MODULAR FEATURES OF SUPERSTRING SCATTERING AMPLITUDES

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SCATTERING AMPLITUDES AND BEYOND

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General setting

To what extent do duality and supersymmetry constrain theories with a large amount of supersymmetry?

- Maximal supergravity/Type II string theory;
- ½-maximal supergravity/heterotic, type I string theory.

The low energy expansion of string amplitudes

Consider narrowly-focused aspects of the low energy expansion of closed string theory obtained from maximally supersymmetric closed string scattering amplitudes.

- **Explicit features of low order Type II string perturbation theory**

With: Eric D'Hoker; Pierre Vanhove; Omer Gurdogan

Mathematical connections: **Modular invariants of Riemann surfaces; Multiple-Zeta Values** and their **elliptic generalisations; Segal Modular Forms, etc**

Part of a broader programme

earlier work involving:
Stephen Miller; Don Zagier; Boris Pioline; Jorge Russo; Rudolfo Russo; Carlos Mafra; Oliver Schlotterer; Anirban Basu; Sav Sethi; Michael Gutperle, …...

- **Non-perturbative features of string amplitudes**

Constraints imposed by SUSY, Duality, Unitarity

Connects perturbative with non-perturbative effects

Modular Forms; Automorphic forms for higher-rank groups; ….

Coefficients of BPS interactions encoding BPS microstate-counting

- **Ultra-violet properties of supergravity**

How do field theory UV divergences arise in low energy limit?

Four-Graviton Scattering in Type IIB String Theory

$$A^{(4)}(\epsilon_r, k_r; \Omega) = \mathcal{R}^4\, T^{(4)}(s,t,u;\Omega)$$

$s = -2\, k_1 \cdot k_2$
$t = -2\, k_1 \cdot k_4$
$u = -2\; k_1 \cdot k_3$

$\mathcal{R}$ linearized curvature $\sim k_\mu\, k_\nu\, \epsilon_{\rho\sigma}$

One complex modulus $\Omega = \Omega_1 + i\Omega_2$ $\quad \Omega_2 = \frac{1}{g} = e^{-\phi}$

inverse string coupling constant

Symmetric function of Mandelstam invariants s, t, u (with $s+t+u=0$).

Has an expansion in power series of $\sigma_2 = s^2+t^2+u^2$ and $\sigma_3 = s^3+t^3+u^3$

(NON-ANALYTIC PIECES ARE ESSENTIAL, BUT WILL BE IGNORED IN THIS TALK)

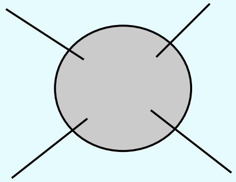
$$T(s,t,u;\Omega) = \sum_{p,q} \mathcal{E}_{(p,q)}(\Omega)\, \sigma_2^p\, \sigma_3^q \quad \to \sim s^{2p+3q} + \ldots$$

Coefficients are $SL(2,\mathbb{Z})$ -invariant functions of scalar fields (moduli, or coupling constants).

To what extent can we determine these coefficients?

Boundary data: String Perturbation Theory $\quad \Omega_2 \to \infty \quad (g \to 0)$

TREE-LEVEL ("VIRASORO" AMPLITUDE)



$$A_0^{(4)}(\epsilon_r, k_r) = g^{-2}\,\mathcal{R}^4\, T_0^{(4)}(s,t,u)$$

$\sigma_n = s^n + t^n + u^n$

$$T_0^{(4)} = \frac{1}{stu}\,\frac{\Gamma(1-\alpha' s)\,\Gamma(1-\alpha' t)\,\Gamma(1-\alpha' u)}{\Gamma(1+\alpha' s)\,\Gamma(1+\alpha' t)\,\Gamma(1+\alpha' u)} = \frac{3}{\sigma_3}\exp\left[\sum_{n=1}^{\infty}\frac{2\zeta(2n+1)}{2n+1}{\alpha'}^{2n+1}\sigma_{2n+1}\right]$$

