right)$$ **: "Complex relative distance"**

# Multi-bion solution

Fujimori, Kamata, TM, Nitta, Sakai(17)

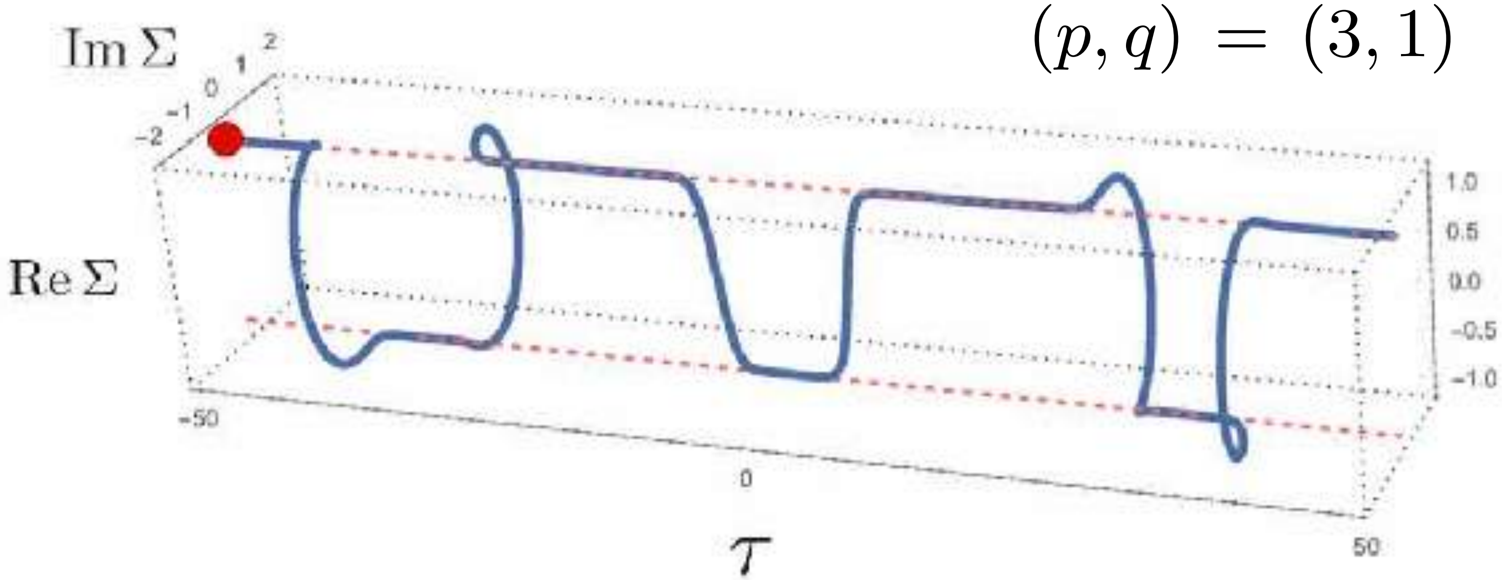
## Multi-bion solution

$$\varphi = e^{i\phi_c}\frac{f(\tau-\tau_c)}{\sin^2\alpha}, \qquad \tilde{\varphi} = e^{-i\phi_c}\frac{f(\tau-\tau_c)}{\sin^2\alpha}$$

$$f(\tau) = \mathrm{cs}(\Omega\tau, k) \equiv \mathrm{cn}(\Omega\tau, k)/\mathrm{sn}(\Omega\tau, k)$$

- classified by integers (*p*,*q*)
- *p* is the number of bions
- *q* determines shape

$$(p,q) = (3,1)$$




$$S \approx pS_{\text{bion}} + 2\pi i\epsilon l$$

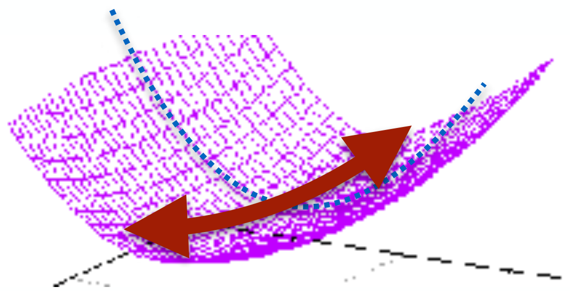
$$S_{\text{bion}} = \frac{2m}{g^2} + 2\epsilon\log\frac{\omega+m}{\omega-m}$$

- **An infinite tower of multi-bion solutions**
- **Contributions are calculated by (multi) quasi-moduli integral**
- **Contributions from real and complex bions cancel for SUSY**

# Quasi-moduli integral

Behtash, Dunne, Schafer, Sulejmanpasic, Unsal (15)
See also Aoyama, Kikuchi (92) for Valley method

**Quasi moduli parameters (Nearly massless modes)**



**= kink distance** $\tau_r$ **& relative phase** $\phi_r$

➡ Existence of these modes means 1-loop det. is not enough…

**Contributions from real and complex bions**

$$\frac{Z_1}{Z_0} \approx \int d\tau_r d\phi_r \, \exp[-V_{\rm eff}(\tau_r, \phi_r)]$$

**Effective potential in quasi-moduli space**

$$V_{\rm eff}(\tau_r, \phi_r) = -\frac{4m}{g^2} \cos\phi_r \, e^{-m\tau_r} + 2\epsilon m \tau_r$$

# Lefschetz Thimble integral

Relation of thimbles and resurgence is well explained in
Cherman, Dorigoni, Dunne, Unsal (13) Cherman, Dorigoni, Unsal (14)

- **Thimble decomposition**

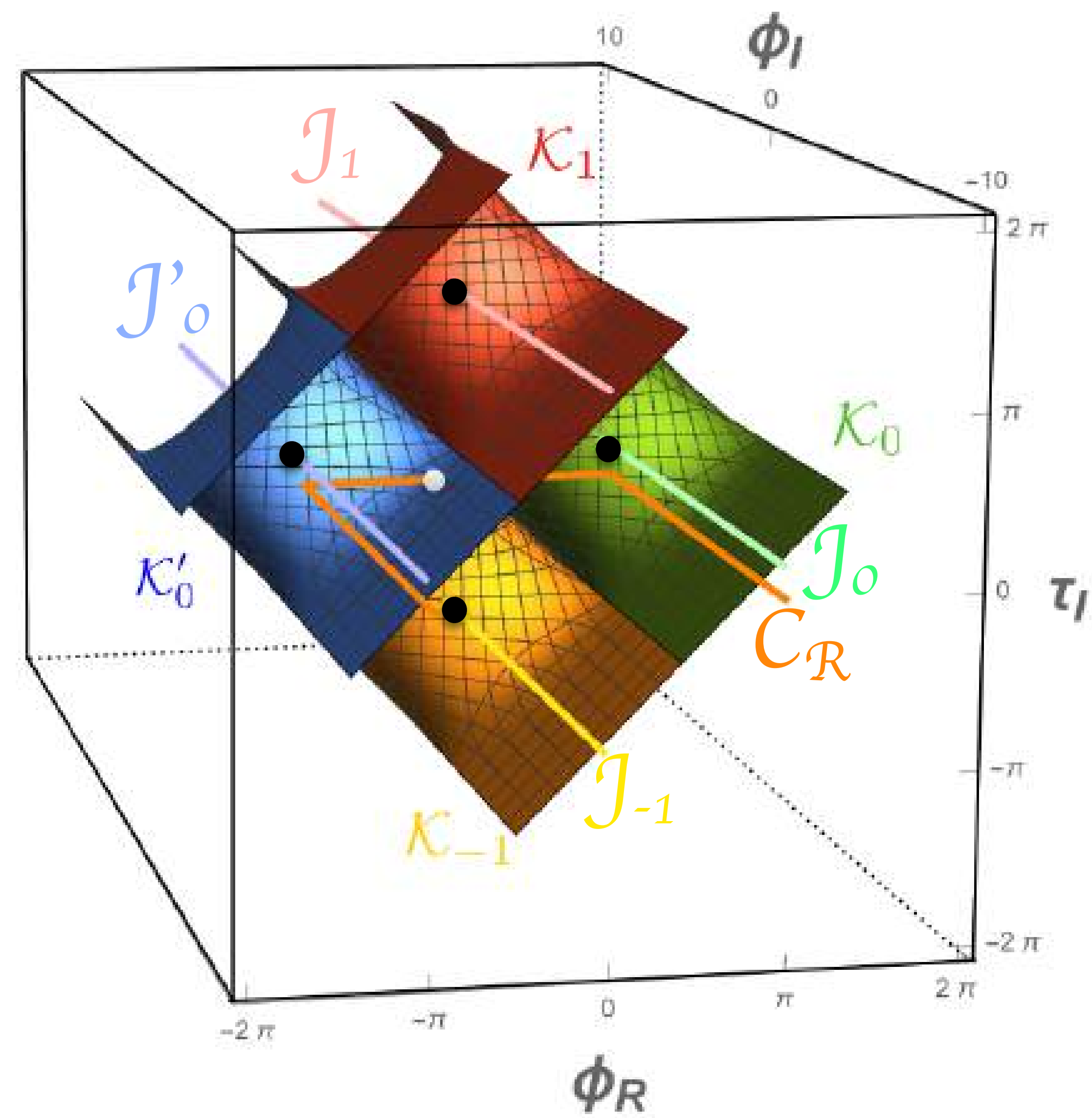
$$\mathcal{C}_{\mathbb{R}} = \sum_{\sigma} n_\sigma \mathcal{J}_\sigma$$

$n_\sigma = \langle \mathcal{C}_{\mathbb{R}}, \mathcal{K}_\sigma \rangle$ intersection number

$\mathcal{J}_\sigma$ : upward flow → Thimble

$\mathcal{K}_\sigma$ : downward flow → Dual thimble

3D projected space of
4D space $(\tau_r, \phi_r) \in \mathbb{C}^2$




$\theta = \arg[g^2] < 0$

- **4D space of complex parameters**

$$(\tau_r, \phi_r) \in \mathbb{C}^2$$

- **(Dual) thimble : 2D surface**

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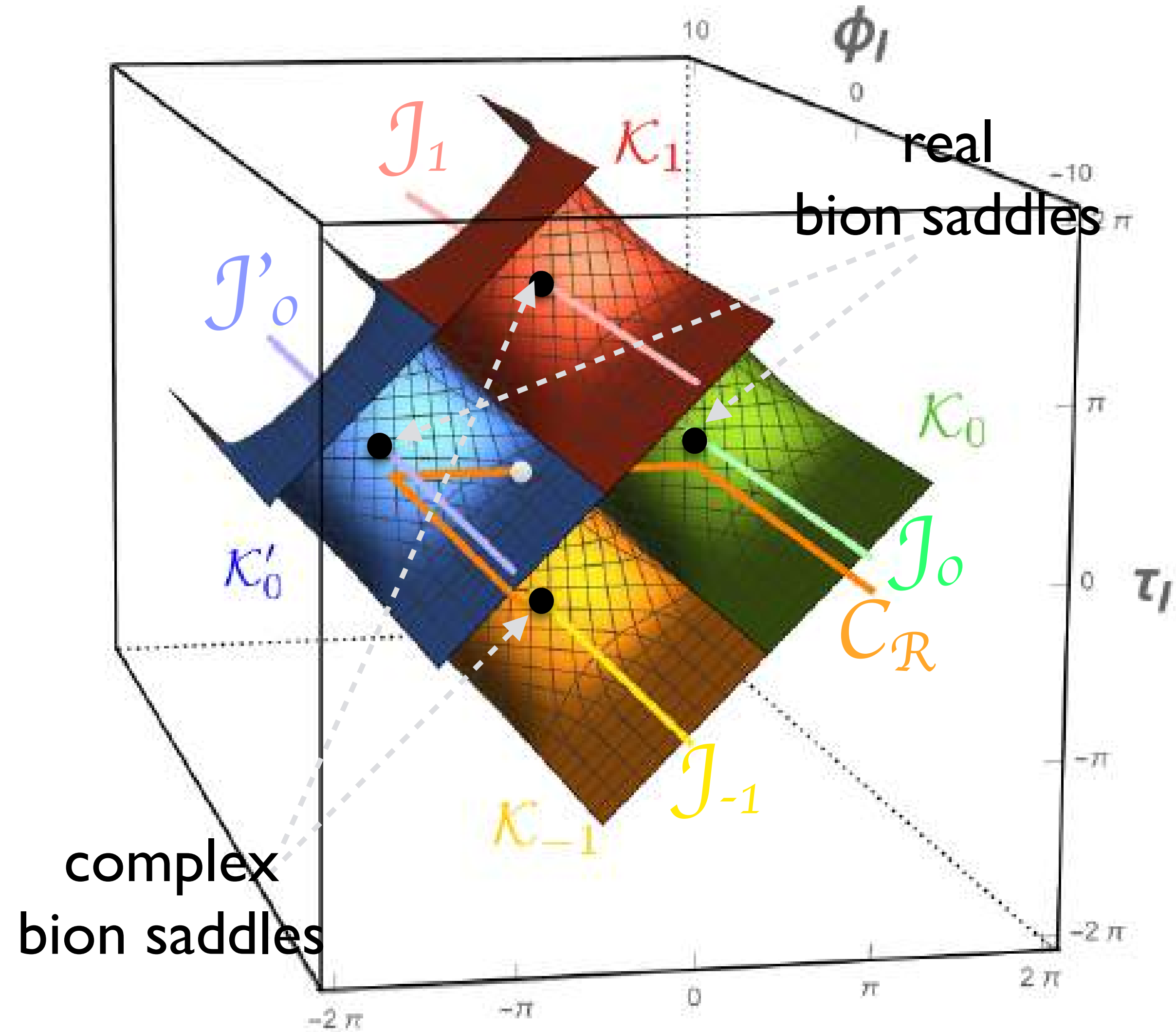


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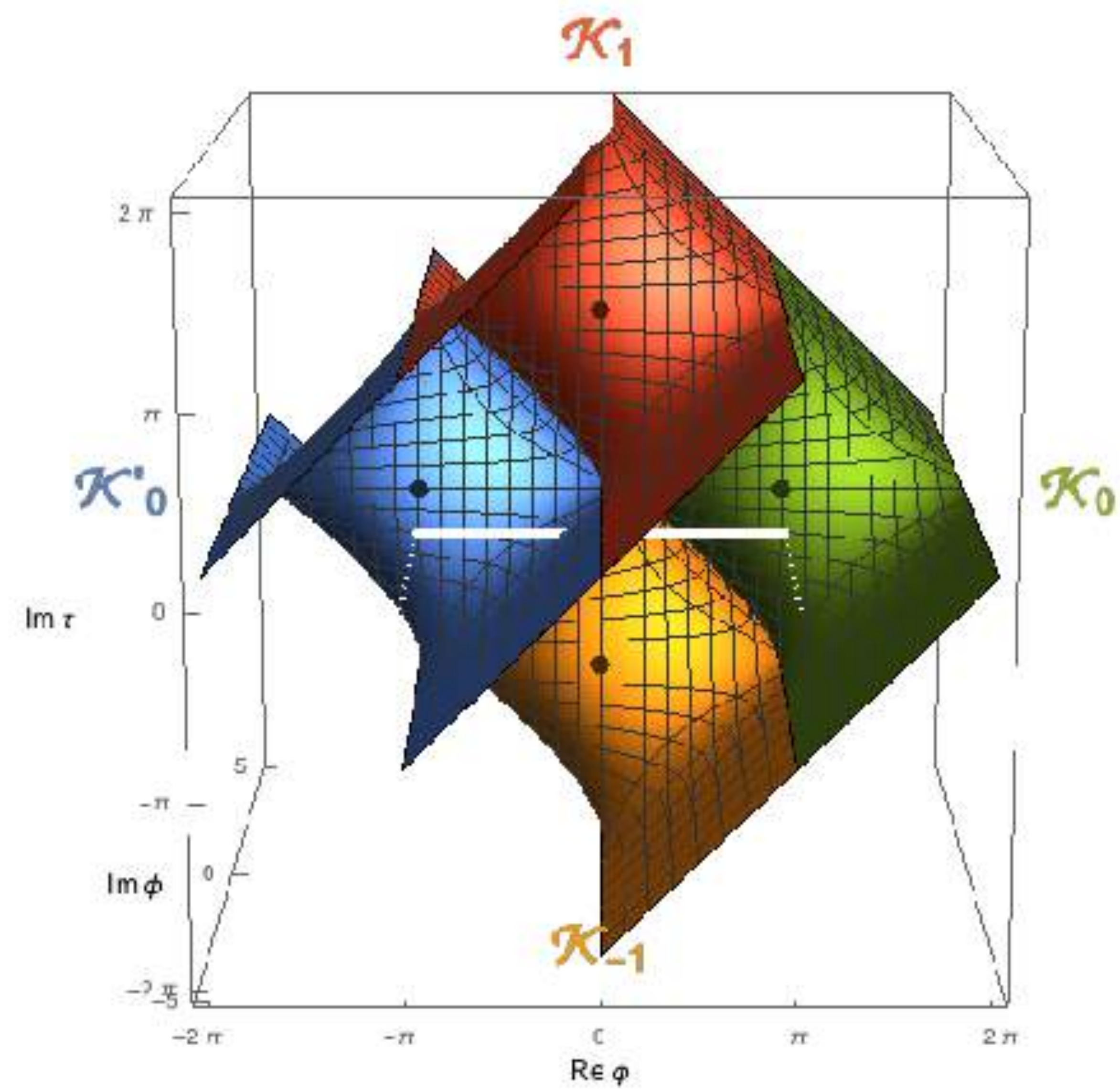


$\theta = \arg[g^2] < 0$

# Lefschetz Thimble integral

Fujimori, Kamata, TM, Nitta, Sakai(16)

$$Z_{\mathrm{q.m.}} = \sum_{\sigma} n_\sigma Z_\sigma$$

- **Integral along Thimble $\boldsymbol{J_\sigma}$**

$$Z_\sigma = \int_{\mathbb{R}} d\tau' \int_{i\mathbb{R}} d\phi' \, e^{-V} = \frac{i}{2m} \left( \frac{g^2 e^{i\theta}}{2m} \right)^{2\epsilon} e^{-2\pi i \epsilon \sigma} \, \Gamma(\epsilon)^2$$

- **Intersection number of original contour & dual thimble $\boldsymbol{K_\sigma}$**

$$(n_{-1}\,, n_0\,, n_1) = \begin{cases} (\,-1\,,\, 1\,,\, 0\,) & \text{for } \theta = -0 \qquad \arg[g^2] < 0 \\ (\,0\,,\, -1\,,\, 1\,) & \text{for } \theta = +0 \qquad \arg[g^2] > 0 \end{cases}$$

**Stokes phenomena** ⇨ **Imaginary ambiguity**

# Contributions from complex bions

Fujimori, Kamata, TM, Nitta, Sakai(16)(17)

**Bion contribution to ground-state energy** $\delta\epsilon \equiv \epsilon - 1$

$$E_{\rm bion} = -2m\left(\frac{g^2}{2m}\right)^{2(\epsilon-1)} \frac{\sin\epsilon\pi}{\pi}\Gamma(\epsilon)^2 e^{-\frac{2m}{g^2}} \times \begin{cases} e^{\pi i\epsilon} & \text{for } \theta = -0 \\ e^{-\pi i\epsilon} & \text{for } \theta = +0 \end{cases}.$$

$$= -2m e^{-\frac{2m}{g^2}}\delta\epsilon + 4m\left(\gamma + \log\frac{2m}{g^2} \pm \frac{i\pi}{2}\right)e^{-\frac{2m}{g^2}}\delta\epsilon^2 + \mathcal{O}(\delta\epsilon^3)$$

For $p$ bions we obtain

$$E_p = -2m e^{-\frac{2pm}{g^2}}\delta\epsilon + 4mp^2\left(\gamma + \log\frac{2m}{g^2} \pm \frac{i\pi}{2}\right)e^{-\frac{2pm}{g^2}}\delta\epsilon^2$$

# Ground-state Energy in CP$^1$ QM

Fujimori, Kamata, TM, Nitta, Sakai(17)

$$E^{(2)} = g^2 - m\frac{\coth\frac{m}{g^2}}{\sinh^2\frac{m}{g^2}}\left[\frac{\mathrm{Ei}\left(\frac{2m}{g^2}\right)+\mathrm{Ei}\left(-\frac{2m}{g^2}\right)}{2} - \gamma - \log\frac{2m}{g^2}\right] = \sum_{p=0}^{\infty} e^{-\frac{2pm}{g^2}} E_p^{(2)}$$

Trans-series

- Perturbative part (asymptotic form)

$$E_0^{(2)} \approx g^2 - 2m\sum_{n=1}^{\infty}(n-1)!\left(\frac{g^2}{2m}\right)^n$$

$$A_l \sim -\frac{1}{2^{l-1}}\frac{\Gamma(l+2(1-\epsilon))}{\Gamma(1-\epsilon)^2}$$

Perturbative coefficients via Bender-Wu method

- Non-perturbative $p$-bion part (asymptotic form)

$$E_{\mathrm{np}}^{(2)} \approx -2m\sum_{p=1}^{\infty} e^{-\frac{2mp}{g^2}}\left[(p+1)^2\sum_{n=1}^{\infty}(n-1)!\left(\frac{g^2}{2m}\right)^n + (p-1)^2\sum_{n=1}^{\infty}(n-1)!\left(-\frac{g^2}{2m}\right)^n - 2p^2\left(\gamma+\log\frac{2m}{g^2}\right)\right]$$

# Ground-state Energy in $CP^1$ QM

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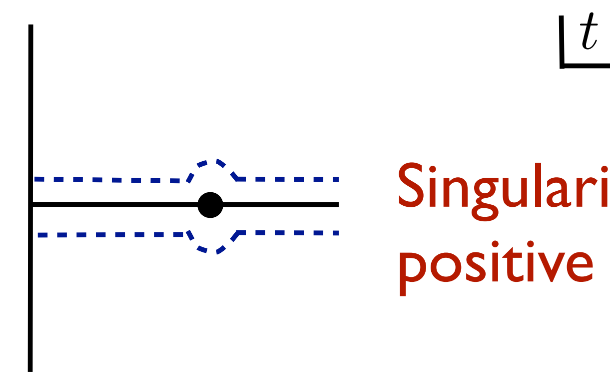
Trans-series

- Perturbative part

$$E_0^{(2)} \;=\; g^2 + 2m\int_0^\infty dt\,\frac{e^{-t}}{t-\frac{2m}{g^2\pm i0}}$$




Singularity on positive real axis

- Non-perturbative $p$-bion part

$$E_p^{(2)} = 2m\int_0^\infty dt e^{-t}\left[\frac{(p+1)^2}{t-\frac{2m}{g^2\pm i0}}+\frac{(p-1)^2}{t+\frac{2m}{g^2}}\right] + 4mp^2\left(\gamma+\log\frac{2m}{g^2}\pm\frac{i\pi}{2}\right)$$

# Ground-state Energy in $\mathbb{C}P^1$ QM

Fujimori, Kamata, TM, Nitta, Sakai(17)

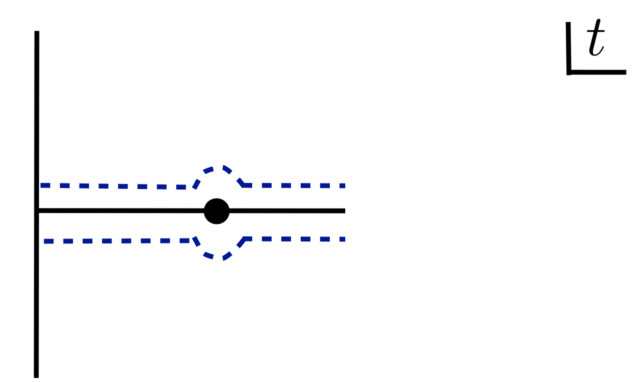
$$E^{(2)} = g^2 - m\frac{\coth\frac{m}{g^2}}{\sinh^2\frac{m}{g^2}}\left[\frac{\mathrm{Ei}\left(\frac{2m}{g^2}\right)+\mathrm{Ei}\left(-\frac{2m}{g^2}\right)}{2} - \gamma - \log\frac{2m}{g^2}\right] = \sum_{p=0}^{\infty} e^{-\frac{2pm}{g^2}} E_p^{(2)}$$

Trans-series

- Perturbative part

$$E_0^{(2)} = g^2 + 2m\int_0^\infty dt\,\frac{e^{-t}}{t - \frac{2m}{g^2 \pm i0}}$$

Perturbative contribution around
0-bion background




- Non-perturbative $p$-bion part

$$E_p^{(2)} = 2m\int_0^\infty dt e^{-t}\left[\frac{(p+1)^2}{t - \frac{2m}{g^2 \pm i0}} + \frac{(p-1)^2}{t + \frac{2m}{g^2}}\right] + 4mp^2\left(\gamma + \log\frac{2m}{g^2} \pm \frac{i\pi}{2}\right)$$

Perturbative contribution around
$p$-bion background

$p$-bion semiclassical
contribution
= the quasi-moduli integral !

