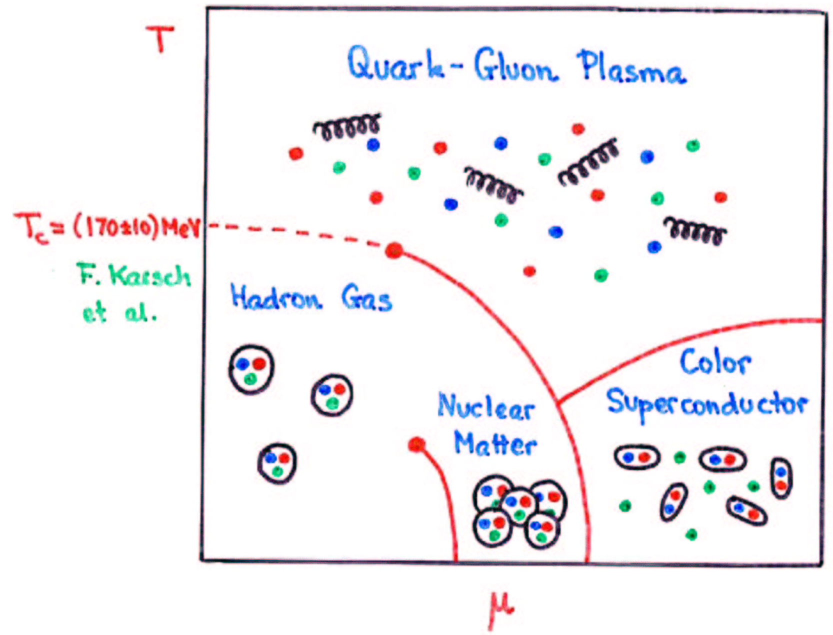
# Simulating dense Matter: Progress and Problems

U.-J. Wiese* (Bern University)

- Lattice QCD at non-zero quark density
- General nature of the complex action problem
- Recent progress for small $\mu$ (RHIC regime)
  - Multiparameter reweighting
  - Taylor expansion around $\mu = 0$
  - Analytic continuation in $\nu = i\mu$
- Strategy for solving complex action problems
- Meron-Cluster algorithm
  - Static quarks at arbitrarily large $\mu$
  - D-Theory for 2-d O(3) model with $\mu \neq 0$
- Conclusions

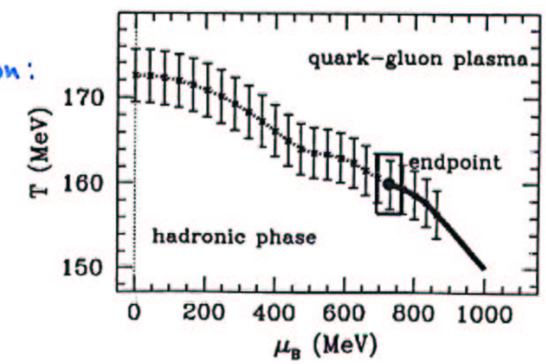
* on leave from MIT

## The QCD Phase Diagram




First step into $\mu$-direction:

Systematic errors need more detailed investigation.




Z. Fodor, S.D. Katz, nucl-th/0201071

Fig. 4: The T-$\mu$ diagram. Direct results are given with errorbars. Dotted line shows the

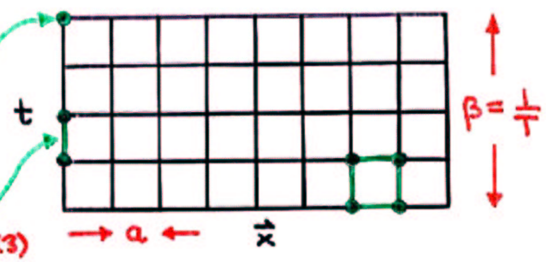
# Lattice Regularization

Quark fields:

$\bar{\Psi}(x), \Psi(x)$

Gluon fields:

$A_\mu(x) \to U_\mu(x) \in SU(3)$




Partition function as a path integral:

$$Z = \mathrm{Tr}\,\exp(-\beta(\mathbb{H} - \mu \mathbb{B}))$$

$$= \int \mathcal{D}A\, \mathcal{D}\bar{\Psi}\, \mathcal{D}\Psi \exp(-S[A, \bar{\Psi}, \Psi])$$

$$= \int \mathcal{D}A \det \mathcal{M}[A] \exp(-S[A])$$

Boundary conditions in Euclidean time:

$$\bar{\Psi}(\vec{x}, \beta) = e^{-\beta\mu}\, \bar{\Psi}(\vec{x}, 0), \quad \Psi(\vec{x}, \beta) = e^{\beta\mu}\, \Psi(\vec{x}, 0)$$

$$A_\mu(\vec{x}, \beta) = {}^{\Omega}A_\mu(\vec{x}, 0)$$

Gluon action:

$$S[A] = \int_0^\beta dt \int d^3x \, \frac{1}{2g^2} \mathrm{Tr}\, F_{\mu\nu} F_{\mu\nu} \to \sum_{\Box} \frac{1}{g^2} \mathrm{Tr}\, U_{\Box}$$

Fermion determinant:

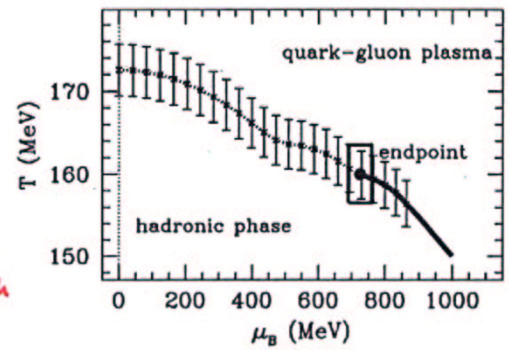
$\det \mathcal{M}[A] \in \mathbb{C}$ for $\mu \neq 0 \Rightarrow$ importance sampling fails

(however: D. Hong, S. Hsu, hep-ph/0202236)

# Simulating at $(T, \mu) \approx (T_c, 0)$

Multiparameter reweighting Z. Fodor, S.D. Katz, nucl-th/0201071

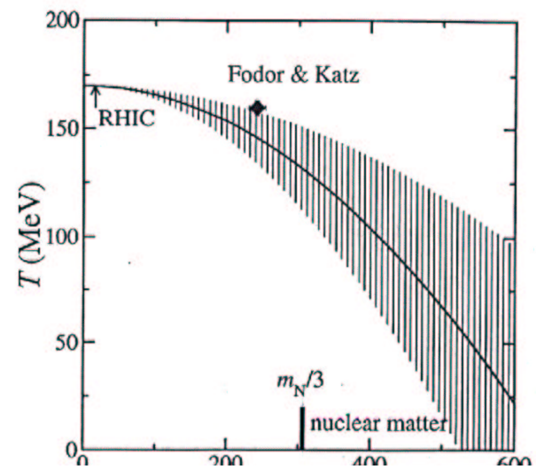
limited to small lattices: $4^4, 6^3 \cdot 4, 8^3 \cdot 4$




location of endpoint should be checked with other methods

Taylor expansion around $\mu = 0$: S. Hands, F. Karsch, et al.

robust method for small $\mu$, not limited to small volumes



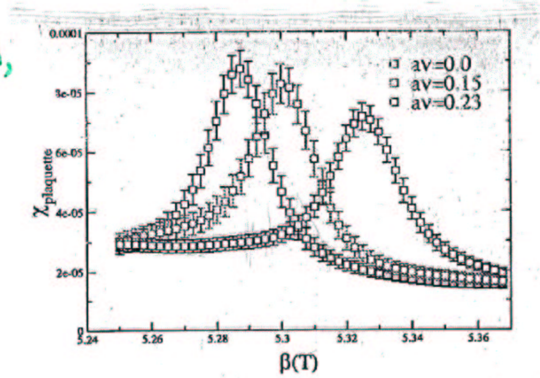

At RHIC $T_c$ is practically the same as for $\mu = 0$.