Tree-level SUPERGRAVITY

R^4 $\quad d^4 R^4$ $\quad d^6 R^4$ $\quad d^8 R^4$

$$= \frac{3}{\sigma_3} + 2\zeta(3)\,{\alpha'}^3 + \zeta(5)\,{\alpha'}^5\sigma_2 + \frac{2\,\zeta(3)^2}{3}\,{\alpha'}^6\,\sigma_3 + \frac{\zeta(7)}{2}\,{\alpha'}^7\,\sigma_2^2$$

$$+ \frac{2\,\zeta(3)\,\zeta(5)}{3}\,{\alpha'}^8\,\sigma_2\,\sigma_3 + \frac{\zeta(9)}{4}\,{\alpha'}^8\,\sigma_2^3 + \frac{2}{27}(2\,\zeta(3)^2 + \zeta(9))\,{\alpha'}^9\,\sigma_3^2 + \ldots$$

$s^k R^4 \sim d^{2k} R^4$

$d^{10} R^4$ $\quad d^{12} R^4$

$\sigma_2 = s^2 + t^2 + u^2$

$\sigma_3 = s^3 + t^3 + u^3$

INFINITE SERIES of $d^{2k} R^4$ terms. COEFFICIENTS ARE POWERS OF **ODD RIEMANN** ζ **VALUES** WITH RATIONAL COEFFICIENTS

Generalisation to N-particle scattering involves **MULTI-ZETA VALUES**.

Zeta values and multiple-zeta values — Very brief review

Zeta values:

- Special values of POLYLOGARITHMS $Li_a(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^a}$ $\zeta(a) = Li_a(1)$

Even zeta values $\zeta(2n) = c_n\, \pi^{2n}$ Odd zeta values $\zeta(2n+1)$ transcendental?

Multi-zeta values (MZV's)

- Special values of MULTIPLE POLYLOGARITHMS $Li_{a_1,\ldots,a_r}(z_1,\ldots,z_r) = \sum_{0<k_1<\cdots<k_r} \prod_{\ell=1}^{r} \left(\frac{z_\ell}{k_\ell}\right)^{a_\ell}$

$$\zeta(a_1,\ldots,a_r) = Li_{a_1,\ldots,a_r}(1,\ldots,1)) = \sum_{0<k_1<\cdots<k_r} \prod_{\ell=1}^{r} k_\ell^{-a_\ell}$$

"weight" $w = \sum_{\ell=1}^{r} a_\ell$ "depth" r

- MZV are numbers with algebraic properties inherited from the algebraic properties of multiple polylogarithms – "STUFFLE" and "SHUFFLE" relations.

  e.g. first non-trivial (irreducible) case is weight $w = 8$

$$350\,\zeta(3,5) = 875\,\zeta(6,2) + 240\,\zeta(2)^4 - 1400\,\zeta(3)\,\zeta(5)$$

- THE DIMENSION d_w OF THE SUBSPACE OF MZV'S OF WEIGHT w OVER $\mathbb{Q}$

$$\sum_{w=0}^{\infty} d_w\, x^w = \frac{1}{1-x^2-x^3}$$

N-PARTICLE TREE AMPLITUDES

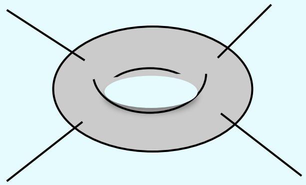
OPEN-STRING TREES: For $N > 4$ coefficients of higher derivative interactions of order α'^n
(Yang-Mills) are MULTIPLE ZETA VALUES with weight n (Stieberger, Broedel, Mafra, Schlotterer)

CLOSED-STRING TREES: For $N > 4$ coefficients are **Single-Valued MZV's** (svMZV's) (Brown)
(gravity) (Schlotterer, Stieberger)

- Special values of **single-valued** multiple polylogarithms – NO MONODROMIES (generalisations of BLOCH-WIGNER dilogarithm $\mathrm{Im}\,(\mathrm{Li}_2(z) + \log(1-z)\,\log|z|)$)
- Kills even zeta values $\zeta_{sv}(2n) = 0$ Also $\zeta_{sv}(2n+1) = 2\zeta(2n+1)$- ODD ZETA'S ONLY
- First non-trivial case is $\zeta_{sv}(3,5,3) = 2\zeta(3,5,3) - 2\zeta(3)\zeta(3,5) - 10\zeta(3)^2\zeta(5)$
  weight $w = 11$
- Role of the KLT construction?

