INHOMOGENEOUS HARTREE DYNAMICS

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1. Motivation
2. Hartree approx.; Hartree ensembles
3. Numerical simulations in 1+1 D φ^4 model
   - thermalization
   - kinks
4. What have we learned

MOTIVATION

classical approximation can be quite reasonable

nicely incorporates non-perturbative phenomena

skyrmions, monopoles, sphalerons *

but

- how to start it up – initial conditions?
- particle production in skyrmion decay

  (effective chiral dynamics in QCD transition)
- it breaks down: classical thermalization

  $n_{\mathbf{k}} = \frac{T}{\omega_{\mathbf{k}}}$ EJR problem

need to connect with quantum world

simplest approximation that incorporates *:

inhomogeneous Hartree

HARTREE APPROXIMATION

Heisenberg e.o.m.

$$(\partial_t^2 - \nabla^2 + \mu^2)\hat{\varphi} + \lambda\hat{\varphi}^3 = 0$$

gaussian approx. $\langle\hat{\varphi}\rangle = \varphi$

$$\langle T\hat{\varphi}_1\hat{\varphi}_2\rangle = \varphi_1\varphi_2 - iG_{12}$$

$$G_{12\cdots n} = 0, \qquad n > 2$$

$$\langle\lambda\hat{\varphi}^3\rangle \to \lambda\varphi^3 + 3\lambda\varphi C$$

$$C(x,x) \equiv -iG(x,x)$$

Closed set of eqns:

$$(\partial_t^2 - \nabla^2 + \mu^2 + \lambda\varphi^2 + 3\lambda C)\varphi = 0$$

$$(\partial_t^2 - \nabla^2 + \mu^2 + 3\lambda\varphi^2 + 3\lambda C)_1 G_{12} = \delta_{12}$$

Compare with Φ-derivable approx.

$\Phi =$ 

in terms of mode functions $f_{\mathbf{k}}(\vec{x},t)$:

$$\hat{\varphi}(\vec{x},t) = \varphi(\vec{x},t) + \sum_{\mathbf{k}}[\hat{b}_{\mathbf{k}} f_{\mathbf{k}}(\vec{x},t) + \text{h.c.}]$$

$$\langle\hat{b}_{\mathbf{k}}\rangle = 0, \quad \langle\hat{b}_{\mathbf{k}}^\dagger\hat{b}_{\mathbf{l}}\rangle = n^0_{\mathbf{k}}\delta_{\mathbf{k}\mathbf{l}}$$

$$C = \sum_{\mathbf{k}}(1 + 2n^0_{\mathbf{k}})|f_{\mathbf{k}}|^2$$

Effective hamiltonian

$$f_{\mathbf{k}} = \frac{1}{\sqrt{2}}(f_{\mathbf{k}1} - i f_{\mathbf{k}2})$$

$$\xi_{\mathbf{k}a} = \sqrt{n^0_{\mathbf{k}} + \tfrac{1}{2}}\, f_{\mathbf{k}a}, \quad \eta_{\mathbf{k}a} = \dot{\xi}_{\mathbf{k}a}, \qquad a = 1,2$$

$$\xi^2 = \sum_{\mathbf{k}}(\xi_{\mathbf{k}1}^2 + \xi_{\mathbf{k}2}^2), \quad \text{etc.} \qquad \pi = \dot{\varphi}$$

$$H_{\text{eff}} = \int d^3x \Big[\tfrac{1}{2}\big(\pi^2 + \eta^2 + (\nabla\varphi)^2 + (\nabla\xi)^2\big) + \tfrac{1}{2}\mu^2(\varphi^2 + \xi^2) + \tfrac{1}{4}\lambda\big(\varphi^4 + 6\varphi^2\xi^2 + 3(\xi^2)^2\big)\Big]$$

on a spatial lattice with $M = N^3$ sites

$O(2M)$ symmetry

$M(2M-1)$ conserved 'angular momenta'

stationary states

$\varphi(\vec{x},t) = v$ $\qquad f_{k}(\vec{x},t) = \dfrac{e^{i\vec{k}\cdot\vec{x} - i\omega_k t}}{\sqrt{2\omega_k V}}$

$$C = \frac{1}{V}\sum_{k}\left(n_k + \tfrac{1}{2}\right)\frac{1}{\omega_k}$$

Hartree eqns $\Rightarrow$ $\omega_k^2 = k^2 + m^2$

$$m^2 = \mu^2 + 3\lambda v^2 + 3\lambda C$$

$v^2 = 0$ 'symmetric phase'

$ = \dfrac{m^2}{2\lambda}$ 'broken phase'

for any n_k

vacuum $n_k = 0$ subtract C to make finite

1+1 D: $\delta\mu^2$

3+1 D: $\delta\mu^2$ but not quite $\delta\lambda$

thermal $n_k = \dfrac{1}{e^{\omega_k/T} - 1}$ $\Rightarrow$ 1st order PT

but does the system thermalize?

QUASIPARTICLE ENERGY & DISTRIBUTION FCN

$$S_k(t) \equiv \frac{1}{V}\int d^3x \int d^3y \; e^{-i\vec{k}\cdot\vec{x}} \langle \hat{\varphi}(\vec{x}+\vec{y},t)\,\hat{\varphi}(\vec{y},t)\rangle_{\text{conn}}$$

$$U_k(t) \equiv \frac{1}{V}\int d^3x \int d^3y \; e^{-i\vec{k}\cdot\vec{x}} \langle \hat{\pi}(\vec{x}+\vec{y},t)\,\hat{\pi}(\vec{y},t)\rangle_{\text{conn}}$$

quasiparticle behavior $\qquad S_k = \left(n_k + \tfrac{1}{2}\right)\dfrac{1}{\omega_k}$

