4-loop Logarithms in the pressure of (hot) QCD

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KITP + MIT, 6/18/02

$$p \overset{V\to\infty}{=} -f = \frac{T}{V} \ln \int \mathcal{D}[A_\mu^a, \bar\psi_f, \psi_f]\, e^{-\int_0^{1/T} d\tau \int d^3x \left(\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \bar\psi_f \not{D} \psi_f \right)}$$

- 4d : lat, pert, resu
- 4d+3d setup, status
- highlights, techniques, open calc's

→ PRL 86 (2001) 10
PRD 65 (2002) 45008
+ more soon

Just compute an integral:

$$e^{-f(T,\Lambda_{\overline{MS}},N_c,N_f)\frac{V}{T}} = \int \mathcal{D}[A_\mu^a, \bar\psi_f, \psi_f]\, e^{-\int_0^{1/T} d\tau \int d^3x\, \mathcal{L}_E(A,\bar\psi,\psi)}$$

$$\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \bar\psi_f \not{D} \psi_f$$

4D lattice MC

discretize : $N_t N_s^3$, a , $\beta \leftarrow \sim \frac{1}{g^2(a)} \sim -\beta_0 \ln(a\Lambda_L)$

fixes temp. $N_t a = 1/T$

the only dim-ful param (has to be fixed, by e.g. m_π)

have to extrapolate to two limits:

thermodynamic, $V \to \infty \Leftrightarrow N_t \ll N_s$

continuum, $a \to 0 \Leftrightarrow \beta \to \infty$

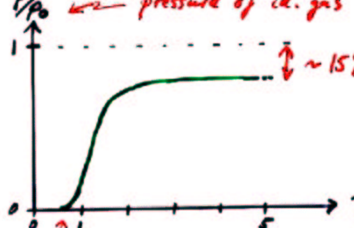
in practice, works only for $T \lesssim$ a few T_c

[Boyd et.al., Karsch et.al., Okamoto et.al., ...]

extrapolate: $V \to \infty$ ($N_\sigma \to \infty$), $a \to 0$ ($N_\tau \to \infty$ @ fixed T)

lattice results (qual.):

p/p_0 ← pressure of id. gas (Stefan-Boltzmann)



~15%

$p_0 = (16 + \frac{21}{2} N_f)\frac{\pi^2 T^4}{90}$

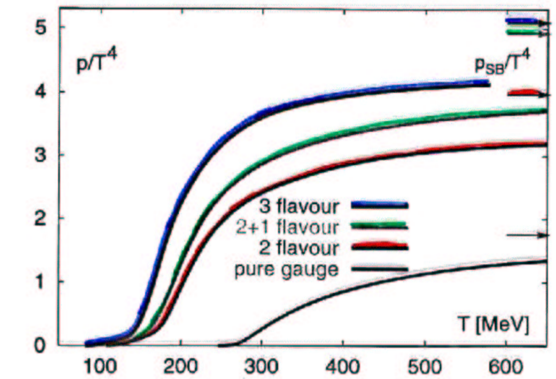
T/T_c ← $T_c = 1.14\,\Lambda_{\overline{MS}}$ ($N_f = 0$)

normalization (int. const.)

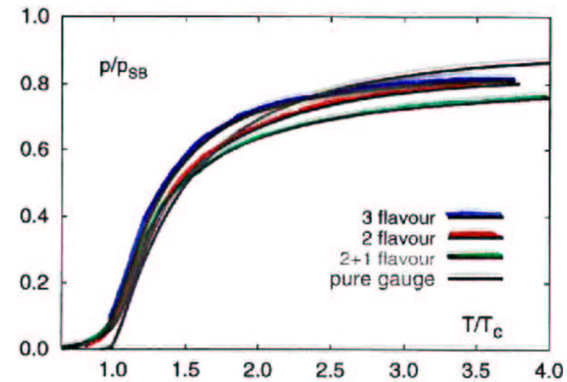
$N_f = 0$ well under control

$N_f > 0$ under (rapid) development

$p(T)$ on the lattice



[from Karsch et.al.]

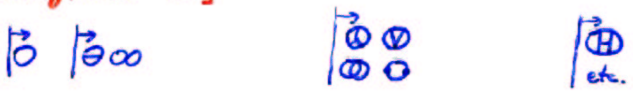


4d pert. calc.

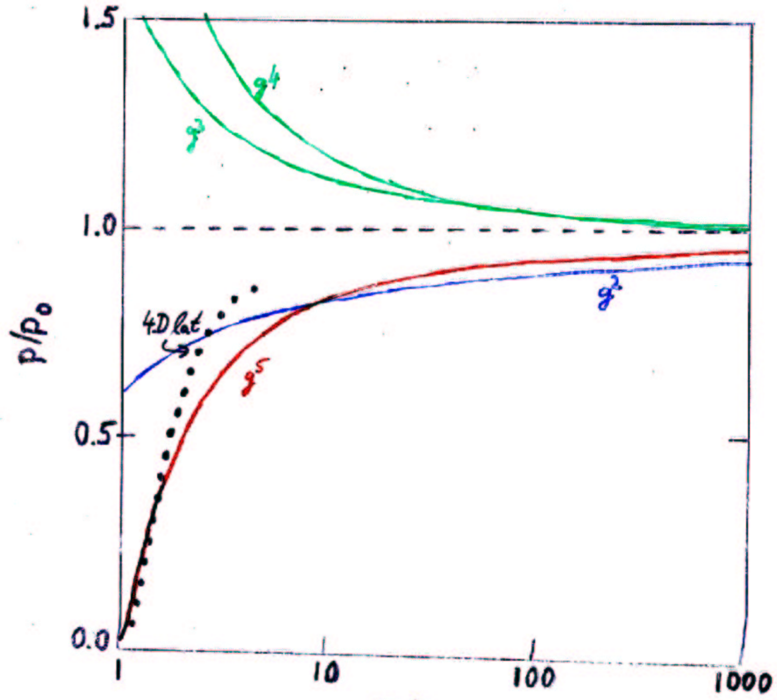
"Linde sea"

$$p/p_0 = 1 + c_2 g^2 + c_3 g^3 + (c_4' \ln g + c_4) g^4 + c_5 g^5 + \mathcal{O}(g^6 \ln g, g^6)$$