HOW DOES THIS GENERALIZE TO HIGHER GENUS ??

Genus one



$$\mathcal{A}_1^{(4)}(\epsilon_r, k_r) = \frac{\pi}{16}\mathcal{R}^4 \int_{\mathcal{M}_1} \frac{d\tau^2}{y^2} \mathcal{B}_1(s,t,u;\tau)$$

Integral over complex structure $\tau = x + iy$

$$\mathcal{B}_1(s,t,u;\tau) = \frac{1}{y^4}\int_{\Sigma^4} \prod_{i=1}^{4} d^2z \, \exp\left(-\frac{\alpha'}{2}\sum_{i<j} k_i \cdot k_j \, G(z_i, z_j)\right)$$

Vertex operator Corr. function

Green function

Low energy expansion - integrate powers of the genus-one Green function over the torus and over the modulus of the torus – difficult!

(MBG, D'Hoker, Russo, Vanhove)

Expanding in a power series in momenta gives (with $\alpha' = 4$)

$$\frac{1}{w!}\frac{1}{y^4}\int_{\Sigma^4}\prod_{i=1}^{4} d^2 z_i \left(\sum_{0<i<j\le 4} s_{ij} G(z_i - z_j)\right)^w = \sum_i \sigma_2^{p_i}\, \sigma_3^{q_i}\, j^{(p_i,q_i)}(\tau)$$

$$\sum_i (2p_i + 3q_i) = w$$

Coefficients of higher derivative interactions

Modular invariants for surface

Feynman diagrams on toroidal world-sheet

Coefficients of higher derivative interactions:

$$\Xi^{(p,q)} = \int_{\mathcal{M}_1} \frac{d^2\tau}{y^2}\, j^{(p,q)}(\tau)$$

“Modular Graph Functions” (D’Hoker, MBG, Vanhove)

$j^{(p,q)}(\tau)$ is sum of world-sheet Feynman diagrams.

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

Each of these is a modular function - invariant under $SL(2,\mathbb{Z})$

$$a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1$$

The Green function on a torus of complex structure $\tau = x + iy$:

$$G(z) = -\ln\left|\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right|^2 - \frac{\pi}{2y}(z - \bar{z})^2 \qquad z = u + \tau v$$

doubly periodic function

$$= \sum_{(m,n)\neq(0,0)} \hat{G}(m,n) e^{2\pi i(mu - nv)} + 2\,\ln\left(2\pi\,|\eta(\tau)|^2\right)$$

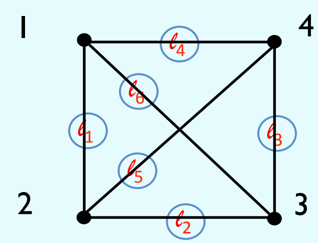
Momentum-space Propagator: integer world-sheet momenta $m, n \in \mathbb{Z}$

$$\hat{G}(m,n) = \frac{y}{|m\tau + n|^2}$$

General contribution to 4-particle amplitude: $i\,,\, j = 1, 2, 3, 4$

Modular function

$$D_{\ell_1,\ell_2,\ell_3,\ell_4;\ell_5,\ell_6} =$$



ℓ_S labels number of propagators on line S

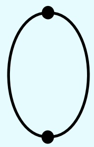
“Weight” $w = \ell_1 + \ell_2 + \cdots + \ell_6$ contributes to $D^{2w}\,\mathcal{R}^4$

World-sheet Feynman diagrams

Multiple sums:

Non-holomorphic SL(2) Eisenstein series

e.g.
1 loop

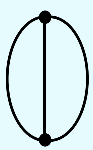
$d^4\mathcal{R}^4$  $= \sum_{(m,n)\neq(0,0)} \frac{y^2}{|m\tau+n|^4} \equiv E_2(\tau)$

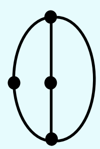
D_2

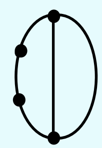
$$E_s(\tau) = \sum_{(m,n)\neq(0,0)} \frac{y^s}{|m\tau+n|^{2s}}$$

e.g.
2 loop $\quad C_{a,b,c}$ sequence $\quad w = a+b+c \quad (w-1)$ vertices $\quad D^{2w}\mathcal{R}^4$

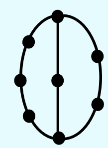
(two-loop diagrams)


$C_{1,1,1} \equiv D_3$
$d^4\mathcal{R}^4$


$C_{2,2,1} \equiv D_{1,1,1,1;1}$
$d^{10}\mathcal{R}^4$


$C_{3,1,1} \equiv D_{2,1,1,1}$
$d^{10}\mathcal{R}^4$

.....


$C_{4,3,2}$
$d^{18}\mathcal{R}^4$

.....

$$C_{a,b,c}(\tau) = \sum_{\substack{(m_r,n_r)\neq(0,0)\\ \sum_i m_i = 0 = \sum_j n_j}} \frac{y^{a+b+c}}{|m_1\tau+n_1|^{2a}|m_2\tau+n_2|^{2b}|m_3\tau+n_3|^{2c}}$$

Direct analysis looks forbidding. But these functions satisfy simple Laplace equations with Laplacian $\Delta = y^2\,(\partial_x^2 + \partial_y^2)$

Simple examples of LAPLACE EQUATIONS :

Eisenstein series

$w = 3$ $$\Delta\,(C_{1,1,1} - E_3) = 0$$

SOLUTION: $$C_{1,1,1} = E_3 + \zeta(3)$$ (also Zagier)

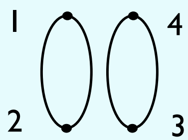
$w = 4$ $$(\Delta - 2)\,C_{2,1,1} = 9E_4 - E_2^2$$

INHOMOGENEOUS LAPLACE EIGENVALUE EQUATIONS

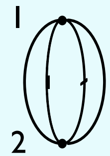
$w = 5$ $$(\Delta - 6)\,C_{3,1,1} = \frac{6}{5}E_5 + \frac{\zeta(5)}{10} + 16E_5 - 4E_2\,E_3\,.$$

$w > 5$ Degeneracy – simultaneous inhomogeneous Laplace eigenvalue equations.

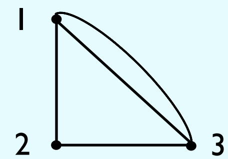
COEFFICIENTS OF $d^8\mathcal{R}^4$ (WEIGHT-4)



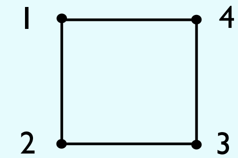
$D_2^2 = E_2^2$



D_4

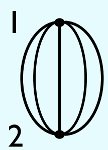


$D_{2,1,1} \equiv C_{2,1,1}$

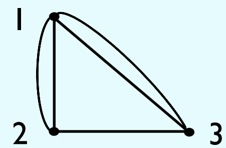


$D_{1,1,1,1}$

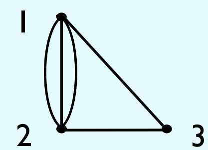
COEFFICIENTS OF $d^{10}\mathcal{R}^4$ (WEIGHT-5)