$\qquad U_k = \left(n_k + \tfrac{1}{2}\right)\omega_k$

define $\quad \overset{n_k + \frac{1}{2}}{\cancel{\omega_k}}{}^{2} = S_k U_k$

$\qquad \overset{\omega_k^2}{\cancel{n_k + \tfrac{1}{2}}} = U_k / S_k$

also out of equilibrium

Example: spinodal instability

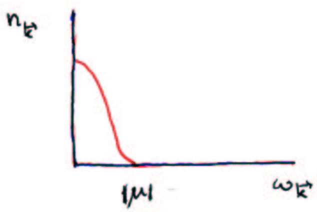
$\varphi(\mathbf{x},t) \equiv 0,\ f_{\mathbf{k}}(\mathbf{x},t) \equiv f_{\mathbf{k}}(t)e^{i\mathbf{k}\mathbf{x}}$ homogeneous

$$[\partial_t^2 + \mathbf{k}^2 + \mu^2 + 3\lambda C(t)]f_{\mathbf{k}}(t) = 0$$

$t = 0$:

$$f_{\mathbf{k}} = \frac{1}{\sqrt{2\omega_k V}}, \quad \partial_t f_{\mathbf{k}} = -i\omega_k f_{\mathbf{k}}, \quad n^0_{\mathbf{k}} = 0$$

leads to



(approx.)

no thermalization

no scattering contribution

seem to have lost good qualities of classical approximation

$\langle\cdots\rangle$: quantum mechanical expectation value
$\Rightarrow$ homogeneous

classical: average over realizations, typically non-homogeneous, which contain scattering

mean field $\langle\hat{\varphi}\rangle = \overline{\varphi_c}$ average classical field

quantum mechanical: ‘realizations’?

coherent states

1 d.o.f.

$$|pq\rangle = e^{z\hat{a}^\dagger}|0\rangle, \quad z = \frac{1}{\sqrt{2\omega}}(\omega q + ip)$$
$$\hat{a}|pq\rangle = z\,|pq\rangle$$
$$\hat{\rho} = \int \frac{dp\,dq}{2\pi}\,\rho(p,q)\,|pq\rangle\langle pq|$$

E.g.

$$\hat{\rho} \propto \exp\left[-\beta\,\omega\,\hat{a}^\dagger\hat{a}\right]$$
$$\rho(p,q) \propto \exp\left[-\left(e^{\beta\omega}-1\right)\frac{1}{2\omega}\left(\omega^2 q^2 + p^2\right)\right]$$

2 pt fcn written as mean-field average

Hartree ensemble approximation:

- write initial state in terms of 'realizations'

$$\hat{\rho} = \int D\pi\,D\varphi\,\rho[\pi,\varphi]\,|\pi\varphi\rangle\langle\pi\varphi|$$

- Hartree for each $|\pi\varphi\rangle\langle\pi\varphi|$ separately

- average over initial φ and π

Scattering regained (especially in broken phase)

consider incoming two-particle state
wave packets $\psi_{1,2}$:

$$|\psi_1\psi_2\rangle = \hat{a}^\dagger[\psi_1]\hat{a}^\dagger[\psi_2]|0\rangle$$
$$\hat{a}^\dagger[\psi] = \sum_{\mathbf{k}} \psi_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}$$
$$C(\mathbf{x},t;\mathbf{x},t) = C_{\text{vac}} + |\psi_1(\mathbf{x},t)|^2 + |\psi_2(\mathbf{x},t)|^2$$
$$\psi(\mathbf{x},t) = \sum_{\mathbf{k}} \psi^*_{\mathbf{k}} f_{\mathbf{k}}(\mathbf{x},t)$$

linearize in 'broken phase', $\varphi = v + \varphi'$

$$(\partial_t^2 - \nabla^2 + m^2)\varphi' = -3\lambda\left(|\psi_1|^2 + |\psi_2|^2\right)$$
$$(\partial_t^2 - \nabla^2 + m^2 + 6\lambda v\varphi')\psi_{1,2} = 0$$

similar to classical electrodynamics

Thermalization tests in 1+1 dimensions

Numerical simulations
initial states out of equilibrium

See if the Bose-Einstein distribution emerges after some time $n_k \to 1/(e^{\beta\omega_k} - 1)$?

Caveat: expect equipartition according to $H_{\rm eff}$ at large times
Corresponding n_k depend on conserved charges

spacetime lattice, N spatial sites,
spatial volume $L = Na$, cutoff $k_{\rm cut} = \pi/a$

vacuum $\varphi(x,t) = v$
▪ 'broken phase' $v \neq 0$ (should go to zero as $L \to \infty$ due to non-perturbative effects (kinks))

renormalized mass

$$\begin{aligned} m^2 &= \mu^2 + 3\lambda v^2 + 3\lambda C \\ &= 2\lambda v^2 \qquad \text{in 'broken phase'} \end{aligned}$$

weak coupling, expect good quasiparticle description

▪ 'symmetric phase' $v = 0$

Initial state I: "Parisi"

$$n_k^0 = 0, \quad \varphi_x = v, \quad \pi_x = Am \sum_{j=1}^{j\max} \cos(2\pi jx/L - \psi_j)$$

$\psi_j \in [0, 2\pi)$ flat distribution, $k_{\rm max}/m = \pi/4$
$E/Lm^2 = 0.5$ in mean field only
BE thermalization would give $T/m \approx 1.1$

$mL = 32$, $\lambda/m^2 = 1/12$, $1/am = 8$, $N = 256$

Results:

- energy goes from mean field towards modes

- initial approach to BE with $T/m \approx 1.1$, $\mu/m \approx 0.2 \to 0$, $\tau m = 15 - 20$

- late time contamination by equipartition, time scale $O(10^4)$ slowing down

- approximate equipartition at huge times $O(10^6)$

- damping rate of mean field zero mode of the order of 'plasmon rate'; 'Twin Peaks' phenomenon in $d = 1$