[Shuryak '78, Kapusta '79, Toimela '83, Arnold/Zhai '94, Kastening/Zhai '95]



series nonanalytic in g^2 (Debye screening 'mass', HTL resu., expand in $m/T \sim g$, $\ln m \to \ln g$ (IR)) and wildly oscillating :



LINDE : $\mathcal{O}(g^6)$ is nonpert !

consider 

propg $((2\pi n T)^2 + \vec{p}^{\,2} + m^2)^{-1}$

leading IR from $n = 0$

$\Rightarrow$ IR div for $\ell > 3$ as $g^6 T^4 \left(\frac{g^2 T}{m}\right)^{\ell - 3}$

((recall $m_L \sim gT$, $m_T \sim g^2 T$ earliest))

$\Rightarrow$ complete failure of pert. th., all diags $\ell > 3$ contrib

But convergence of series to g^5

is bad ! Higher c_i's are large !

take e.g. $N_c = 3$, $N_f = 6$, $\bar{\mu} = T$

$$F = -79 \frac{\pi^2}{90} T^4 \left[1 - 0.08 + 0.10 + (0.03 - 0.02) + (0 - 0.08) + \mathcal{O}(g^6) \right]$$

((even $\bar{\mu}$-dependence does not improve with higher orders))

resummation

[Anderson, Braaten, ... '99-'02]
[Blaizot, Iancu, Rebhan '99-'01]
[Peshier '99] [Parwani '00]

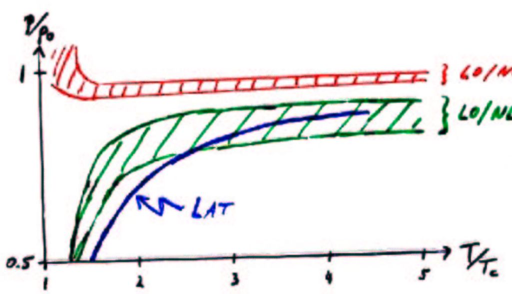
use effective masses, tadpole improvements, ...

But neglect / suppress IR (long-distance) effects

$\rightsquigarrow$ work only for short-distance dominated observables.

NB in the case of f, there is an intricate cancellation of long-dist. effects. $\rightsquigarrow$ see next sect's

But one needs a priori knowledge ...



4d + 3d setup

scale hierarchy πT, gT, ... @ $T \gg T_c$ $\left(g(Q\sim T) \sim \frac{1}{\ln T/\Lambda_{\overline{MS}}} \right)$

$\rightsquigarrow$ effective theories ('int. out heavy modes', dim. red.)

4d QCD $\longrightarrow$ 3d adj. H

$$\mathcal{L}_{3d} = \tfrac{1}{4} F_{ij}^2 + (D_i A_0)^2 + m_D^2 A_0^2 + \lambda_A A_0^4 + \dots$$

small coupling allows <u>perturbative</u> reduction

$$\frac{g_3^2}{T} = \frac{8\pi^2}{11 \ln(6.742\, T/\Lambda_{\overline{MS}})}$$

$$x \equiv \frac{\lambda_A}{g_3^2} = \frac{3}{11 \ln(5.371\, T/\Lambda_{\overline{MS}})}$$

$$y \equiv \frac{m_D^2}{g_3^4} = \frac{3}{8\pi^2 x} + \frac{9}{16\pi^2} + \mathcal{O}(x)$$

(NLO, FAC, $N_f=0$, $N=3$)
challenge: do NNLO

3d adj H is confining $\rightsquigarrow$ nonperturbative

$\mathcal{O}(1)$ $\mathcal{O}(g^2)$ $\longleftarrow$ hard modes $\longrightarrow$ $\mathcal{O}(g^4)$ [Braaten/Nieto]

$$\frac{p(T)}{p_0(T)} = 1 - \tfrac{5}{2} x - \frac{45}{8\pi^2} \left(\frac{g_3^2}{T} \right)^3 \left(\mathcal{F}_{\overline{MS}}(x,y) - 24 \frac{4}{(4\pi)^4} \left[\ln \frac{\bar{\mu}_3}{T} + \delta \right] \right)$$

$\delta \approx 1.35 \times 10^{-4}$

$$= -\frac{1}{V g_3^6} \ln \int \mathcal{D}[A_i^a, A_0^a]\, e^{-\int d^3x\, \mathcal{L}_{3d}}$$

- factorization scale $\bar{\mu}_3$ cancels in pressure
- <u>measure</u> long-distance piece $\mathcal{F}$?!

'measure' $\mathcal{F}_{\overline{MS}}$ via 3D MC ?

first, take derivatives: $\partial_y \mathcal{F}_{\overline{MS}} = \langle A_0^2 \rangle_{\overline{MS}}$

$\partial_x \mathcal{F}_{\overline{MS}} = \langle A_0^4 \rangle_{\overline{MS}} + \dots$

since $y(x)$

integrate back $\mathcal{F}_{\overline{MS}} = \mathcal{F}_{\overline{MS}}(y_0) + \int_{y_0}^{y} dy \left(\partial_y \mathcal{F}_{\overline{MS}} + \frac{dx}{dy} \partial_x \mathcal{F}_{\overline{MS}} \right)$

choose y_0 to get int. const perturbatively

e.g. at $T_0 = 10^{11} \Lambda_{\overline{MS}} \hat{=} y_0 = 3.86$

now real hard work is hidden in details:

- relate L, $\overline{MS}$ schemes

  → compute diags in lat. reg. scheme   3loop ✓✓ 4loop ...

- determine $\mathcal{F}_{\overline{MS}}(y_0)$ accurately

  a) pert. + $\Delta \mathcal{F}_{\overline{MS}}(y_0) = e_0 \frac{d_A C_A^3}{(4\pi)^4} (1 + \mathcal{O}(x_0))$

  b) int. out A_0 ($m_D \sim gT$)

  $\Delta \mathcal{F}_{\overline{MS}} = -\frac{1}{V g_3^6} \ln \int \mathcal{D}[A_i^a] \, e^{-\int d^3x \left(\frac{1}{4} F_{ij}^2 + \dots \right)}$

- MC + numeric back-integration involved also ...