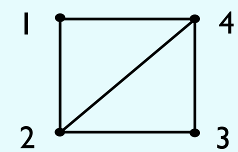
D_5



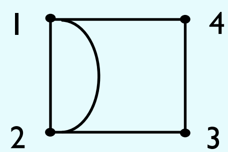
$D_{2,2,1}$



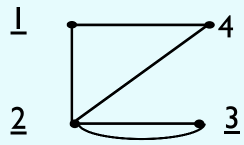
$D_{3,1,1}$



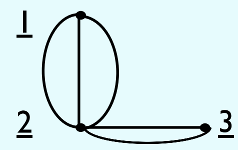
$D_{1,1,1,1;1} \equiv C_{2,2,1}$



$D_{2,1,1,1} \equiv C_{3,1,1}$



$D_{1,1,1}\, D_2$



$D_3\, D_2$

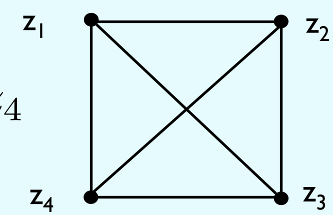
RELATION TO SINGLE-VALUED ELLIPTIC MULTIPLE POLYLOGARITHMS

(D'Hoker, MBG, Gurdogan, Vanhove)

A MODULAR GRAPH FUNCTION IS A SINGLE-VALUED ELLIPTIC MULTIPLE POLYLOGARITHM EVALUATED AT A SPECIAL VALUE OF ITS ARGUMENT

A typical Modular Graph Function:

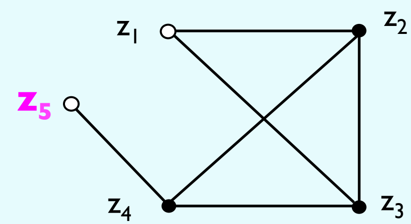
$$D(\tau) = \int d^2z_1 \dots d^2z_4$$



i.e. Split one vertex of a modular graph function, leaving two vertices UNINTEGRATED

Now Consider $\tilde{D}(\zeta;\tau) = \int d^2z_2\, d^2z_3\, d^2z_4$

with $\zeta = \exp(2\pi i(z_5 - z_1))$



It is easy to see that $D(\tau) = \tilde{D}(1;\tau)$

- $\tilde{D}(\zeta;\tau)$ IS SINGLE VALUED (IN ζ) ELLIPTIC MULTIPLE POLYLOGARITHM
- GENERALISATION OF SINGLE-VALUED ELLIPTIC POLYLOGARITHM OF ZAGIER (1990)

$$D_{a,b}(\zeta;\tau) = \frac{(2i\tau_2)^{a+b-1}}{2i\pi} \sum_{(m,n)\neq(0,0)} \frac{e^{2i\pi(nu-mv)}}{(m\tau+n)^a(m\bar{\tau}+n)^b}$$

Modular graph functions of arbitrary weight

Special values of SINGLE-VALUED ELLIPTIC MULTIPLE POLYLOGARITHMS

POLYNOMIAL RELATIONSHIPS

As with MZV's, these elliptic functions satisfy a fascinating set of polynomial relationships – we have found a few of these (with great difficulty!)

A general modular graph function has a $q = e^{2\pi i\tau}$ expansion with a finite number of powers of τ_2

- In the $\tau_2 \to \infty$ limit

$$D_{\dots}(q,\bar{q}) \to \sum_{-w+1}^{w} c_w^k \, \tau_2^k + O(e^{-\tau_2})$$

Determined by explicit analysis

LAURENT SERIES

- The coefficients of the Laurent series c_w^k are rational multiples of MULTIPLE ZETA VALUES.

e.g.

$$C_{1,1,5}(y) = \frac{34}{8513505}y^7 + \frac{2}{945}\zeta(3)y^4 + \frac{17}{252}\zeta(5)y^2 + \frac{23}{105}\zeta(3)^2 y + \frac{1391}{560}\zeta(7) - \frac{3\zeta(3)\zeta(5)}{y} + \frac{953\zeta(9)+144\zeta(3)^3}{32y^2} - \frac{1701\zeta(3)\zeta(7)+120\zeta(5)^2}{32y^3} + \frac{324\zeta(3,5,3) - 324\zeta(3)\zeta(3,5) + 22299\zeta(11) + 8460\zeta(5)\zeta(3)^2}{320y^4} - \frac{891\zeta(5)\zeta(7)+702\zeta(9)\zeta(3)}{16y^5} + \frac{7209}{128}\frac{\zeta(13)}{y^6},$$