## Analytic Continuation in $\nu = i\mu$

P. de Forcrand, O. Philipsen.

Crossover:

$$\left.\frac{\partial \chi}{\partial T}\right|_{\mu, T_c} = 0$$

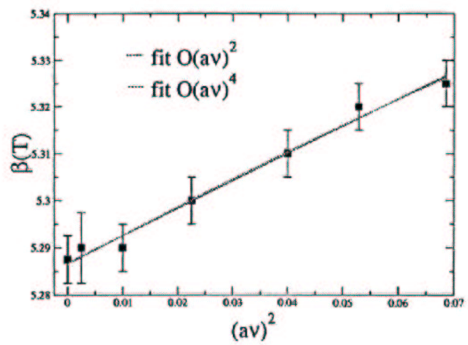



Analyticity:

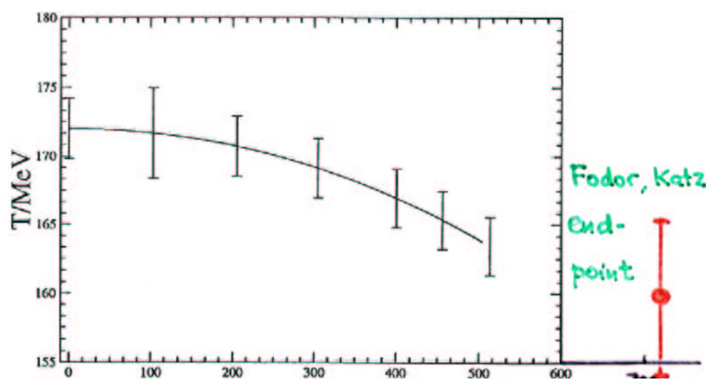
$T_c(\mu)$ is an analytic function

Continuation:

Simulate $\nu = i\mu$ ($\det \mathcal{M}[A] \in \mathbb{R}$) and analytically continue to $\mu$




Three different methods give consistent results for small $\mu$.




Fodor, Katz

end-point

## What is the Problem?

Idea of importance sampling: generate a Markov chain of configurations $[n]$ according to their Boltzmann weight, fails because $\mathrm{Sign}[n]\, e^{-S[n]} \in \mathbb{C}$ cannot be interpreted as a probability.

Naive "solution": include $\mathrm{Sign}[n]$ in observables:

$$\langle O \rangle_f = \frac{1}{Z_f} \sum_n O[n]\, \mathrm{Sign}[n]\, e^{-S[n]}$$

$$= \frac{\langle O\, \mathrm{Sign} \rangle_b}{\langle \mathrm{Sign} \rangle_b}, \quad Z_f = \sum_n \mathrm{Sign}[n]\, e^{-S[n]}$$

and apply importance sampling to "bosonic" ensemble:

$$Z_b = \sum_n e^{-S[n]}$$

This fails because

$$\langle \mathrm{Sign} \rangle_b = \frac{1}{Z_b} \sum_n \mathrm{Sign}[n]\, e^{-S[n]} = \frac{Z_f}{Z_b} \sim e^{-\beta V \Delta f}$$

is exponentially small.

# A General Strategy (not always applicable)

General strategy: Cancel analytically all negative contributions $\mathrm{Sign}[n] = -1$ with other positive contributions $\mathrm{Sign}[n'] = 1$, so that effectively $\mathrm{Sign} = 0$ for a cancelling pair of configurations or $\mathrm{Sign} = 1$ for an uncancelled configuration $\Rightarrow \mathrm{Sign}^2 = \mathrm{Sign}$

The statistical error estimate:

$$\frac{\Delta \mathrm{Sign}}{\langle \mathrm{Sign} \rangle_b} = \frac{\sqrt{\langle \mathrm{Sign}^2 \rangle_b - \langle \mathrm{Sign} \rangle_b^2}}{\sqrt{N}\, \langle \mathrm{Sign} \rangle_b} \sim \frac{1}{\sqrt{N}\sqrt{\langle \mathrm{Sign} \rangle_b}} \sim e^{\beta V \Delta f/2}$$