- control of higher order operators in $\mathcal{L}_{eff}$:

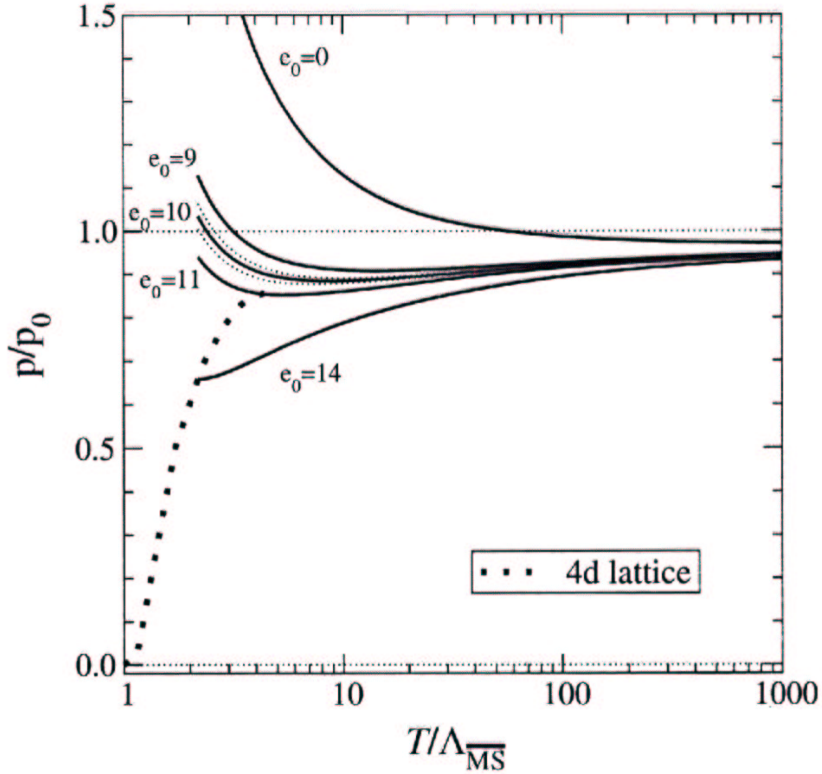
  → assure that $\langle A_0^4 \rangle$-contrib. small.

status:

4D hard modes (3 loop)

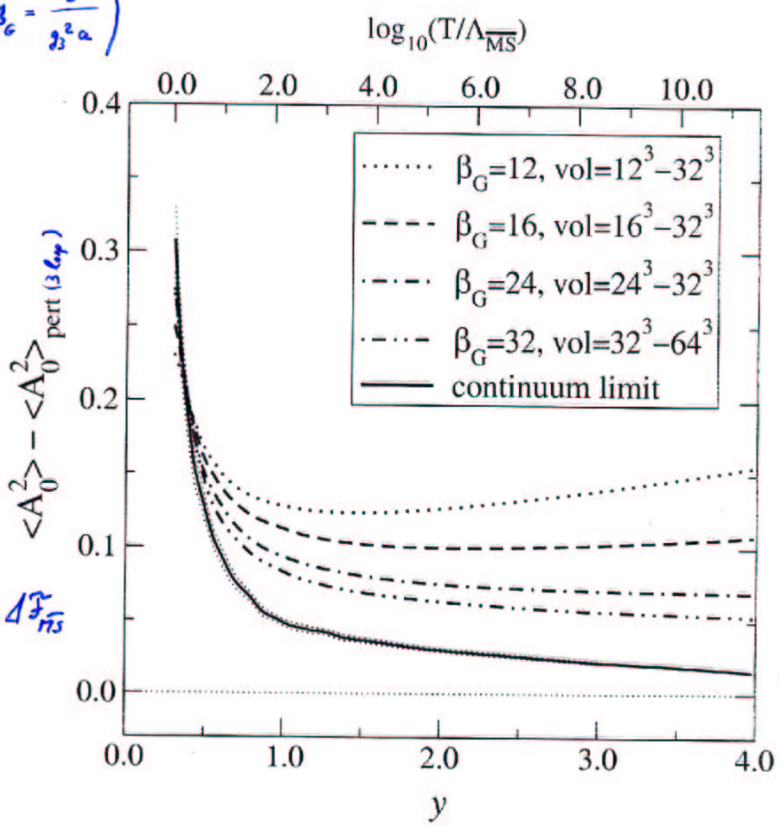
+ 3D lattice MC ($\langle A_0^2 \rangle$ only)

+ free param. in int. const. (e_0)



(observe cancellation of long-dist effects @ $T \gtrsim 30 \Lambda_{\overline{MS}}$)

$\left(\beta_G = \frac{6}{g_3^2 a}\right)$



$\partial_y \Delta F_{\overline{MS}}$

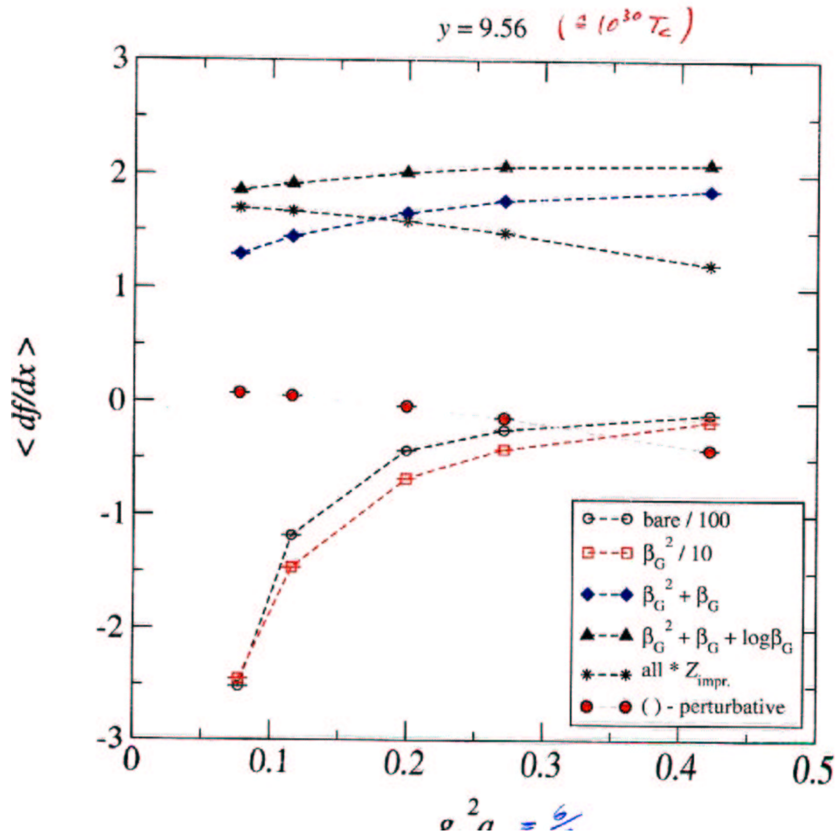
$\left\langle \frac{\mathrm{Tr} A_0^2}{g_3^2} \right\rangle \sim \underset{1/a}{\bigcirc} + \underset{\ln a}{\bigcirc} + \underset{\mathcal{O}(a)}{\ldots}$