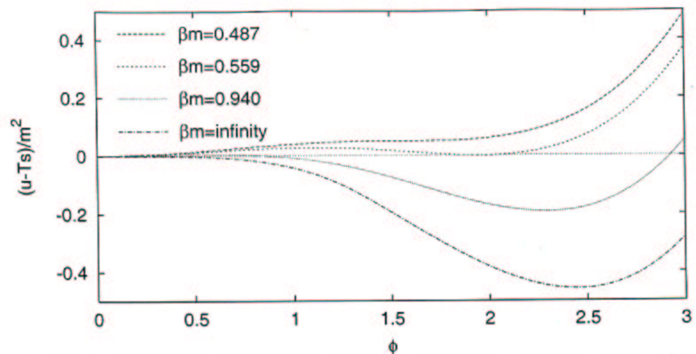


FIG. 4. Finite temperature effective potential $f/\lambda = (u - Ts)/\lambda$ versus φ for various values of $\beta m(\varphi_c, 0)$. The potential is again normalized to zero at $\varphi = 0$.

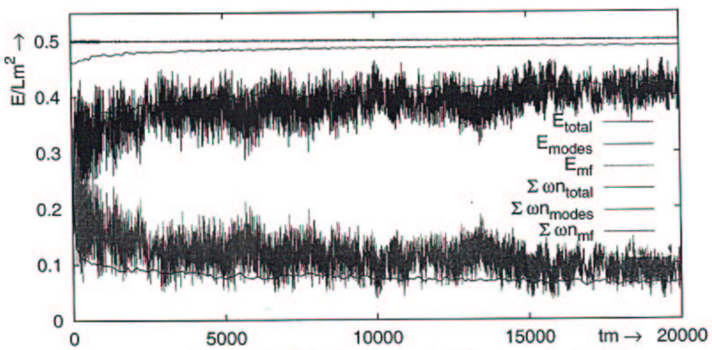


FIG. 5. The total energy density E/Lm^2 (horizontal line at 0.5), energy density of the mean field (lower band) and of the modes (higher band). Also plotted are the various energy densities in the quasiparticle interpretation, $\sum_k n_k\omega_k/Lm^2$.

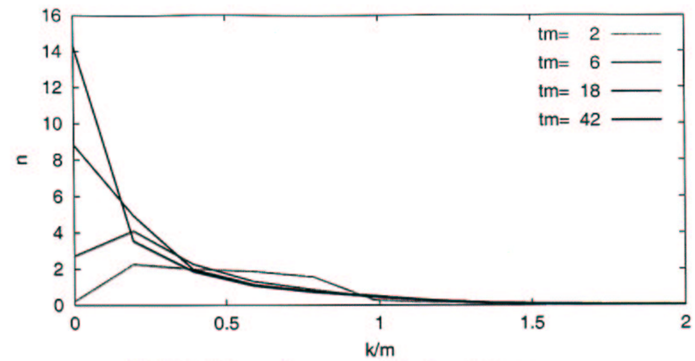


FIG. 6. Particle number n_k versus k/m for early times.

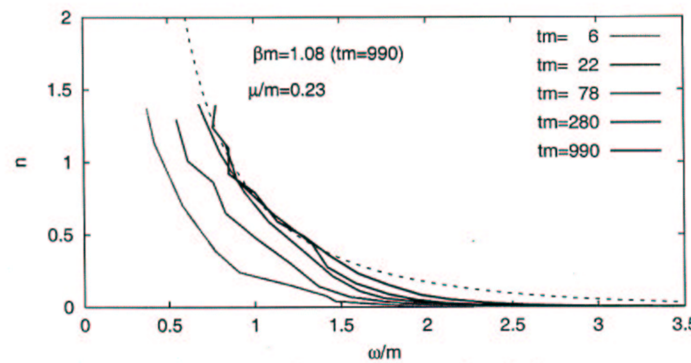


FIG. 7. Particle number n_k (modes only) versus ω_k for early times.

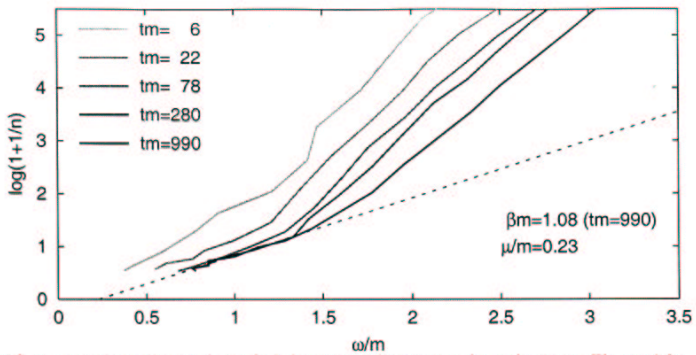


FIG. 8. Particle number $\log(1+1/n_k)$ (modes only) versus ω_k for early times. The straight line is a Bose-Einstein fit for the latest time, over $\omega/m < 1.2$.