(Zerbini)

$(y = \tau_2)$

Weight-11 svMZV $\frac{9}{8}\zeta_{sv}(3,5,3) - \frac{225}{16}\zeta_{sv}(5)\zeta_{sv}(3)^2 - \frac{9573}{256}\zeta_{sv}(11)$

- Specific polynomials (over $\mathbb{Q}$) of modular graph functions with a given weight have vanishing Laurent series (MAKING USE OF MZV ALGEBRAIC RELATIONS).

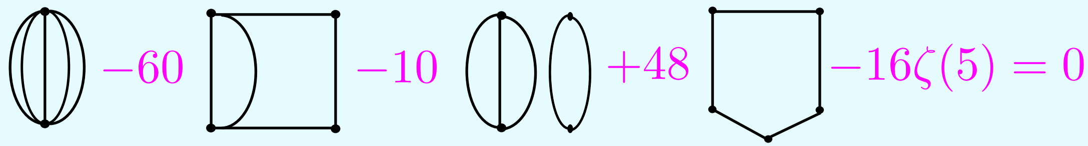
- It turns out (in every example we have studied):

  WHEN THE LAURENT SERIES OF THE POLYNOMIAL VANISHES THE EXACT POLYNOMIAL VANISHES.

IS THIS ALWAYS THE CASE – IS THIS A THEOREM?

Examples of Polynomial Relationships

e.g. weight 5 $D_5 - 60\,C_{3,1,1} - 10\,E_2\,C_{1,1,1} + 48\,E_5 - 16\,\zeta(5) = 0$



polynomial of weight 5 in functions of different depth (different no. of loops).

e.g. weight 6

$$-3\,D_{411} + 109\,C_{222} + 408\,C_{321} + 36\,C_{411} + 18\,C_{211}\,E_2 + 12\,E_3^2 - 211\,E_6 + 12\,E_3\zeta_3 = 0$$

polynomial of weight 6 in functions of different depth.

Other explicit low-weight examples are known (D'Hoker, Kaidi; Basu)

Modular graph functions of any given weight satisfy polynomial relations with rational coefficients

Elliptic generalisation of the rational polynomial relations between multiple polylogarithms and single-valued MZV's

Obvious Questions:

- What is the basis of modular graph functions?
- Some (presumably) related issues in open string loop amplitudes (Broedel, Mafra, Matthes, Schlotterer), which involve "holomorphic" elliptic multiple polylogarithms (Brown, Levin).
- Are these related by a "KLT" type of relation - c.f open and closed tree-level amplitudes.
- Is there an elliptic analogue of the iterated integral representation of multiple polylogarithms
- Important generalisation to modular graph forms

Integration over fundamental domain

Genus-one expansion coefficients :

Integrating over τ - using the earlier relations - gives the one-loop expansion:

$$A_1^{(4)} = \frac{\pi}{3}\left(1 + 0\,\sigma_2 + \frac{\zeta(3)}{3}\,\sigma_3 + 0\,\sigma_2^2 + \frac{116\,\zeta(5)}{5}\,\sigma_2\,\sigma_3 \ldots\right)\mathcal{R}^4$$

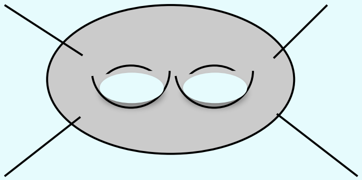
$\mathcal{R}^4$ $\quad d^4\,\mathcal{R}^4$ $\quad d^6\,\mathcal{R}^4$ $\quad d^8\,\mathcal{R}^4$ $\quad d^{10}\,\mathcal{R}^4$

+ non-analytic threshold piece

These coefficients are analogous to the tree-level coefficients:

What is the connection between them ?

