Transport coefficients and spectral functions from the lattice ?

Gert Aarts
(OSU)

hep-ph/0203177
with José María Martínez Resco
JHEP

hep-ph/0108125
PLB

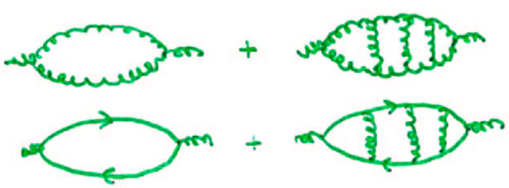
Outline

- transport coefficients
- spectral functions ⟷ lattice correlators
- shear viscosity $\lambda\phi^4$
  SU(N)
- relevance for other problems: thermal dilepton rate
- classical field approximation: spectral function and the plasmon

Transport coefficients

- RHIC : non ideal hydrodynamics
  (ideal hydro works extremely well...)
- field theory calculation : highly nontrivial
  what has been computed ?
  (using Kubo relations and diagrams)
  - scalars : shear viscosity (Jeon)

  - QCD : color conductivity (Martinez Resco and Valle Basagoiti)



- other transport coefficients in gauge theories
  (viscosity, electrical conductivity, flavor diffusion)
  computed using kinetic theory to leading log (Arnold, Moore, Yaffe)

Fully nonperturbative computation of transport coefficients with lattice QCD ?

- first attempt : Karsch & Wyld (1987)
- continued : Nakamura et.al. (1996-..)

How : Kubo relation : shear viscosity

$$\eta = \frac{1}{20} \lim_{\omega \to 0} \frac{1}{\omega} \int d^4x \, e^{i\omega t} \langle [\pi_{ij}(\vec{x},t), \pi_{ij}(\vec{0},0)] \rangle$$

with $\pi_{ij} = T_{ij} - \frac{1}{3}\delta_{ij} T_{kk}$, T_{ij} energy-momentum tensor

spectral function :

$$\rho_{\pi\pi}(x-y) = \langle [\pi_{ij}(x), \pi_{ij}(y)] \rangle$$

$$\rho_{\pi\pi}(\omega,\vec{p}) = \int d^4x \, e^{ip.x} \rho_{\pi\pi}(x)$$

$$\Rightarrow \quad \eta = \frac{1}{20} \frac{\partial}{\partial\omega} \rho_{\pi\pi}(\omega) \Big|_{\omega=0}$$

relation with euclidean correlator :

$$G^E_{\pi\pi}(\tau) = \int d^3x \, \langle \pi_{ij}(\vec{x},\tau) \pi_{ij}(\vec{0},0) \rangle \qquad 0 < \tau < 1/T$$

$$= T \sum_n e^{i\omega_n \tau} G^E_{\pi\pi}(\omega_n)$$

Dispersion relation :

$$G^E_{\pi\pi}(\omega_n) = \int \frac{d\omega}{2\pi} \frac{\rho_{\pi\pi}(\omega)}{\omega - i\omega_n}$$

$$\Rightarrow \boxed{G^E_{\pi\pi}(\tau) = \int_0^\infty \frac{d\omega}{2\pi} K(\tau,\omega)\,\rho_{\pi\pi}(\omega)}$$

with kernel:
$$K(\tau,\omega) = T\sum_n e^{i\omega_n\tau}\frac{2\omega}{\omega^2+\omega_n^2}$$
$$= e^{\omega\tau}n(\omega) + e^{-\omega\tau}[1+n(\omega)]$$

lattice program:

- compute $G^E_{\pi\pi}(\tau)$ numerically
- reconstruct $\rho_{\pi\pi}(\omega)$ from integral equation
  - using ansatz for $\rho_{\pi\pi}(\omega)$ (Karsch & Wyld)
  - using Maximal Entropy Method
- find $\eta \sim \frac{d}{d\omega}\rho_{\pi\pi}(\omega)|_{\omega=0}$

obvious questions :

- $\rho_{\pi\pi}(\omega)$ at high T ?
- $G^E_{\pi\pi}(\tau)$ at high T ?
- how does η, or more generally $\rho_{\pi\pi}(\omega)$ at $\omega \ll T$, manifest itself in $G^E_{\pi\pi}(\tau)$?