# Ground-state Energy in $CP^1$ QM

Fujimori, Kamata, TM, Nitta, Sakai(17)

$$E^{(2)} = g^2 - m\frac{\coth\frac{m}{g^2}}{\sinh^2\frac{m}{g^2}}\left[\frac{\mathrm{Ei}\left(\frac{2m}{g^2}\right)+\mathrm{Ei}\left(-\frac{2m}{g^2}\right)}{2} - \gamma - \log\frac{2m}{g^2}\right] = \sum_{p=0}^{\infty} e^{-\frac{2pm}{g^2}} E_p^{(2)}$$

Trans-series

- Perturbative part

$$E_0^{(2)} = g^2 + 2m\int_0^\infty dt\frac{e^{-t}}{t-\frac{2m}{g^2\pm i0}} \rightarrow \mp 2mi\pi$$

Imaginary ambiguity of perturbation is cancelled by that of 1-bion semiclassical contribution

- Non-perturbative $p$-bion part

$p$=1 bion

$$E_p^{(2)} = 2m\int_0^\infty dt e^{-t}\left[\frac{(p+1)^2}{t-\frac{2m}{g^2\pm i0}} + \frac{(p-1)^2}{t+\frac{2m}{g^2}}\right] + 4mp^2\left(\gamma + \log\frac{2m}{g^2} \pm \frac{i\pi}{2}\right)$$

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Trans-series

- Perturbative part

$$E_0^{(2)} = g^2 + 2m\int_0^\infty dt \frac{e^{-t}}{t - \frac{2m}{g^2 \pm i0}}$$

- Non-perturbative $p$-bion part

Cancelled

$(p$-1)-bion  $p$-bion

$$E_p^{(2)} = 2m\int_0^\infty dt e^{-t}\left[\frac{(p+1)^2}{t-\frac{2m}{g^2\pm i0}} + \frac{(p-1)^2}{t+\frac{2m}{g^2}}\right] + 4mp^2\left(\gamma + \log\frac{2m}{g^2} \pm \frac{i\pi}{2}\right)$$

Imaginary ambiguity of perturbation around ($p$-1)-bion is cancelled by that of semiclassical contribution of $p$-bion ! And it repeats to infinite $p$…

**This is an example of the exact resurgent structure, in which exact cancellation of imaginary ambiguities to all orders of trans-series is observed.**

# 2D $CP^{N-1}$ sigma model on $R \times S^1$

Fujimori, Kamata, TM, Nitta, Sakai(18)

- **SUSY-deformed Lagrangian**

$$L \;=\; L_{\mathcal{N}=(2,2)\,\mathbb{C}P^{N-1}} \;-\; \frac{\delta\epsilon}{2\pi R} m \frac{1-|\varphi|^2}{1+|\varphi|^2} \qquad \left( L = 2\pi R \quad m = \frac{1}{NR} \right)$$

deformation term

- **$Z_N$-twisted b.c.**

$$\varphi(y+2\pi R) = e^{2\pi i m R}\varphi(y) \qquad \psi_{\ell,r}(y+2\pi R) = e^{2\pi i m R}\psi_{\ell,r}(y)$$

- **Bion solutions**

Real: $$\varphi = \sqrt{\frac{\omega^2}{\omega^2 - m^2}}\frac{e^{imy+i\phi_0}}{\sinh\omega(x-x_0)} \qquad \left( \psi_{\ell,r} = 0 \quad \omega^2 = m^2 + \frac{mg^2\delta\epsilon}{\pi R} \right)$$

Complex: $$\varphi = \sqrt{\frac{\omega^2}{\omega^2 - m^2}}\frac{ie^{imy+i\phi_0}}{\cosh\omega(x-x_0)} \qquad \tilde{\varphi} = \sqrt{\frac{\omega^2}{\omega^2 - m^2}}\frac{ie^{imy-i\phi_0}}{\cosh\omega(x-x_0)}$$

# 2D CP$^{N-1}$ sigma model on R × S$^1$

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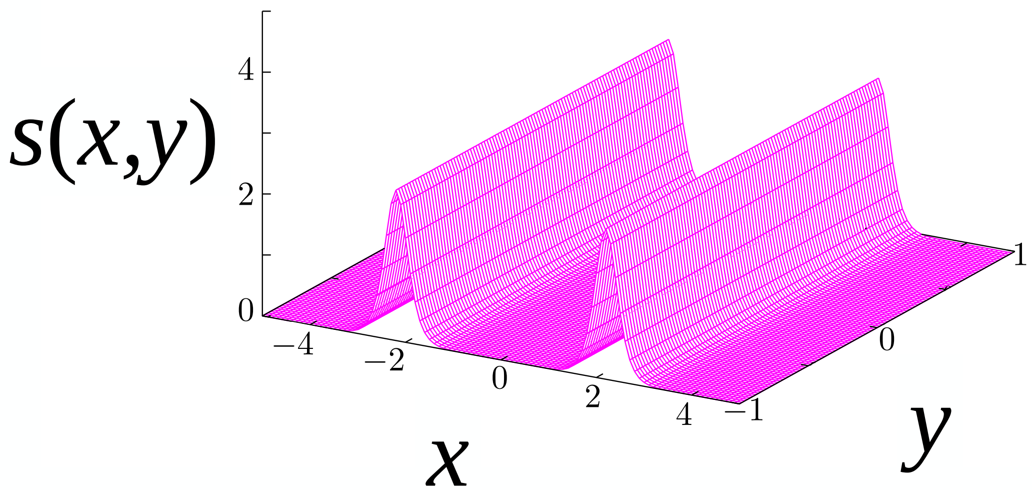
- **Bion solutions**

composed of two $1/N$ fractional instantons




homogeneous in compactified direction

# 2D CP$^{N\text{-}1}$ sigma model on R × S$^1$

Fujimori, Kamata, TM, Nitta, Sakai(18)

- **Quasi-moduli integral (*N*=2)**

**Bare effective action** $$S(x_r,\phi_r) = \frac{4\pi mR}{g^2} - \frac{8\pi mR}{g^2}\cos\phi_r e^{-mx_r} + 2mx_r$$

**Bion contribution** $$Z_{\rm bion} = 2\pi\beta\int_{\mathcal{M}} dx_r d\phi_r\, \mathcal{J}\, \frac{\det\Delta_F}{\det'\Delta_B}\, e^{-S} \qquad (\text{small } g^2)$$

quantum fluctuation

# 2D CP$^{N\text{-}1}$ sigma model on R × S$^1$

Fujimori, Kamata, TM, Nitta, Sakai(18)

- **Quasi-moduli integral (*N*=2)**

**Bare effective action** $$S(x_r,\phi_r) = \frac{4\pi mR}{g^2} - \frac{8\pi mR}{g^2}\cos\phi_r e^{-mx_r} + 2mx_r$$

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⇩ **sum over KK modes of quantum fluctuation via zeta function regularization**

**KK summed 1-loop det.**

$$\mathcal{J}\frac{\det'\Delta_B}{\det\Delta_B^0}\bigg|_{n=0} \approx \left(\frac{4m^2R}{g_R^2}\right)^2 + \cdots, \qquad \frac{\det\Delta_F}{\det\Delta_F^0} \approx e^{-2mx_r} + \cdots$$

$$\log\frac{\det'\Delta_B}{\det\Delta_B^0}\bigg|_{\rm KK} = 2\sum_{n=1}^{\infty}\left[X_n + Y_n\cos\phi_r\, e^{-mx_r} + \mathcal{O}(g^2)\right]$$

$$X_n = \log\frac{\frac{n}{R}-m}{\frac{n}{R}+m}, \qquad Y_n = \frac{4mR}{n} + \mathcal{O}(n^{-2})$$

# 2D $CP^{N-1}$ sigma model on $R \times S^1$

Fujimori, Kamata, TM, Nitta, Sakai(18)

- **Quasi-moduli integral ($N$=2)**

**Bare effective action**

$$S(x_r,\phi_r) = \frac{4\pi m R}{g^2} - \frac{8\pi m R}{g^2}\cos\phi_r e^{-m x_r} + 2 m x_r$$

**Bion contribution**

$$Z_{\text{bion}} = 2\pi\beta \int_{\mathcal{M}} dx_r d\phi_r\, \mathcal{J}\, \frac{\det\Delta_F}{\det'\Delta_B}\, e^{-S} \qquad (\text{small } g^2)$$

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$$\sum_{n=1}^{\infty} X_n = -2mR\log R\Lambda_0 + \log\frac{\Gamma(1+mR)}{\Gamma(1-mR)}, \qquad \sum_{n=1}^{\infty} Y_n = 4mR\log R\Lambda_0 + \cdots$$

# 2D CP$^{N\text{-}1}$ sigma model on R × S$^1$

Fujimori, Kamata, TM, Nitta, Sakai(18)

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**Bare effective action**

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⇩ **sum over KK modes of quantum fluctuation via zeta function regularization**

**Renormalized effective action**

$$S_R(x_r,\phi_r)=S(x_r,\phi_r)-\log\frac{\det\Delta_F}{\det\Delta_F^0}+\log\frac{\det'\Delta_B}{\det\Delta_B^0}$$

# 2D CP$^{N\text{-}1}$ sigma model on R × S$^1$

Fujimori, Kamata, TM, Nitta, Sakai(18)

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**sum over KK modes of quantum fluctuation via zeta function regularization**

**Renormalized effective action**

$$S_R(x_r, \phi_r) = \frac{4\pi m R}{g_R^2} - \frac{8\pi m R}{g_R^2}\cos\phi_r e^{-m x_r} + 2 m x_r$$

$$\frac{1}{g_R^2} = \frac{1}{g^2} - \frac{1}{\pi}\log|R\Lambda_0| \qquad \Lambda_{\mathbb{C}P^1} = \Lambda_0 e^{-\frac{\pi}{g^2}}$$

**dynamical scale**

# 2D CP$^{N\text{-}1}$ sigma model on R × S$^1$

Fujimori, Kamata, TM, Nitta, Sakai(18)

- **Quasi-moduli integral (*N*=2)**

**Bare effective action**
$$S(x_r, \phi_r) = \frac{4\pi m R}{g^2} - \frac{8\pi m R}{g^2} \cos\phi_r e^{-m x_r} + 2 m x_r$$

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$$Z_{\rm bion} = 2\pi\beta \int_{\mathcal{M}} dx_r d\phi_r \, \mathcal{J} \, \frac{\det \Delta_F}{\det' \Delta_B} \, e^{-S} \qquad ({\rm small}\ g^2)$$

⇩ sum over KK modes of quantum fluctuation via zeta function regularization

**Vacuum energy**
$$E_{\rm bion} \approx |R\Lambda|^2 ({\rm Re} \pm i{\rm Im})$$
$$= e^{-2\pi/g_R^2} ({\rm Re} \pm i{\rm Im})$$

**Renormalon-like Imaginary ambiguity**

**It has to be cancelled by perturbative Borel resummation**

# 2D $CP^{N-1}$ sigma model on R × $S^1$

Fujimori, Kamata, TM, Nitta, Sakai(18)

- **Quasi-moduli integral (*N*=2)**

**Bare effective action** $$S(x_r,\phi_r) = \frac{4\pi m R}{g^2} - \frac{8\pi m R}{g^2}\cos\phi_r e^{-m x_r} + 2 m x_r$$

**Bion contribution** $$Z_{\rm bion} = 2\pi\beta \int_{\mathcal{M}} dx_r d\phi_r\, \mathcal{J}\, \frac{\det\Delta_F}{\det'\Delta_B}\, e^{-S} \qquad (\text{small } g^2)$$

**sum over KK modes of quantum fluctuation via zeta function regularization**

**Vacuum energy** $$E_{\rm bion} \approx |R\Lambda|^2 (\mathrm{Re} \pm i\mathrm{Im}) = e^{-2\pi/g_R^2}(\mathrm{Re} \pm i\mathrm{Im})$$

**Renormalon-like Imaginary ambiguity**

**Resurgence with renormalized coupling**

## Question: is it related to the IR-renormalon on $\mathbb{R}^2$ ?

1. There have been intensive studies on this subject.

   Ishikawa, Morikawa, Nakayama, Shibata, Suzuki, Takaura (19) Yamazaki, Yonekura (19)
   Ishikawa, Morikawa, Shibata, Suzuki (20) Morikawa, Takaura (20)
   Fujimori, TM, Nishimura, Nitta, Sakai, in progress
   See Marino, Reis (19)(20) for renormalons and resurgence in other models,

2. In any case, the key is whether the compactified theory is smoothly connected to the theory on $\mathbb{R}^2$.

3. This is the reason why we next discuss adiabatic continuity of $\mathbb{Z}_N$-QFT by use of 't Hooft anomaly and lattice simulation

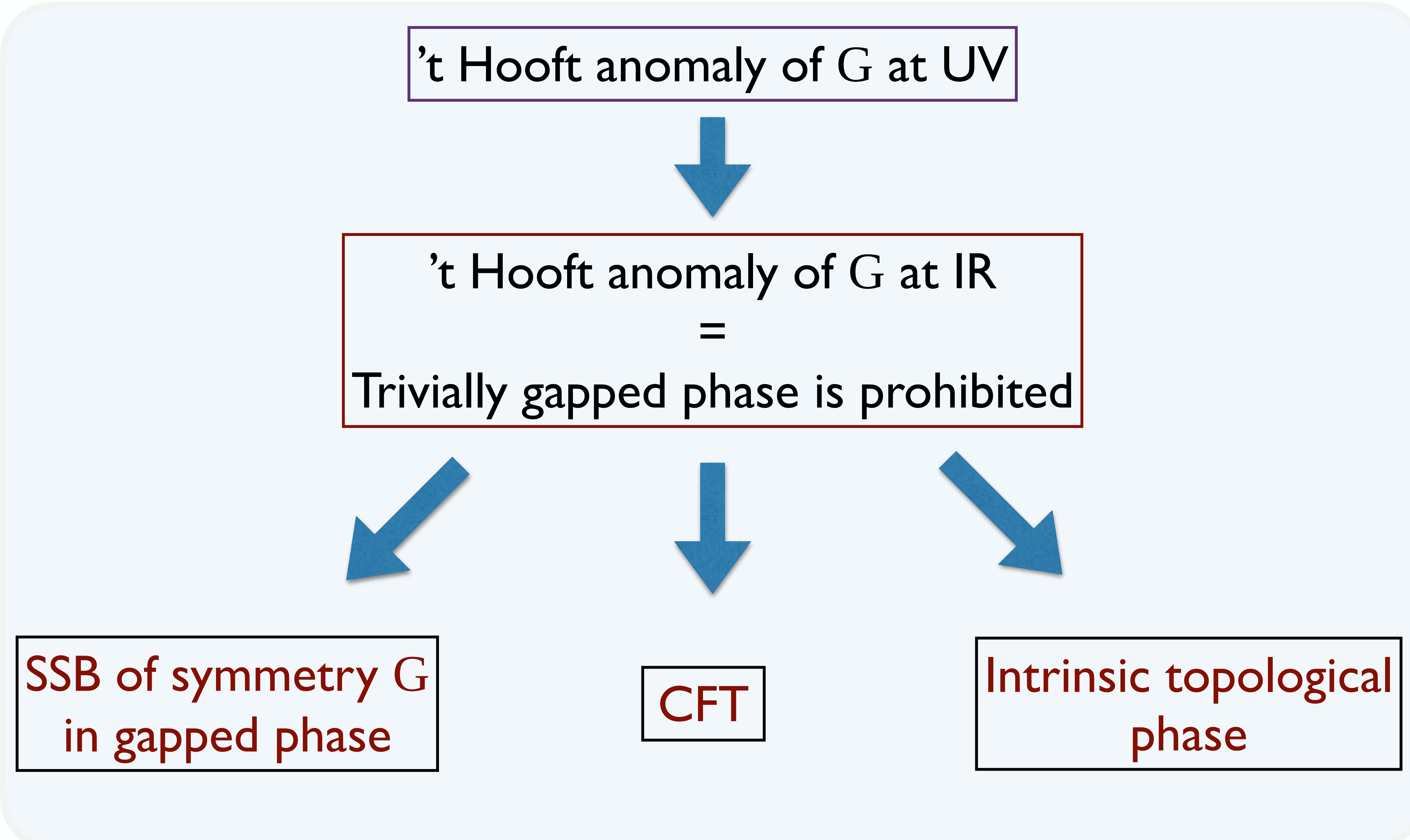
**So, let us start a journey to the continuity !**

# 2. Review of anomaly and lattice (for $Z_N$-twisted 4D gauge theories)

Tanizaki, TM, Sakai (17)
Tanizaki, Kikuchi, TM, Sakai (17)
Iritani, Itou, TM (15)

See also Shimizu, Yonekura(17)

# Use of 't Hooft anomaly matching



# Use of 2D 't Hooft anomaly matching

't Hooft anomaly of G at UV

't Hooft anomaly of G at IR
=
Trivially gapped phase is prohibited

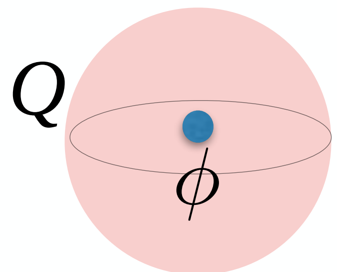
SSB of discrete sym.
(gapped)

CFT
(gapless)

# $\mathbb{Z}_N$ 1-form symmetry in 4D

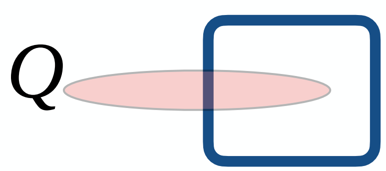
Gaiotto, Kapustin, Seiberg, Willett (14)

- 0-form symmetry = usual global symmetry,
  whose charged object is 0-dim point-like operator




$$\phi \to e^{i\frac{2\pi}{N}}\phi$$ $\mathbb{Z}_N$ 0-form symmetry

- 1-form symmetry = invariance under transf. by closed 1-form $\varepsilon^{(1)}$,
  whose charged object is 1-dim line operator




$$W(C) = \mathrm{tr}\left[iP\exp\int_C a\right]$$

Wilson loop in SU($N$) YM

$$W(C) \to \exp\left(\frac{i}{N}\int_C \epsilon^{(1)}\right) W(C) \qquad a \to a + \epsilon^{(1)}/N$$

$$= e^{\frac{2\pi i \mathbb{Z}}{N}} W(C)$$ $\mathbb{Z}_N$ 1-form symmetry

**SU($N$) Yang-Mills theory has $\mathbb{Z}_N$ 1-form center symmetry at low-$T$**

# How to gauge $Z_N$ 1-form symmetry

Kapustin, Seiberg (14) Aharony, Seiberg, Tachikawa (13)

How to gauge such $Z_N$ 1-form symmetry

→ Background gauge field for $Z_N$ 1-form symmetry
= Pair of U(1) 2-form and 1-form gauge fields $(B, C)$

$$NB = dC$$

generalization of $NA = d\phi$ ($A$:U(1) gauge, $\phi$:Higgs) in U(1)→$Z_N$ Higgsed vacuum

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generalization of $NA = d\phi$ ($A$:U(1) gauge, $\phi$:Higgs) in U(1)→$Z_N$ Higgsed vacuum

- $Z_N$-gauged action with these U(1) fields $\tilde{a} = a + \frac{1}{N}C$ $\tilde{G} = d\tilde{a} + i\tilde{a}\wedge\tilde{a}$

$$S = \frac{1}{2g^2}\int \mathrm{tr}[(\tilde{G}-B)\wedge *(\tilde{G}-B)] + \frac{i\theta}{8\pi^2}\int \mathrm{tr}[(\tilde{G}-B)\wedge(\tilde{G}-B)] + \underline{S_{\mathrm{TFT}}}$$

includes discrete theta parameter $p$

We note $Z_N$ 1-form symmetry itself has no 't Hooft anomaly, but CP symmetry may be broken → Mixed 't Hooft anomaly

# SU($N$) Yang-Mills theory with $\theta$=π

Gaiotto, Kapustin, Komargodski, Seiberg (17)