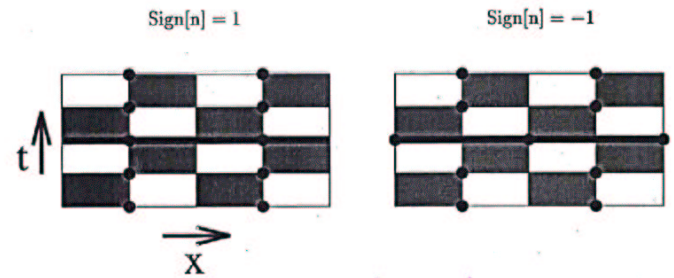
Naively, one would still need $N \sim e^{\beta V \Delta f}$ configurations, but now one can apply true importance sampling and generate only the uncancelled configurations with $\mathrm{Sign} = 1$.

Note that $\langle n_i | e^{-\epsilon H} | n_{i+1} \rangle \in \mathbb{R}_{\geq 0}$ if $H$ is diagonal in the basis $|n\rangle$. Of course, if we could diagonalize the Hamiltonian analytically, we would not even worry about

# Meron-Cluster Algorithm

Chandrasekharan, Wiese
Phys. Rev. Lett. 83 (1999) 3116

Connect fermionic variables by bonds to form clusters and then flip all variables in a cluster with probability $\frac{1}{2}$.




Flipping a meron-cluster leads to a cancellation of signs.

Configuration space is enlarged by bond variables, which represent constraints on fermionic variables.

| weight | configuration | break-ups |
|---|---|---|
| $\exp\left(-\frac{\epsilon G}{2}\right)$ | | A, D, E |
| $\cosh\left(\frac{\epsilon}{2}\right)$ | | A, B, C |
| $\sinh\left(\frac{\epsilon}{2}\right)$ | | B, D |

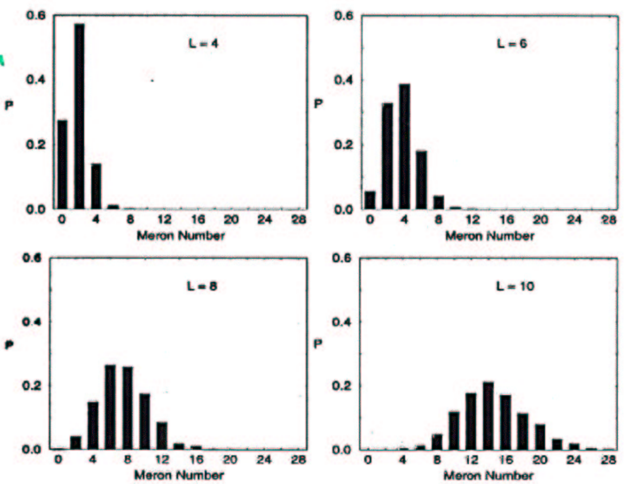
Table 1: *Cluster break-ups of various plaquette configurations together with their relative probabilities $A, B, \ldots, E$. The dots represent occupied sites and the fat lines are the cluster connections.*

# Improved Estimator for Sign

A configuration containing $N_c$ clusters is a member of a subensemble of $2^{N_c}$ equally probable configurations. One can analytically average over the subensemble:

$$\langle \mathrm{Sign} \rangle_{2^{N_c}} = \begin{cases} 0 & \text{if there are some meron-clusters} \\ 1 & \text{in the zero-meron sector} \end{cases}$$

