HTL Perturbation Theory to Two Loops

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QCD and Gauge Theory Dynamics in the RHIC Era
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Outline

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Introduction/Motivation

- The weak coupling expansion of the QCD free energy, $\mathcal{F}$, has been calculated to order $\alpha_s^{5/2}$. [1,2,3]
- At temperatures expected at RHIC energies, $T \sim 0.3$ GeV, the running coupling constant $\alpha_s(2\pi T)$ is approximately 1/3.
- The successive terms contributing to $\mathcal{F}$ can strictly only form a decreasing series if $\alpha_s \lesssim 1/20$ which corresponds to $T \sim 10^5$ GeV.

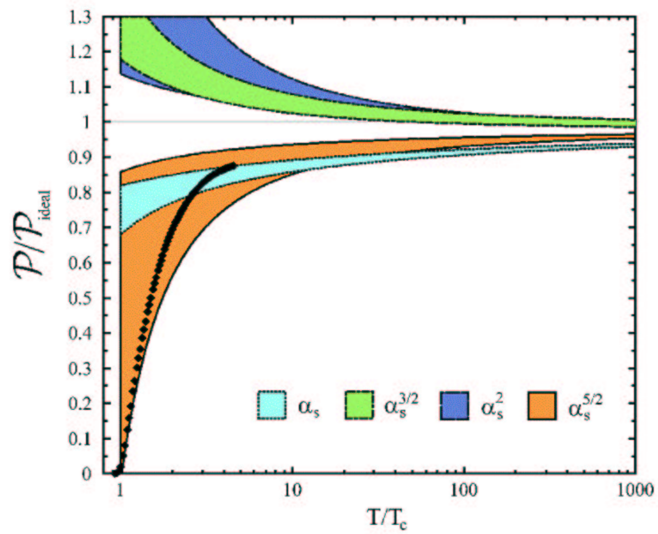


Figure 1. Perturbative QCD pressure vs temperature. ($\pi T \leq \mu \leq 4\pi T$)

4-d lattice results from G. Boyd et al, 95/96.

[1] P. Arnold and C. Zhai, 94/95. [2] B. Kastening and C. Zhai, 95. [3] E. Braaten and A. Nieto, 96.

Other Approaches

- Borel transform, Padé approximates, Self-similar approximates

  (Hatsuda, Kastening, Parwani, Yukalov, Yukalova, . . .)

  - Can be used to construct more stable sequences of successive approximations
  - Can only be applied when several orders in weak-coupling are known
  - Real series in g and $\log g$
  - Has to be redone for each observable

- Quasiparticle Models

  (Gorenstein, Heinz, Peshier, . . .)

  - Nice physical picture
  - Not systematic
  - Not gauge invariant

- 4-d lattice

  (Boyd, Karsch, . . .)

  - Only able to probe temperatures a few times T_c
  - Light dynamical quarks require lots of computing power
  - Not able to study systems with non-zero baryon density
  - Limited to static quantities

Other Approaches cntd

- 3-d lattice (dimensional reduction)

  (Braaten, Nieto, Kajantie, Laine, Rummukainen, Schroder, . . .)

  - Limited to static quantities
  - Only able to study systems with very small baryon density $\mu \lesssim 2\pi T$
  - Light dynamical quarks require additional computing power

- Φ-derivable approximations

  (Luttinger, Ward, Baym, Blaizot, Iancu, Rebhan, Braaten, Petitgirard, . . .)

  - Limited to static quantities
  - Not gauge invariant unless vertex functions are included in variation
  - Hard to systematically improve because this

Our approach

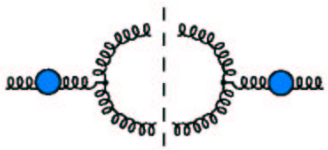
Use a simpler variational approximation based on a single variational parameter m_D which can be interpreted as a quasiparticle mass. It is similar it spirit to *optimized perturbation theory* (Stevenson), *variational perturbation theory* (Kleinert, Sisakian, Solovtsov, Shevchenko, . . .), or *linear delta expansion* (Duncan, Moshe, . . .).

Our variational ansatz are the Hard-Thermal-Loops propagators and vertices. The propagators and vertices are related by Ward-Takahashi identities so that all results are manifestly gauge invariant.

Necessity of HTL Resummation

- Finite-temperature gauge fields act as if they have a mass $m \sim gT$. [4]

- Consider the following diagram for a soft gluon decaying into two gluons.

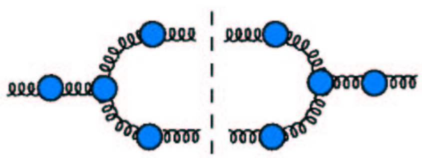


It gives

$$\mathrm{Im}\,\Pi \sim g(gT)^2 \tag{1}$$

with a **gauge dependent coefficient**.

- To fix the problem we must resum all effects of order gT. Instead of computing the graph above, we should compute this graph



Gluon damping rate result:[5]

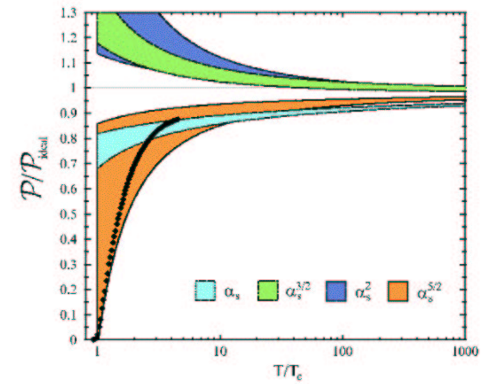
$$\gamma = 6.635\,\frac{g^2 N_c T}{24\pi} \tag{2}$$

[4] V.V. Klimov, 82; H.A. Weldon, 82. [5] E. Braaten and R. Pisarski, 90.

Resummed Perturbation Theory

The canonical to deal with this is to consider the scales T and gT separately, assuming a large separation between them.

$$\begin{array}{ccccl} d_2g^2 & & d_4g^4 & & (T) \\ & d_3g^3 & & d_5g^5 & (gT) \\ \hline d_2g^2 & d_3g^3 & d_4g^4 & d_5g^5 & \end{array}$$

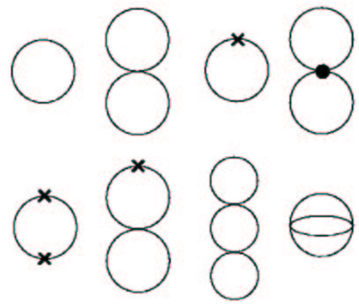


Screened or HTL Perturbation Theory

$$\begin{array}{cccccccl} a_2g^2 & a_3g^3 & a_4g^4 & a_5g^5 & a_6g^6 & \ldots & & \text{(one-loop)} \\ b_2g^2 & & b_4g^4 & b_5g^5 & b_6g^6 & \ldots & & \text{(two-loop)} \\ & & c_4g^4 & c_5g^5 & c_6g^6 & \ldots & & \text{(three-loop)} \\ \hline d_2g^2 & d_3g^3 & d_4g^4 & d_5g^5 & d_6g^6 & \ldots & & \end{array}$$

Finite-temperature Scalar Theory

- The resummed perturbative expansion has been performed to $O(g^5)$. [6]
- This series is as ill-behaved as its analogue in QCD.