- CP transformation

$$S = \frac{1}{2g^2}\int \mathrm{tr}[(\tilde{G}-B)\wedge *(\tilde{G}-B)] + \frac{i\theta}{8\pi^2}\int \mathrm{tr}[(\tilde{G}-B)\wedge(\tilde{G}-B)] + S_{\mathrm{TFT}}$$

$\theta \to -\theta \qquad p \to -p \qquad$ with $\qquad \theta = \pi$

$$Z[A,B] \to Z[A,B]\exp\left[-\frac{i}{4\pi}\int \mathrm{tr}\{\tilde{G}\wedge\tilde{G}\} - \frac{iN(2p-1)}{4\pi}\int B\wedge B\right]$$

$$= Z[A,B]\exp\left[-2\pi i\mathbb{Z}\frac{2p-1}{N}\right]$$

- **If *N* is even, it obviously has mixed ’t Hooft anomaly**
- **If *N* is odd, " 2*p*-1=0 mod *N* " is possible, but it’s inconsistent with the anomaly-absent condition "2*p*=0 mod *N*" for $\theta$=0**

# SU($N$) Yang-Mills theory with $\theta=\pi$

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- CP transformation

$$S = \frac{1}{2g^2}\int \mathrm{tr}[(\tilde{G}-B)\wedge *(\tilde{G}-B)] + \frac{i\theta}{8\pi^2}\int \mathrm{tr}[(\tilde{G}-B)\wedge(\tilde{G}-B)] + S_{\mathrm{TFT}}$$

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$$= Z[A,B]\exp\left[-2\pi i\mathbb{Z}\frac{2p-1}{N}\right]$$

**Mixed 't Hooft anomaly and Global inconsistency indicate SSB of either of CP or Z$_N$ 1-form symmetry as long as the system is in a gapped phase.**

# SU($N$) Yang-Mills theory with $\theta=\pi$ on $R_3 \times S_1$

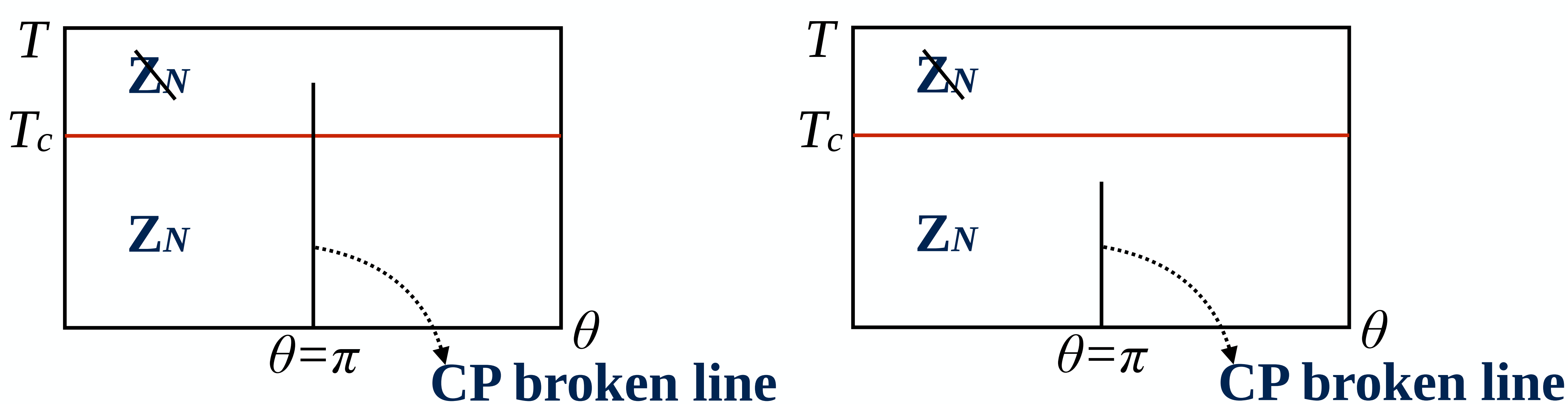
Gaiotto, Kapustin, Komargodski, Seiberg (17)

$$Z[A, B^{(1)}, B^{(2)}] \to Z[A, B^{(1)}, B^{(2)}] \exp\left[-\frac{iN(2p-1)}{2\pi} \int B^{(2)} \wedge B^{(1)} \wedge L^{-1} dx^4\right]$$

$$\downarrow$$

$$\int B^{(2)} \wedge B^{(1)}$$

**Mixed 't Hooft anomaly and Global inconsistency indicate spontaneous breaking of either of CP or $Z_N$ 1-form symmetry even at finite-temperature (trivially gapped phase is forbidden)!**



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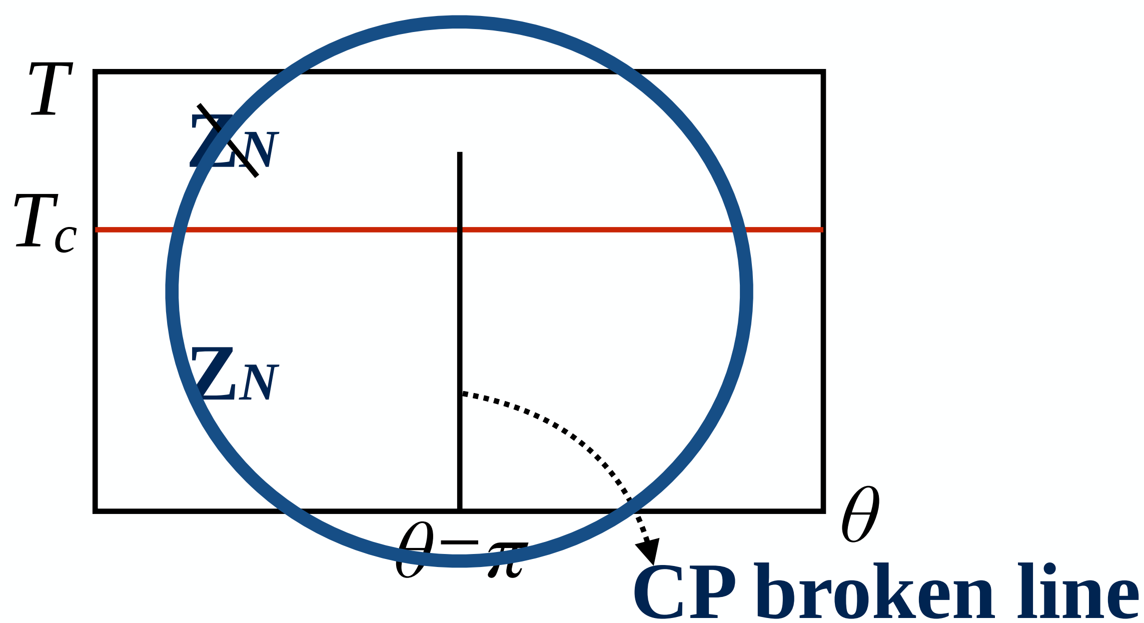


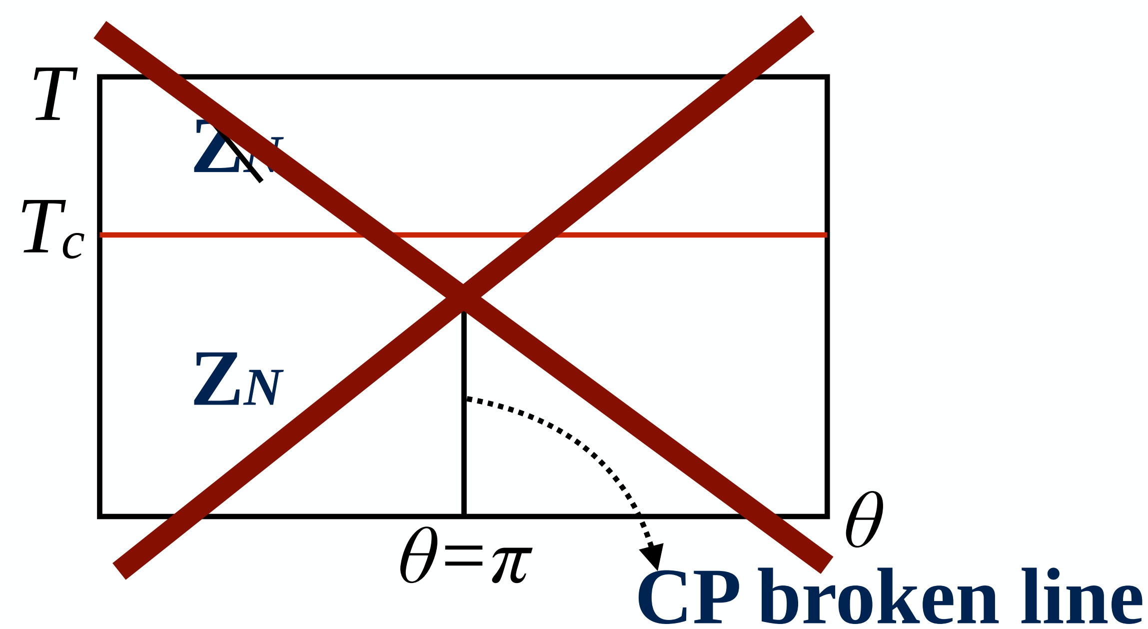

# Anomaly matching for $N_C = N_F = N$ QCD

Tanizaki, Kikuchi(17)
Tanizaki, TM, Sakai (17)

$$S = \frac{1}{2g^2}\int \mathrm{tr}(G_c \wedge *G_c) + \int \mathrm{d}^4x\, \mathrm{tr}\left\{\overline{\Psi}\gamma_\mu D_\mu(a)\Psi\right\}$$

- Flavor and chiral symmetries $\dfrac{SU(N)_L \times SU(N)_R \times U(1)_V \times (\mathbb{Z}_{2N})_A}{(\mathbb{Z}_N)_{\mathrm{color}} \times (\mathbb{Z}_N)_L \times (\mathbb{Z}_N)_R \times \mathbb{Z}_2}$

- We here concentrate only on subgroups of the symmetries

vector-like $\dfrac{SU(N)_{\mathrm{flavor}}}{(\mathbb{Z}_N)_{\mathrm{color-flavor}}}$ axial $(\mathbb{Z}_{2N})_{\mathrm{axial}}$

- Procedure of gauging the vector-like symmetry is similar to that in YM

After gauging $SU(N)_{\mathrm{flavor}}$, we encounter $Z_N$ 1-form symmetry as a result of the quotient $(Z_N)_{\mathrm{color\text{-}flavor}}$ $\longrightarrow$ $(B,C)$ fields are needed

# Anomaly matching for $N_C = N_F = N$ QCD

Tanizaki, Kikuchi(17)
Tanizaki, TM, Sakai (17)

Let us study mixed 't Hooft anomaly between

vector-like $\dfrac{SU(N)_{\mathrm{flavor}}}{(\mathbb{Z}_N)_{\mathrm{color-flavor}}}$ axial $(\mathbb{Z}_{2N})_{\mathrm{axial}}$

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Let us study mixed 't Hooft anomaly between

vector-like $\dfrac{SU(N)_{\mathrm{flavor}}}{(\mathbb{Z}_N)_{\mathrm{color-flavor}}}$ axial $(\mathbb{Z}_{2N})_{\mathrm{axial}}$

To show it, we gauge the vector sym. and perform $(\mathrm{Z}_{2N})_{\mathrm{axial}}$ transf.

$$S_{\mathrm{gauged}} = \frac{1}{2g^2}\int \mathrm{tr}\left\{(\mathcal{G}_{\mathrm{c}}+B)\wedge *(\mathcal{G}_{\mathrm{c}}+B)\right\} + \int \mathrm{d}^4x\, \mathrm{tr}\left\{\overline{\Psi}\gamma_\mu D_\mu(\widetilde{a},\widetilde{A})\Psi\right\}$$

$(\mathrm{Z}_{2N})_{\mathrm{axial}}$ transformation

$$\Delta S = \frac{\mathrm{i}}{4\pi}\int \mathrm{tr}\left\{(\mathcal{G}_{\mathrm{c}}+B)\wedge(\mathcal{G}_{\mathrm{c}}+B)\right\} + \frac{\mathrm{i}}{4\pi}\int \mathrm{tr}\left\{(\mathcal{G}_{\mathrm{f}}+B)\wedge(\mathcal{G}_{\mathrm{f}}+B)\right\} = -\frac{\mathrm{i}2N}{4\pi}\int B\wedge B = -\frac{4\pi i}{N}\mathbb{Z}$$

thus we have a mixed 't Hooft anomaly

$$\Rightarrow \quad \mathcal{Z}[(A,B)] \mapsto \mathcal{Z}[(A,B)]\exp\left(-\frac{2\mathrm{i}N}{4\pi}\int B\wedge B\right)$$

Either of symmetries should be broken: consistent with chiral SSB

# Anomaly matching for $N_C = N_F = N$ QCD

Tanizaki, Kikuchi(17)
Tanizaki, TM, Sakai (17)

Let us study mixed 't Hooft anomaly between

vector-like $\frac{SU(N)_{\mathrm{flavor}}}{(\mathbb{Z}_N)_{\mathrm{color-flavor}}}$ axial $(\mathbb{Z}_{2N})_{\mathrm{axial}}$

To show it, we gauge the vector sym. and perform $(\mathbb{Z}_{2N})_{\mathrm{axial}}$ transf.

$$S_{\mathrm{gauged}} = \frac{1}{2g^2}\int \mathrm{tr}\Big\{(\mathcal{G}_{\mathrm{c}} + B)\wedge *(\mathcal{G}_{\mathrm{c}} + B)\Big\} + \int \mathrm{d}^4x\, \mathrm{tr}\left\{\overline{\Psi}\gamma_\mu D_\mu(\widetilde{a}, \widetilde{A})\Psi\right\}$$

$(\mathbb{Z}_{2N})_{\mathrm{axial}}$ transformation

$$\Delta S = \frac{\mathrm{i}}{4\pi}\int \mathrm{tr}\Big\{(\mathcal{G}_{\mathrm{c}}+B)\wedge(\mathcal{G}_{\mathrm{c}}+B)\Big\} + \frac{\mathrm{i}}{4\pi}\int \mathrm{tr}\Big\{(\mathcal{G}_{\mathrm{f}}+B)\wedge(\mathcal{G}_{\mathrm{f}}+B)\Big\} = -\frac{\mathrm{i}2N}{4\pi}\int B\wedge B = -\frac{4\pi i}{N}\mathbb{Z}$$

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$$\Rightarrow \quad \mathcal{Z}[(A,B)] \mapsto \mathcal{Z}[(A,B)]\exp\left(-\frac{2\mathrm{i}N}{4\pi}\int B\wedge B\right)$$

However, this 't Hooft anomaly disappears in compactified theory…

# $Z_N$-QCD theory on $R^3 \times S^1$

- **Introducing SU($N$)$_{\text{flavor}}$ $Z_N$ holonomy = $Z_N$-QCD**

Kouno, et.al. (12~)
Iritani, Itou, TM (15)
Cherman et.al. (16~)

$$\Omega = \mathrm{e}^{\mathrm{i}\phi}\mathrm{diag}[1, \omega, \omega^2, \ldots, \omega^{N-1}] \qquad \omega \equiv e^{\frac{2\pi i}{N}}$$

Equivalent to flavor-dependent $Z_N$ twisted boundary condition

$$\Psi(\boldsymbol{x}, x^4 + L) = \Psi(\boldsymbol{x}, x^4)\Omega \quad \xrightarrow{L\to\infty} \quad N\text{-flavor QCD on } R^4$$

➡ we have a new intertwined $Z_N$ 0-form symmetry

$$\underbrace{\Psi \mapsto \Psi S}_{\text{flavor rotation}} \; + \; \underbrace{\Omega \to \omega\Omega}_{Z_N \text{ 0-form transf.(center)}} \qquad S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

named as color-flavor center symmetry $(\mathbb{Z}_N)_S$

# Anomaly matching for $Z_N$-QCD theory on $R^3 \times S^1$

Tanizaki, TM, Sakai (17)    Tanizaki, Kikuchi, TM, Sakai (17)

Let us gauge flavor and $(Z_N)_S$ , then perform $(Z_{2N})_{\mathrm{axial}}$ transformation

$$\mathcal{Z}_\Omega[(A_K, B^{(1)}, B^{(2)})] = \mathcal{Z}[(A_K + B^{(1)} + A_{\mathrm{cl}}, B^{(2)} + B^{(1)} \wedge L^{-1}\mathrm{d}x^4)]$$

$$\mathcal{Z}_\Omega[(A_K, B^{(1)}, B^{(2)})] \mapsto \mathcal{Z}_\Omega[(A_K, B^{(1)}, B^{(2)})] \exp\left(-\frac{2\mathrm{i}N}{2\pi}\int B^{(2)} \wedge B^{(1)}\right)$$

Mixed anomaly survives !

**Mixed 't Hooft anomaly among**

| $(\mathbb{Z}_N)_S$ | $U(1)^{N-1}/(\mathbb{Z}_N)_{\mathrm{color-flavor}}$ | $(\mathbb{Z}_{2N})_{\mathrm{axial}}$ |
|---|---|---|
| color-flavor center sym. | flavor sym. | axial sym. |

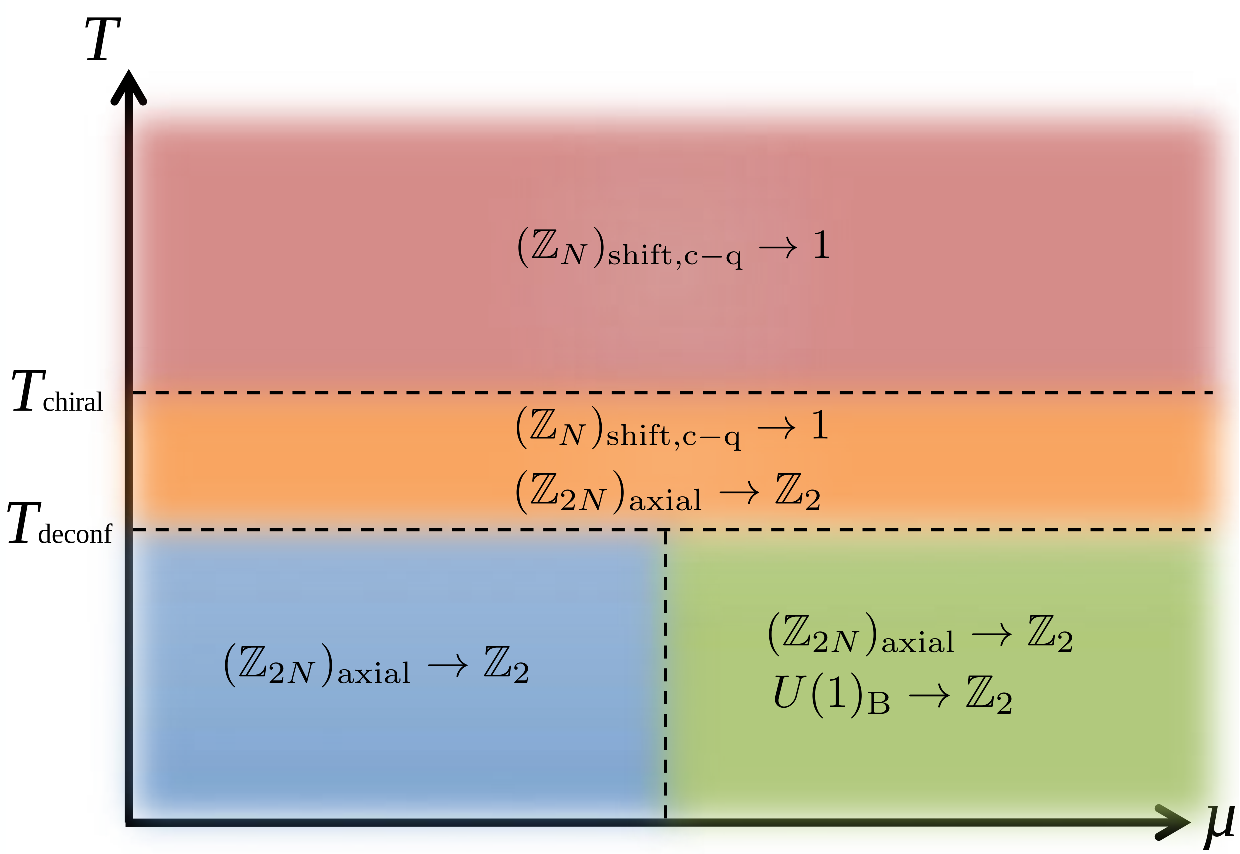
**one of them should be broken if gap exists**

**$Z_N$-twisted boundary condition works to make 't Hooft anomaly survives in compactified theory.**

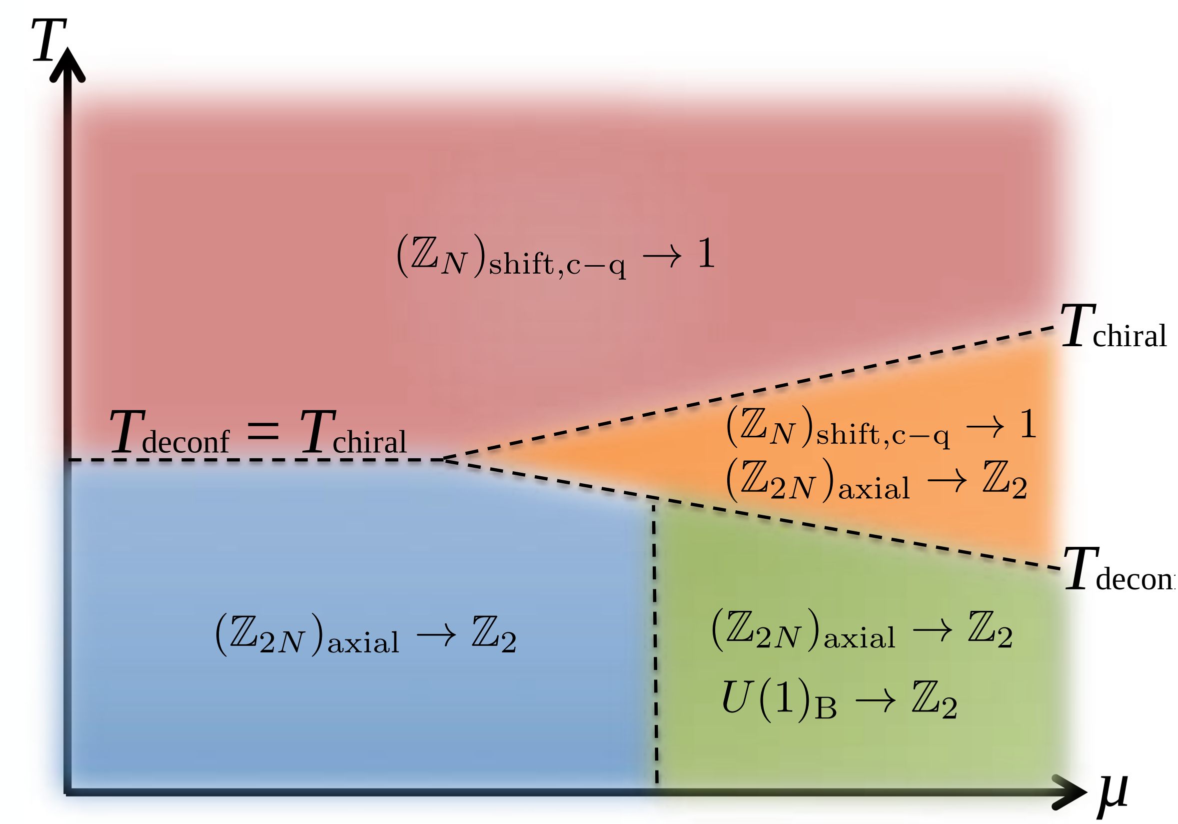
# Anomaly matching $Z_N$-QCD theory on $R^3 \times S^1$