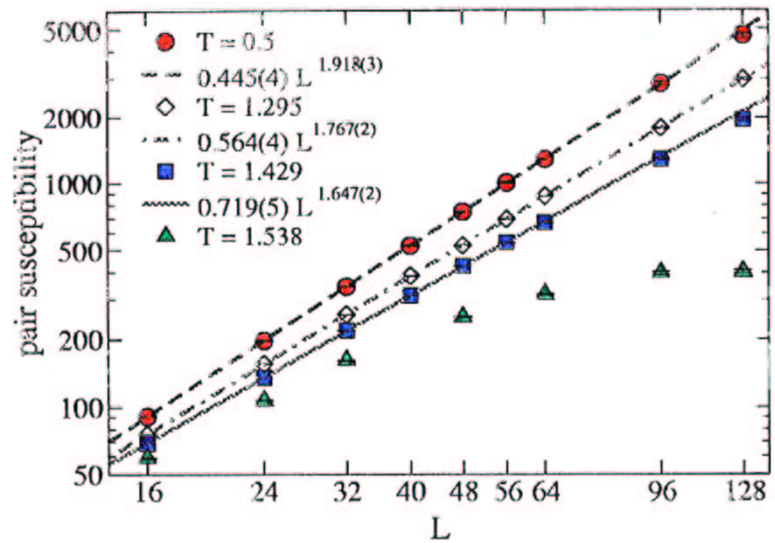
Distribution of the number of meron-clusters for increasing physical volumes

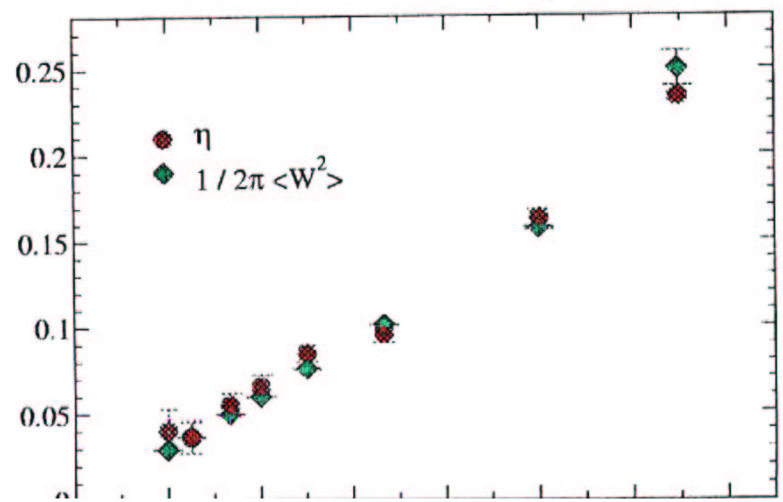



Typical observables get contributions only from few-meron sectors. Restricting the simulation to these sectors solves the sign problem completely. In practice improvement

# Attractive Hubbard Model

Chandrasekharan, Osborn



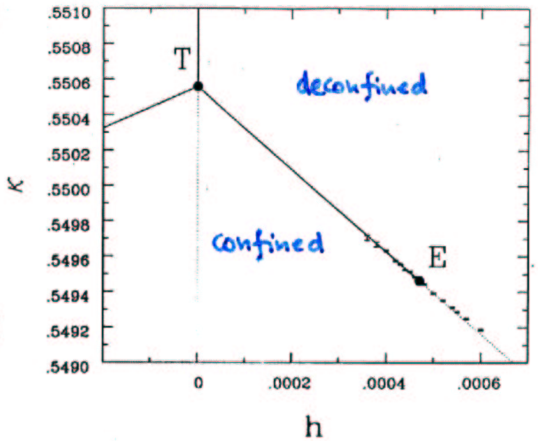



Alford, Chandrasekharan, Cox, Wiese

# QCD with Heavy Quarks at $\mu \neq 0$

Partition function: $Z = \int \mathcal{D}A\, e^{-S[A]}\, e^{h \int d^3x\, \Phi(x)},\ h = e^{\beta(\mu - M)}$

Polyakov loop: $\Phi(x) = \mathrm{Tr}\, \mathcal{P} \exp \int_0^\beta dt\, A_t(\vec{x},t) \in \mathbb{C}$