Genus Two



Amplitude is explicit but difficult to study. (D'Hoker, Gutperle, Phong)

Low energy expansion: (D'Hoker, MBG, Pioline, R. Russo)

Result:

$$A_2^{(4)} = g_s^2 \left(\frac{4}{3}\zeta(4)\,\sigma_2 R^4 \;\; +4\zeta(4)\sigma_3 R^4 + \ldots\right)$$

$d^4\,R^4$ $\qquad$ $d^6\,R^4$

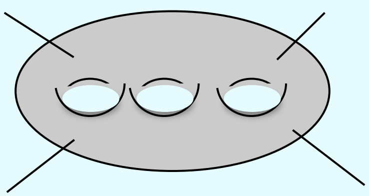
$$\int_{\mathcal{M}_2} \frac{|d^3\Omega|^2}{\det(\mathrm{Im}\Omega)^3}$$

(Genus-two modular invariant)

Related to a weakly holomorphic Jacobi form analogous to the K3 elliptic genus.

Performing an integral over genus-two fundamental domain is non-trivial!

Genus Three



Technical difficulties analysing 3-loops. Gomez and Mafra evaluated the leading low energy behaviour using pure spinor formalism, giving

$$A_3^{(4)} = g_s^4 \left(\frac{4}{27}\,\zeta(6)\,\sigma_3 + \ldots\right)\mathcal{R}^4$$

$d^6\,R^4$

Higher Genera

New problems - No explicit expression

Non-perturbative extension

e.g. Type IIB

$\Omega = \Omega_1 + i\Omega_2$

$\Omega_2 = \frac{1}{g} = e^{-\phi}$

$$A^{(4)}(s,t,u;\Omega) = \mathcal{R}^4\, T(s,t,u;\Omega)$$

$$T(s,t,u;\Omega) = \sum_{p,q} \mathcal{E}_{(p,q)}(\Omega)\, \sigma_2^p\, \sigma_3^q \quad \rightarrow \sim s^{2p+3q} + \ldots$$

$SL(2,\mathbb{Z})$ invariant functions

Using:

- Nonlinear supersymmetry
- Duality between M-theory (quantum 11-dimensional supergravity on two-torus) and and string theory compactified on a circle

$\Delta_\Omega = \Omega_2^2(\partial^2_{\Omega_1} + \partial^2_{\Omega_2}) \longrightarrow$

$$\Delta_\Omega\, \mathcal{E}_{(0,0)}(\Omega) = \frac{3}{4}\, \mathcal{E}_{(0,0)}(\Omega) \qquad \mathcal{R}^4$$

$$\Delta_\Omega\, \mathcal{E}_{(1,0)}(\Omega) = \frac{15}{4}\, \mathcal{E}_{(1,0)}(\Omega) \qquad D^4\, \mathcal{R}^4$$

With b.c: Power behaviour as $\Omega_2 \to \infty$

Solutions: Non-holomorphic Eisenstein series

NON-HOLOMORPHIC EISENSTEIN SERIES

$$E_s(\Omega) = \sum_{\gcd(p,q)=1} \frac{\Omega_2^s}{|p+q\Omega|^{2s}} = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} (\mathrm{Im}\,\gamma\Omega)^s$$

Poincare series – manifest $SL(2,\mathbb{Z})$

Parabolic subgroup $\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$

- $SL(2,\mathbb{Z})$ invariant (generalises to higher rank duality groups)
- Solution of LAPLACE EIGENVALUE EQN.

$$\Delta_\Omega\, E_s(\Omega) = s(s-1)\, E_s(\Omega)$$

- Fourier series $$E_s(\Omega) = 2\sum_{k=0}^{\infty} \mathcal{F}_k(\Omega_2)\cos(2\pi i k\Omega_1)\,.$$
- ZERO MODE $k=0$ - TWO POWER-BEHAVED TERMS (perturbative) :

$$\mathcal{F}_0 = \Omega_2^s + \frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\zeta(2s)\Gamma(s)}\Omega_2^{1-s}$$