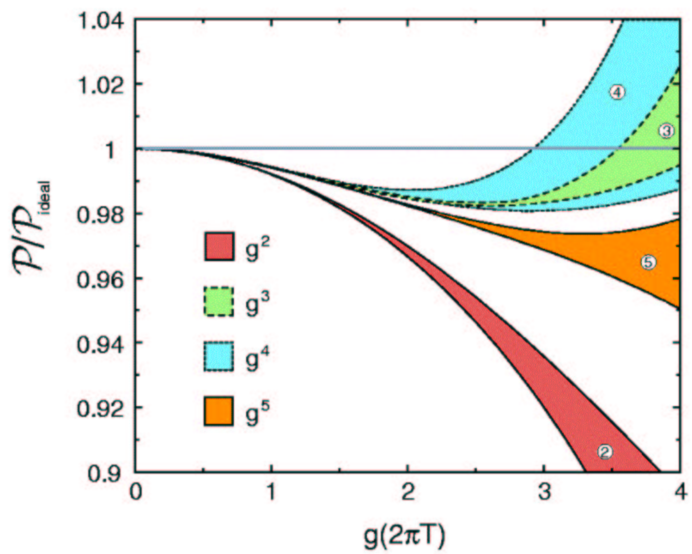


Figure 2. Successive perturbative approximations to the pressure of an $N=1$, ϕ^4 scalar field theory as a function of $g(2\pi T)$.

[6] P. Arnold and C. Zhai, 94/95; R.R. Parwani and H. Singh, 95; E. Braaten and A. Nieto, 95.

Screened perturbation theory

- Within screened perturbation theory, a mass, which can be treated as a variational parameter, is added to the Lagrangian and the loop expansion is recomputed.[7,8]

$$\begin{aligned}\mathcal{L}_{\rm SPT} &= -\mathcal{E}_0 + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2(1-\delta)\phi^2 \\ &\quad -\frac{\delta}{24}g^2\phi^4 + \Delta\mathcal{L} + \Delta\mathcal{L}_{\rm SPT}\end{aligned} \tag{3}$$

We can split this into free, interaction, and counterterm parts

$$\begin{aligned}\mathcal{L}_{\rm free} &= -\mathcal{E}_0 + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \\ \mathcal{L}_{\rm int} &= \delta(-\frac{1}{24}g^2\phi^4 + \frac{1}{2}m_1^2\phi^2) \\ \mathcal{L}_{\rm ct} &= \Delta\mathcal{L} + \Delta\mathcal{L}_{\rm SPT}\end{aligned} \tag{4}$$

The additional counterterms that are required have the form

$$\Delta\mathcal{L}_{\rm SPT} = -\Delta\mathcal{E}_0 - \frac{1}{2}\Delta m^2(1-\delta)\phi^2 \tag{5}$$

with

$$\begin{aligned}\Delta\mathcal{E}_0 &= Z_E\,(1-\delta)^2\,m^4 \\ \Delta m^2 &= (Z_\phi Z_m - 1)m^2\end{aligned} \tag{6}$$

- KPP performed a two-loop calculcation for $N=1$, and a three-loop calculation only in the large-N limit.

[7] F. Karsch, A. Patkós, and P. Petreczky, 97. [8] S. Chiku and T. Hatsuda, 98.

- Together with Eric Braaten and Jens Andersen I have computed the full $N = 1$ three-loop calculation of the thermodynamic functions including the contribution from the finite-temperature massive "basketball" diagram.[9]

$$
\begin{aligned}
(4\pi)^2 \mathcal{P}_{3-\text{loop}} \; = \; & \frac{1}{8}\left[4J_0T^4 + 4J_1m^2T^2 + 2J_2m^4 - m^4\right] \\
& -\frac{1}{8}\alpha\left[J_1T^2 - (L+1)m^2\right]\left[J_1T^2 + 2J_2m^2 + (L-1)m^2\right] \\
& +\frac{1}{48}\alpha^2\Big[3J_2\left(J_1T^2 - (L+1)m^2\right)^2 + \left(3(3L+4)J_1^2 + 6K_2 + 4K_3\right)T^4 \\
& -\left(12L^2 + 28L - 12 - \pi^2 - 4C_1\right)J_1m^2T^2 \\
& +\left(5L^3 + 17L^2 + \tfrac{41}{2}L - 23 - \tfrac{23}{12}\pi^2 - \psi''(1) + C_0\right)m^4\Big]
\end{aligned}
\qquad (7)
$$

with

$$
L \equiv \log\frac{\mu^2}{m^2} \qquad C_0 = 39.429 \qquad C_1 = -9.8424 \qquad (8)
$$

$$
J_n(\beta m) \equiv \frac{4\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{2}-n)}\beta^{4-2n}\int_0^\infty dk\,\frac{k^{4-2n}}{(k^2+m^2)^{1/2}}\,\frac{1}{e^{\beta(k^2+m^2)^{1/2}}-1}
$$

and $K_2(\beta m), K_3(\beta m)$ being two- and three-dimensional integrals which are too messy to list here.

[9] ABS, 01.

Choosing $m(g,T)$

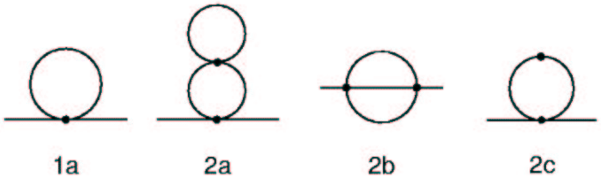
Once the evaluation of the contributing diagrams is completed for a general screening mass, m, the dependence of m on g and T must be specified.