Tanizaki, Kikuchi, TM, Sakai (17)

We can make constraints on finite-$(T, \mu)$ phase diagram of $Z_N$-QCD




conjecture 1

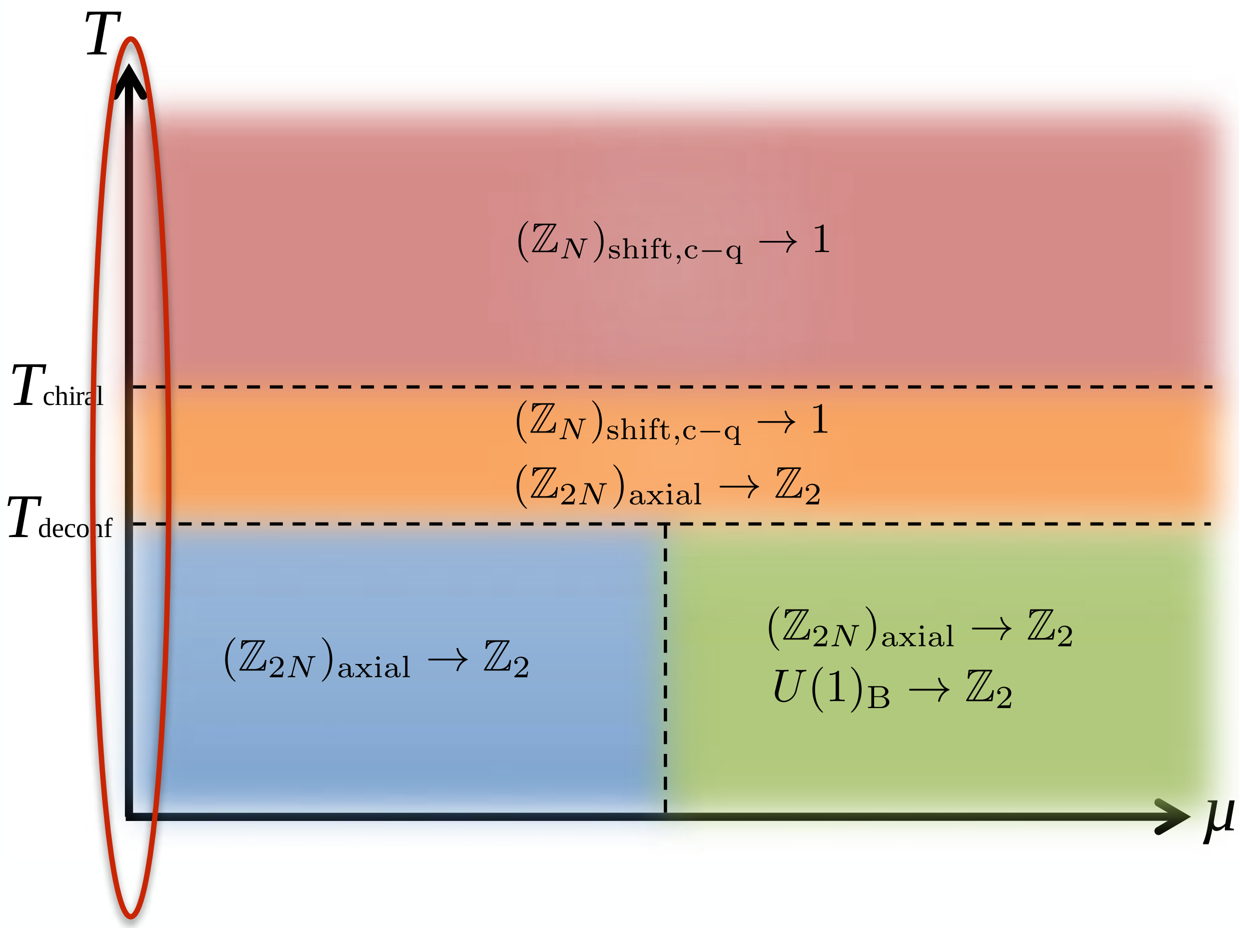



conjecture 2

# Anomaly matching $Z_N$-QCD theory on $R^3 \times S^1$

Tanizaki, Kikuchi, TM, Sakai (17)

We can make constraints on finite-$(T, \mu)$ phase diagram of $Z_N$-QCD

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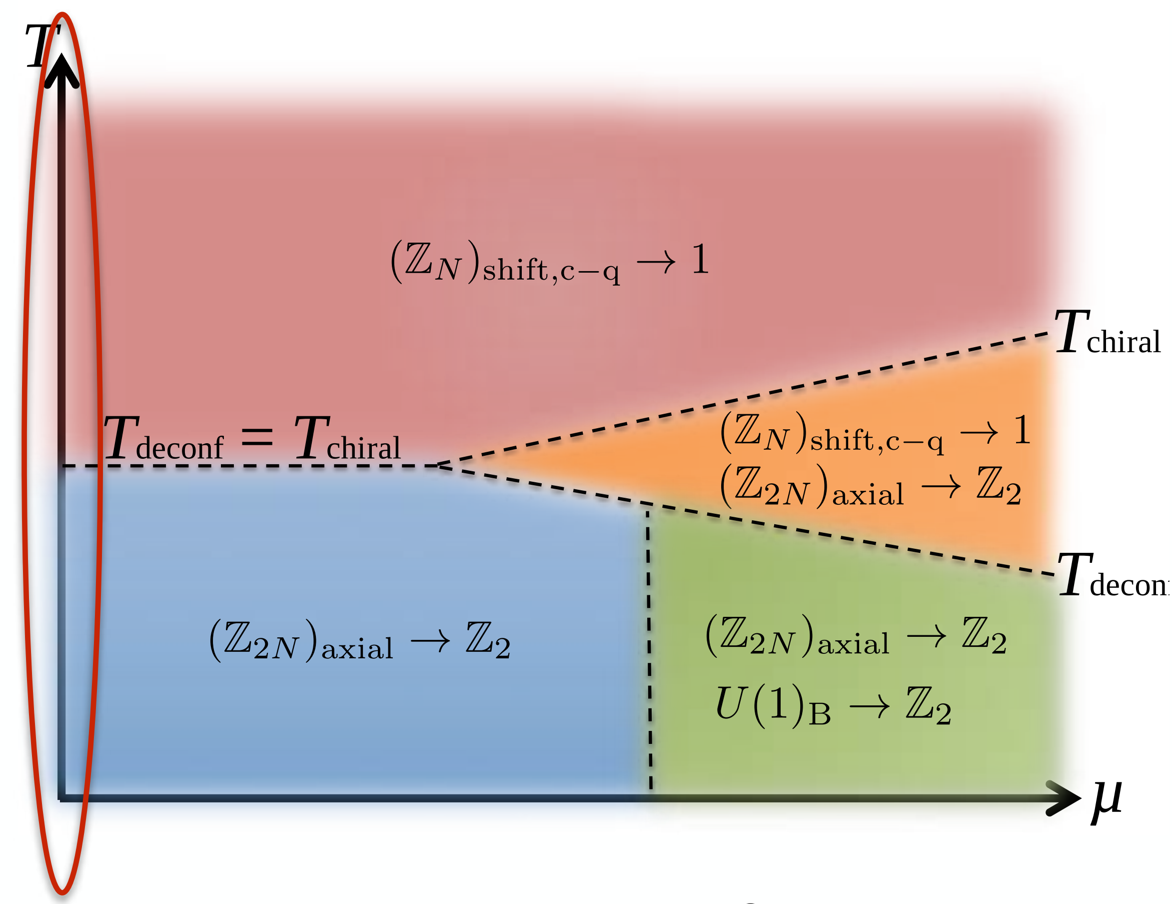


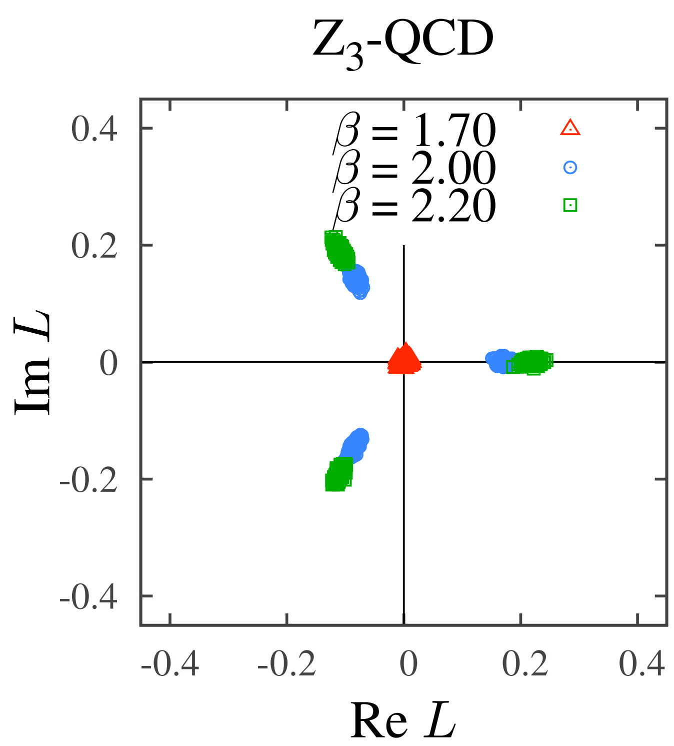

conjecture 2

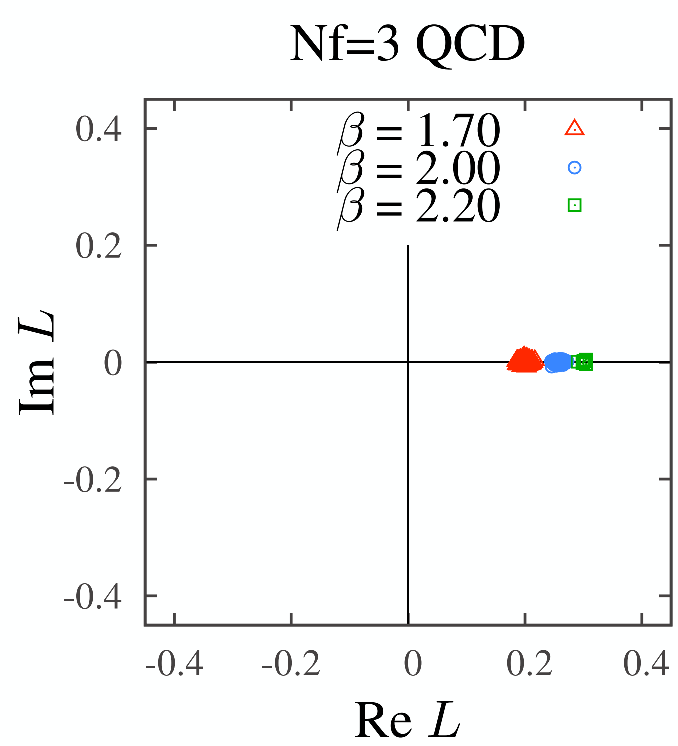
We can compare these with the lattice results Iritani, Itou, TM (15)

# Comparison with lattice $Z_3$-QCD on $R^3 \times S^1$

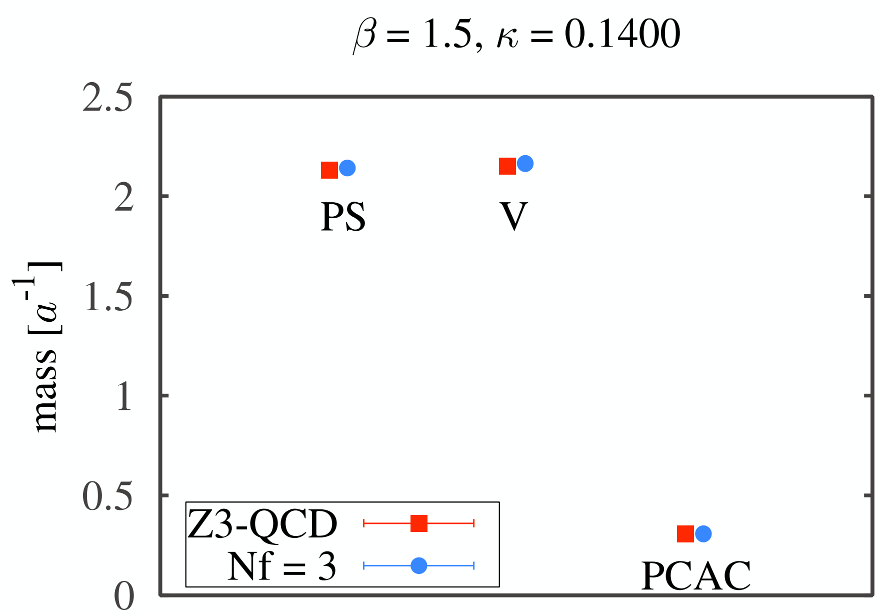
Iritani, Itou, TM (15)

- Polyakov-loop distribution plot




- Hadron spectrum




Low-T : around the origin ~ Z3-sym

High-T : equiv. 3 vacua ~ SSB of Z3

→ **Z3 at the action level**

Low-T : on the real axis

High-T : on the real axis

→ **Explicit Z3 breaking**

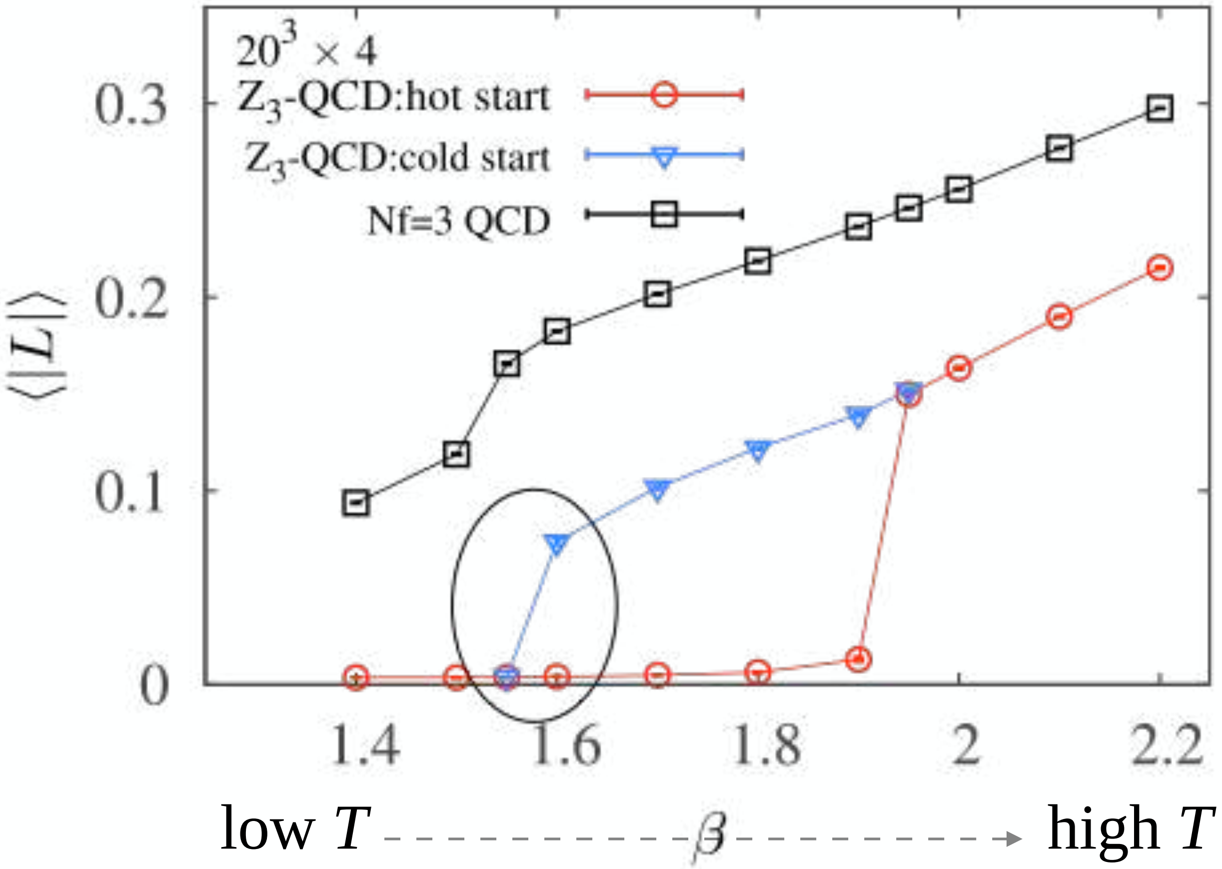
At zero temperature, meson spectrum agrees with that of usual QCD

# Comparison with lattice $Z_3$-QCD on $R^3 \times S^1$

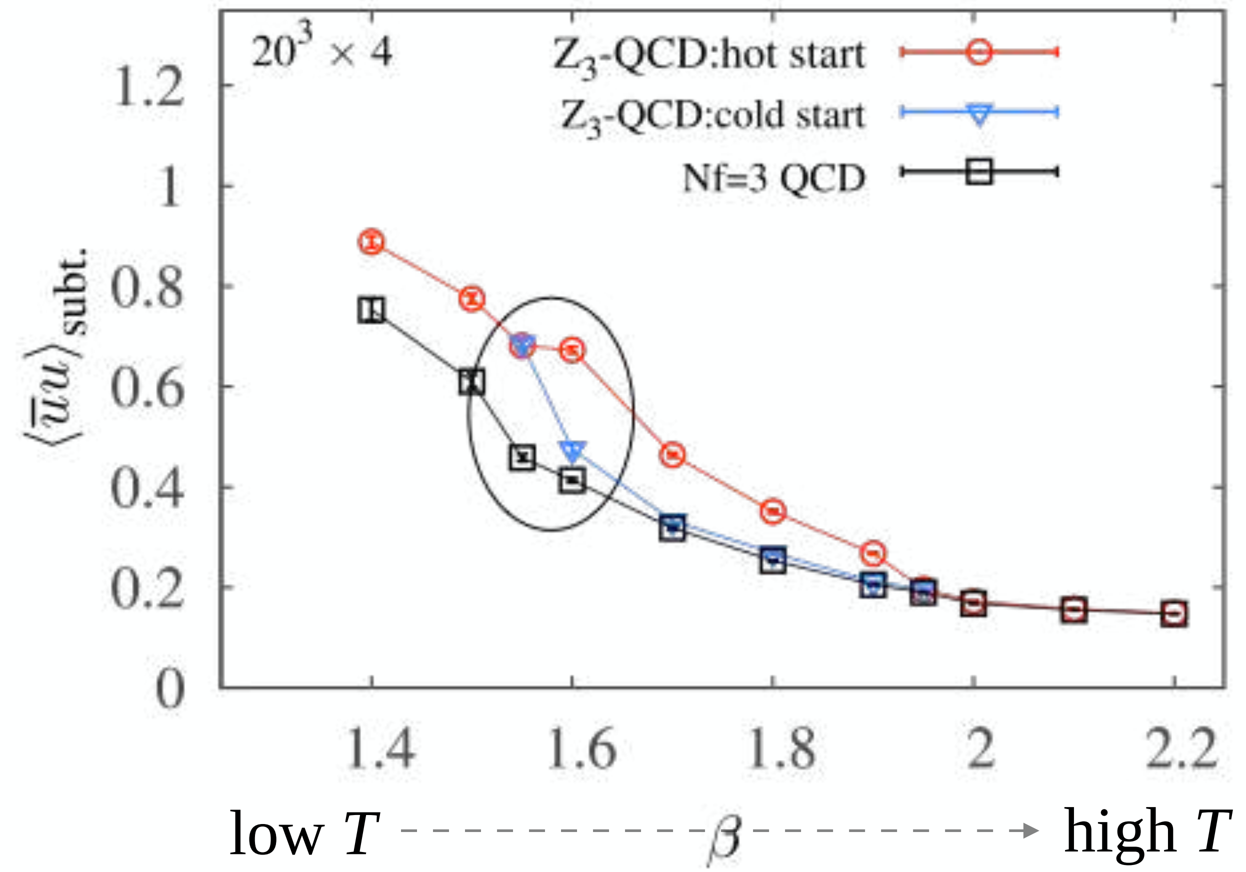
Iritani, Itou, TM (15)

$\beta$ dependence of EV of Polyakov-loop & chiral condensate

**Polyakov loop: order parameter of $(Z_N)_S$**




**Chiral condensate: order para. of $(Z_{2N})_{axial}$**




Chiral transition never occurs below the center transition temperature.
It is consistent with the absence of trivially-gapped phase, or the result of the 't Hooft anomaly matching!