How does η, or more generally $\rho_{\pi\pi}(\omega)$ at $\omega \ll T$, manifest itself in $G^E_{\pi\pi}(\tau)$?

easy :
$$G_{\pi\pi}(\tau) = \int_0^\infty \frac{d\omega}{2\pi} K(\tau,\omega)\,\rho_{\pi\pi}(\omega)$$
$$K(\tau,\omega) = n(\omega)e^{-\omega\tau} + [1+n(\omega)]e^{\omega\tau}$$

$$\Rightarrow \omega \ll T: \quad K(\tau,\omega) = \underbrace{\frac{2T}{\omega}}_{\tau\text{-independent}} + \underbrace{\mathcal{O}(\omega/T)}_{\tau\text{-dependence}}$$

$$\Rightarrow G_{\pi\pi}(\tau) = \int_0^{\omega_\Lambda} \frac{d\omega}{2\pi}\frac{2T}{\omega}\rho_{\pi\pi}(\omega) \qquad \omega_\Lambda \ll T$$

constant contribution
determined by integral of $\rho_{\pi\pi}(\omega)/\omega$

low-frequency region $\leftrightarrow$ soft dynamics
$\sim$ transport coefficient
$\Updownarrow$
single constant term in $G(\tau)$ $\sim \int d\omega\, \rho(\omega)/\omega$

$\rho_{\pi\pi}(\omega)$ at high T ?

$\lambda\phi^4$:

$$\pi_{ij} = \partial_i\phi\,\partial_j\phi - \tfrac{1}{3}\delta_{ij}\,\partial_k\phi\,\partial_k\phi$$

$$\rho_{\pi\pi}(x-y) = \langle[\pi_{ij}(x),\pi_{ij}(y)]\rangle$$

$\Rightarrow$ one-loop expression :

$$\rho_{\pi\pi}(\omega) = \frac{4}{3}\int\frac{d^4k}{(2\pi)^4}\,|\vec{k}|^4\, n(k^0)\,\rho(k^0,\vec{k})\left[\rho(k^0+\omega,\vec{k}) - \rho(k^0-\omega,\vec{k})\right]$$

with one-particle spectral function $\rho(x-y) = \langle[\phi(x),\phi(y)]\rangle$

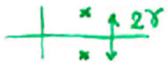
- quasi particles :

HTL plasmon mass $\quad m^2 = \frac{\lambda T^2}{24}(1+\dots)$

finite width $\quad \gamma_k = -\frac{\mathrm{Im}\,\Sigma_R(\omega_k,\vec{k})}{2\omega_k}$

$$= \gamma\,\frac{T}{\omega_k}\,B(k/T) \qquad \gamma = \frac{\lambda^2 T}{1536\pi}$$

- finite width essential when $\omega\to 0$:



pinch singularities (Jeon)

$$\rho_{\pi\pi}(\omega) \sim \frac{\omega}{\gamma}\,T^4 \qquad (\omega\to 0)$$

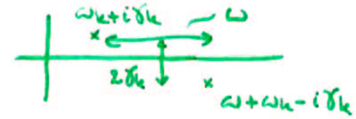
$$\eta \sim \frac{T^4}{\gamma} \sim \frac{T^3}{\lambda^2}$$

- ladder diagrams contribute at same order (Jeon)

each additional rung $\sim \frac{\lambda^2 T}{\gamma} \sim 1$

But : interested in all ω's [much easier]

$\omega \simeq \gamma$



$\omega \gg \gamma$ no pinch singularity, HTL calculation sufficient
no need for ladder series

$\omega \gtrsim T$ bare calculation

- $\omega \gg \gamma$ one-particle spectral function $\rho(p) = 2\pi\,\varepsilon(p^0)\,\delta(p^2-\omega_p^2)$ $\quad \omega_p = \sqrt{p^2+m^2}$

$$\Rightarrow \rho_{\pi\pi}(\omega) = \Theta(\omega-2m)\,\frac{(\omega^2-4m^2)^{5/2}}{48\pi\omega}\left[n(\omega/2)+\tfrac{1}{2}\right]$$

- threshold from HTL
- large frequencies: $\rho_{\pi\pi} = \frac{\omega^4}{96\pi}$ zero temperature decay

- $\omega \lesssim m$ pinching poles

one-particle spectral function

$$\rho_{BW}(p) = \frac{1}{2\omega_p}\left[\frac{2\gamma_p}{(p^0-\omega_p)^2+\gamma_p^2} - \frac{2\gamma_p}{(p^0+\omega_p)^2+\gamma_p^2}\right]$$