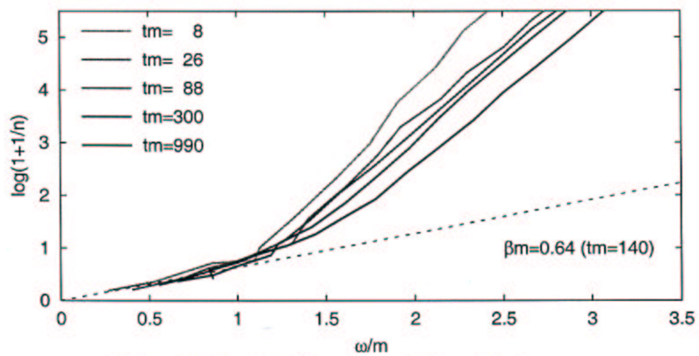


FIG. 9. As in Fig. 8, including the mean field contribution.

$$n = \frac{1}{e^{\beta(\omega-\mu)} - 1} \quad \Rightarrow \quad \log\left(1+\frac{1}{n}\right) = \beta(\omega-\mu)$$

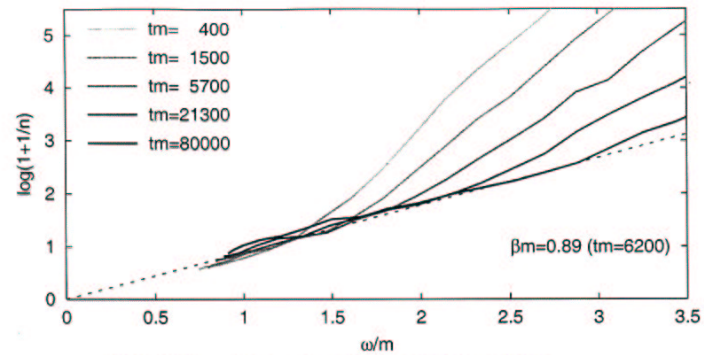


FIG. 10. The particle numbers (modes only) for later times.

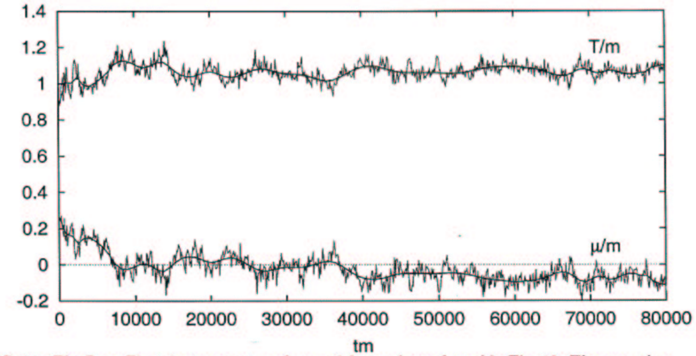


FIG. 11. The Bose-Einstein temperature for particle numbers plotted in Fig. 10. The smoother lines are drawn to guide the eye.

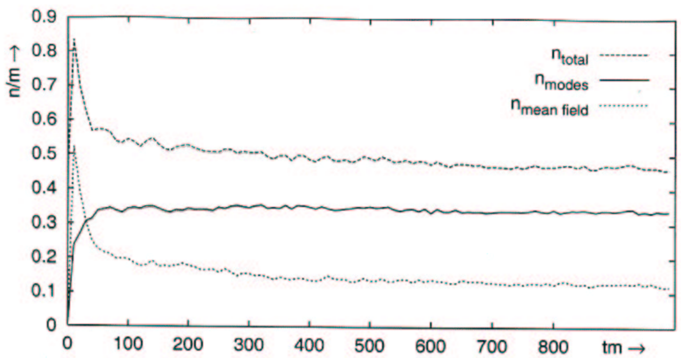


FIG. 12. Particle densities $n/m = \sum_k n_k/Lm$.

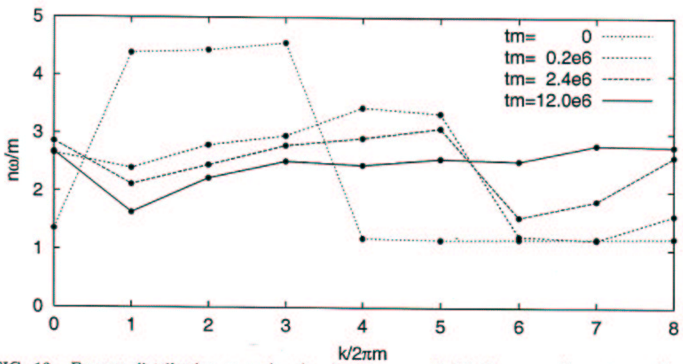


FIG. 13. Energy distribution $n_k\omega_k/m$ (modes + mean field) for a small system with $N = 16$, $Lm = 1$, $E/Lm^2 = 36$.

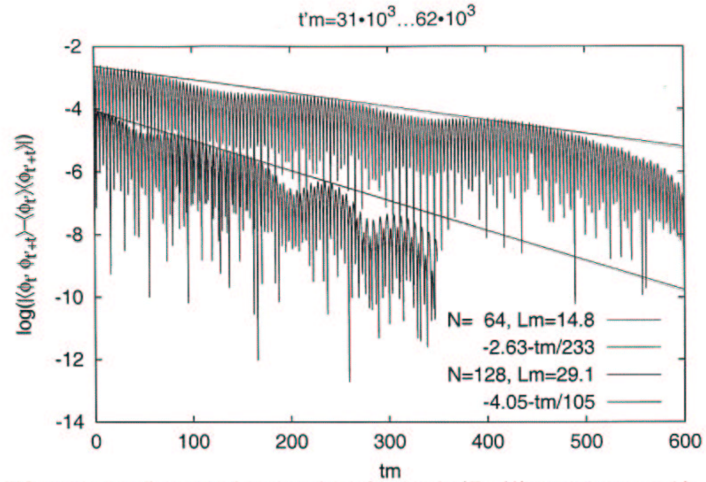


FIG. 14. Numerically computed auto-correlation functions $\log|F_{0\mathrm{mf}}(t)|$ versus time tm_T, with m_T the temperature dependent mass. The coupling is weak, $\lambda/m_T^2 = 0.11$ and the temperature $T/m_T \approx 1.4$ for the smaller volume (with significant deviations from the Bose-Einstein distribution) and ≈ 1.6 for the larger volume (reasonably BE).