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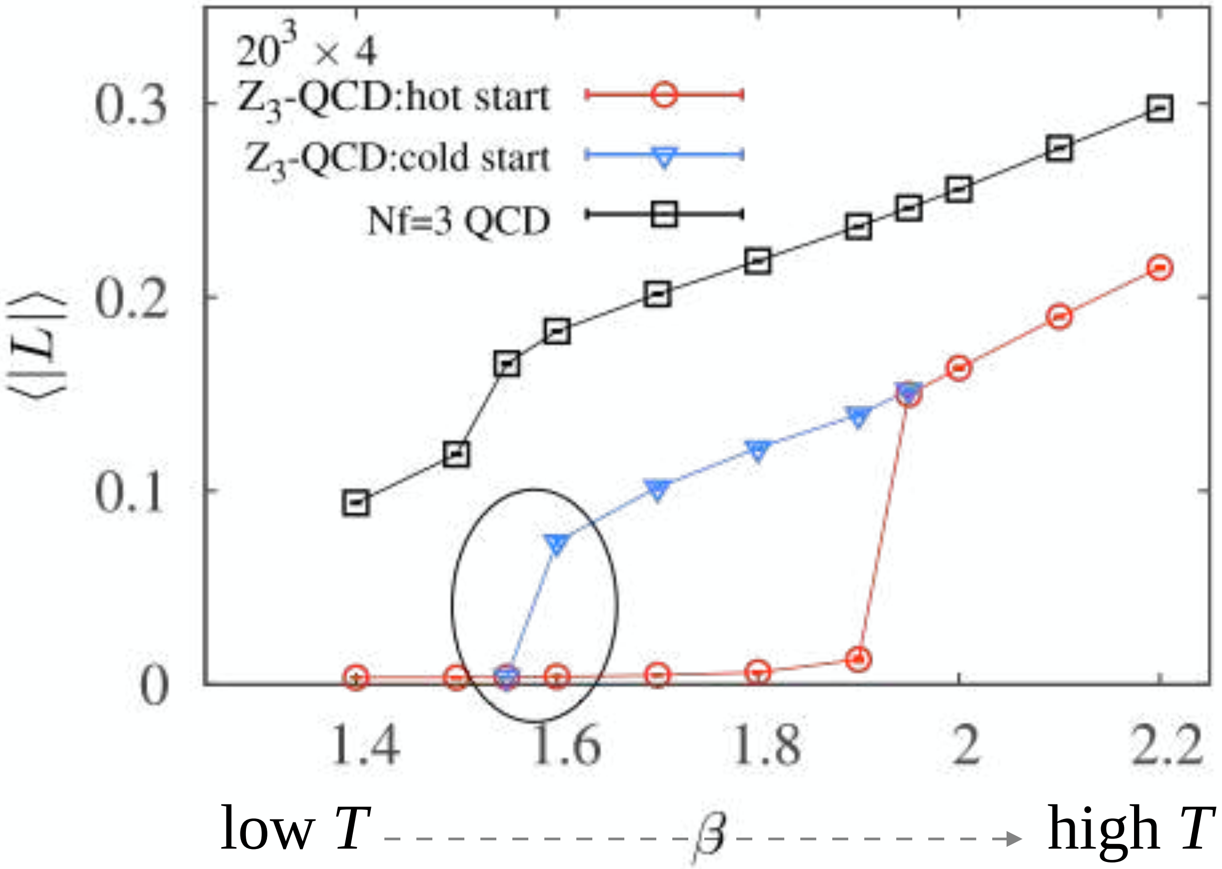
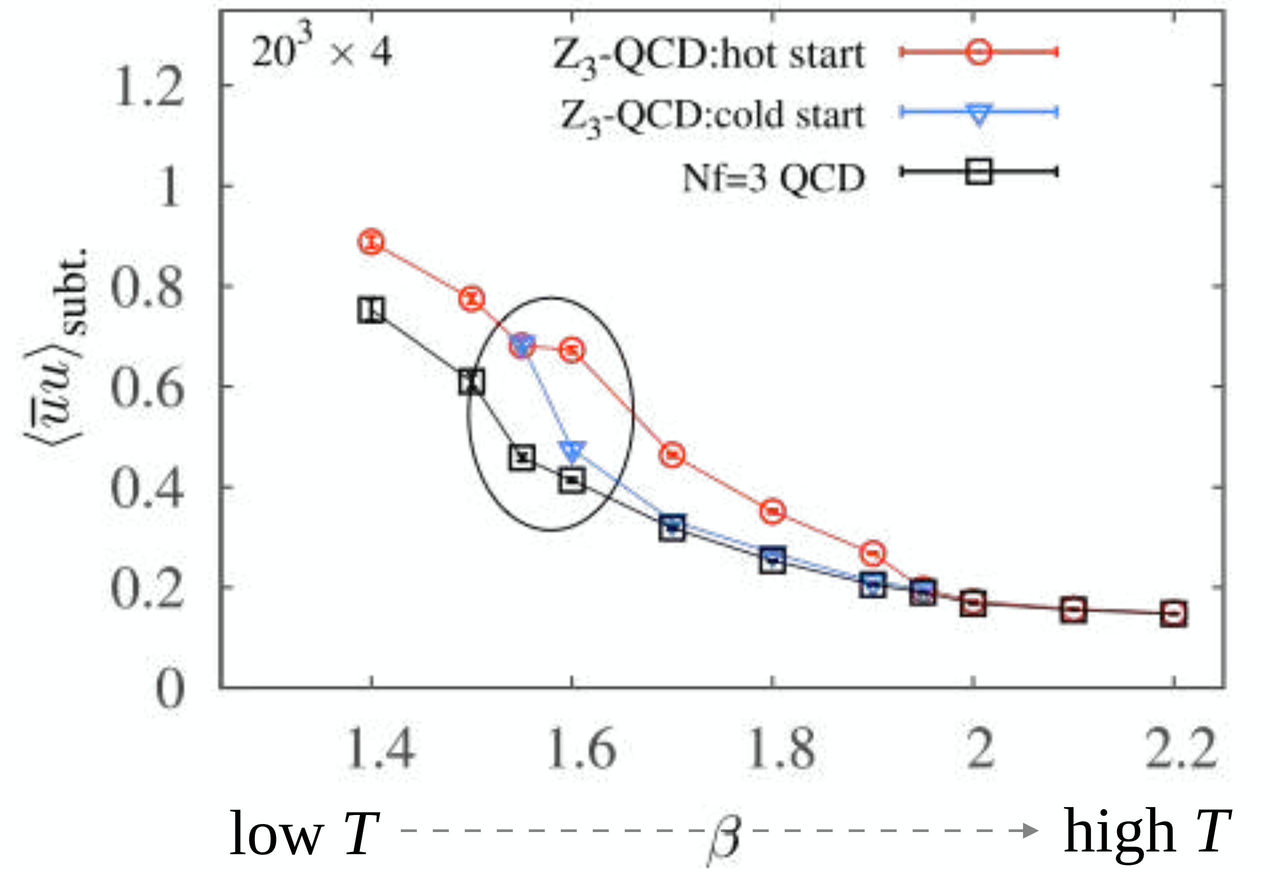



**Chiral condensate: order para. of $(Z_{2N})_{axial}$**




For feasible way to realize adiabatic continuity in $Z_N$-QCD, see Cherman, Schafer, Unsal (16): heavy adjoint quark is a key

# 3. Adiabatic continuity in $\mathrm{CP}^{N\text{-}1}$ model on $\mathrm{R} \times \mathrm{S}^1$

Tanizaki, TM, Sakai (17)
Fujimori, Itou, TM, Nitta, Sakai (19)(20)

# Anomaly matching for $\mathrm{CP}^{N\text{-}1}$ models with $\theta=\pi$

Komargodski, Sharon, Thorngren, Zhou (17)

We first study mixed 't Hooft anomaly on $\mathrm{R}^2$ between

flavor $SU(N)/\mathbb{Z}_N$ time reversal $\mathsf{T}$

To show it, we gauge flavor sym. and perform $\mathrm{T}$ transformation

$$\mathcal{Z}_\pi[\mathsf{T}\cdot(A,B)]\exp\left(-\mathrm{i}k\int\mathsf{T}\cdot B\right)=\mathcal{Z}_\pi[(A,B)]\exp\left(-\mathrm{i}k\int B\right)\mathrm{e}^{\mathrm{i}(2k-1)\underline{\int B}}$$

$\int B \to \frac{2\pi\mathbb{Z}}{N}$

For even $N$, it has a mixed 't Hooft anomaly for $\theta=\pi$
For odd $N$, it has global inconsistency between $\theta=0,\pi$

➡ It indicates spontaneous $\mathrm{T}$ breaking at $\theta=\pi$ on $\mathrm{R}^2$

On $\mathrm{R}\times\mathrm{S}^1$ this mixed 't Hooft anomaly disappears…..

# $Z_N$-twisted $CP^{N-1}$ model at $\theta=\pi$ on $R\times S^1$

Tanizaki, TM, Sakai (17)

$Z_N$ twisted boundary condition in $S^1$ direction

$$\vec{z}(x^1, x^2 + L) = \Omega \vec{z}(x^1, x^2) \qquad \Omega = \mathrm{diag}(1, \omega, \ldots, \omega^{N-1})$$

➡ we have intertwined $Z_N$ 0-form shift symmetry $(\mathbb{Z}_N)_S$

$\underline{\vec{z} \to S\vec{z}}$ (flavor rotation) & $\underline{\Omega \to \omega\Omega}$ ($Z_N$ 0-form transf.)

➡ we gauge $(\mathbb{Z}_N)_S$ by $U(1)$ 1-form field $B^{(1)}$ then perform $T$ transf.

➡ $$\mathcal{Z}_{\pi,\Omega}[\mathsf{T}\cdot B^{(1)}] = \mathcal{Z}_{\pi,\Omega}[B^{(1)}]\exp\left(-\mathrm{i}\int B^{(1)}\right)$$

Mixed anomaly survives !

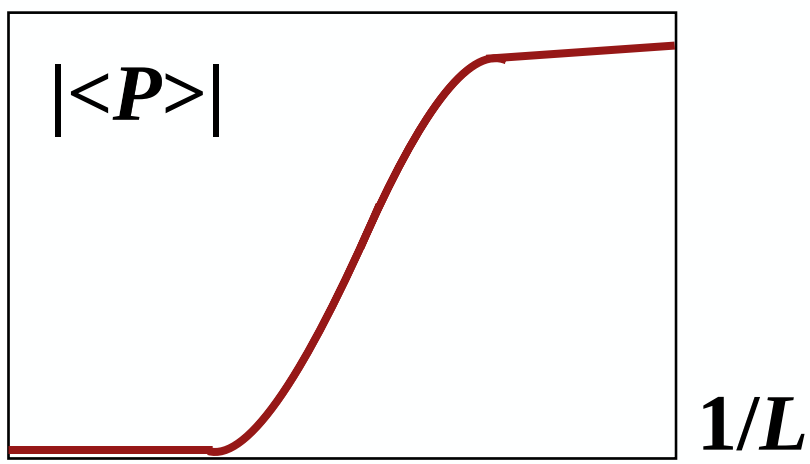
Mixed anomaly (global inconsistency) btwn $Z_N$ and $T$ survives on $R\times S^1$.
It is a necessary condition of adiabatic continuity, but not conclusive…

# Lattice simulation for $Z_N$-twisted $CP^{N-1}$ model

Fujimori, Itou, TM, Nitta, Sakai (19)(20)

What we want to check is the following conjecture:

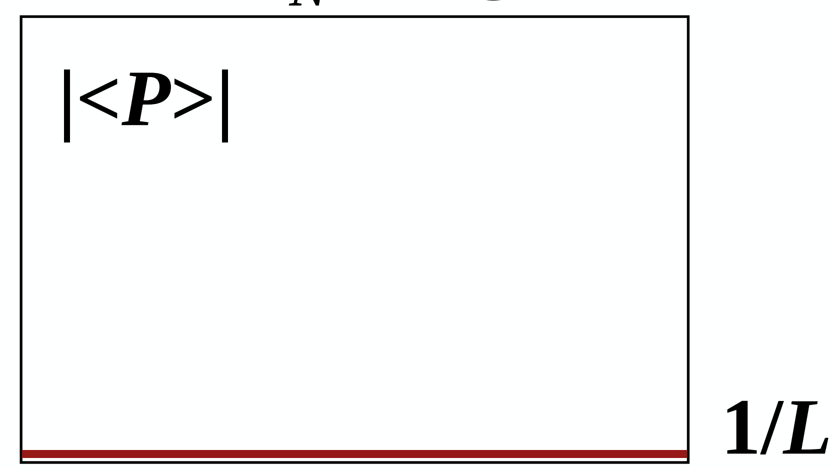
**PBC**




**$|\langle P \rangle| \not\approx 0$ for large $1/L$**

Fujimori, Itou, TM, Nitta, Sakai (19)

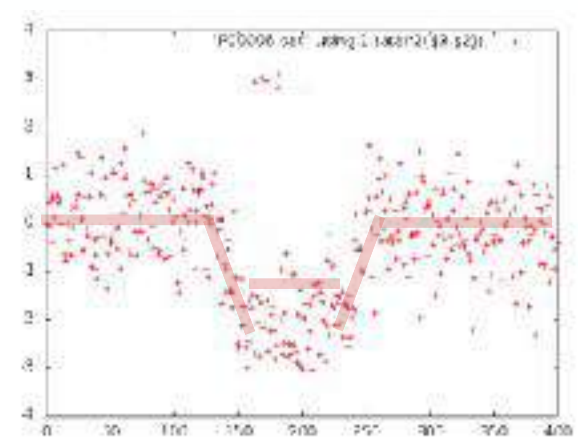
**$Z_N$-TBC**




**still $|\langle P \rangle| \sim 0$ for large $1/L$**

We will show very suggestive results on fractional instantons and adiabatic continuity

Fujimori, Itou, TM, Nitta, Sakai (20)



# Setup of lattice simulation

cf.) Berg,Luscher(81), Campostrini,et.al.(92), Alles,et.al.(00), Flynn,et.al.(15), Abe,et.al.(18)

- **Lattice formulation** $S = -N\beta \sum_{n,\mu} \left( \bar{z}_{n+\mu} \cdot z_n \lambda_{n,\mu} + \bar{z}_n \cdot z_{n+\mu} \bar{\lambda}_{n,\mu} - 2 \right)$

  Vector field $\phi$ is introduced: $\begin{aligned} \phi_{2j} &= \Re[z_{n,j}], \quad \phi_{2j+1} = \Im[z_{n,j}], \quad j = 0, \cdots, N-1 \\ \phi^R_\mu &= \Re[\lambda_\mu], \quad \phi^I_\mu = \Im[\lambda_{n,\mu}], \end{aligned}$

  ➡ $s_\phi = -N\beta\phi \cdot F_\phi = -N\beta |F_\phi| \cos\theta$

  Over heat-bath algorithm is adopted to update this $\theta$

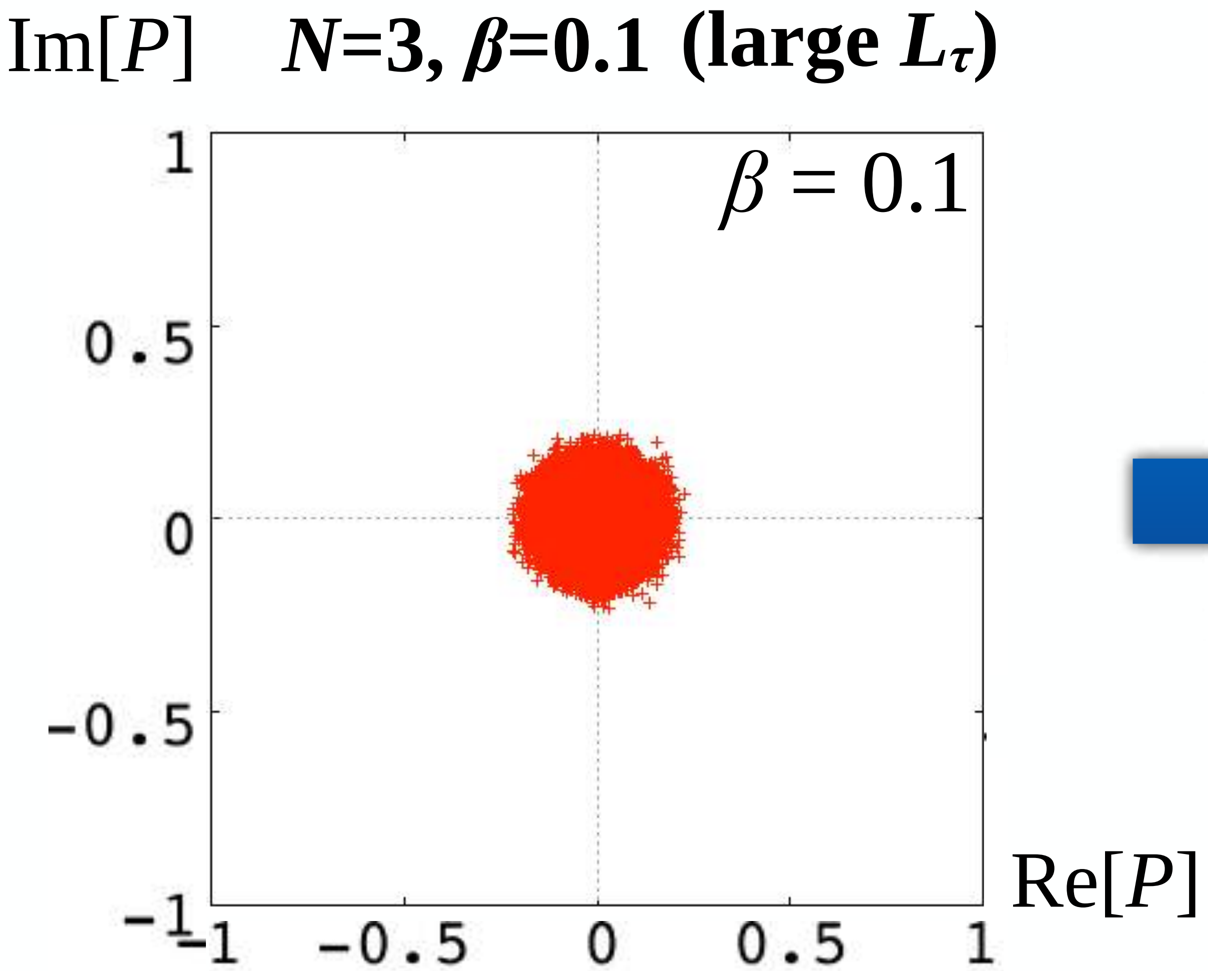
- **Parameters and quantities**

  $N_x = 200$ $N_\tau = 8$, $\beta$= 0.1-4.0, $N$ = 3-20, $N_{\text{sweep}}$ = 800,000

  - Distribution and expectation values of Polyakov loop
  - "Pseudo" entropy density $s = L_\tau(\langle T_{xx}\rangle - \langle T_{\tau\tau}\rangle)$

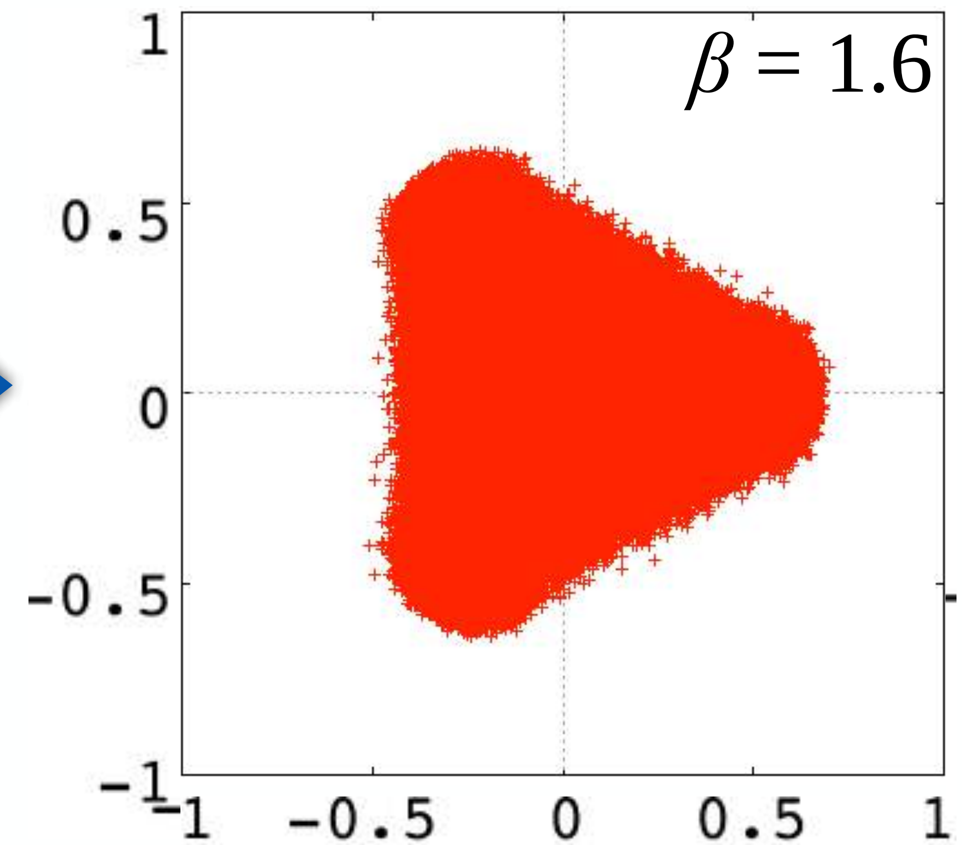
# Distribution plot of Polyakov loop

$N$=3, $\beta$=0.1 (large $L_\tau$)




$|\langle P \rangle| \sim 0$

$N$=3, $\beta$=1.6 (small $L_\tau$)