$\sim \beta_G \quad \ln\beta_G \quad \mathcal{O}(1/\beta_G)$

precision of simulations: resolve even 4loop log div's !

$\left\langle \frac{(\mathrm{Tr} A_0^2)^2}{g_3^4} \right\rangle \sim \underset{1/a^2}{8} + \ldots$

$\sim \beta_G^2 + \ldots$



integration const

schematically: $\mathcal{F}^{\text{theory}} = -\ln \int e^{-S_{\text{theory}}}$

$$\mathcal{F}^{4d\,QCD}(T) = \mathcal{F}^{4d\,QCD}_{\text{pert}}(T) + \mathcal{F}^{3d\,adj\,H}(T)$$

$$= \mathcal{F}^{4d\,QCD}_{\text{pert}}(T) + \underbrace{\mathcal{F}^{3d\,adj\,H}(T_0)} + \int_T^{T_0} \langle \mathrm{Tr} A_0^2 + (\mathrm{Tr} A_0^2)^2 \rangle^{3d\,adj\,H}$$

$$\xrightarrow{T_0 \to \infty} \mathcal{F}^{3d\,adj\,H}_{\text{pert}}(T_0)$$

$$= \mathcal{F}^{3d\,adj\,H}_{3\text{loop}}(T_0) + e_0$$

or A_0 'heavy': $m_D \sim gT$

eff. theory for g^2T modes: 3d pure YM

$$= \mathcal{F}^{4d\,QCD}_{\text{pert}}(T) + \mathcal{F}^{3d\,adj\,H}_{\text{pert}}(T) + \mathcal{F}^{3d\,YM}$$

in principle measurable

$$\mathcal{F}^{3d\,YM} = \ln \int e^{-S_{3d\,YM}}$$

$$Vg^6\left[\left(\tfrac{1}{ag^2}\right)^3 + \ldots + \left(\ln \tfrac{1}{ag^2} + 1\right) + \mathcal{O}(a) \right] = \ln \int e^{-\int \frac{1}{4g^2}\mathcal{F}^2}$$

div's computable in L reg

$\partial_{g^2} \rightsquigarrow$ const $\sim \langle \mathrm{Tr}\, \mathcal{F}^2 \rangle$

(L $\leftrightarrow$ $\overline{\text{MS}}$ 4loop)

N.B. higher-order operators

$\frac{g^3}{T^{3/2}} f^{abc} \mathcal{F}^a_{ij} \mathcal{F}^b_{jk} \mathcal{F}^c_{ki}$, $\frac{g^2}{T^2} (D\mathcal{F})^2$ [Chapman '94]

... g^{12} (dim. analysis)

pert. setup: skeletons + SD

(the 'trivial' part of every computation)

- sy factors
  (ex:  → CL rules n/a)
- get all diags
  (QGRAF [Nogueira], FeynArts [Denner/Hahn]
  n/a for 0-pt-fcts)
- effective classification → SKELETONS ! (2PI)

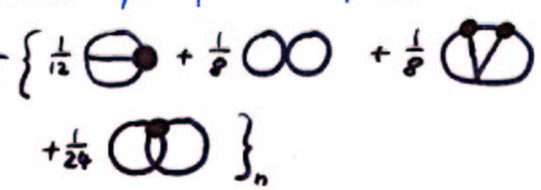
skeleton expansion [Luttinger/Ward] [Baym] ... proof ...

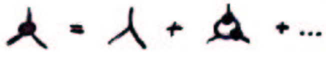
$$\mathcal{F}[D] = \sum_i c_i \left(\mathrm{tr} \ln D_i^{-1} + \mathrm{tr}\, \Pi_i[D]\, D_i \right) - \Phi[D]$$

$c_{\text{boson}} = \frac{1}{2}$, $c_{\text{fer}} = -1$ 2PI

$$\delta_{D_i} \Phi[D] = c_i \Pi[D]$$

new: closed exact eqn for n-loop Φ

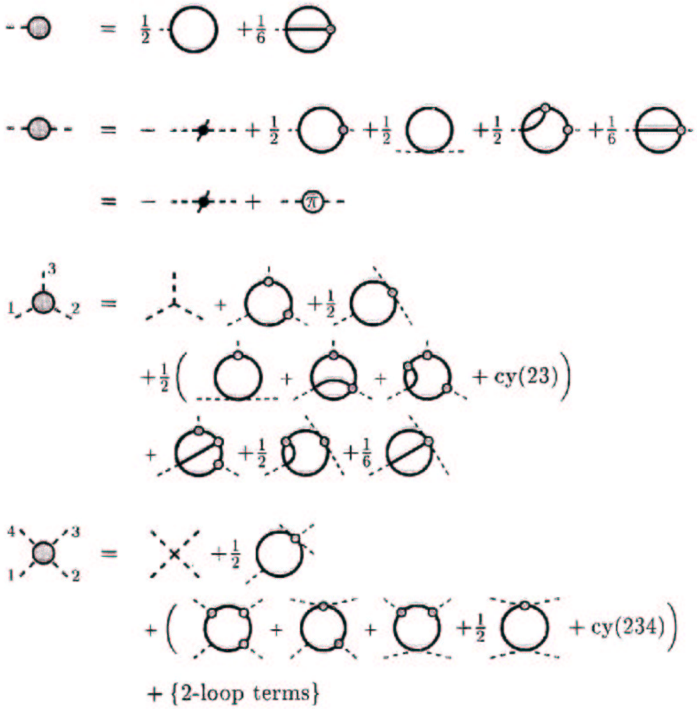
$\Phi_n[\Delta] = \frac{1}{n-1}$ 

+ SD eqs: 

etc.