poles at $p^0 = \pm(\omega_p \pm i\gamma_p)$

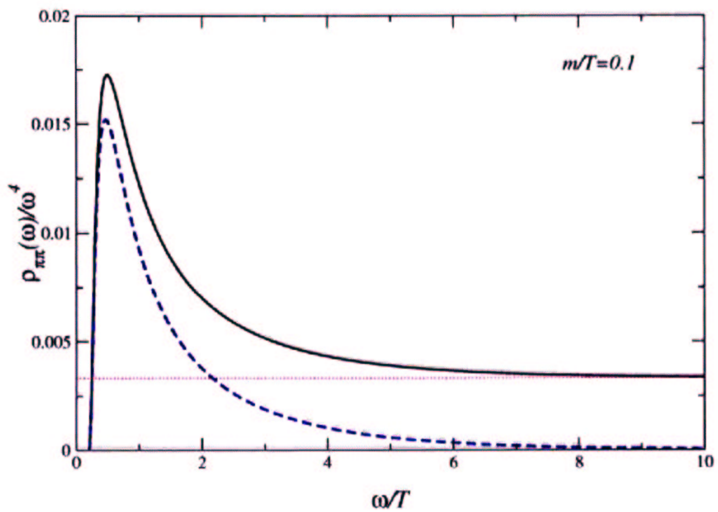
$$\Rightarrow \rho_{\pi\pi}(\omega) = -\frac{8}{3}\int\frac{d^3k}{(2\pi)^3}\,\frac{|\vec{k}|^4}{\omega_k^2}\,n'(\omega_k)\,\frac{\omega\gamma_k}{\omega^2+4\gamma_k^2}$$

- anticipated expression :
  - distance between the poles $\omega^2+4\gamma_k^2$
  - $\omega\to 0$: $\rho_{\pi\pi}(\omega) \sim \omega/\gamma_k$

$$\rho_{\pi\pi}(\omega) = \theta(\omega-2m)\,\frac{(\omega^2-4m^2)^{5/2}}{48\pi\omega}\left[n(\omega/2)+\frac{1}{2}\right]$$

ω sufficiently large

no pinch singularity

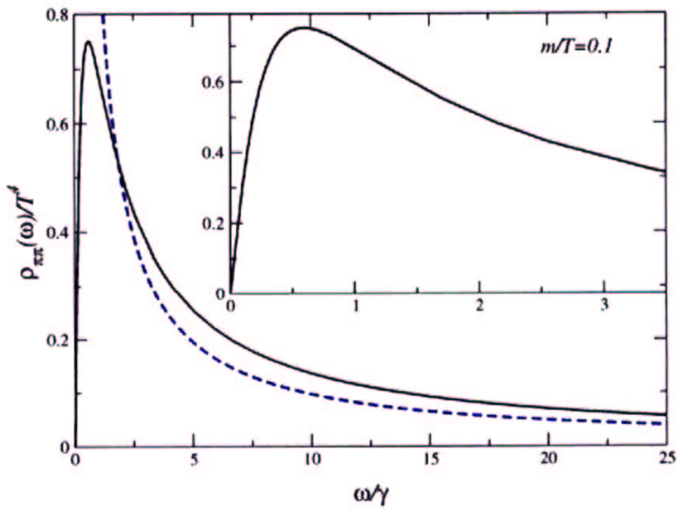


ω small: pinching poles

$$\rho_{\pi\pi}(\omega) = -\frac{8}{3}\int\frac{d^3k}{(2\pi)^3}\,\frac{|\mathbf{k}|^4}{\omega_k^2}\,n'(\omega_k)\,\frac{\omega\gamma_k}{\omega^2+4\gamma_k^2}$$

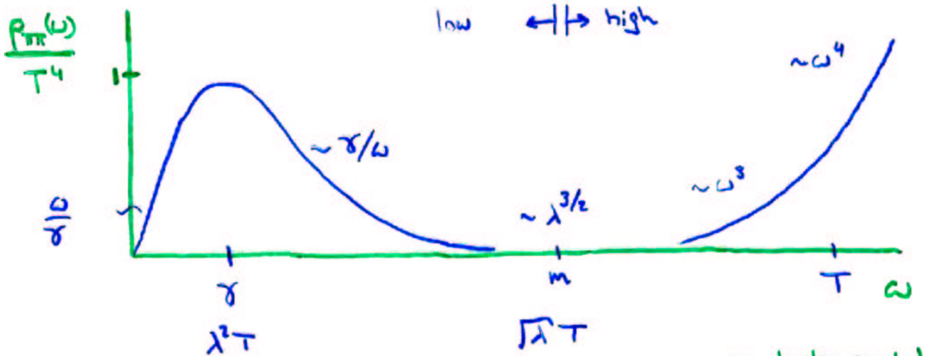
with width $\gamma_k = \gamma\,\frac{T}{\omega_k}\,B(k/T, m/T)$ (Wang & Heinz)

$$\text{-------} = \frac{8\zeta(3)}{\pi^2}\,\frac{\gamma}{\omega} \qquad (\omega\gg\gamma,\ B=1)$$



viscosity: $\sim \frac{\partial}{\partial\omega}\rho_{\pi\pi}(\omega)\Big|_{\omega=0}$