$|\langle P \rangle| \sim 0$

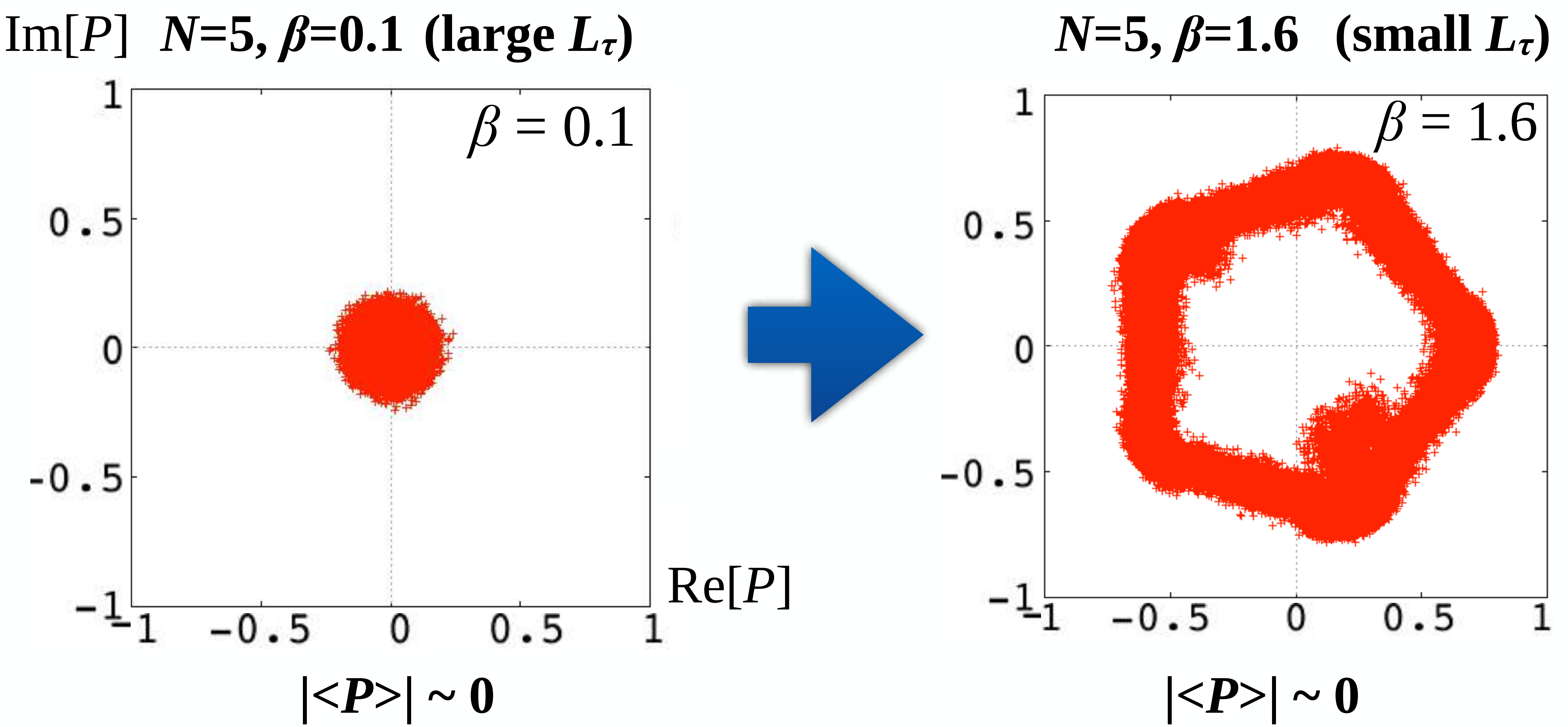
Low-$\beta$ : concentrates around origin

→ $Z_N$ symmetry at action level

High-$\beta$ : forms regular-polygon shape

→ $Z_N$ symmetry at quantum level

# Distribution plot of Polyakov loop




Low-$\beta$ : concentrates around origin

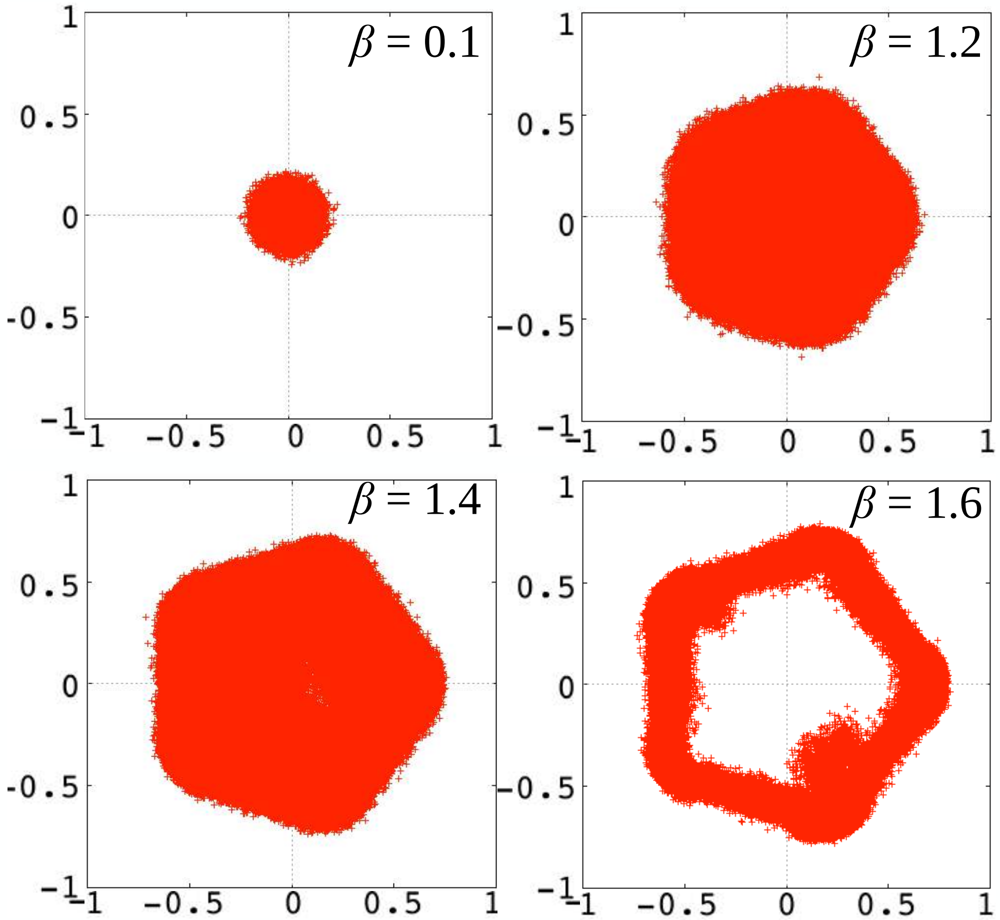
→ $Z_N$ symmetry at action level

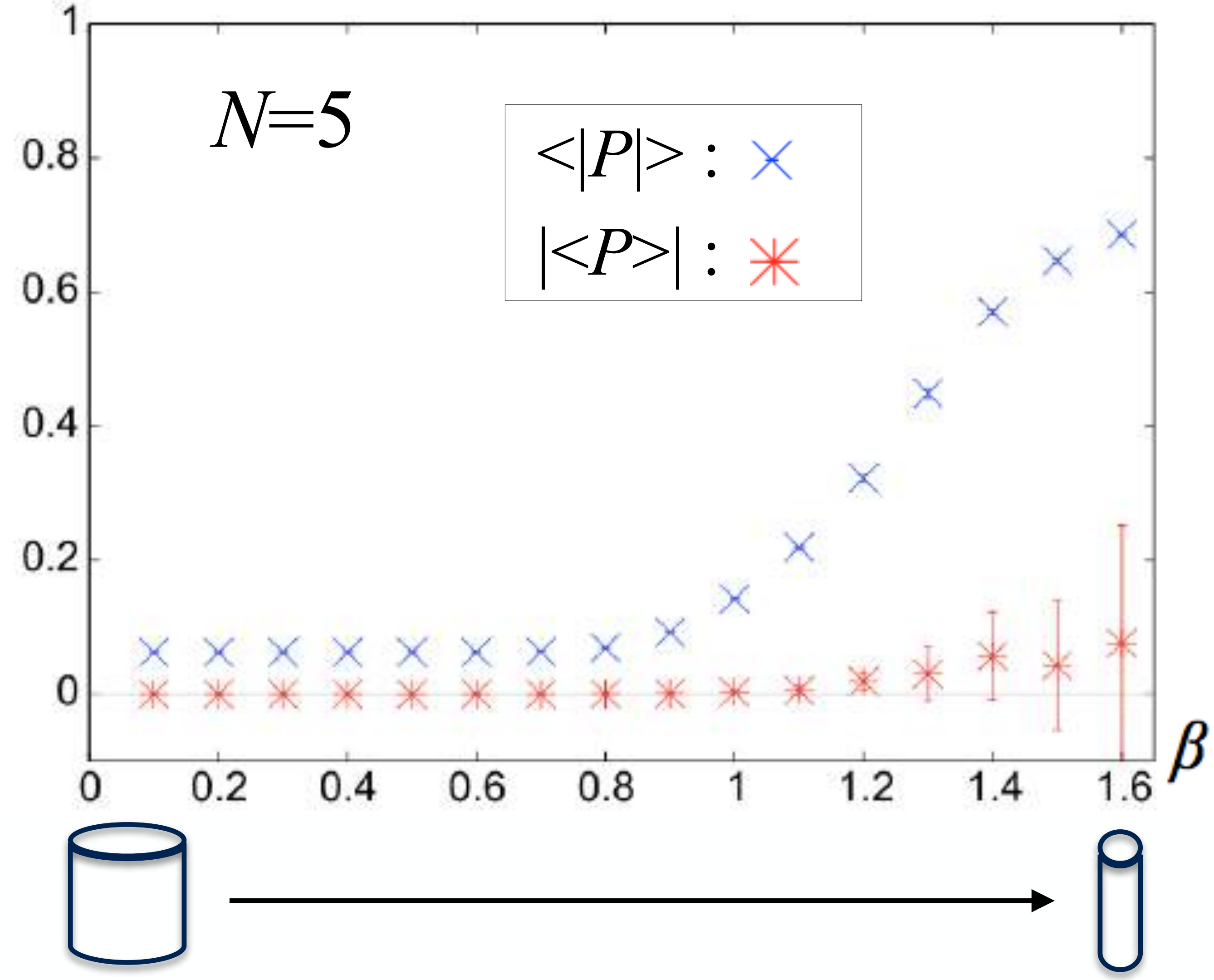
High-$\beta$ : forms regular-polygon shape

→ $Z_N$ symmetry at quantum level

# EV of Polyakov loop $|\langle P \rangle|$

**$N$ = 5**



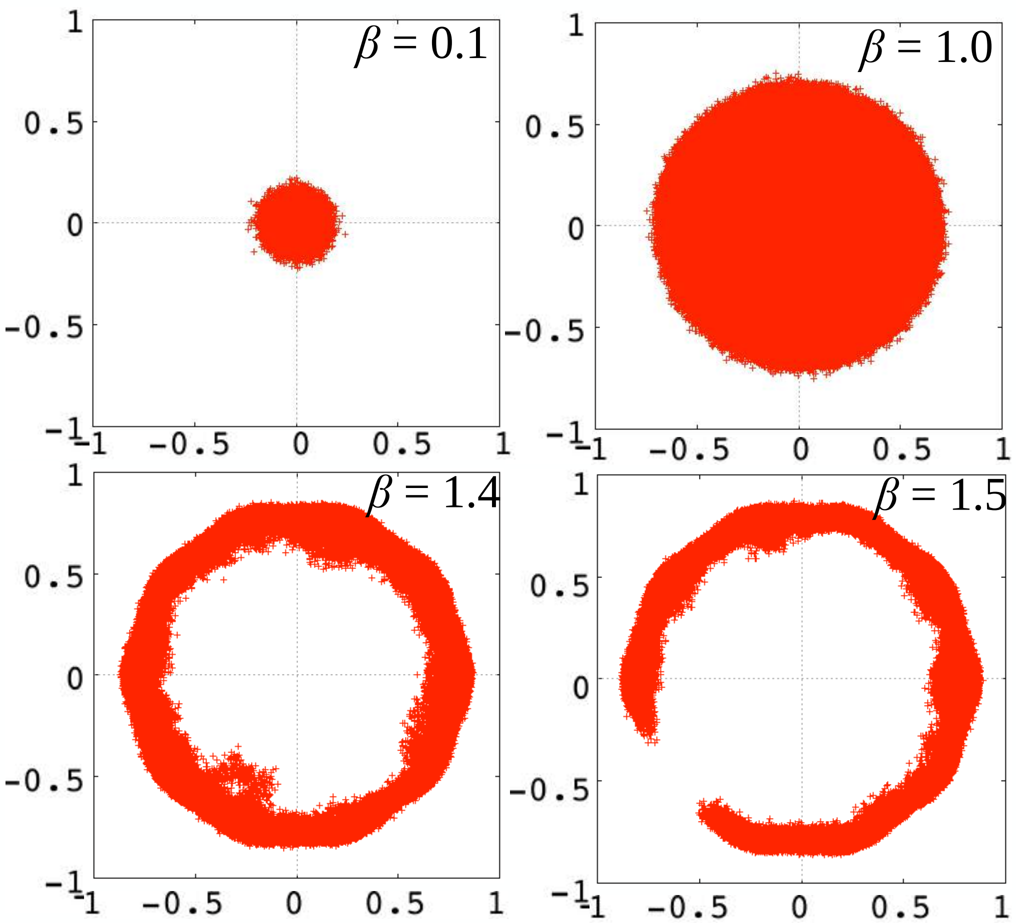



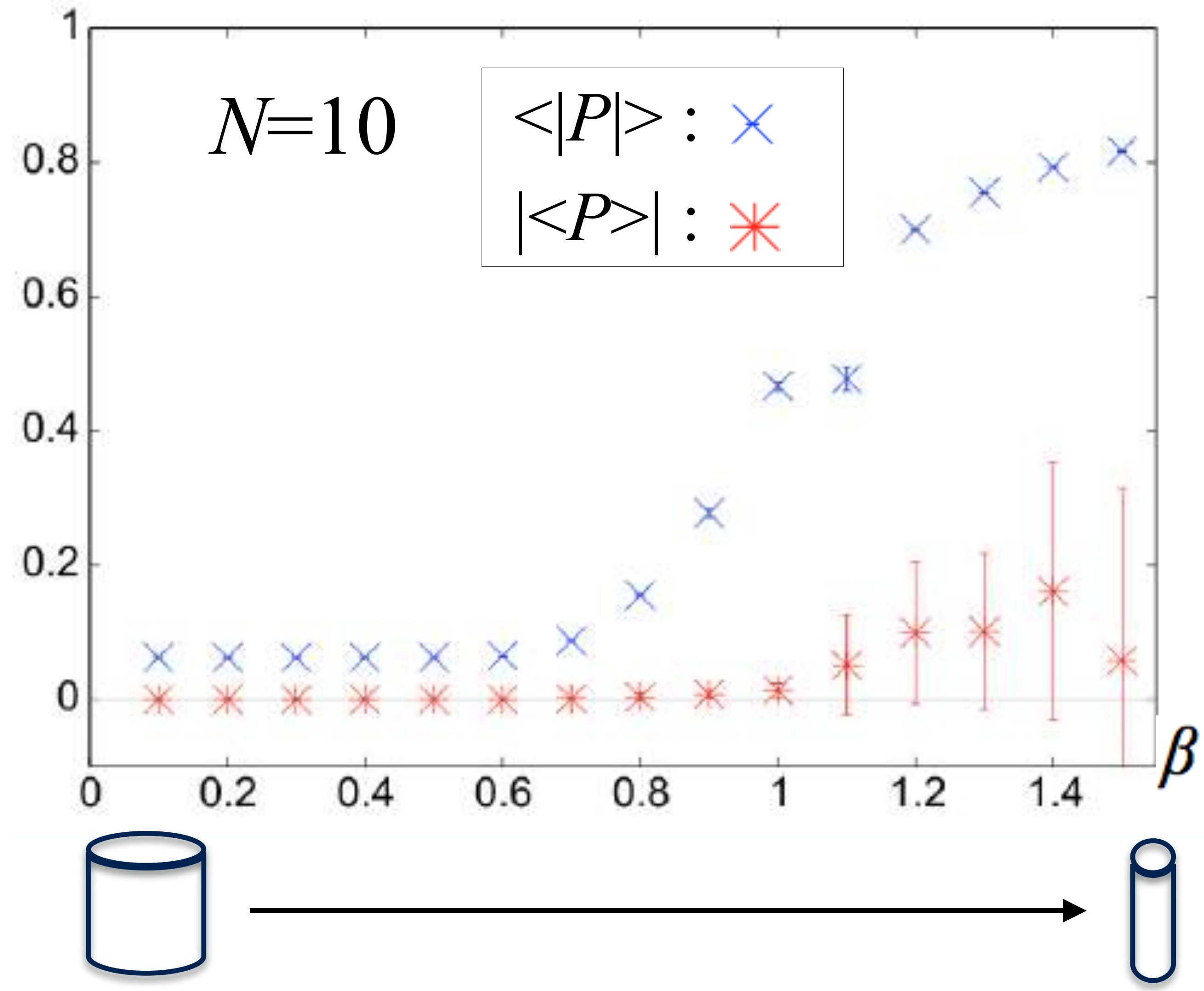

- Low $\beta$ → $|\langle P \rangle| = 0$ : distribution around origin
- High $\beta$ → $|\langle P \rangle| \sim 0$ : distribution forms regular polygons

$\langle P \rangle$ is still small even above characteristic $\beta$ defined by $\langle |P| \rangle$
This peculiar behavior indicates $Z_N$ stability

# EV of Polyakov loop $|\langle P\rangle|$

**$N$ = 10**

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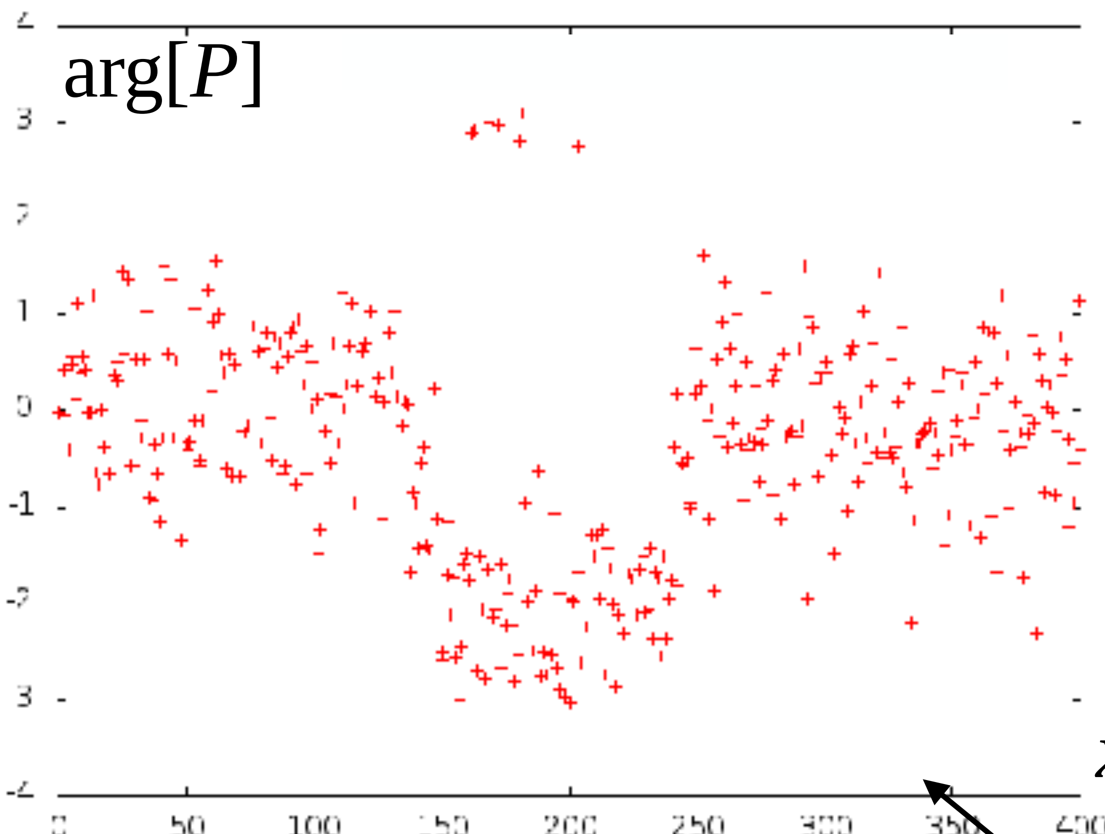
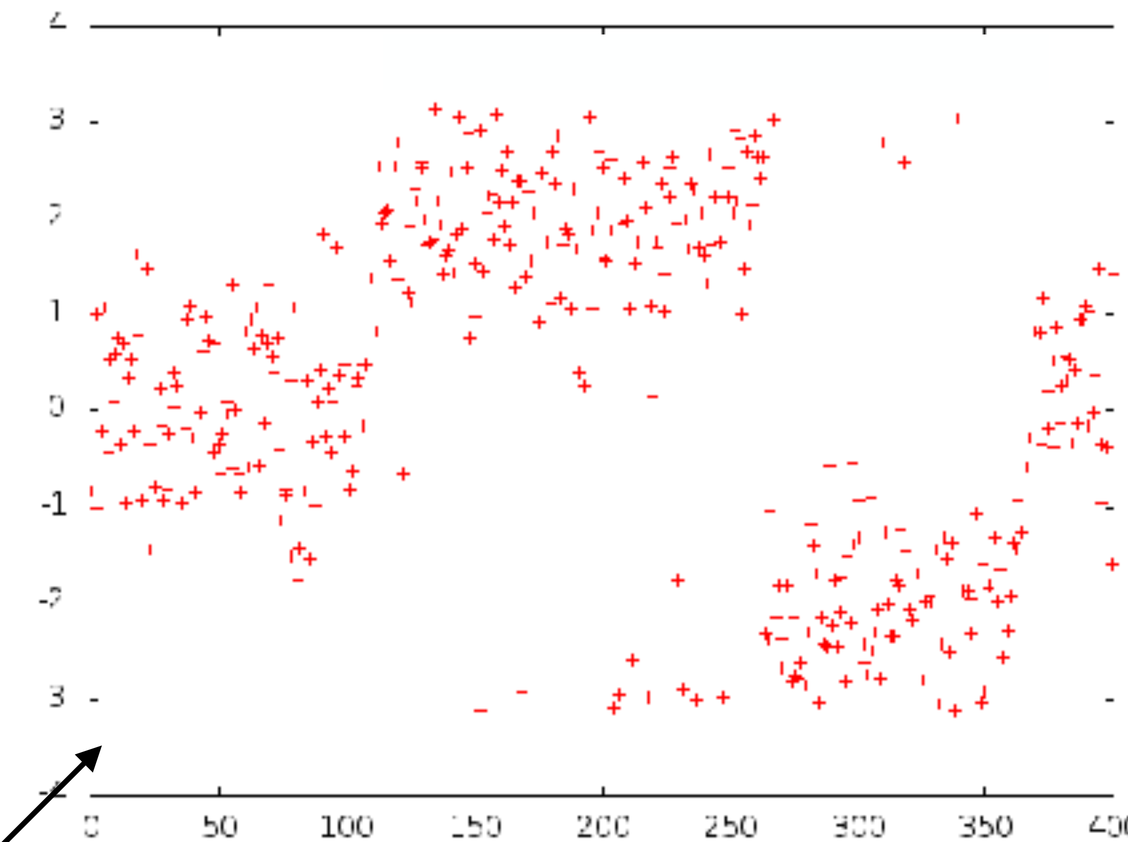
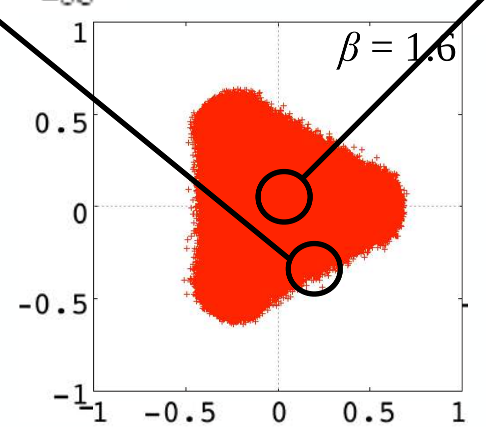
# Fractional instantons