SD eqs (generic $\varphi^3+\varphi^4$ theory)

$$\Gamma_n^{1PI} = \delta_\phi^{\,n-1}\, S'[\phi + D[\phi]\,\delta_\phi]\Big|_{\phi=0} \qquad \langle\varphi\rangle$$



...

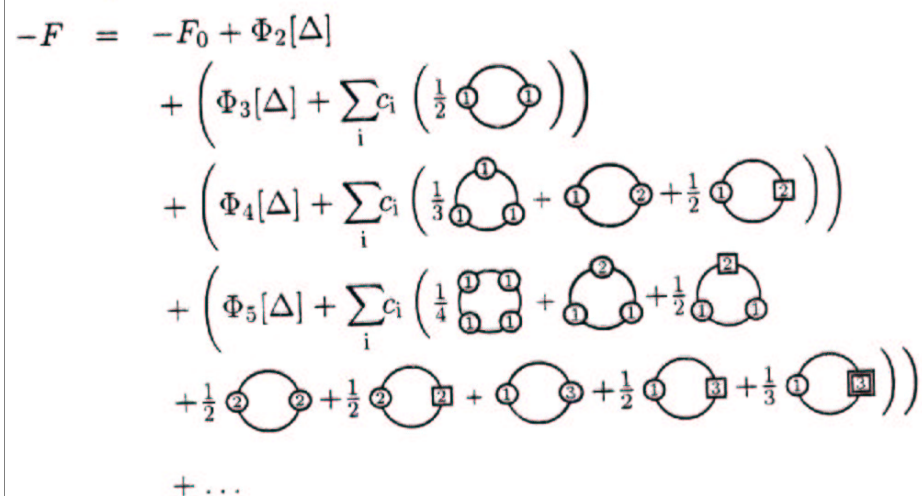
non-pert. tool in principle!

here: used to organize p.t.

$$F[D] = \sum_i c_i \left(\mathrm{Tr}\ln D_i^{-1} + \mathrm{Tr}\Pi_i[D]D_i\right) - \Phi[D]$$

loop exp.

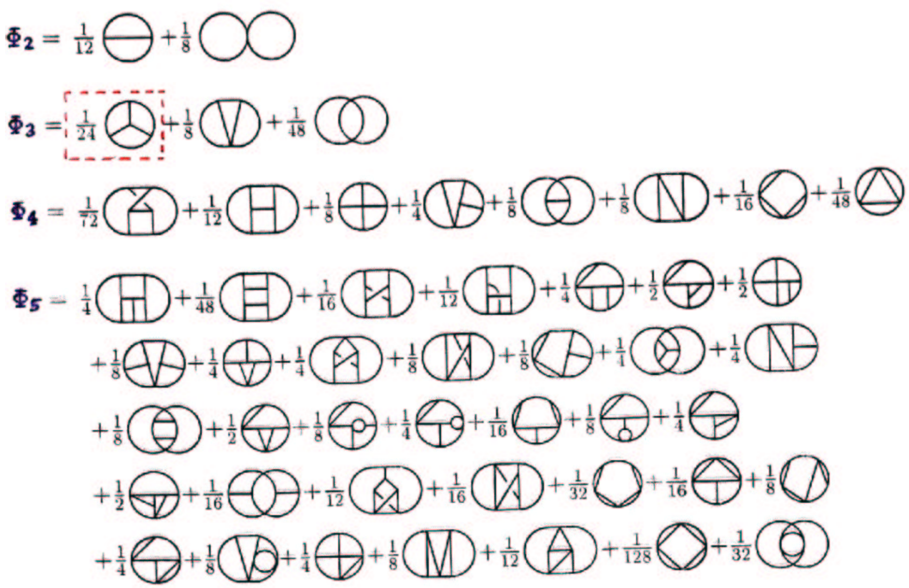
$$\delta_{D_i}\Phi[D] \;=\; c_i\Pi[D]$$



use

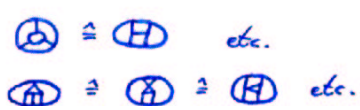
+ SD eqs

to get all diags ?



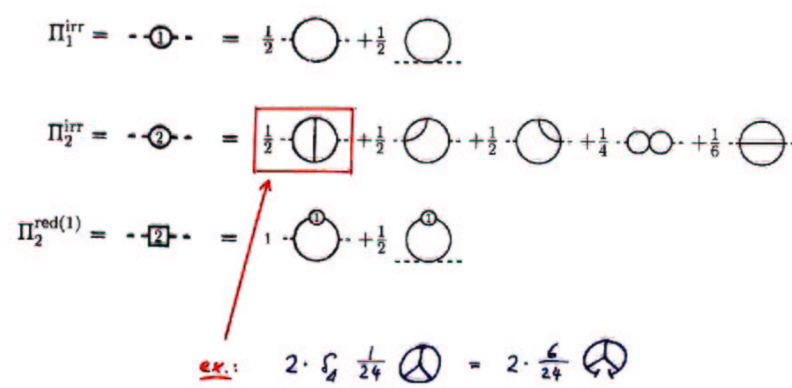
...