(1 initial configuration)

Initial state II: "Bose – Einstein (BE)"

$$\rho[\pi,\varphi] \propto \prod_k \exp\left\{-(e^{\omega_k/T_0}-1)\,\frac{1}{2\omega_k}\left[\pi_k^2+\omega_k^2(\varphi_k-v)^2\right]\right\}$$

closer to equilibrium as initially all momentum modes φ_k, π_k are populated
however initial mass in ω_k is $m \neq m(T_0)$

$T_0/m = 1,\ 5;\ \lambda/m^2 = 1/4,\ 1/12,\ Lm = 25.6,\ 1/am = 10$

Results:

- initial thermalization towards BE with $m(T)$, $\tau_{\rm BE} m \approx 20$

- drift towards classical equipartition with $\tau_{\rm cl} m \approx 2500$

- may reduce number of modes $\ll N$

- zero # of modes, i.e. classical, similar? yes, but $O(10)$ quicker towards equipartition

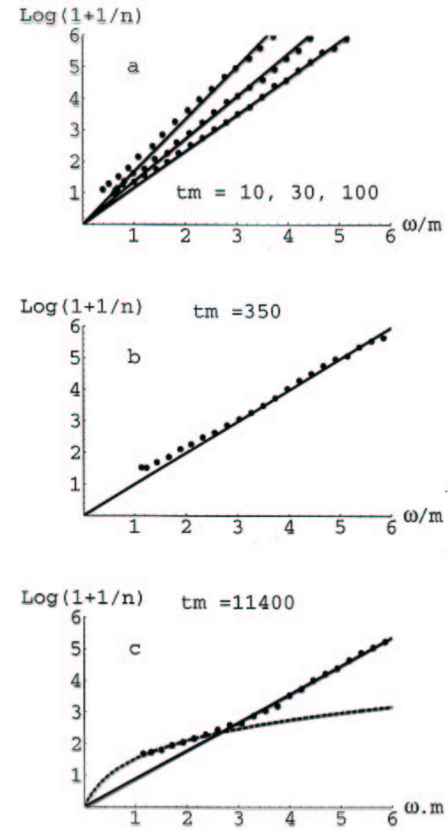


Figure 1: *Particle densities as a function of energy, plotted as* $\log(1+1/n)$. *In Figs. a-c the initial* $T_0/m = 1$. *The model parameters are:* $\lambda/m^2 = 1/2v^2 = 1/4$, $Lm = 25.6$, $1/am = 10$.

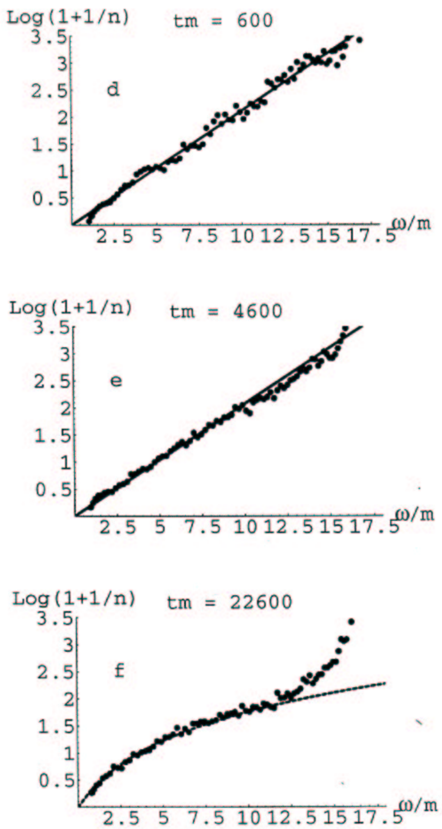


Figure 2: *Particle densities as a function of energy, plotted as* $\log(1+1/n)$. *The initial temperature* $T_0/m = 5$. *The model parameters are:* $\lambda/m^2 = 1/2v^2 = 1/4$, $Lm = 25.6$, $1/am = 10$.

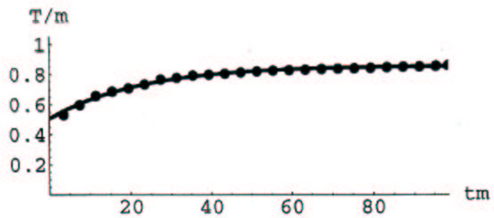


Figure 3: *Time dependence of BE (from the modes only) for the data of Figs. 1a-c.*

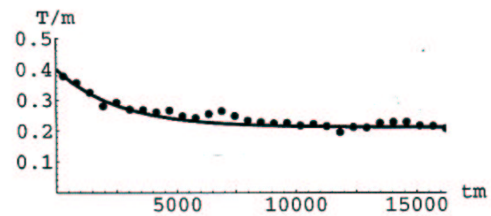
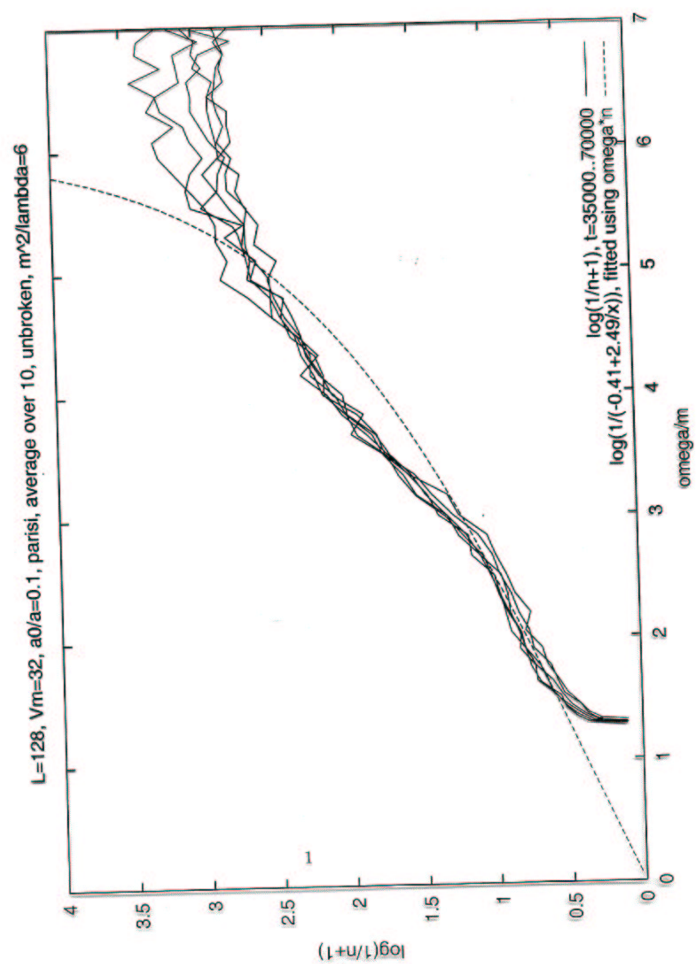