Complete result for spectral function :



- $\omega \sim m$: both contributions match parametrically

$$\rho_{\pi\pi}/T^4 \sim \lambda^{3/2}$$

- higher loops

$\omega \lesssim \gamma$: pinching poles from ladders

changes details, not characteristic shape

$\gamma \lesssim \omega \lesssim m$: $\rho_{\pi\pi}(\omega)/T^4 \sim \gamma/\omega$

three loop diagram contributes as well


$\sim \lambda^2 T/\omega$

$SU(N_c)$ gauge theory : same story, more involved

- $\pi_{ij} = F_i^{a\mu}F_{j\mu}^a - \frac{1}{3}\delta_{ij}F^{ak\mu}F_{k\mu}^a$

- $D_{\mu\nu} = P_{\mu\nu}^T\Delta_T + P_{\mu\nu}^L\Delta_L$

nonabelian diagrams :

higher order

with

$$\Delta_T(P) = -\int\frac{d\omega}{2\pi}\frac{\rho_T(\omega,\vec{p})}{p^0-\omega} \qquad \Delta_L(P) = \frac{1}{\vec{p}^2} + \int\frac{d\omega}{2\pi}\frac{\rho_L(\omega,\vec{p})}{p^0-\omega}$$

$$\Rightarrow \rho_{\pi\pi}(\omega) = \frac{2}{3}(N_c^2-1)\int\frac{d^4k}{(2\pi)^4}\left[n(k^0)-n(k^0+\omega)\right]\times$$
$$\times\Big\{V_1(k,\omega)\,\rho_T(k^0,\vec{k})\,\rho_T(k^0+\omega,\vec{k})$$
$$+V_2(k)\,\rho_L(k^0,\vec{k})\,\rho_T(k^0+\omega,\vec{k})$$
$$+V_3(k)\,\rho_L(k^0,\vec{k})\,\rho_L(k^0,\vec{k})\Big\}$$

with $V_1(k,\omega) = 7|\vec{k}|^4 - 10\,\vec{k}^2k^0(k^0+\omega) + 7k^{02}(k^0+\omega)^2$

etc..

- single particle spectral functions: (HTL)

$$\rho_T(k) = 2\pi Z_T(k)\left[\delta(k^0-\omega_T(\vec{k})) - \delta(k^0+\omega_T(\vec{k}))\right] + \beta_T(k)$$

plasmon (pole) Landau damping (cut)

- transverse gluons (pole-pole) dominate when $\omega \gtrsim T$

$$\rho_{\pi\pi}(\omega) = \frac{N_c^2-1}{4\pi}\omega^4\left[n(\omega/2)+\frac{1}{2}\right]$$

- $\omega \sim gT$ pole-cut contribution dominates
- smaller frequencies: pinching poles from transverse gluons dominate

Smaller frequencies: pinch singularity

- use finite gluon damping rate $\gamma = \dfrac{g^2 N_c T}{4\pi} \ln 1/g$

  [& gauge invariance & ladder diagrams]

- 1-loop viscosity $\eta_{1\,\text{loop}} \sim (N_c^2-1)\dfrac{T^4}{\gamma} \sim \dfrac{(N_c^2-1)}{N_c}\dfrac{T^3}{g^2 \ln 1/g}$ (Pisarski)

- wrong parametrically:
  kinetic theory predicts: $\eta \sim \dfrac{N_c^2-1}{N_c^2}\,\dfrac{T^3}{g^4 \ln 1/g}$ (AMY)

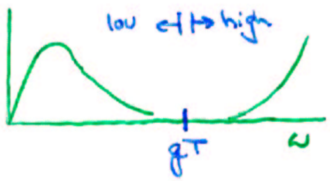
  $\Rightarrow$ contribution from ladders larger than 1 loop result
  & gauge invariant summation of subclass of diagrams
  unsolved problem

- However, does not affect the characteristic shape of $\rho_{\pi\pi}$ much

and does not affect the euclidean correlator

Euclidean correlator

$$G_{\pi\pi}(\tau) = \int_0^\infty \frac{d\omega}{2\pi}\, K(\tau,\omega)\, \rho_{\pi\pi}(\omega)$$



- split contributions

- high:

$$G_{\pi\pi}(\tau) = \frac{\pi^2 T^5}{3\sin^5 u}\Big[(\pi-u)(11\cos u + \cos 5u) + 6\sin u + 2\sin 3u\Big] + \mathcal{O}(m^2/T^2)$$

$u = 2\pi\tau T$ $\qquad m^2/T^2 = g^2$

$$\simeq \frac{1}{8\pi^2}\left[\frac{1}{\tau^5} + \frac{1}{(1/T-\tau)^5} + \frac{2}{(3/2T-\tau)^5} + \frac{2}{(1/2T+\tau)^5}\right]$$