(1) Use the weak coupling perturbative expression.

$$
m^2 = \frac{g^2T^2}{24} = \frac{2\pi^2}{3}\alpha T^2 \qquad (9)
$$

(2) Since the free-energy should be independent of the mass parameter if calculated to all orders, we could require that $\mathcal{F}$ be independent of m. I will call this the variational mass

$$
\frac{\partial \mathcal{F}}{\partial m} = 0 \qquad (10)
$$



$$
m^2 = \frac{1}{2}\alpha\left[J_1T^2 - (L+1)m^2\right] \qquad (11)
$$

(3) Something more phenomenological?

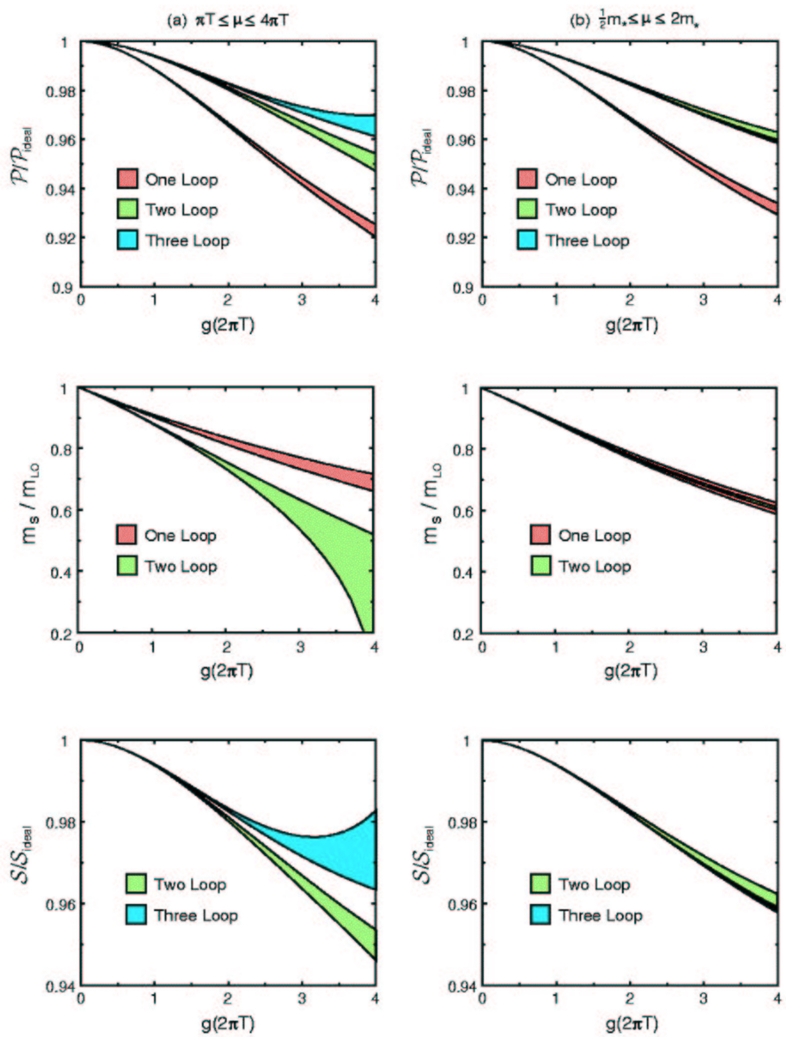


Figure 3. SPT results for the pressure, screening mass, and entropy.

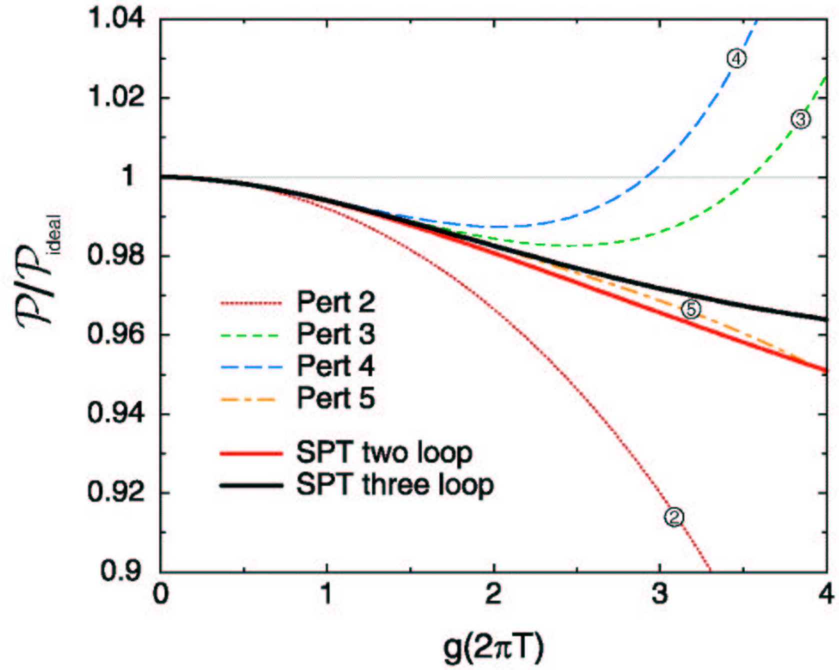


Figure 4. Comparison of SPT results to the successive perturbative approximations.

Mass Expansions in SPT

It is possible to obtain a completely analytic expression if you expand the SPT integrals in powers of $\hat{m} = m/2\pi T$.[11]

$$\begin{aligned}\mathcal{F}_{0+1+2} &= \mathcal{F}_{\text{ideal}}\left\{1 - \frac{5}{4}\alpha + \left[-\frac{59}{12} + \frac{15}{4}L + \frac{5}{4}\gamma - \frac{5}{2}\frac{\zeta'(-3)}{\zeta(-3)} + 5\frac{\zeta'(-1)}{\zeta(-1)}\right]\alpha^2\right.\\ &\quad + \frac{15}{2}\hat{m}\left[1 - \left(5 + 3\gamma + 7L - 8\log\hat{m} - 8\log 2 - 4\frac{\zeta'(-1)}{\zeta(-1)}\right)\alpha\right]\alpha\\ &\quad \left. - \frac{15}{2}\hat{m}^3\left[1 - 6(L+\gamma)\alpha\right] + \frac{5}{8\hat{m}}\alpha^2\right\}. \end{aligned} \tag{12}$$

Similarly the gap equation can be expanded in $\hat{m}$ giving

$$\hat{m}^2 = \frac{1}{6}\alpha\left[1 - 6\hat{m} - 6\hat{m}^2\left(L+\gamma\right)\right]. \tag{13}$$

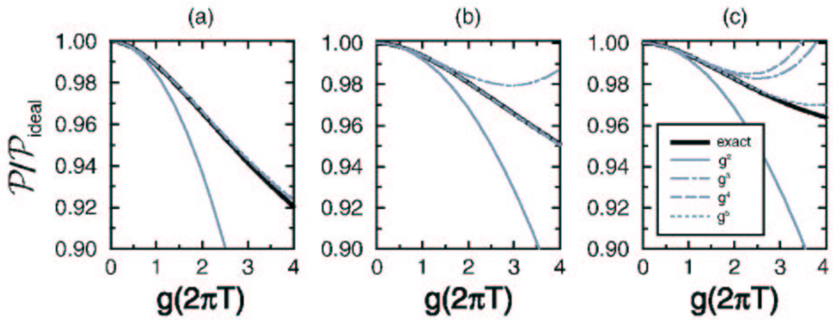


Figure 5. The one-, two and three-loop SPT improved approximations to the pressure as a function of $g(2\pi T)$ for $\mu = 2\pi T$.