equiv. topologies



• skeletons → self Es

$(\Pi = 2!\, \delta_\Delta \Phi)$

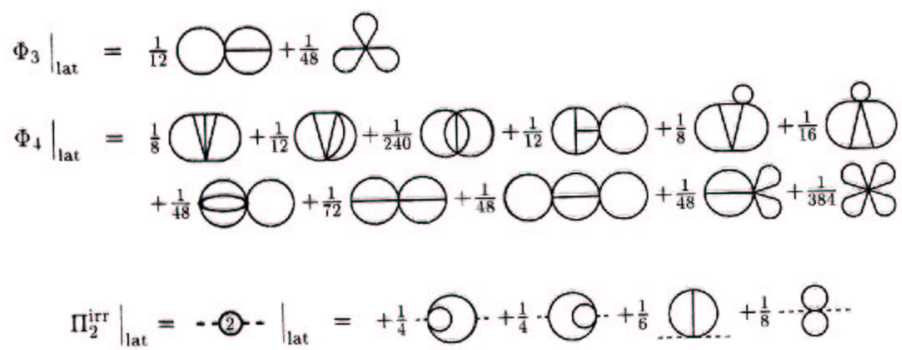


• skeletons → n-pt fcts

$\Gamma_3 = 3!\, \delta_{\Gamma_3} \Phi$

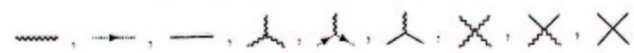
etc.

lattice topologies ≤ 4 loop

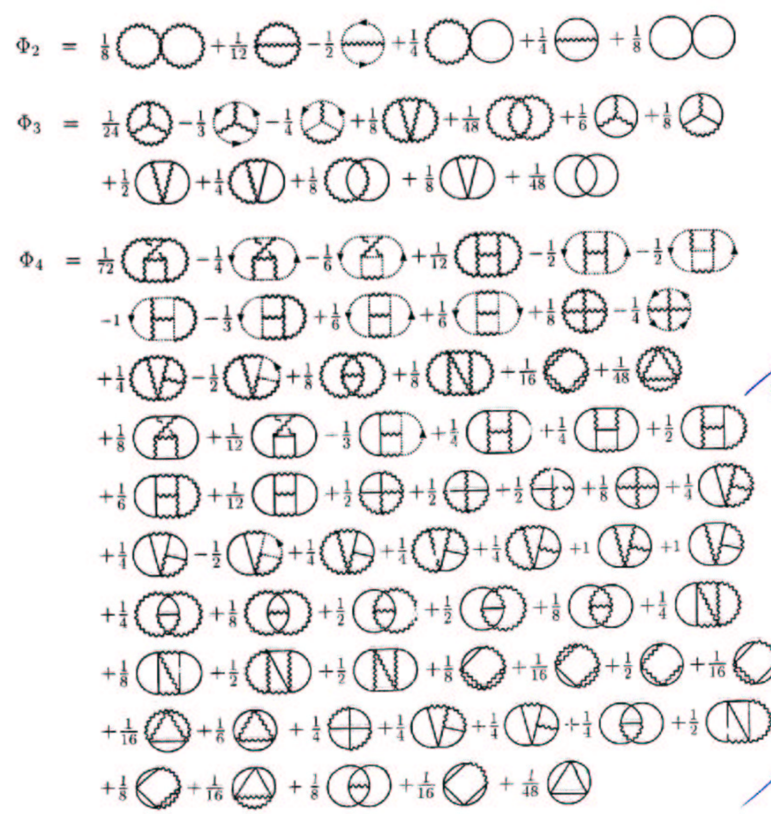


pick your favourite theory

SU(N) + adj Higgs



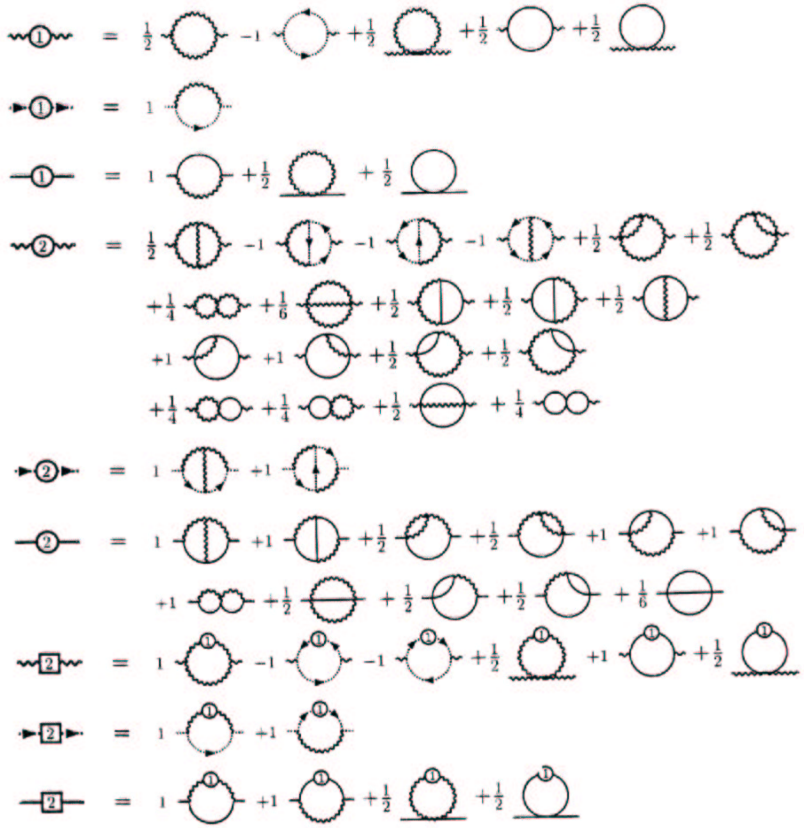
plug into skeletons $\rightsquigarrow$



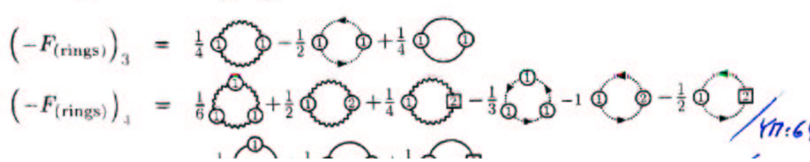
/YM: 18

/45

$(\Pi_{bos} = 2\,\delta_4 \Phi)\ (\Pi_{fer} = -\delta_4 \Phi)$



tough: overlapping UV/IR div's !



YM:64

some technical points of generic interest:

'modern' pert. comp.