Figure 4: *Time dependence of classical (from the modes and mean field) temperatures for the data of Figs. 1a-c.*

symmetric phase



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symmetric phase

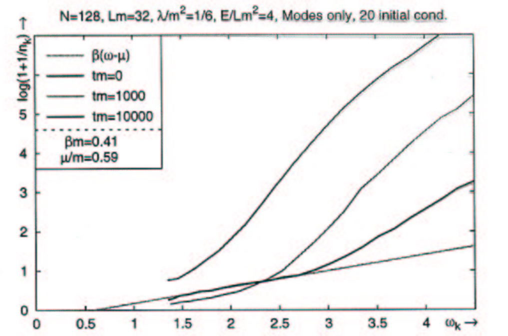


Figure 3: dist parisi early

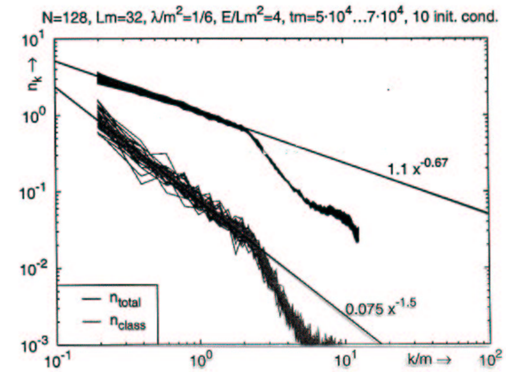


Figure 4: dist parisi late

4

- symmetric phase "Parisi" & "BE"

summary so far

- broken phase

  approximate thermalization toward BE

  classical contamination $\tau_{cl} \gtrsim 100\, \tau_{BE}$

- symmetric phase

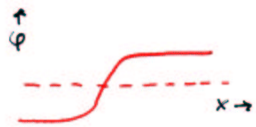
  need larger coupling and energy density

  to see "thermalization" similar to broken phase

KINKS

- at rest:

  $\varphi(x) = v \tanh\left(\frac{m}{2} x\right)$



classical soln

use as initial mean field in Hartree approx.

- kink-antikink test – need small coupling

  ('cooling': friction term in mean-field eq.)

  - kink shape gets smeared out translational zero mode

  - difficult to implement in $k\bar{k}$ coll.

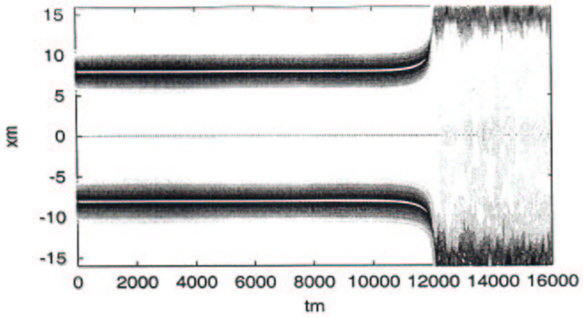
$M_{clas} = \frac{m^2}{3\lambda} = 4m$

$\lambda = m^2/12$

$M_{cool} = 3.8\, m$

- kink-antikink collisions, initial cond. $t=0$:

$$\varphi(x,t) = v\left\{\tanh\left[\frac{m}{2}\gamma(x-x_1-ut)\right] - \tanh\left[\frac{m}{2}\gamma(x-x_2+ut)\right] - 1\right\}$$