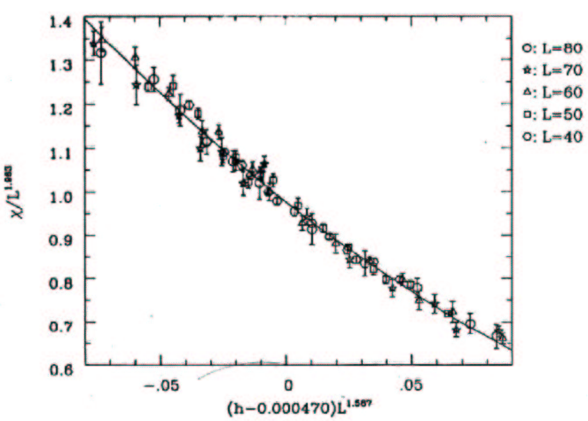



Potts model approximation

$\Phi(x) \in \mathbb{Z}(3)$

Original Swendsen-Wang cluster algorithm naturally extends to a meron-cluster algorithm.

QCD phase diagram with critical endpoint E.



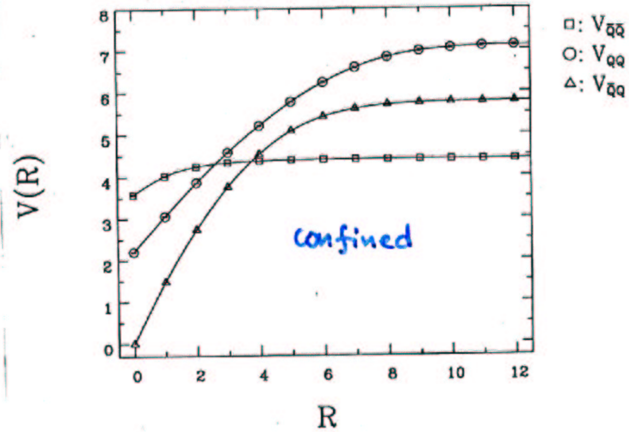

QCD at the critical endpoint is in the universality class of the 3-d Ising model.

Related work:
Karsch and Stickan,
Engels, Kaczmarek, Karsch

# Potentials between Static Sources

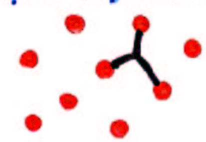
Potentials follow from Polyakov loop correlators:

$\langle \Phi(0)^* \Phi(R) \rangle = \exp(-\beta V_{\bar{Q}Q}(R))$

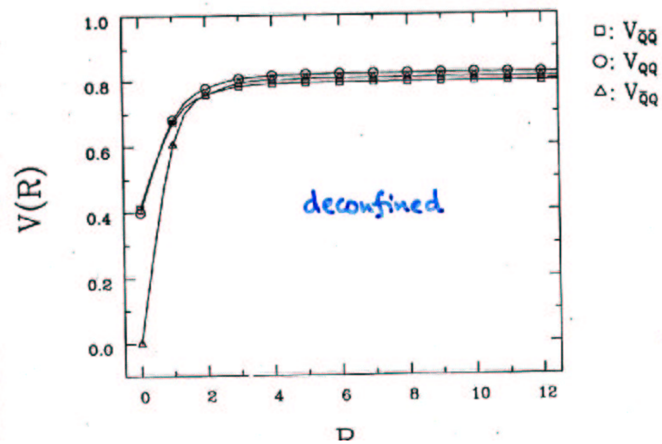



External antiquarks form mesons with background quarks:

while external quarks form baryons



and thus cost more energy in the confined regime.