[9] AS, 01.

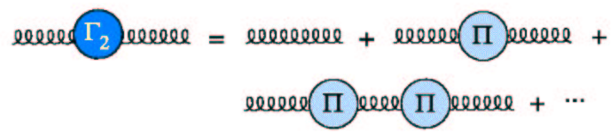
Hard-thermal-loop perturbation theory

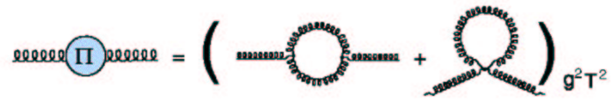
Hard-thermal-loop (HTL) perturbation theory is a reorganization of the perturbative series for QCD which is similar in spirit to screened perturbation theory.

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_{\text{HTL}} + \Delta\mathcal{L}_{\text{HTL}}(g, m_D^2(1-\delta)) \tag{14}$$

The HTL "improvement" term is

$$\mathcal{L}_{\text{HTL}} = -\frac{1}{2}(1-\delta)m_D^2\,\text{Tr}\left(G_{\mu\alpha}\left\langle\frac{y^\alpha y^\beta}{(y\cdot D)^2}\right\rangle_y G^\mu{}_\beta\right) \tag{15}$$





HTLpt Feynman rules - Minkowski space

Gluon Self-energy

The HTL gluon self-energy tensor is

$$\Pi^{\mu\nu}(p) = m_D^2 \left[\mathcal{T}^{\mu\nu}(p,-p) - n^\mu n^\nu\right] \tag{16}$$

with

$$\mathcal{T}^{\mu\nu}(p,-p) = \left\langle y^\mu y^\nu \frac{p\cdot n}{p\cdot y}\right\rangle \qquad y = (1, \hat{\mathbf{y}}) \tag{17}$$

$\Pi^{\mu\nu}$ satisfies

$$\begin{aligned} p_\mu \Pi^{\mu\nu}(p) &= 0 \\ g_{\mu\nu}\Pi^{\mu\nu}(p) &= -m_D^2 \end{aligned} \tag{18}$$

$\Pi^{\mu\nu}$ can be expressed in terms of two scalar functions

$$\begin{aligned} \Pi^{\mu\nu}(p) &= -\Pi_T(p) T_p^{\mu\nu} - \frac{1}{n_p^2}\Pi_L(p) L_p^{\mu\nu} \\ T_p^{\mu\nu} &= g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} - \frac{n_p^\mu n_p^\nu}{n_p^2} \qquad L_p^{\mu\nu} = \frac{n_p^\mu n_p^\nu}{n_p^2} \end{aligned} \tag{19}$$

Both self-energy functions can be written in terms of $\mathcal{T}^{00}$:

$$\begin{aligned} \Pi_T(p) &= \frac{3m_g^2}{(d-1)n_p^2}\left[\mathcal{T}^{00}(p) - 1 + n_p^2\right] \\ \Pi_L(p) &= 3m_g^2\left[1 - \mathcal{T}^{00}(p)\right] \end{aligned} \tag{20}$$

where

$$\mathcal{T}^{00}(p) = \frac{\Gamma(\frac{3}{2}-\epsilon)}{\Gamma(\frac{1}{2})\Gamma(1-\epsilon)}\int_{-1}^{1} dc\,(1-c^2)^{-\epsilon}\frac{p_0}{p_0 - |\mathbf{p}|c} \tag{21}$$

$d = 3$ limit

If we take the limit $d \to 3$ then $\mathcal{T}^{00}$ becomes

$$\mathcal{T}^{00}(p) = \frac{p_0}{2|\mathbf{p}|}\log\frac{p_0 + |\mathbf{p}|}{p_0 - |\mathbf{p}|} \tag{22}$$

Gluon Propagator

We consider general covariant and Coulomb gauge.

$$\Delta^{-1}(p)^{\mu\nu} = \Delta_\infty^{-1}(p)^{\mu\nu} - \frac{1}{\xi} p^\mu p^\nu$$
$$= \Delta_\infty^{-1}(p)^{\mu\nu} - \frac{1}{\xi}\left(p^\mu - p\cdot n\, n^\mu\right)\left(p^\nu - p\cdot n\, n^\nu\right) \quad (23)$$

In the limit $\xi \to \infty$ the inverse propagator reduces to

$$\Delta_\infty^{-1}(p)^{\mu\nu} = -\frac{1}{\Delta_T(p)} T_p^{\mu\nu} + \frac{1}{n_p^2 \Delta_L(p)} L_p^{\mu\nu} \quad (24)$$

where Δ_T and Δ_L are

$$\Delta_T(p) = \frac{1}{p^2 - \Pi_T(p)} \quad (25)$$
$$\Delta_L(p) = \frac{1}{-n_p^2 p^2 + \Pi_L(p)} \quad (26)$$

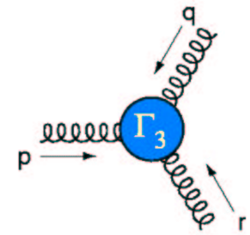
The propagators obtained by inverting the tensors in (23) are

$$\Delta^{\mu\nu}(p) = -\Delta_T(p) T_p^{\mu\nu} + \Delta_L(p) n_p^\mu n_p^\nu - \xi \frac{p^\mu p^\nu}{(p^2)^2} \quad (27)$$
$$= -\Delta_T(p) T_p^{\mu\nu} + \Delta_L(p) n^\mu n^\nu - \xi \frac{p^\mu p^\nu}{\left(n_p^2 p^2\right)^2} \quad (28)$$