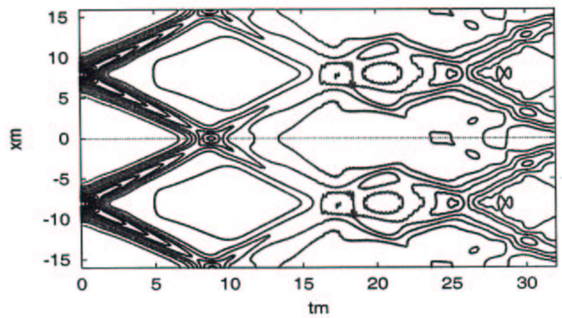
$u = 0$

Classical

$\lambda = \frac{m^2}{1.25}$

$(v = 0.79)$

N=512, Lm=32, 512 modes, E/Lm^2=[0.01:0.2], λ/m^2=1/1.25, u=0.0

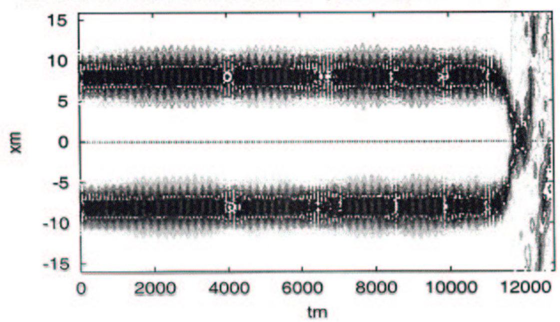


Hartree

$\lambda = \frac{m^2}{1.25}$

$(v = 0.79)$

N=256, Lm=32, 256 modes, E/Lm^2=[0.08:1.6], λ/m^2=1/12, u=0.0



Hartree

$\lambda = \frac{m^2}{12}$

$(v = 2.45)$

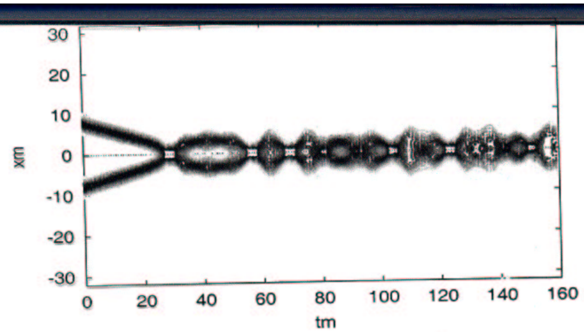
- critical velocity $u_c^{clas} \approx 0.25$

  $u_c^{H} \approx 0.75$

- large γ : can use larger coupling (time delay)
- 'plasma region' production of particles

- energy density of central region $\approx$ indep. of γ (γ = 10 – 20 –)

  decays faster classically than Hartree

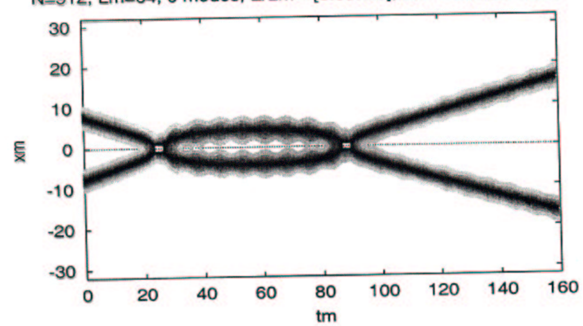


Classical

$u = 0.21$

$(\gamma \approx 1.023)$

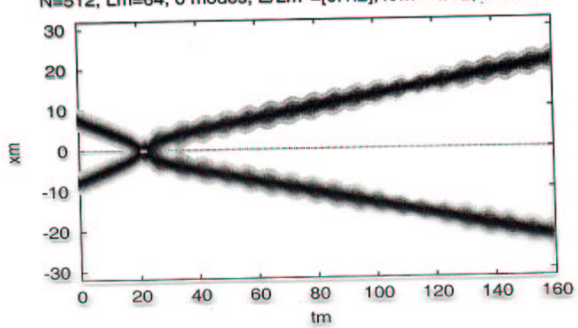
N=512, Lm=64, 0 modes, E/Lm^2=[0.09:1.8], λ/m^2=1/12, u=0.25



$u = 0.25$

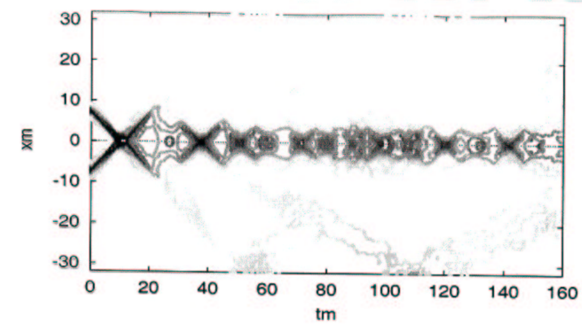
$(\gamma \approx 1.033)$

N=512, Lm=64, 0 modes, E/Lm^2=[0.1:2], λ/m^2=1/12, u=0.30



$u = 0.30$

$(\gamma \approx 1.048)$

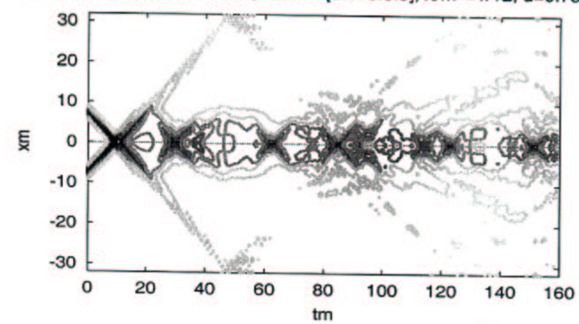


Hartree

$u = 0.70$

$(\gamma \approx 1.40)$

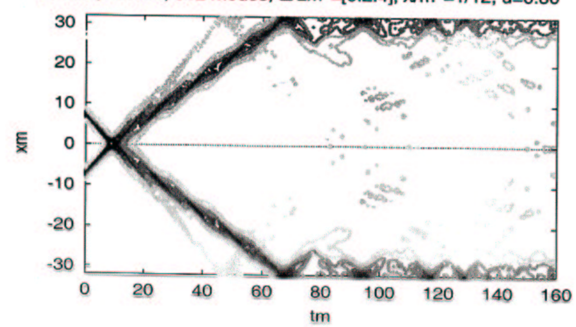
N=512, Lm=64, 512 modes, E/Lm^2=[0.175:3.5], λ/m^2=1/12, u=0.75