# Spin Ladders in a Magnetic Field

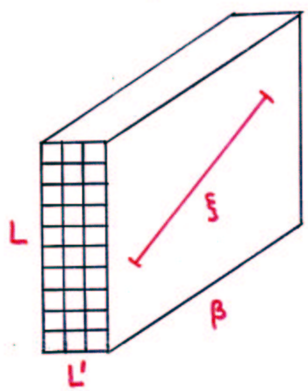
S. Chandrasekharan, B. Scarlet
U.-J.W., cond-mat/9909451

Hamilton operator:

$$H = J \sum_{\langle xy \rangle} \vec{S}_x \cdot \vec{S}_y + \vec{B} \cdot \sum_x \vec{S}_x$$

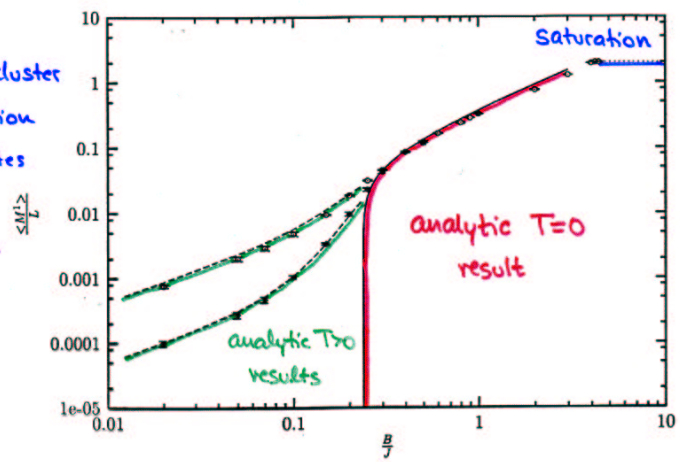
Correlation length:

$$\xi \sim \exp(2\pi \rho_s L'/c) \gg L'$$




Via dimensional reduction a spin ladder in a magnetic field turns into a 2-d field theory with non-zero chemical potential.

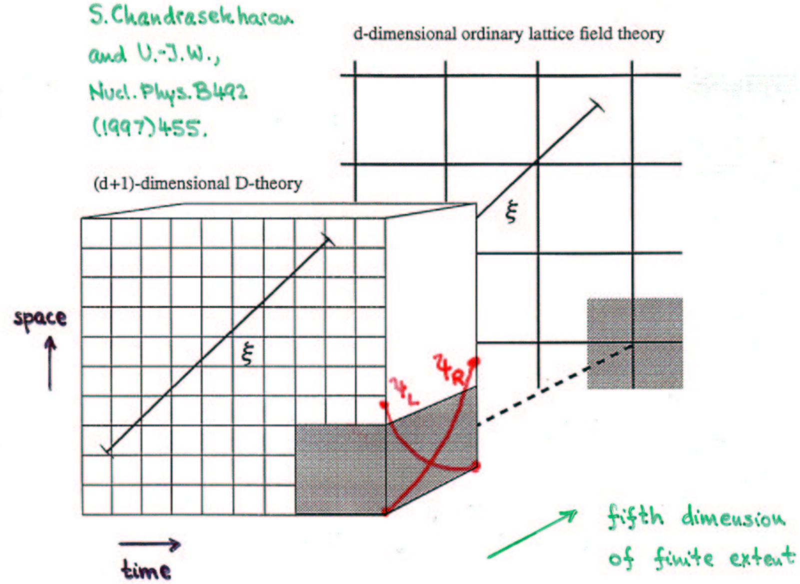
Meron-cluster simulation eliminates sign problem.




# D-Theory: Quantum links

Wilson's classical links are replaced by quantum links, just like classical spins can be replaced by quantum spins.

S. Chandrasekharan and U.-J.W., Nucl. Phys. B492 (1997) 455.




Hadron mass: $m = \frac{1}{\xi}$

Confinement results from squeezing the fifth direction.

A new way to think about QCD via dimensional reduction of discrete variables (D-Theory).

# Conclusions

- Taylor expansion and analytic continuation methods applied to standard lattice QCD can access the small $\mu$ relevant for RHIC.
- Preliminary results with the multiparameter reweighting method suggest that the critical endpoint may be at large $\mu$ (future GSI?).
- Large $\mu$ and small $T$ have a very severe sign problem and require more powerful methods.
- The meron-cluster algorithm has led to a complete solution of severe sign problems in a variety of models.
- The D-theory formulation of field theory may make the meron-cluster algorithm applicable to QCD.