Three-gluon vertex

The three-gluon vertex is

$$i\Gamma_{abc}^{\mu\nu\lambda}(p,q,r) = -g f_{abc} \Gamma^{\mu\nu\lambda}(p,q,r)$$



where

$$\Gamma^{\mu\nu\lambda}(p,q,r) = g^{\mu\nu}(p-q)^\lambda + g^{\nu\lambda}(q-r)^\mu + g^{\lambda\mu}(r-p)^\nu$$
$$+3m_g^2 \mathcal{T}^{\mu\nu\lambda}(p,q,r) \quad (29)$$

and

$$\mathcal{T}^{\mu\nu\lambda}(p,q,r) = -\left\langle y^\mu y^\nu y^\lambda \left(\frac{p\cdot n}{p\cdot y\, q\cdot y} - \frac{r\cdot n}{r\cdot y\, q\cdot y}\right)\right\rangle \quad (30)$$

This tensor is totally symmetric in its three indices and traceless in any pair of indices ($g_{\mu\nu}\mathcal{T}^{\mu\nu\lambda} = 0$) and satisfies the "Ward identity"

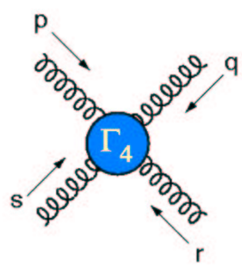
$$q_\mu \mathcal{T}^{\mu\nu\lambda}(p,q,r) = \mathcal{T}^{\nu\lambda}(p+q,r) - \mathcal{T}^{\nu\lambda}(p,r+q) \quad (31)$$

The three-gluon vertex tensor therefore satisfies the Ward identity

$$p_\mu \Gamma^{\mu\nu\lambda}(p,q,r) = \Delta_\infty^{-1}(q)^{\nu\lambda} - \Delta_\infty^{-1}(r)^{\nu\lambda} \quad (32)$$

Four-gluon vertex

The four-gluon vertex is



$$
\begin{aligned}
i\Gamma^{\mu\nu\lambda\sigma}_{abcd}(p,q,r,s) &= \\
& -ig^2\Big\{ f_{abx}f_{xcd}\left(g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\lambda}\right) \\
& \quad -6m_g^2 \mathrm{tr}\left[T^a\left(T^bT^cT^d + T^dT^cT^b\right)\right]\mathcal{T}^{\mu\nu\lambda\sigma}(p,q,r,s)\Big\} \\
& \quad + 2 \text{ cyclic permutations}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{T}^{\mu\nu\lambda\sigma}(p,q,r,s) &= \Big\langle y^\mu y^\nu y^\lambda y^\sigma \Big(\frac{p\cdot n}{p\cdot y\; q\cdot y\; (q+r)\cdot y} \\
& + \frac{(p+q)\cdot n}{q\cdot y\; r\cdot y\; (r+s)\cdot y} + \frac{(p+q+r)\cdot n}{r\cdot y\; s\cdot y\; (s+p)\cdot y}\Big)\Big\rangle
\end{aligned} \quad (33)
$$

This tensor is totally symmetric in its four indices and traceless in any pair of indices ($g_{\mu\nu}\mathcal{T}^{\mu\nu\lambda\sigma} = 0$) and satisfies the "Ward identity"

$$
q_\mu \mathcal{T}^{\mu\nu\lambda\sigma}(p,q,r,s) = \mathcal{T}^{\nu\lambda\sigma}(p+q,r,s) - \mathcal{T}^{\nu\lambda\sigma}(p,r+q,s) \quad (34)
$$

Plasma Effects

(1) Massive quasiparticles

(2) Screening (static $\omega \to 0$ limit of the propagator)

Coulomb exchange - Debye screening

$$
\frac{1}{k^2} \longrightarrow \frac{1}{k^2 + 3m_g^2} \quad (35)
$$

Magnetic exchange - Dynamical screening

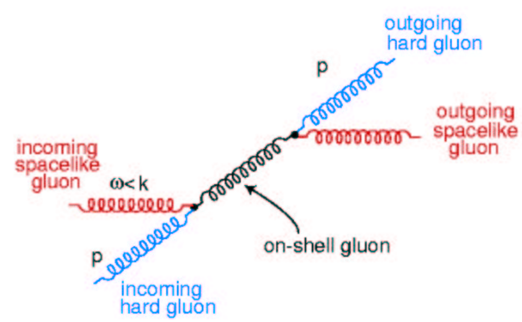
$$
\frac{1}{k^2} \longrightarrow \frac{1}{k^2 + i\frac{3}{4}\pi m_g^2\omega/k} \quad (36)
$$

Screening at scale $r \sim \left(\frac{3\pi}{4}m^2\omega\right)^{-\frac{1}{3}}$.

No screening at $\omega = 0$ $\implies$ "magnetic mass problem".

(3) Landau Damping

Imaginary contribution to propagator for $-k < \omega < k$



One-loop calculation of the QCD free energy

In the imaginary-time formalism, the renormalized one-loop free energy can be written as

$$\mathcal{F}_{\rm HTL} = (N_c^2-1)\left[\sum\!\!\!\!\!\!\!\int_P \mathrm{Tr}\log\Delta^{-1}(P) - \sum\!\!\!\!\!\!\!\int_P \log\Delta_{\rm ghost}^{-1}(P) + \Delta_0\mathcal{E}_0\right]$$

with

$$\Delta_0\mathcal{E}_0 = \frac{N_c^2-1}{128\pi^2\epsilon}m_D^4 \tag{37}$$

After taking into account cancellations from the ghost term and simplifying

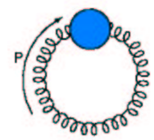
$$\mathcal{F}_{\rm LO} = (N_c^2-1)\left[(d-1)\mathcal{F}_T + \mathcal{F}_L + \Delta_0\mathcal{E}_0\right] \tag{38}$$

where $\mathcal{F}_T$ and $\mathcal{F}_L$ are the unrenormalized transverse and longitudinal free energies respectively.

$$\begin{aligned}\mathcal{F}_T &= \frac{\mu^{3-d}}{2\beta}\sum_n\int_k \log\left[k^2+\omega_n^2+\Pi_T(i\omega_n,k)\right]\\ \mathcal{F}_L &= \frac{\mu^{3-d}}{2\beta}\sum_n\int_k \log\left[k^2-\Pi_L(i\omega_n,k)\right]\end{aligned} \tag{39}$$

Separation into hard and soft contributions

Hard momenta



$$\begin{aligned}\mathcal{F}_g^{(h)} &= \frac{d-1}{2}\sum\!\!\!\!\!\!\!\int_P \log(P^2) + \frac{1}{2}m_D^2\sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2}\\ &- \frac{1}{4(d-1)}m_D^4\sum\!\!\!\!\!\!\!\int_P\left[\frac{1}{(P^2)^2} - 2\frac{1}{p^2P^2} - 2d\frac{1}{p^4}\mathcal{T}_P\right.\\ &\left. + 2\frac{1}{p^2P^2}\mathcal{T}_P + d\frac{1}{p^4}(\mathcal{T}_P)^2\right] + \mathcal{O}(m_D^6)\end{aligned}$$