Pick up two of configurations and look into the $x$-dependence of $\arg[P]$




1/3 fractional antiintanton +
1/3 fractional instanton
= **bion**

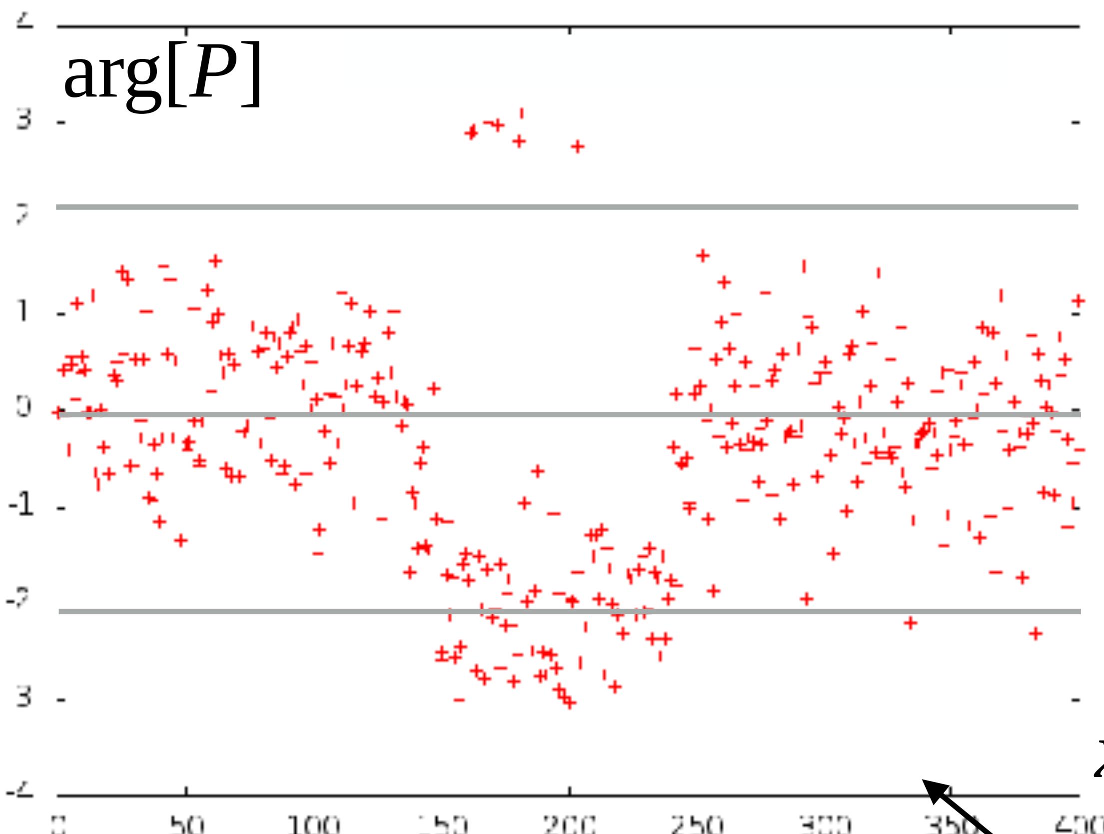


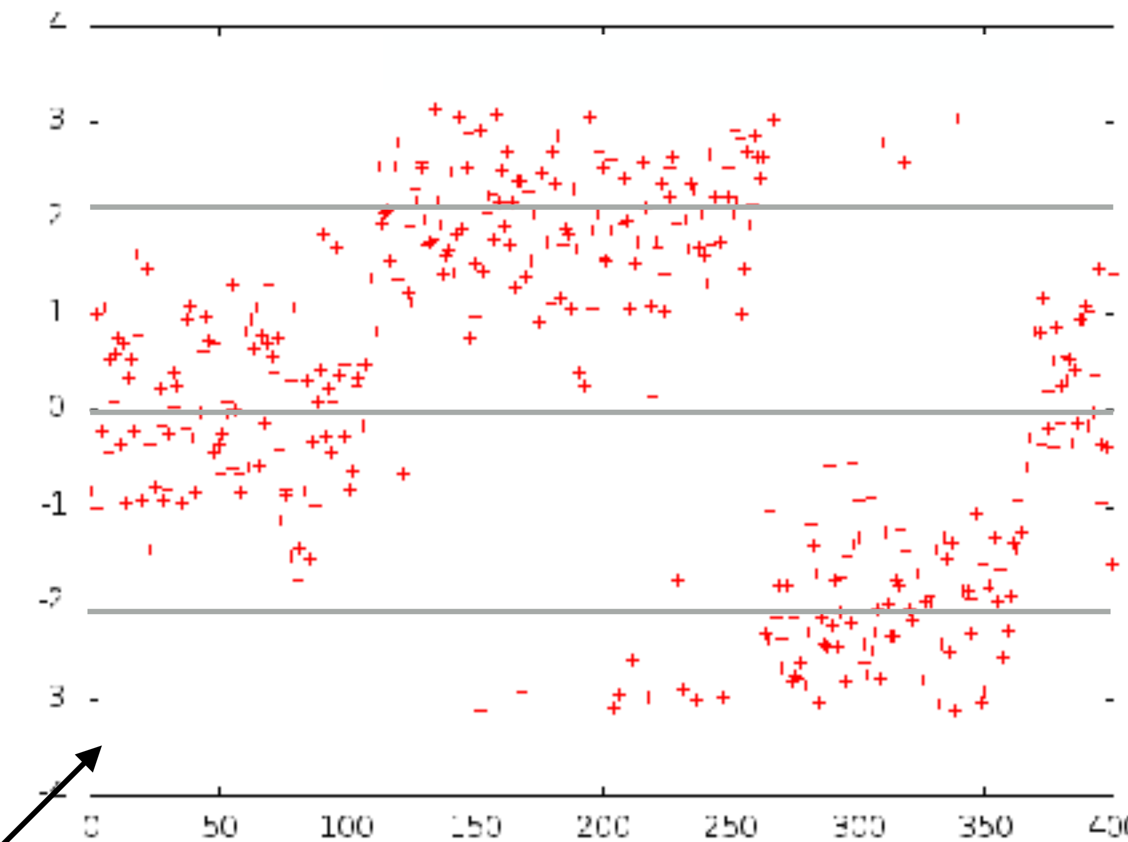

3 × 1/3 fractional intantons
= **single instanton**

Fractional instantons cause transition among $N$ classical minima, which leads to stability of $Z_N$ symmetry and adiabatic continuity

# Fractional instantons

Pick up two of configurations and look into the $x$-dependence of $\arg[P]$

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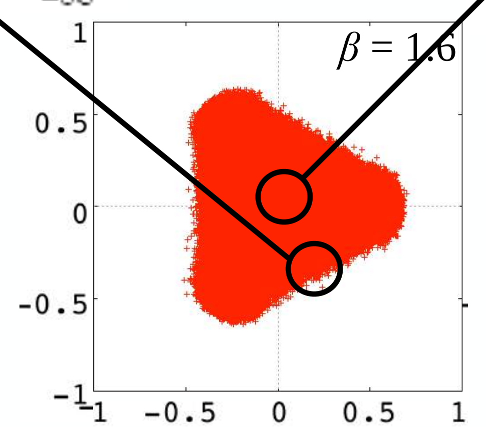


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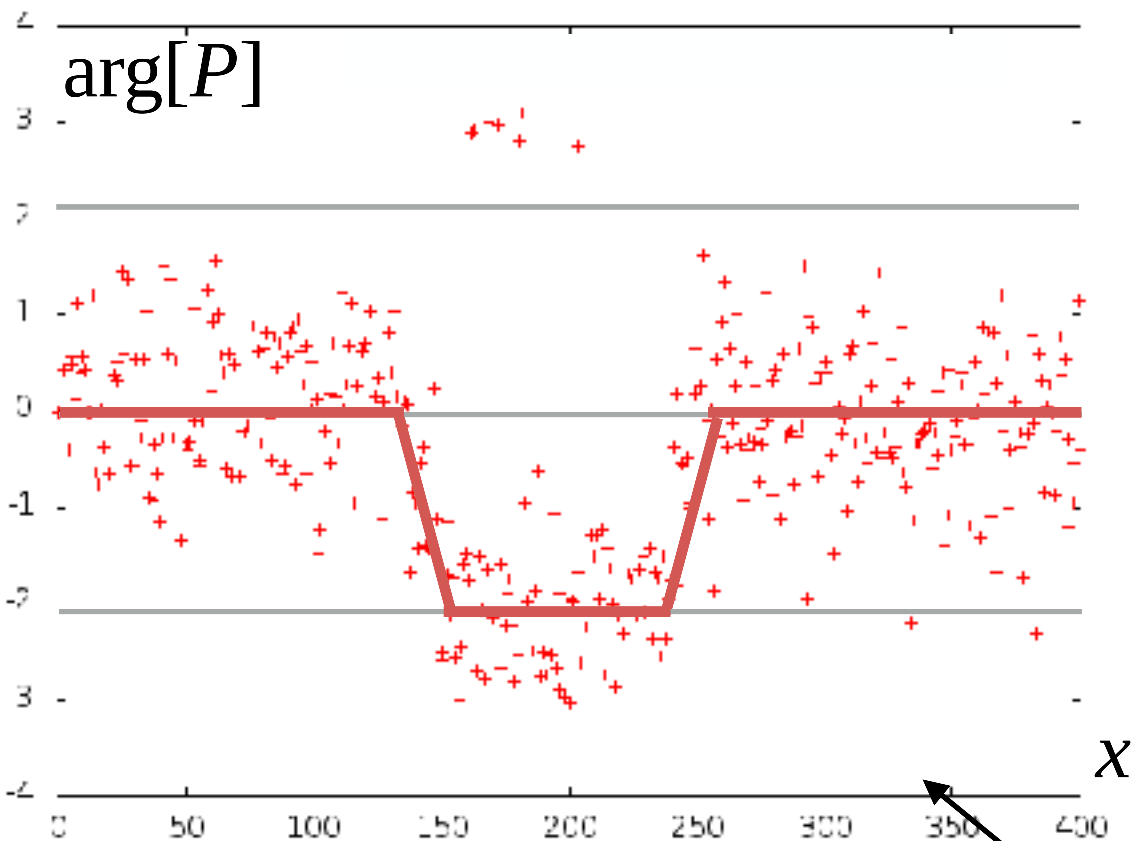


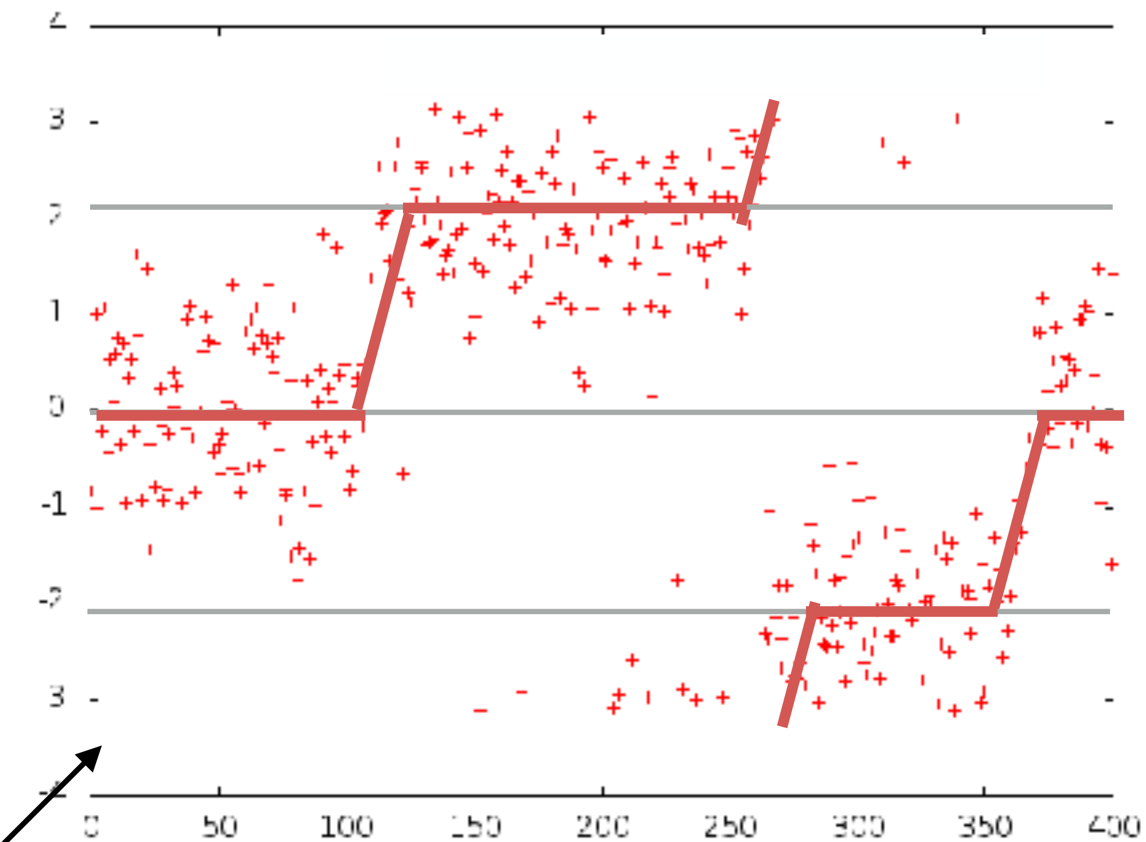


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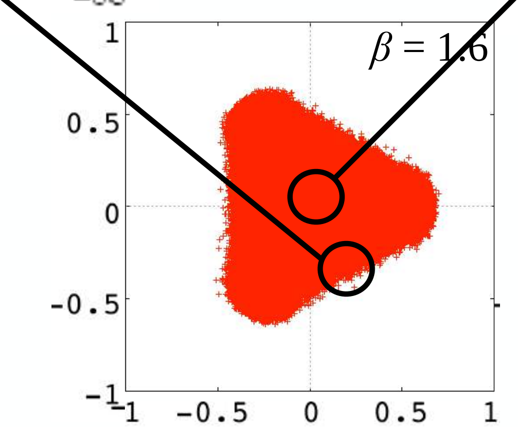
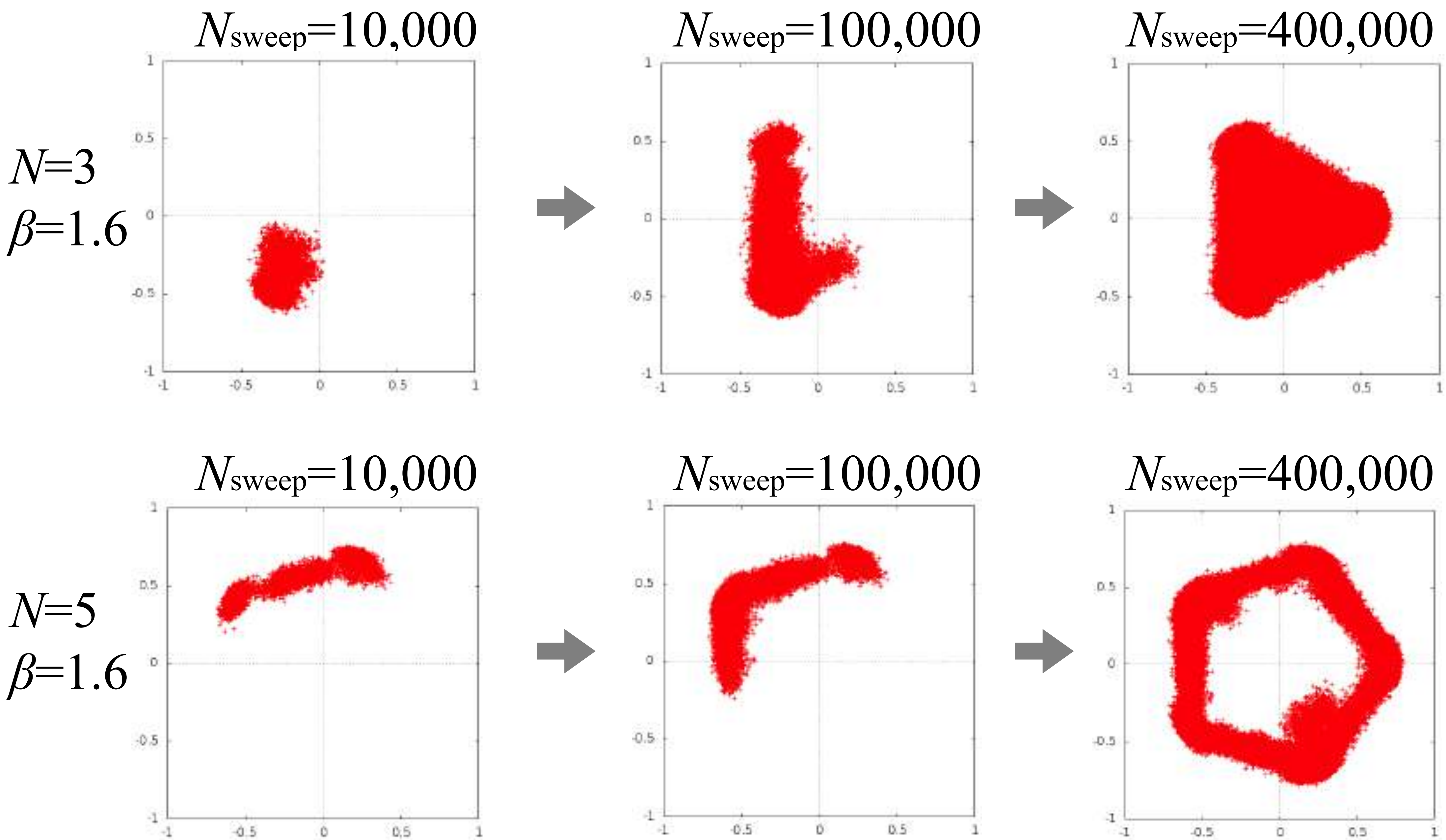



$3 \times 1/3$ fractional intantons
= **single instanton**

See Itou(18) for YM fractional instantons

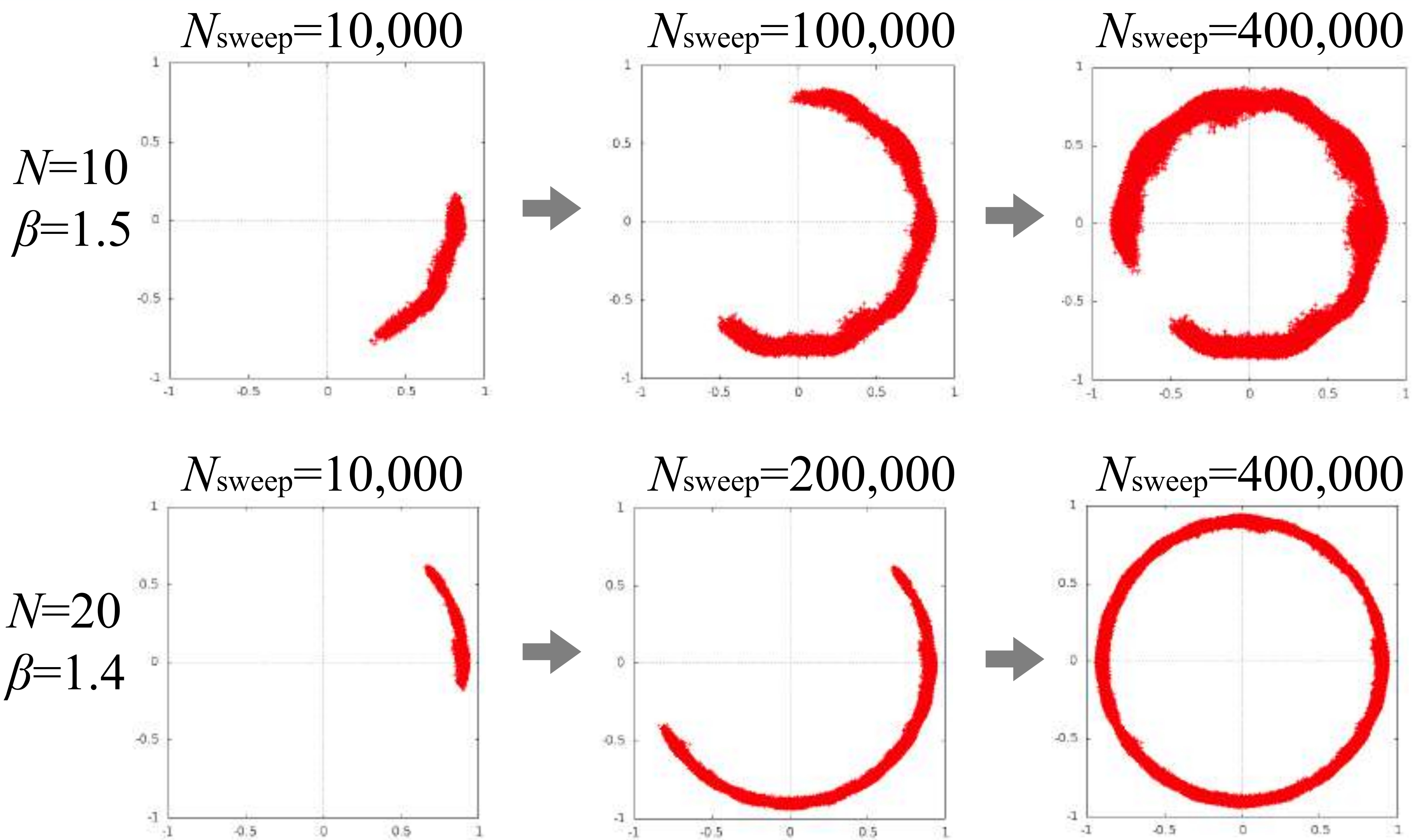
Fractional instantons cause transition among $N$ classical minima, which leads to stability of $Z_N$ symmetry and adiabatic continuity

# Large statistics gives small |<P>|




Even though polygon shape is broken and |<P>| is nonzero for large beta, adopting larger statistics restores polygon shape and leads to small |<P>|.