Maxwell-Boltzmann statistics, $n(\omega) \sim e^{-\omega/T}$

  - simple result: $\sim \dfrac{1}{\tau^5}$ + reflection symmetry $\tau \to 1/T - \tau$

  - central value: $G_{\pi\pi}\left(\tau = \dfrac{1}{2T}\right) = \dfrac{4\pi^2}{45} T^5\left(1 - \dfrac{25}{8\pi^2}\dfrac{m^2}{T^2} + \dots\right)$

$$= \frac{4\pi^2}{45} T^5\left(1 + \mathcal{O}(g^2)\right)$$

- low: $\omega \ll T$ expand kernel

  $K(\tau,\omega) \simeq 2T/\omega$ $\qquad \int d\omega\, \rho(\omega)/\omega$

$$\Rightarrow G_{\pi\pi}(\tau) = -\frac{8}{3}\int\frac{d^3k}{(2\pi)^3}\frac{|k|^4}{\omega_k^2}\, n'(\omega_k)\int\frac{d\omega}{2\pi}\frac{2T}{\omega}\frac{\gamma_k\,\omega}{\omega^2+4\gamma_k^2}$$

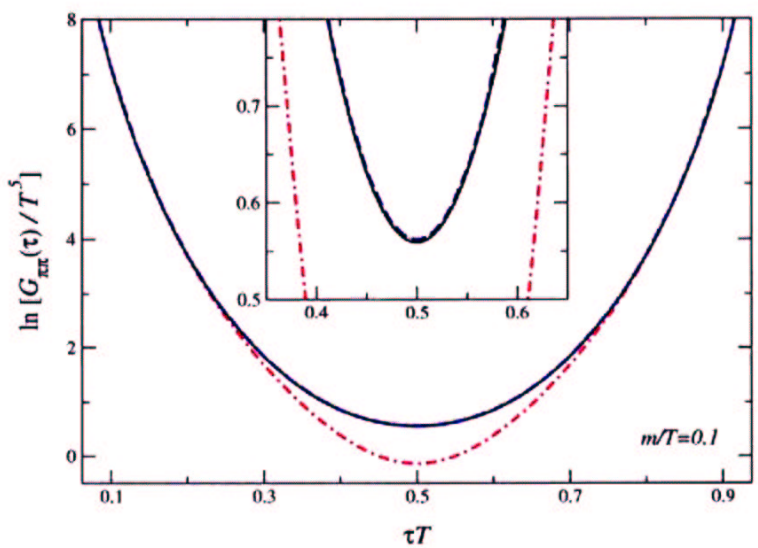
$$= \frac{4\pi^2}{45} T^5\left(1 + \mathcal{O}(g^2,\lambda)\right)$$

Euclidean correlator:

$$G(\tau) \approx \frac{1}{8\pi^2}\left[\frac{1}{\tau^5} + \frac{1}{(1/T-\tau)^5} + \frac{2}{(3/2T-\tau)^5} + \frac{2}{(1/2T+\tau)^5}\right]$$

$$+ \frac{4\pi^2}{45} T^5$$

contribution from interesting low frequencies



Findings:

- spectral function has a characteristic bumpy shape
- fit used in the literature completely wrong
  (we proposed a better one)
- τ dependence dominated by "uninteresting" high frequencies
- small frequency part gives τ independent contribution
  $\sim \int d\omega\, \rho(\omega)/\omega$

$\Rightarrow$ euclidean correlator basically insensitive to transport coefficients

nonperturbative lattice calculation in a weakly coupled theory?

Relevance for other correlators:

thermal dilepton rate

$\mathrm{Im}\, \Pi_R{}^{\mu}{}_{\mu}(\omega, \vec{p})$ — photon polarization

in terms of spectral function:

$$\rho_V(x-y) = \langle [j^\mu(x), j_\mu(y)] \rangle \qquad j^\mu = \bar{\psi}\gamma^\mu\psi$$

recent lattice calculation using Maximal Entropy Method (Karsch et.al.)

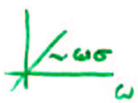
- pinch singularities
- complicated behaviour at very small $\omega \ll T$
- difficult to analyse from the lattice

$\Rightarrow$ $$\rho^{\mu\nu}(x-y) = \langle [j^\mu(x), j^\nu(y)] \rangle$$

$$\rho_V = -\rho^{00} + \rho^{ii}$$

- ρ^{ii} : electrical conductivity

$$\sigma_{em} = \frac{1}{6} \frac{\partial}{\partial\omega} \rho^{ii}(\omega)\Big|_{\omega=0}$$



one loop: $$\rho^{ii}(\omega) = N_c \frac{\omega^2}{2\pi}\left[1-2n_F(\omega/2)\right] + N_c \frac{2\pi}{3} T^2 \omega\delta(\omega)$$

pinching pole

$\Big|$ $\sim \frac{\omega\gamma}{\omega^2+4\gamma^2}$ ladders etc.