Soft momenta

$$\mathcal{F}_g^{(s)} = \frac{1}{2}T\int_{\mathbf{p}}\log(p^2+m_D^2)$$

LO result

$$\begin{aligned}\frac{\Omega_{\rm LO}}{\mathcal{F}_{\rm ideal}} &= 1 - \frac{15}{2}\hat{m}_D^2 + 30\hat{m}_D^3\\ &+ \frac{45}{4}\left(\log\frac{\hat{\mu}}{2} - \frac{7}{2} + \gamma + \frac{\pi^2}{3}\right)\hat{m}_D^4 + \mathcal{O}(\hat{m}_D^6)\end{aligned}$$

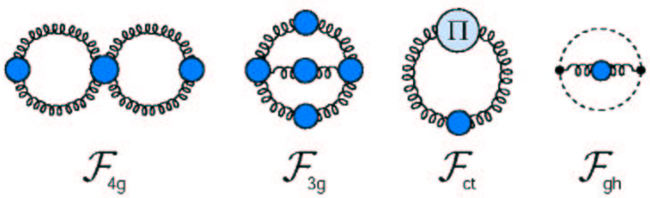
Resummed perturbative result - $\hat{\mu}=1$, $N_c=3$

$$\begin{aligned}\frac{\mathcal{F}_{\rm QCD}}{\mathcal{F}_{\rm ideal}} &= 1 - \frac{15}{4}\frac{\alpha_s}{\pi} + 30\left(\frac{\alpha_s}{\pi}\right)^{3/2} + \frac{135}{2}\left(\log\frac{\alpha_s}{\pi} + 3.51\right)\left(\frac{\alpha_s}{\pi}\right)^2\\ &-799.2\left(\frac{\alpha_s}{\pi}\right)^{5/2} + \mathcal{O}(\alpha_s^3)\end{aligned}$$

Two-loop calculation

The thermodynamic potential at next-to-leading order in HTL perturbation theory is given by

$$\Omega_{\rm NLO} = \mathcal{F}_{\rm LO} + (N_c^2-1)\left[\mathcal{F}_{4g}+\mathcal{F}_{3g}+\mathcal{F}_{\rm ct}+\mathcal{F}_{\rm gh}\right] + \Delta_1\mathcal{E}_0 + \Delta_1 m_D^2 \frac{\partial}{\partial m_D^2}\Omega_{LO}$$



$$\mathcal{F}_{3g} = \frac{N_c}{12}g^2 \sum\!\!\!\!\!\!\int_{PQ} \Gamma^{\mu\lambda\rho}(P,Q,R)\Gamma^{\nu\sigma\tau}(P,Q,R)\Delta^{\mu\nu}(P)\Delta^{\lambda\sigma}(Q)\Delta^{\rho\tau}(R)$$

$$\mathcal{F}_{4g} = \frac{N_c}{8}g^2 \sum\!\!\!\!\!\!\int_{PQ} \Gamma^{\mu\nu,\lambda\sigma}(P,-P,Q,-Q)\Delta^{\mu\nu}(P)\Delta^{\lambda\sigma}(Q)$$

$$\mathcal{F}_{gh} = \frac{N_c}{2}g^2 \sum\!\!\!\!\!\!\int_{PQ} \frac{1}{Q^2}\frac{1}{R^2}Q^\mu R^\nu \Delta^{\mu\nu}(P) = \frac{N_c}{2}g^2 \sum\!\!\!\!\!\!\int_{PQ} \frac{1}{q^2}\frac{1}{r^2}(Q^\mu - Q\cdot n\, n^\mu)(R^\nu - R\cdot n\, n^\nu)\Delta^{\mu\nu}(P)$$

$$\mathcal{F}_{\rm ct} = \frac{1}{2}\sum\!\!\!\!\!\!\int_{P} \Pi^{\mu\nu}(P)\Delta^{\mu\nu}(P)$$

Mass Expansions and separating the hard and soft contributions

Counterterm

The counterterm can be evaluated in exactly the same way as the one-loop graph by considering the contributions from hard and soft modes and expanding in m_D/T up to order m_D^4.

$$m_D^2,\ m_D^3,\ \text{and}\ m_D^4$$

$\mathcal{F}_{3g+4g+gh}$

For the two loop graphs there are three possibilities that need to be considered

(hh) All momenta flowing through the loops are hard.

$$\alpha_s \text{ and } \alpha_s m_D^2$$

(ss) All momenta flowing through the loops are soft.

$$\alpha_s m_D \text{ and } \alpha_s m_D^3$$

(hs) One of the momenta flowing through the loops soft and the others are hard.

$$\alpha_s m_D^2$$

Renormalization

The renormalization contributions at first order in δ are

$$\Delta_1 \mathcal{E}_0 = -\frac{N_c^2-1}{64\pi^2\epsilon} m_D^4 \qquad \text{and} \qquad \Delta_1 m_D^2 = -\frac{11}{4\epsilon}\frac{N_c\alpha_s}{3\pi} \tag{40}$$

NLO result

$$\begin{aligned}\frac{\Omega_{\rm NLO}}{\mathcal{F}_{\rm ideal}} &= 1 - 15\hat{m}_D^3 - \frac{45}{4}\left(\log\frac{\hat\mu}{2} - \frac{7}{2} + \gamma + \frac{\pi^2}{3}\right)\hat{m}_D^4 \\ &+ \frac{N_c\alpha_s}{3\pi}\left[-\frac{15}{4} + 45\hat{m}_D - \frac{165}{4}\left(\log\frac{\hat\mu}{2} - \frac{36}{11}\log\hat{m}_D - 2.001\right)\hat{m}_D^2 \right. \\ &\left. + \frac{495}{2}\left(\log\frac{\hat\mu}{2} + \frac{5}{22} + \gamma\right)\hat{m}_D^3\right]\end{aligned} \tag{41}$$

HTLpt Perturbative Limit: $\hat{m}_D^2 \to \frac{\alpha_s}{\pi}$, $\hat\mu = 1$, $N_c = 3$

$$\begin{aligned}\frac{\mathcal{F}_{\rm NLOpert}}{\mathcal{F}_{\rm ideal}} &= 1 - \frac{15}{4}\frac{\alpha_s}{\pi} + 30\left(\frac{\alpha_s}{\pi}\right)^{3/2} + \frac{135}{2}\left(\log\frac{\alpha_s}{\pi} + 1.70\right)\left(\frac{\alpha_s}{\pi}\right)^2 \\ &+ 27.56\left(\frac{\alpha_s}{\pi}\right)^{5/2} + \mathcal{O}(\alpha_s^3)\end{aligned}$$