ϕ^4 doable on paper : 1 diag $\leftrightarrow$ 1 integral

 $\leftrightarrow \int_{1234} \frac{1}{p^2+m^2} \cdots$

YM on a computer :  $\leftrightarrow 2^9 \cdot 6^6 \approx 25\,M$ integrals !

use FORM [J. Vermaseren]

symbolic programming
knows vectors, tensors, γ_μ, ...
low-level, but fast
size of expression $\Leftrightarrow$ size of harddisk

strategy

A) use Partial Integration [........., Laporta '00] ← ordering

$$0 = \int d^d p\ \partial_{p_\mu} f_{(\mu)}(p)$$

to reduce any integral
to a lin. comb. of (a few) basic scalar master int's
(minimal) set has to be found!
is 'output' of general reduction algorithm

result: $$\int_{p_1}^{(d)} \cdots \int_{p_n}^{(d)} \frac{(p_1 \cdot p_2)^{n_1} \cdots (p_i \cdot p_j)^{n_{ij}} \cdots}{(p_1^2+m_1^2)^{a_1} \cdots ((p_i-p_j)^2+m_{ij}^2)^{a_{ij}} \cdots}$$

$$= \sum_{\text{master int's}} c_i(d) \int_{p_1}^{(d)} \cdots \int_{p_n}^{(d)} \frac{1}{(\ )^1 \cdots (\ )^1 \cdots}$$ — typically (or zero)

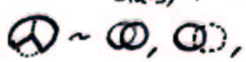
$c_i(d) \sim \frac{\text{poly}(d)}{(d-3)(d-4)\cdots}$

B) compute basic int's to sufficient depth in $\varepsilon = \frac{3-d}{2}$ ← pick

ad A) reduction (here: 4loop vac diags)

~ 5000 lines of FORM code

- 'QED like' case: searched $\mathcal{O}(1M)$ PI rel's
  - ~ 1 week CPU time on 1GHz P3
  - output: database with $\mathcal{O}(100K)$ lines
  - examples:  $= -\frac{d-2}{2(d-3)}\left(\frac{1}{m}\bigcirc\right)^2$

try this: Li_2, Cl_2 [D. Broadhurst] $\longrightarrow$ , $(\bigcirc)^3$

...

\# of basic integrals: $\mathcal{O}(10)$!

- SU(N) + adj. H: generates all diags; color + Lorentz-algebra symmetries, simplifications, ordering
- adding up all diags, get $\sum c_i \,\mathrm{Master}_i$
  - gauge-par. ξ cancels analytically, in d dim.!

status: checking + debugging

N.B. this program has served as a non-trivial check of FORM itself ...

ad B) basic scalar integrals

3d, Eucl, massive

dim. reg., $\overline{MS}$, x-space, Gegenbauer, ...

new: (examples)

 $\sim \frac{1}{\varepsilon} + (1+\ln\bar\mu) + \ldots$

Li_2's [A. Vuorinen]

 $\sim \frac{1}{\varepsilon^2} + \frac{1+\ln\bar\mu}{\varepsilon} + (1+\ln\bar\mu+\ln^2\bar\mu) + \ldots$

 $\sim \frac{1}{\varepsilon^2} + \frac{1+\ln\bar\mu}{\varepsilon} + \ldots$

...

(all coeffs are fcts of masses)

example for reduction algorithm: 3loop f (compare BN '95)

INPUT

$$(3\text{–loop})_{\lambda^0} = \tfrac{1}{6} + \tfrac{1}{8} + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{4} + \left(\tfrac{1}{4}\right)_{\lambda^0}$$

$$f_{3,0}(d) = \frac{1}{m^4}\,\frac{1}{3d-8}\sum_{i=1}^{6}\left\{c_{1,i}(d,\xi)\frac{1}{m^2}\left(\;\right)^3 + c_{2,i}(d,\xi)m^2 + c_{3,i}(d,\xi)m^2\right\}$$

$$= \frac{1}{m^4}\,\frac{1}{3d-8}\left[-\frac{(d-2)(12d^4-155d^3+654d^2-1064d+608)}{8(d-6)(d-4)(2d-7)}\,\frac{1}{m^2}\left(\;\right)^3 + \frac{(d-2)^3(3d-11)}{(d-4)(2d-7)}\,m^2 - \frac{3d^2-18d+16}{2(d-6)(d-4)}\,m^2\right]$$

OUTPUT

$$f_{3,0}(3-2\epsilon) \approx -\left(\frac{(1-\xi)^2}{16\epsilon}+\frac{39+4\pi^2-66\xi+15\xi^2}{48}\right)-\left(\frac{(1-\xi)^2}{16\epsilon}+\frac{21+4\pi^2-36\ln 2-36\xi+9\xi^2}{24}\right)$$
$$+\left(\frac{3(1-\xi)(3-\xi)}{16\epsilon}+\frac{87+4\pi^2-96\xi+21\xi^2}{16}\right)-\left(\frac{3(6-4\xi+\xi^2)}{32\epsilon}+\frac{3(44-32\xi+7\xi^2)}{32}\right)$$
$$+\left(\frac{12-4\xi+\xi^2}{32\epsilon}+\frac{100+32\ln 2-84\xi+15\xi^2}{96}\right)-\left(\frac{1-\xi}{4\epsilon}+\frac{105+4\pi^2-24\xi+3\xi^2}{24}\right)$$
$$= -\left(\frac{89}{24}-\frac{11}{6}\,\ln 2+\frac{\pi^2}{6}\right)+\mathcal{O}(\epsilon)$$

Tektronix Tektronix Tektronix

analytic results $\leq$ 4loop

add up all diags

reduce to scalar integrals

compute those

renormalise

1loop 2loop

$$f_{3d}^{\rm ren}(\mu) = -\frac{\bar g_3^6 d_A}{4\pi}\left\{\frac{\hat m^3}{3}+\hat m^2\left[-\left(\ln\hat\mu+\frac{3}{4}\right)-\frac{\bar\lambda}{4}\right]\right.$$

3loop
$$+\hat m\left[-\left(\frac{89}{24}-\frac{11}{6}\ln 2+\frac{\pi^2}{6}\right)+\frac{\bar\lambda}{4}+\bar\lambda^2\left(\frac{1}{d_A+2}\left(-\frac{3}{2}+\ln 2\right)+\frac{1}{8}\right)\right]$$