- ρ^{00} : restricted by Ward identities

- $\rho^{00}(\omega) = N_c \frac{2\pi}{3} T^2 \omega\delta(\omega)$

charge conservation

$$\rho^{00}(t) = \frac{1}{V}\langle [Q(t), Q(0)] \rangle \qquad \partial_t Q(t) = 0$$

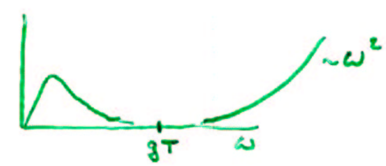
$\Rightarrow$ $\rho^{00}(\omega) \sim \omega\delta(\omega)$ always !

simple test on respecting Ward Id in summation schemes.

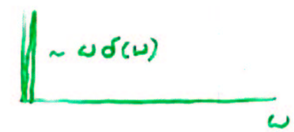
- expect for $\rho_V = \mathrm{Im}\, \Pi_R{}^{\mu}{}_{\mu}$

$$\rho_V = -\rho^{00} + \rho^{ii}$$

$\rho^{ii} \sim$



$\rho^{00} \sim$



challenge for MEM to disentangle the low-frequency domain

(Karsch et al find $\rho_V(\omega) \sim 0$ for $\omega \lesssim 3T$)

Spectral functions and the classical approximation

- classical field approximation:
  nonperturbative approach to real-time dynamics of soft, highly populated fields
- spectral functions:
  - equilibrium QFT

$$\rho(x-y) = i\langle [O(x), O^\dagger(y)]\rangle$$

$$F(x-y) = \tfrac{1}{2}\langle [O(x), O^\dagger(y)]_+\rangle \quad \text{symmetric correlator}$$

KMS condition: $F(p) = -i[n(p^0)+\tfrac{1}{2}]\rho(p)$

- classical fields at finite T

$$\rho_{cl}(x-y) = -\langle \{O(x), O^\dagger(y)\}\rangle_{cl}$$

with Poisson bracket

$$\{A(x), B(y)\} = \int d^d z \left[\frac{\delta A(x)}{\delta\phi(z)}\frac{\delta B(y)}{\delta\pi(z)} - \dots\right]$$

(difficult to compute)

$$S(x-y) = \langle O(x)\, O^\dagger(y)\rangle_{cl} \quad \text{normal correlator}$$

classical KMS condition:

$$S(p) = -i\frac{T}{p^0}\rho_{cl}(p)$$

or in

$$\boxed{\rho_{cl}(t,\vec{x}) = -\frac{1}{T}\partial_t S(t,\vec{x})}$$

Example: 2+1 D scalar field: plasmon

$$H = \int d^2x \left[\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{2}m^2\varphi^2 + \tfrac{\lambda}{4!}\varphi^4\right]$$

one-particle spectral function $\quad (O = O^\dagger = \varphi)$

$$\rho = i\langle [\phi(x), \phi(y)]_-\rangle \qquad \rho_0(p) = 2\pi\delta(p_0^2 - \vec{p}^2 - m^2)$$

$$\Rightarrow \quad S(x-y) = \langle \phi(x)\phi(y)\rangle_{cl}\ ; \quad \rho_{cl}(x-y) = -\langle \{\phi(x), \phi(y)\}\rangle_{cl}$$

KMS: $\rho_{cl} = -\frac{1}{T}\partial_t S \qquad \pi(t,\vec{x}) = \partial_t\phi(t,\vec{x})$

$$\Rightarrow \quad \boxed{\rho_{cl}(t,\vec{x}) = -\frac{1}{T}\langle \pi(t,\vec{x})\,\phi(0,\vec{0})\rangle_{cl}}$$

Numerical calculation:

- spatial lattice $N\times N$, periodic b.c. $\qquad N=128$
- lattice spacing a, $\quad ma = 0.2$
- leapfrog, time step a_0, $\quad a_0/a = 0.1$
- symmetric definition:

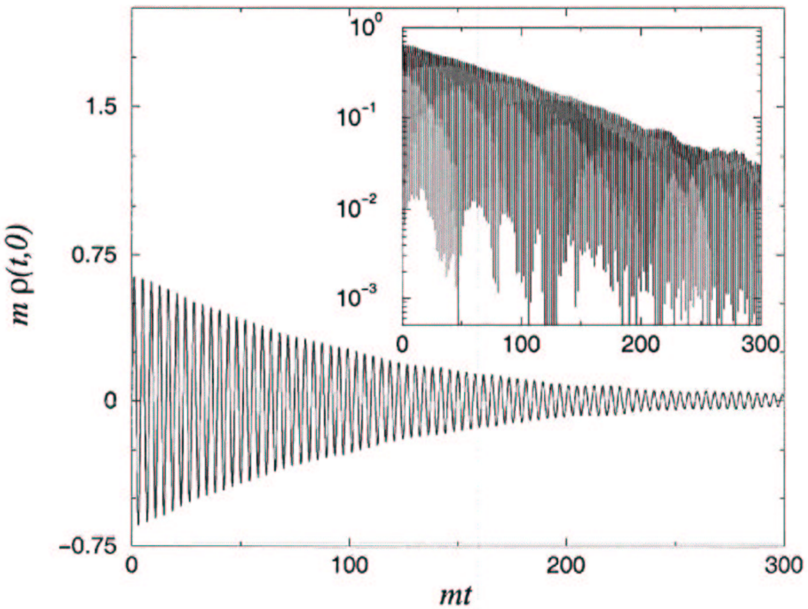
$$\rho_{cl,lat}(t,\vec{x}) = -\frac{1}{T}\langle \pi(t+\tfrac{1}{2}a_0,\vec{x})\,\tfrac{1}{2}[\phi(0,\vec{0}) + \phi(a_0,\vec{0})]\rangle_{cl}$$

- temperature $T = a^2\langle\pi^2\rangle$
- classical theory: λ can be scaled out $(\lambda T/m^2)$
  without loss of generality $\lambda/m = 1$
- • thermal initial configuration
  (HMC, Kramers equation)
  • real-time (Hamiltonian) evolution $\quad \sim 2000\times$

Spectral function in real time

$\rho_{cl}(t, \vec{p}=0)$

temperature $T/m = 7.2$

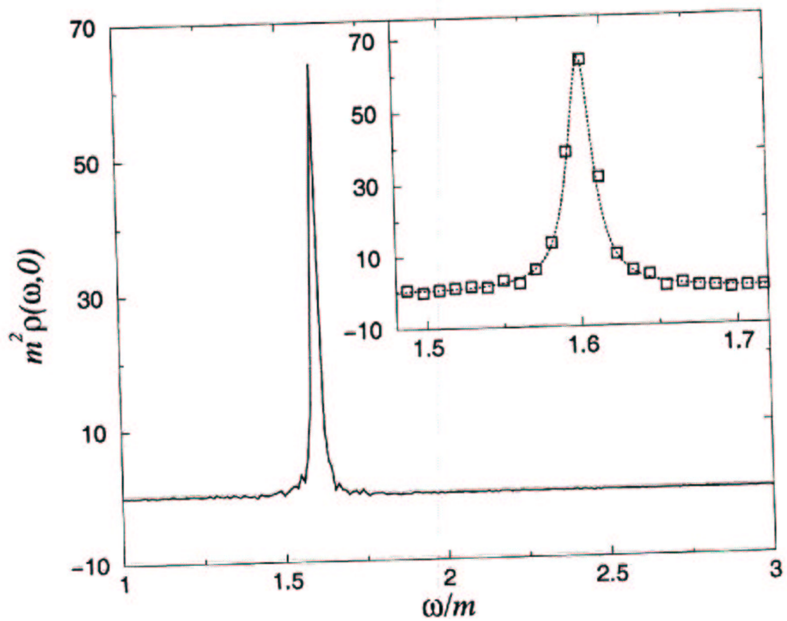


Spectral function vs frequency

$\rho_{cl}(\omega, \vec{p}=0)$ [from sine-transform]

inset: dotted line: Breit-Wigner fit

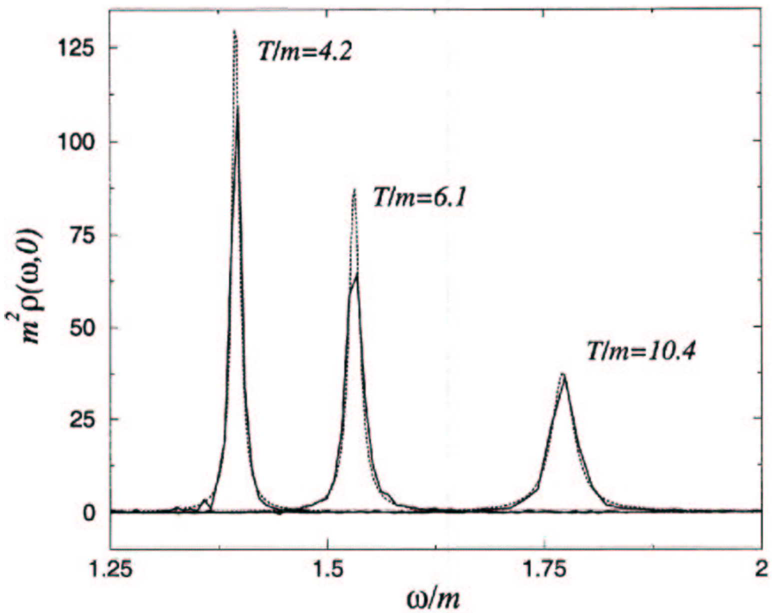
$$\rho_{BW}(\omega) = \frac{2\omega\Gamma}{(\omega^2 - M^2)^2 + \omega^2\Gamma^2}$$