Resummed perturbative result - $\hat\mu = 1$, $N_c = 3$

$$\begin{aligned}\frac{\mathcal{F}_{\rm QCD}}{\mathcal{F}_{\rm ideal}} &= 1 - \frac{15}{4}\frac{\alpha_s}{\pi} + 30\left(\frac{\alpha_s}{\pi}\right)^{3/2} + \frac{135}{2}\left(\log\frac{\alpha_s}{\pi} + 3.51\right)\left(\frac{\alpha_s}{\pi}\right)^2 \\ &- 799.2\left(\frac{\alpha_s}{\pi}\right)^{5/2} + \mathcal{O}(\alpha_s^3)\end{aligned}$$

Gap equation

The gap equation which determines m_D is obtained by differentiating $\Omega_{\rm NLO}$ with respect to m_D and setting this derivative equal to zero yielding:

$$\begin{aligned}&\hat{m}_D^2\left[1 + \left(\log\frac{\hat\mu}{2} - \frac{7}{2} + \gamma + \frac{\pi^2}{3}\right)\hat{m}_D\right] = \\ &\frac{N_c\alpha_s}{3\pi}\left[1 - \frac{11}{6}\left(\log\frac{\hat\mu}{2} - \frac{36}{11}\log\hat{m}_D - 3.637\right)\hat{m}_D \right. \\ &\left. + \frac{33}{2}\left(\log\frac{\hat\mu}{2} + \frac{5}{22} + \gamma\right)\hat{m}_D^2\right].\end{aligned} \tag{42}$$

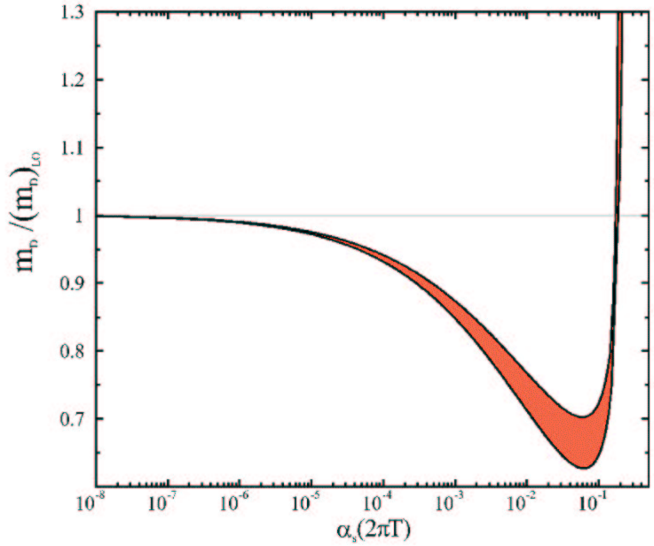


Figure 6. Solution to gap equation for m_D as a function of $\alpha_s(2\pi T)$.

Final HTLpt LO and NLO Results for Free Energy (Pressure)

Plugging the numerical solution to the gap equation above into the expression for $\Omega_{\rm NLO}$ we obtain

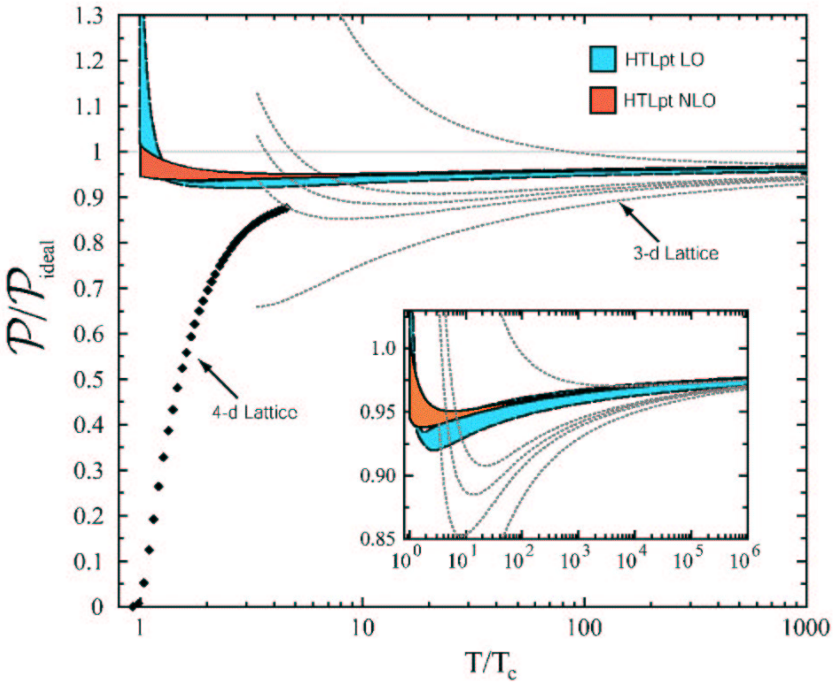


Figure 7. Comparison of HTLpt LO and NLO results with 3-d and 4-d lattice results.

3-d lattice results from K. Kajantie et al, 01.

4-d lattice results from G. Boyd et al, 95/96.

Perturbative Expansion of Gap Equation

$$\hat{m}_D = \left(\frac{\alpha_s}{\pi}\right)^{\frac{1}{2}}\left[1+\frac{3}{2}\left(\log\frac{\alpha_s}{\pi}-\frac{17}{3}\log\frac{\hat{\mu}}{2}+2.10\right)\left(\frac{\alpha_s}{\pi}\right)^{\frac{1}{2}}\right] \qquad (43)$$