# Large statistics gives small |<P>|

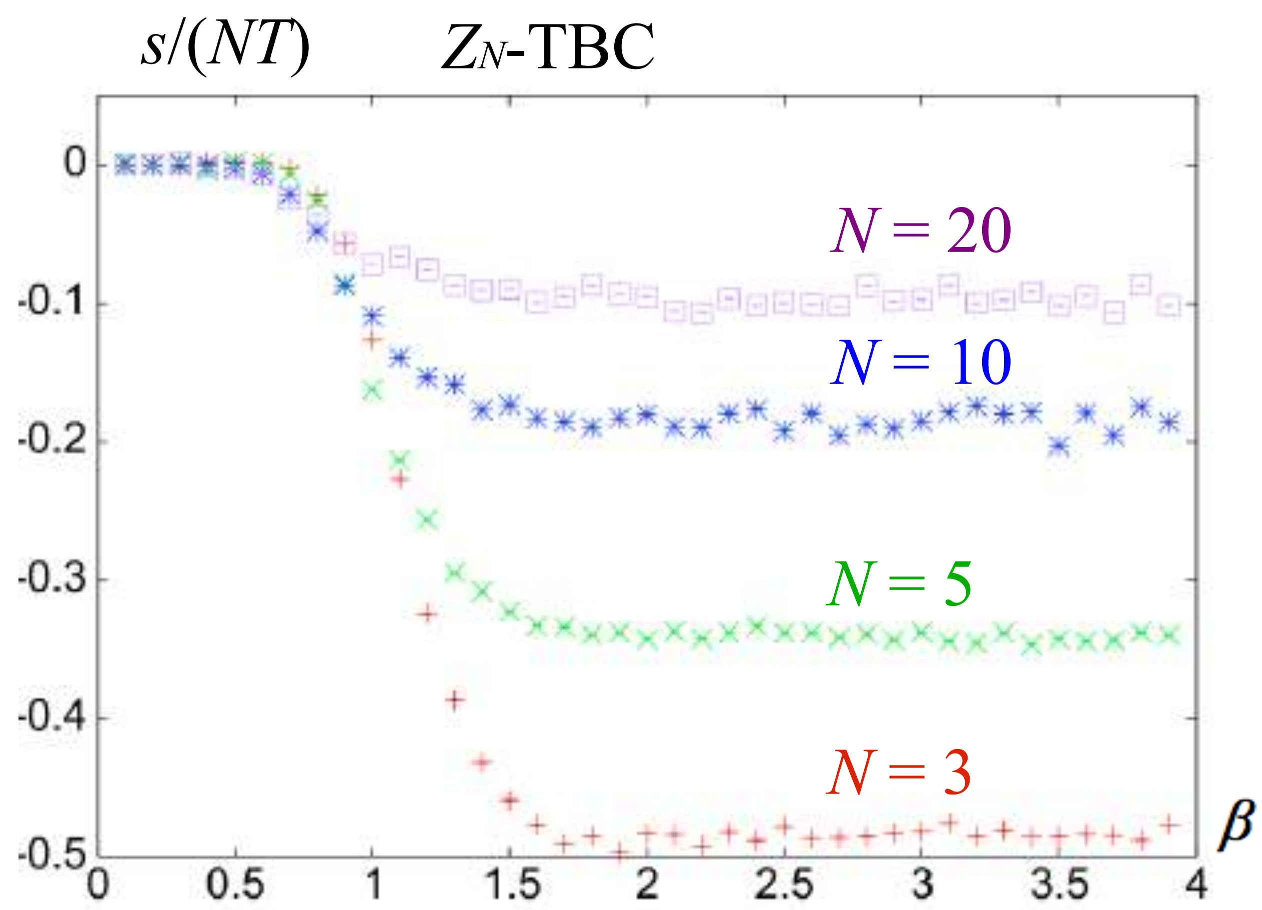



Even though polygon shape is broken and |<P>| is nonzero for large beta, adopting larger statistics restores polygon shape and leads to small |<P>|.

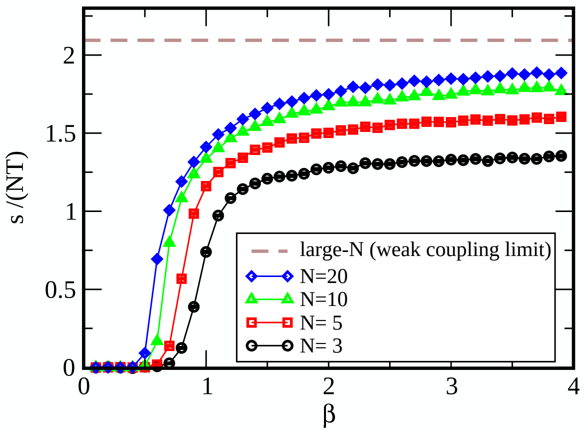
# "Pseudo entropy density"

Quantity corresponding to thermal entropy density for PBC

$$s = \langle T_{xx} - T_{\tau\tau} \rangle / T, \qquad \text{with} \;\; T \equiv 1/L_\tau$$




cf.) Thermal entropy density for PBC




- It takes a negative value, unlike thermal entropy for PBC
- There is no transition, which is consistent with adiabatic continuity
- In large N, it is likely to become zero for the whole beta region, consistent with the volume independence

cf.) Sulejmanpasic (16)

All of these results imply $\mathbb{Z}_N$ stability
even at small compactification circumference
and supports adiabatic continuity conjecture !

# 4. Other theories with $Z_N$-twisted b.c.

Hongo, TM, Tanizaki (18)
TM, Tanizaki, Unsal (19)

Yuya will discuss them in detail in the next talk.

# $SU(3)/U(1)^2$ flag sigma model on $R \times S^1$

Hongo, TM, Tanizaki(18)

$$S = -\sum_{\ell=1}^{3} \int \left[ \frac{1}{2g} |(d + i a_\ell)\phi_\ell|^2 - \frac{i\theta_\ell}{2\pi} d a_\ell \right]$$

$$\boldsymbol{\phi}_\ell = (\phi_{1,\ell}, \phi_{2,\ell}, \phi_{3,\ell})^t \in \mathbb{C}^3 \ (\ell = 1, 2, 3)$$

$$\overline{\boldsymbol{\phi}}_\ell \cdot \boldsymbol{\phi}_k = \delta_{\ell k} \qquad a_1 + a_2 + a_3 = 0$$

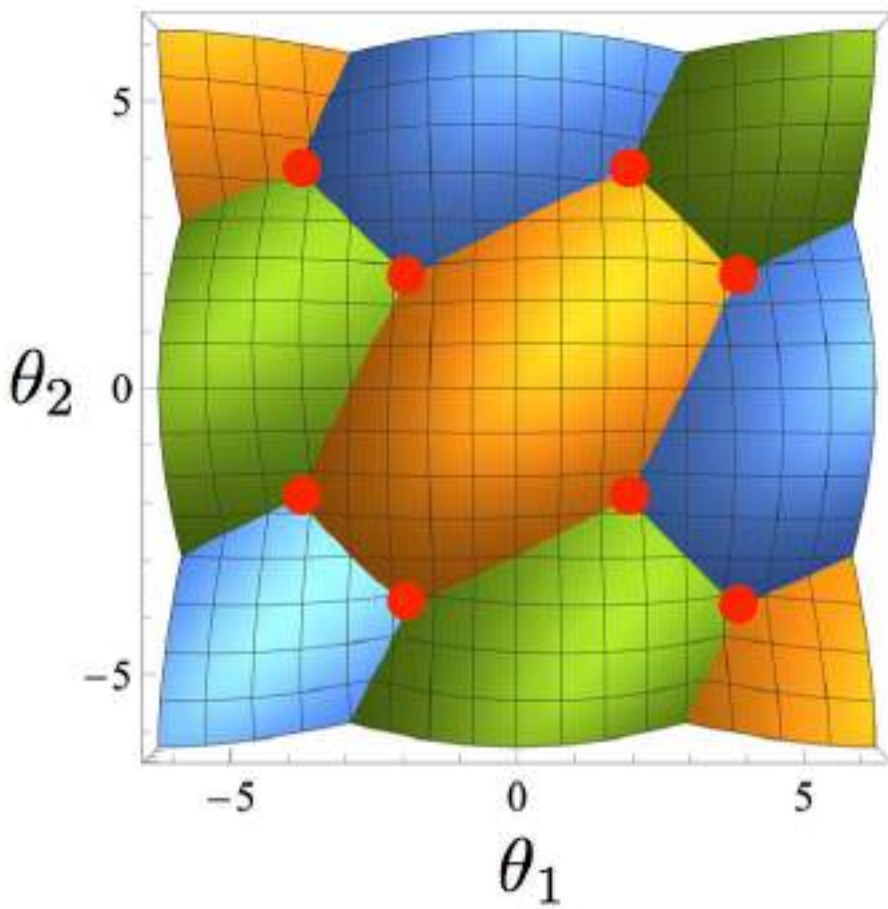
- Another extension of spin chain systems
- Another generalization of $O(3)$ or $CP^1$ nonlinear sigma model

Bykov (11), Lajko, et.al. (17) Ohmori,Seiberg,Shao(18)

# Phase diagram on $R\times S^1$ and $R^2$

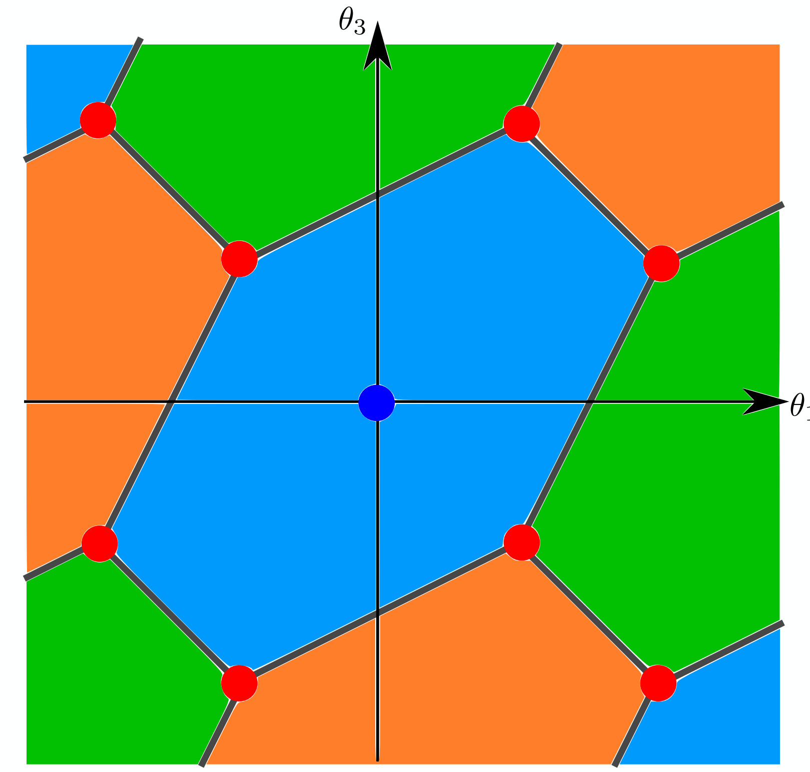
Hongo, TM, Tanizaki(18)

$R\times S^1$




**consistent !**

$R^2$




Tanizaki, Sulejmanpasic (18)

- 't Hooft anomaly survives in $Z_3$-twisted flag sigma model on $R \times S^1$
- DIGA with fractional instantons on $R \times S^1$ gives phase structure consistent with the conjectured one on $R^2$

$Z_N$-twisted theory correctly keeps 2d vacuum structure !

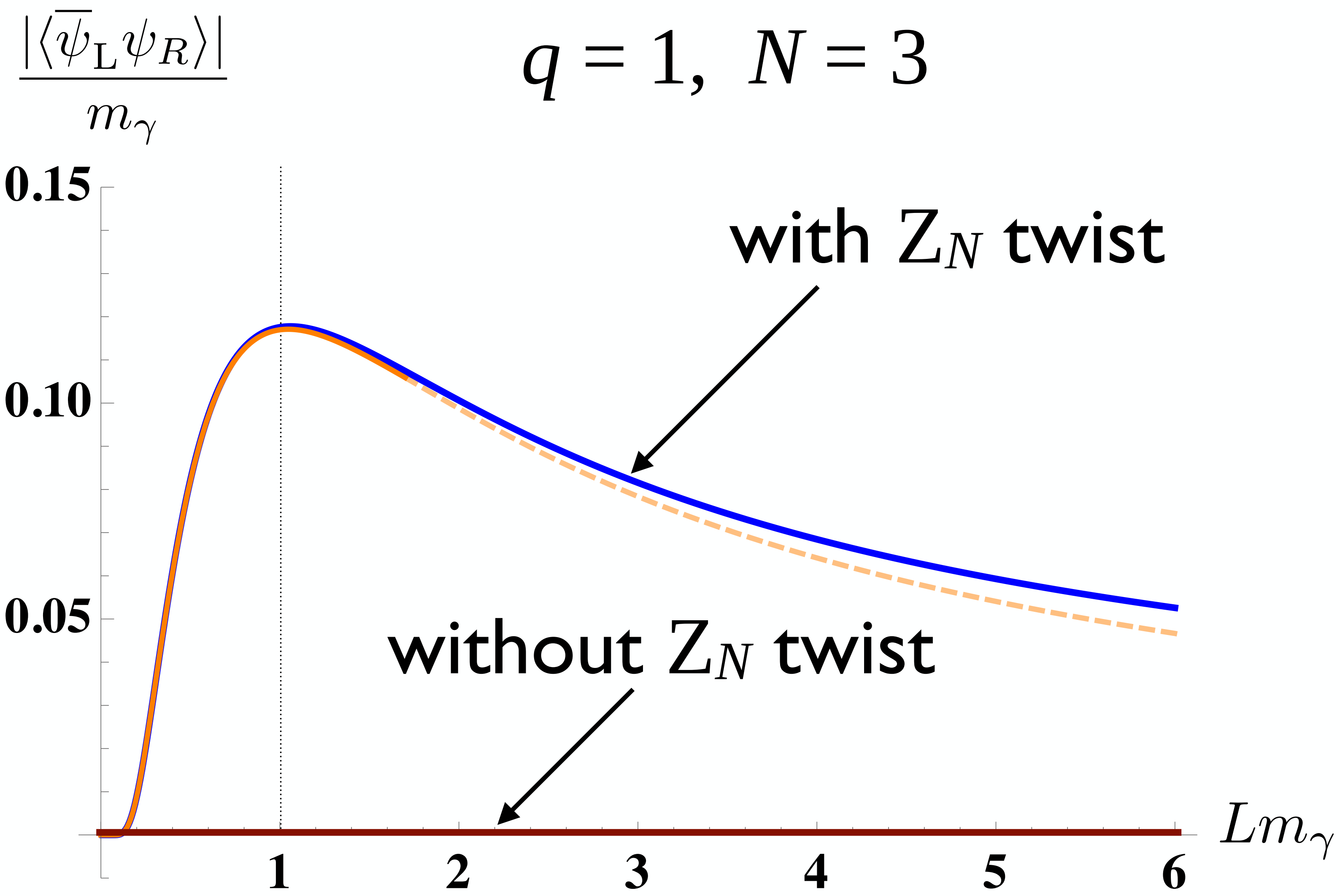
# Charge-$q$ $N$-flavor Schwinger model on $\mathrm{R} \times \mathrm{S}^1$

TM, Tanizaki, Unsal (19)

$$S = \frac{1}{2e^2}\int_{M_2} |\mathrm{d}a|^2 + \frac{\mathrm{i}\theta}{2\pi}\int_{M_2} \mathrm{d}a + \sum_{f=1}^{N}\int_{M_2} \mathrm{d}^2x\, \overline{\psi}^f \gamma^\mu (\partial_\mu + q\mathrm{i}a_\mu)\psi^f$$

- $q$=2 on domain-wall of $\mathcal{N}$=1 SU(2) SYM   Anber, Poppitz (18)
- $\mathrm{O}\bar{1}$ - $\overline{\mathrm{D}1}$ system : $q$=2, $N$=8   Sugimoto, Takahashi (04)  Armoni, Sugimoto (18)

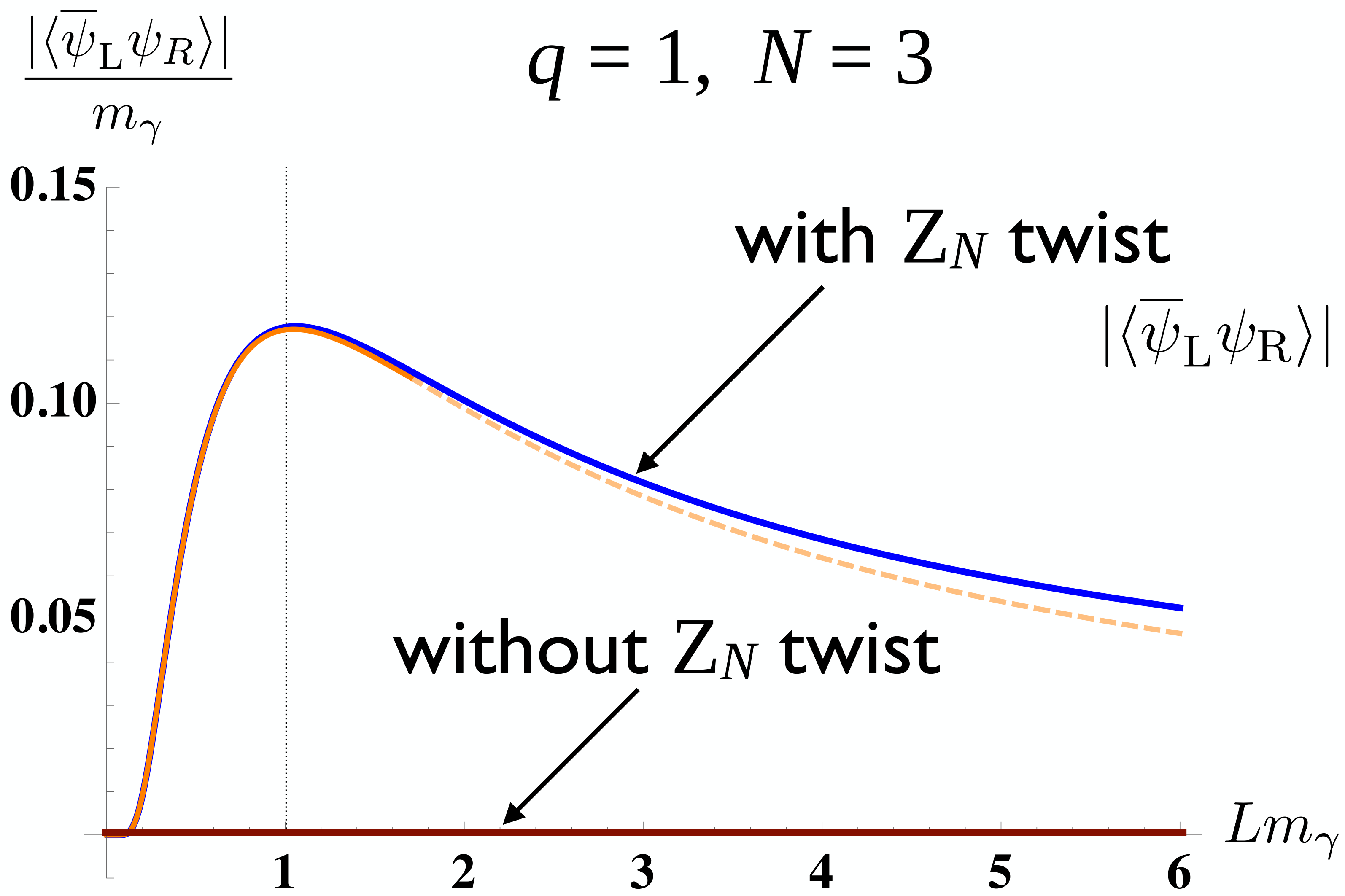
# Chiral condensate in $\mathrm{R} \times \mathrm{S}^1$




- 't Hooft anomaly survives in $\mathrm{Z}_N$-twisted Schwinger model on $\mathrm{R} \times \mathrm{S}^1$
- Chiral condensate of $\mathrm{Z}_N$-twisted model exhibits WZW scaling dim.

$\mathrm{Z}_N$-twisted theory correctly keeps 2-dimensional CFT properties !

# Chiral condensate in $\mathrm{R} \times \mathrm{S}^1$




$$|\langle \overline{\psi}_{\mathrm{L}} \psi_{\mathrm{R}} \rangle| \rightarrow \left(\frac{1}{L}\right)^{1-\frac{1}{N}}$$

SU(N)$_1$ WZW scaling dim.

- ’t Hooft anomaly survives in $Z_N$-twisted Schwinger model on $\mathrm{R} \times \mathrm{S}^1$
- Chiral condensate of $Z_N$-twisted model exhibits WZW scaling dim.

$Z_N$-twisted theory correctly keeps 2-dimensional CFT properties !

# Summary

- Bion contributions at field theoretical levels with $Z_N$-twisted boundary condition yield renormalon-like imaginary ambiguity.
- The 't Hooft anomalies survive in the compactified theory with $Z_N$-twisted boundary condition.
- Lattice simulation exhibits $Z_N$ stability even at small radius, which could imply adiabatic continuity in 2D.
- Other results also indicate that $Z_N$ twisted b. c. leads to the adiabatic continuity of the vacuum and phase structures in 2D.

# Other resurgence projects

- Exact results of 3D N=2 Chern-Simons with matters via localization exhibit very clear resurgent structure. Fujimori, Honda, Kamata, TM, Sakai (18) inspired by Aniceto, Russo, Schiappa(14) Gukov, Marino, Putrov(16) Honda(16)

- Quantum phase transition in 3D N=4 QED can be elucidated by use of thimble analysis and trans-series expansion. Fujimori, Honda, Kamata, TM, Sakai, Yoda, in progress inspired by Russo, Tierz (16)

- Relation between Stokes phenomena in exact-WKB and standard resurgent analysis is clarified, where equivalence of several quantization conditions are shown explicitly. Sueishi, Kamata, TM, Unsal, to appear in arXiv shortly

- Schwinger effect under time-dependent strong electric field is analyzed by exact-WKB method, whose results are consistent with those of steepest descent method. Taya, Fujimori, TM, Nitta, Sakai in progress

- Resurgent structure of non-relativistic quantum system is being studied in a similar manner to the relativistic one. Fujimori, Kamata, TM, Nitta, Sakai in progress inspired by Marino, Reis (19)