4loop
$$+(8c_8\ln\hat\mu+c_9)$$
$$+\bar\lambda\left(\frac{1}{d_A+2}\left(\frac{5}{8}(8-\pi^2)\ln\hat\mu+c_7\right)+\ln^2\hat\mu-\frac{53}{12}\ln\hat\mu-\frac{101-6\pi^2-93\ln 2}{72}\right)$$
$$+\bar\lambda^2\left(\frac{1}{d_A+2}\left(-\ln^2\hat\mu+\frac{32-\pi^2}{8}\ln\hat\mu-\frac{4+\pi^2}{16}+c_4+c_5\right)+\frac{1}{8}\right)$$
$$\left.+\bar\lambda^3\left(-\frac{d_A+8}{(d_A+2)^2}\,\frac{\pi^2}{24}\,(\ln\hat\mu+4c_1)+\frac{1}{d_A+2}\left(\frac{1}{4}-\frac{\ln 2}{2}\right)-\frac{1}{24}\right)\right\}$$

AV

some coeff's (still) unknown

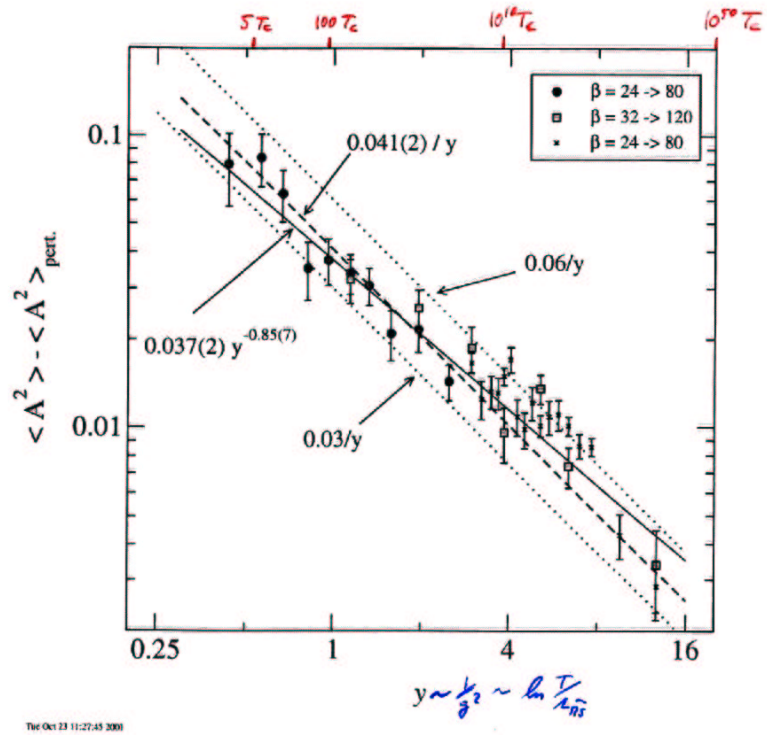
... in progress

$c_8 \Rightarrow g^6\ln g$ -term in p_{4d} ?

$\sim(-a+b)$ on next slide

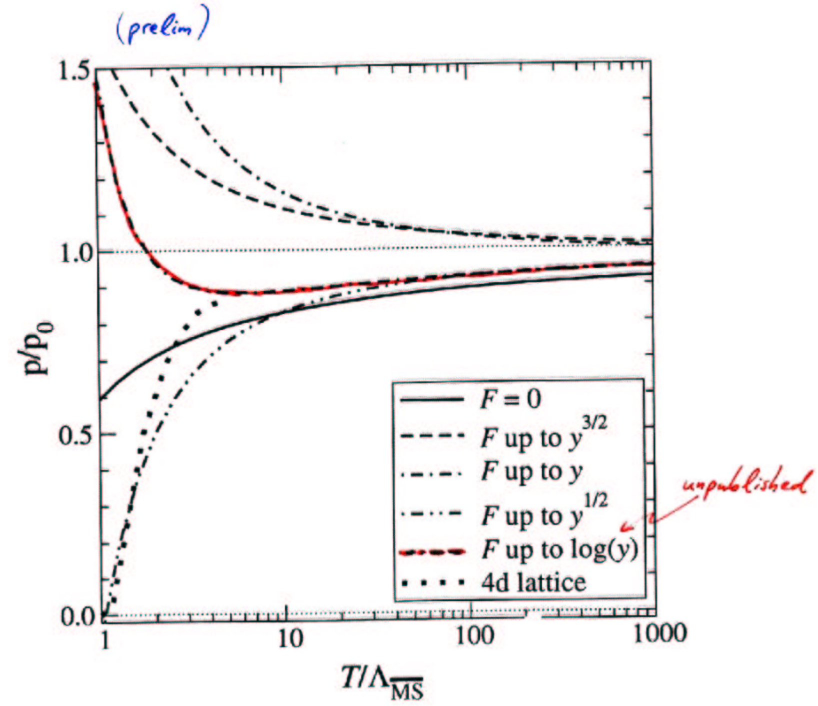
measurement of 4-loop c_g [K. Rummukainen]

$\approx 6 \cdot 10^{15}$ flops



$$\Rightarrow \frac{8 \cdot 3^3}{2(4\pi)^4}(-a+b) = 0.041 \quad \rightsquigarrow \quad -a+b = 9.5$$

(prelim)



$$\frac{p}{p_0} = 1 + \ldots + (\underline{a+b})\, g^6 \ln g + c\, g^6$$

LAT + LAT pt : $\underline{-a+b}$ known

here: assume a (and c) = 0

non-pert. const.

UV div of 3d SU(N) + adj H

Conclusions

- p is known poorly over a huge T-interval.
  4D lattice: up to $\sim 5T_c$
  pert. theory: down from $T=\infty$ to ..?
- effective theory separates scales,
  provides clear separation of pert and non-pert
  contrib's.
- A combined perturbative + 3D lattice MC
  seems to offer a (<u>the</u>?) solution
  (hard work to do; in progress...)
- 3d eff. th. brings major benefits
  superren. / universality / cost → utilize these!
  $n_f \cdots \mu \cdots \quad \cdots + i g \mu_f A_0^3$
- useful spin-off's include
  skeleton setup: efficient + reliable @ high orders
  integration: automatized reduction
  + new analytical results