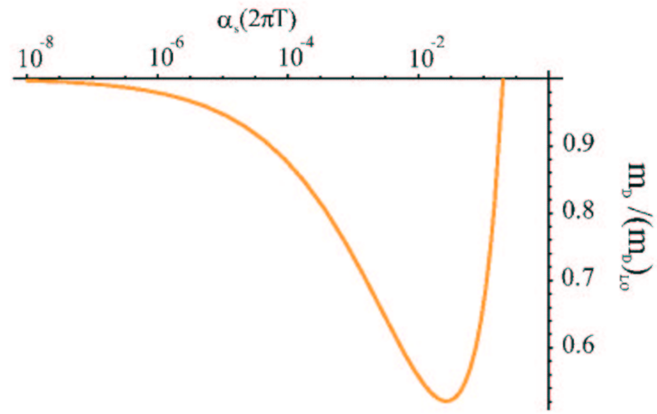
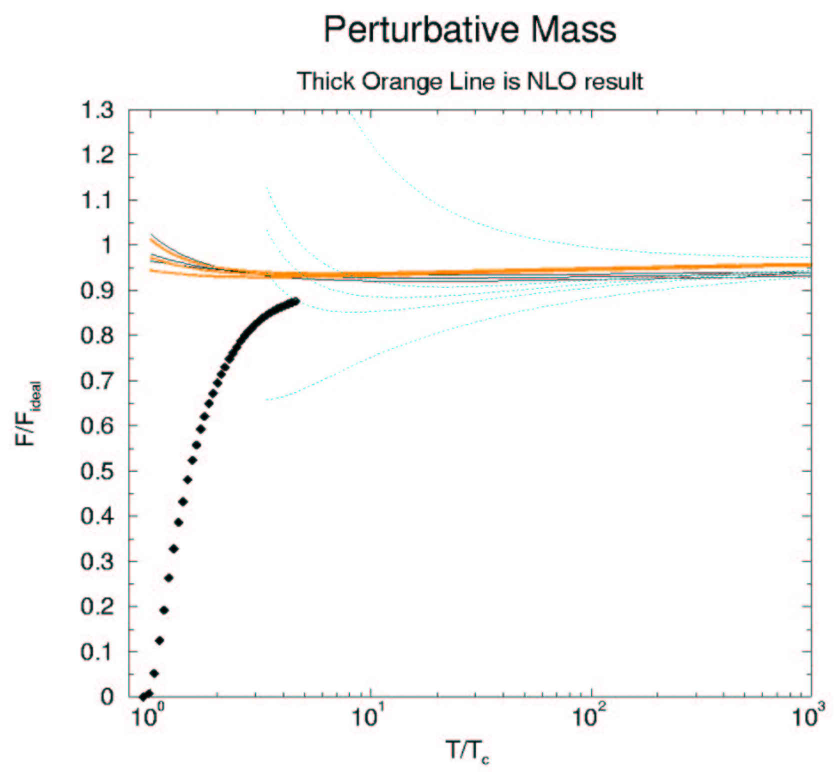
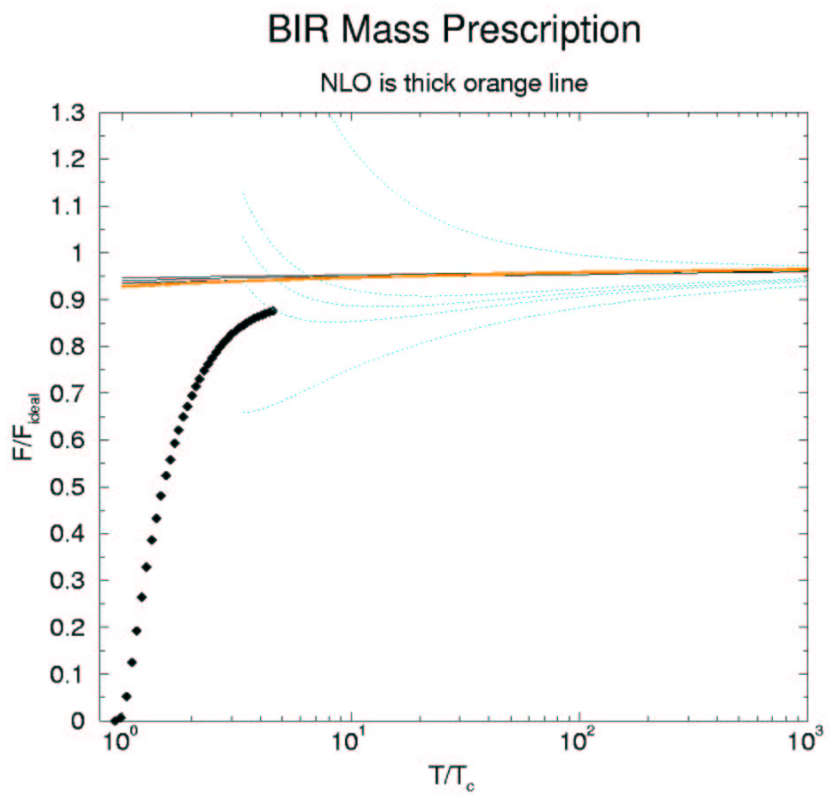


Figure 8. Perturbative m_D given above as a function of $\alpha_s(2\pi T)$.

Inserting this into the expression for $\Omega_{\rm NLO}$ we obtain the following result which includes all HTLpt contributions to the free energy up to $\mathcal{O}(\alpha_s^{5/2})$.

$$\frac{\mathcal{F}_{\rm NLOpert}}{\mathcal{F}_{\rm ideal}} = 1-\frac{15}{4}\frac{\alpha_s}{\pi}+30\left(\frac{\alpha_s}{\pi}\right)^{3/2}+\frac{135}{2}\left(\log\frac{\alpha_s}{\pi}+1.70\right)\left(\frac{\alpha_s}{\pi}\right)^2$$
$$+796.1\left(1+0.701\log\frac{\alpha_s}{\pi}+0.127\log^2\frac{\alpha_s}{\pi}\right)\left(\frac{\alpha_s}{\pi}\right)^{5/2}+\mathcal{O}(\alpha_s^3)$$





$$\hat{m}_D^2 = \frac{\alpha_s}{\pi}\left(1 - 6\hat{m}_D\right) \tag{44}$$

Conclusions and Outlook

Scalar field theory

- We have calculated the three-loop free energy (pressure), two-loop screening mass, and three-loop entropy for a massless scalar quantum field theory using screened perturbation theory.
- The results of screened perturbation theory calculation show that the convergence of the thermodynamic functions is dramatically improved over resummed perturbation theory.

QCD

- We have obtained a completely analytic expression for the QCD free energy at NLO in HTLpt.
- Final result is free from singularites, only requiring vacuum and mass renormalizations.
- The NLO result from HTLpt does not agree with 3-d or 4-d lattice predictions. Correction is 45% of lattice result at 5 T_c. Correction is 59% of lattice result at 1000 T_c.
- However, the approximation seems to be under control (converged) and the variation with respect to the renormalization scale is very small.
- Expansion in $\hat{m}$?
- Physics? Magnetic screening, Toplogical modes, Z_{N} symmetry
- This method can be extended to include quarks (under way), finite chemical potential and real-time processes.