Spectral functions for different temperatures

dotted lines are Breit-Wigner fits

$\rho(M) \sim 2/M\Gamma$



Perturbative expectation:

the plasmon

$$\rho = \frac{-\mathrm{Im}\,\Sigma_R}{(\omega^2-\vec{p}^2-m^2-\mathrm{Re}\,\Sigma_R)^2+(\mathrm{Im}\,\Sigma_R)^2}$$

$\Sigma_R = \mathrm{Re}\,\Sigma_R + i\,\mathrm{Im}\,\Sigma_R$ retarded self energy

weak coupling:

$$\Gamma \equiv -\mathrm{Im}\,\Sigma_R/\omega \ll \sqrt{\vec{p}^2+m^2+\mathrm{Re}\,\Sigma_R}$$ narrow width

$\rightsquigarrow$ Breit-Wigner spectral function (at zero momentum)

$$\rho_{BW}(\omega) = \frac{2\omega\Gamma}{(\omega^2-M^2)^2+\omega^2\Gamma^2}$$ $\rightsquigarrow$ Fig

$\sim$

Perturbation theory (one-loop resummed) [2+1 D]

- M^2:  $M^2 = m^2 + \frac{\lambda}{2}\int\frac{d^2p}{(2\pi)^2}\frac{T}{\vec{p}^2+M^2}$ (classical)

  $\rightsquigarrow$ lattice gap equation for M

- Γ:  onshell $2\to2$ scattering (quantum = classical)

  $$\Gamma = c\,\frac{\lambda^2T^2}{M^3} \qquad c = \frac{3-2\sqrt{2}}{32\pi} \sim 0.0017$$

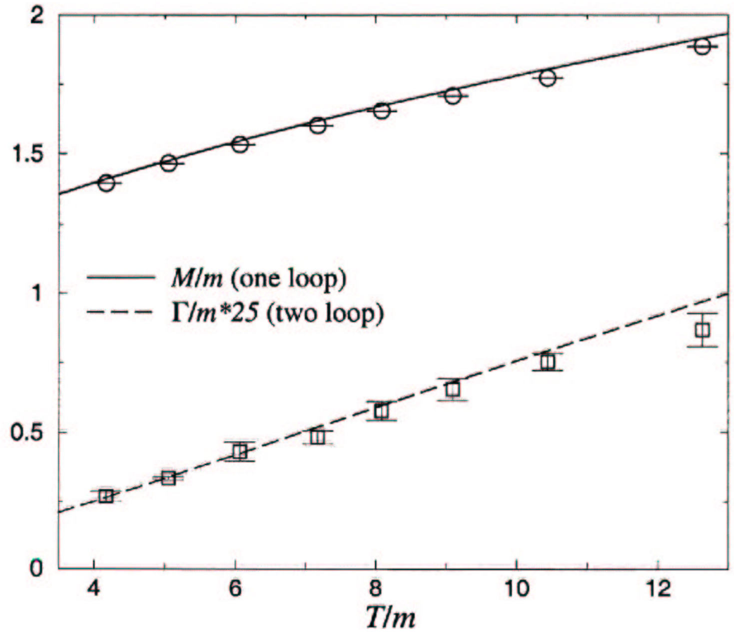
  [new calculation]

$\Rightarrow$ Compare with nonperturbative plasmon mass and

Temperature dependence of perturbative and nonperturbative classical plasmon mass M and width Γ

[data points from BW-fit, error from jackknife]

Perturbation theory is applicable



Conclusions

- transport coefficients:

  analytical calculation in field theory nontrivial

- from the lattice:

  euclidean correlators insensitive to transport coefficients and to details of soft dynamics ($\omega \ll T$) in general

- generic feature:

  challenge for Maximal Entropy Method

  ex: thermal dilepton rate

- spectral function directly in real time:

  classical approximation

  ex: plasmon