$u = 0.75$

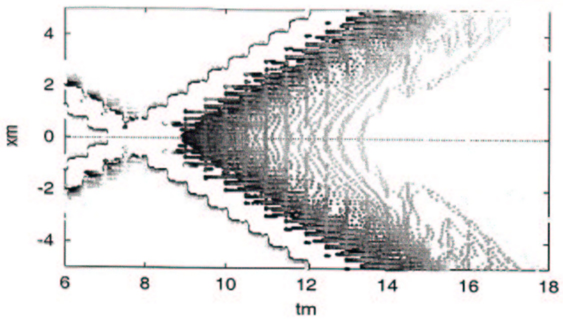
$(\gamma \approx 1.51)$

N=512, Lm=64, 512 modes, E/Lm^2=[0.2:4], λ/m^2=1/12, u=0.80



$u = 0.80$

$(\gamma \approx 1.67)$

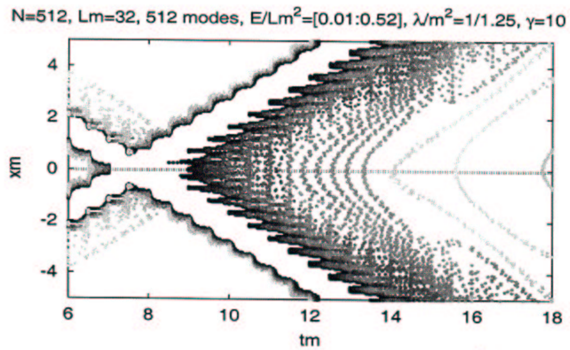


Hartree

$\lambda = \frac{m^2}{1.25}$

$\gamma = 20$

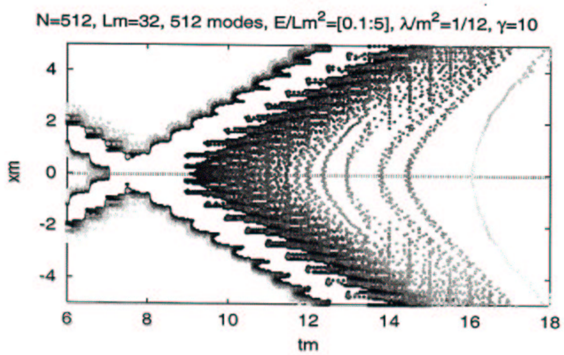
$(u = 0.9987)$



$\lambda = \frac{m^2}{1.25}$

$\gamma = 10$

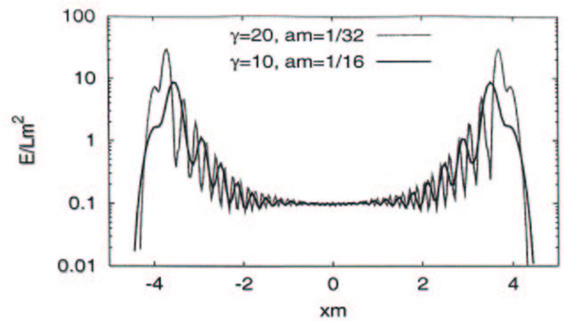
$(u = 0.9950)$



$\lambda = \frac{m^2}{12}$

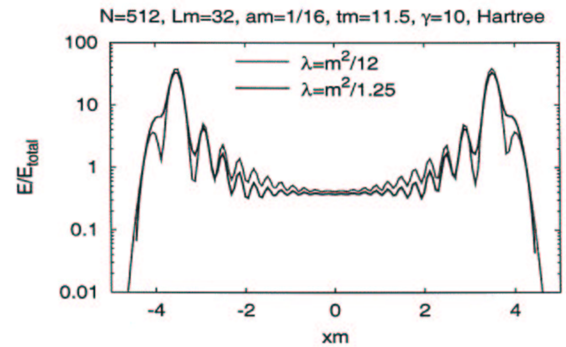
$\gamma = 10$

$(u = 0.9950)$



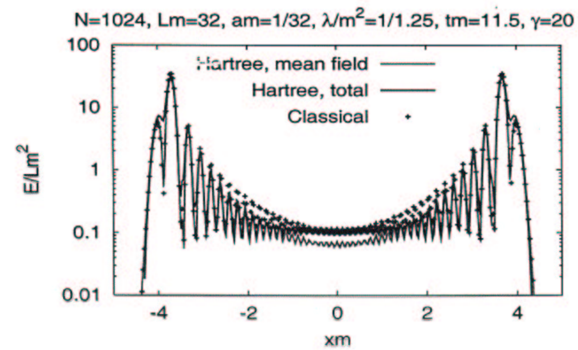
Hartree

$\gamma = 20$
and
$\gamma = 10$



Hartree

$\lambda = \frac{m^2}{1.25}$
and
$\lambda = \frac{m^2}{12}$



Classical
and
Hartree
mean fld
and total

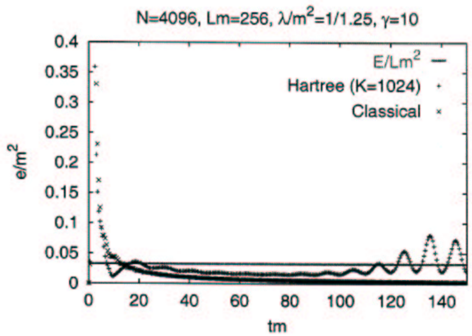


Figure 1: energy density at $x = 0$ versus time

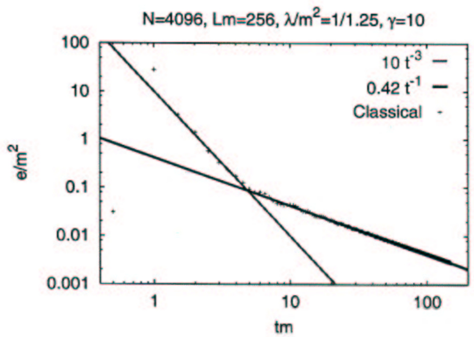


Figure 2: classical energy density at $x = 0$ versus time

WHAT HAVE WE LEARNED?

- glimpse of non-perturbative qft dynamics

  basic question: do we need inhomogeneous realizations for non-perturbative stuff

  still open

- numerical effort constant $\times N^{2d+1}$ in $d+1$ D

  limit # of modes $\rightarrow$ CONSTANT $\times N^{d+1}$

  used in 3+1 D (preliminary - much the same as 1+1 D, $\lambda = 1$ looks like weak coupling)

- new light on initial conditions for classical approximation

  BE ensemble $T \ll \frac{1}{a}$ avoid EJR problems

  use for simulations studying heavy ion coll.