- NON-ZERO MODES $k>0$ - D-INSTANTON SUM

$$\mathcal{F}_k = \frac{2\pi^s}{\zeta(2s)\Gamma(s)}|k|^{s-\frac{1}{2}}\,\sigma_{2s-1}(k)\,\Omega_2^{\frac{1}{2}}\,K_{s-\frac{1}{2}}(2\pi|k|\Omega_2)$$
$$\sim \frac{\pi^{s-\frac{1}{2}}}{\zeta(2s)\Gamma(s)}|k|^{s-1}\,\sigma_{2s-1}(k)\,e^{-2\pi|k|\Omega_2}\,(1+O(\Omega_2^{-1}))$$

divisor sum
$$\sigma_n(k) = \sum_{p|k} p^n$$

Low order interaction coefficients

α'^3 $\quad \mathcal{E}_{(0,0)} = 2\zeta(3)\, E_{\frac{3}{2}}(\Omega)$

MBG, Gutperle, Vanhove

$$2\zeta(3)\, g_s^{-\frac{1}{2}}\, E_{\frac{3}{2}}(\Omega) = 2\zeta(3)\, g_s^{-2} + 4\zeta(2)\, g_s^{0} + \sum_{k\neq 0}(\dots)\, \sigma_2(k)\, e^{-2\pi|k|\Omega_2} e^{2\pi i k \Omega_1}$$

Perturbative terms: tree-level genus-one D-instantons

NON-RENORMALISATION BEYOND 1-LOOP FOR R^4 $\quad \frac{1}{2} - BPS$

α'^5 $\quad \mathcal{E}_{(1,0)} = \zeta(5)\, E_{\frac{5}{2}}(\Omega)$

$$\zeta(5)\, g_s^{\frac{1}{2}}\, E_{\frac{5}{2}}(\Omega) = \zeta(5)\, g_s^{-2} + \frac{4}{3}\zeta(4)\, g_s^{2} + \sum_{k\neq 0}(\dots)\, \sigma_4(k)\, e^{-2\pi|k|\Omega_2} e^{2\pi i k \Omega_1}$$

Perturbative terms: tree-level genus-two D-instantons

NO GENUS-ONE TERM (no ¼-BPS states)

NON-RENORMALISATION BEYOND 2 LOOPS FOR $d^4\, \mathcal{R}^4$ $\quad \frac{1}{4} - BPS$

NEXT ORDER $g_s\,\mathcal{E}_{(0,1)}(\Omega)\,\sigma_3\,\mathcal{R}^4 \quad O(\alpha'^6\,d^6\,\mathcal{R}^4)$

$\mathcal{E}_{(0,1)}(\Omega)$ NOT an Eisenstein series but satisfies INHOMOGENEOUS Laplace equation

MBG, Vanhove

$$(\Delta_\Omega - 12)\,\mathcal{E}_{(0,1)}(\Omega) = -\left(2\zeta(3)\,E_{\frac{3}{2}}(\Omega)\right)^2$$

→ The square of the coefficient of R^4

This equation was conjectured by consideration of duality between two-loop eleven-dimensional supergravity compactified on a two-torus and type IIB compactified on a circle.

Structure also motivated by (but not directly derived from) supersymmetry.

(See also, Yifan Wang, Xi Yin, 2015)

THE SOLUTION OF THIS EQUATION HAS SOME WEIRD AND WONDERFUL (PUZZLING) FEATURES.

ZERO MODE OF SOLUTION (zero net D-instanton number):

$$g\,\mathcal{E}_{(0,1)}\big|_{\substack{\text{zero}\\\text{mode}}} = \frac{2}{3}\,\zeta(3)^2\,g^{-2} + \frac{4}{3}\zeta(2)\,\zeta(3)\;g^0 + 4\zeta(4)\,g^2 + \frac{4}{27}\,\zeta(6)\,g^4 + \sum_k c_k e^{-4\pi k/g}$$

GENUS ZERO — GENUS ONE — GENUS TWO — GENUS THREE — SUM OF D-INSTANTONS

PRECISE AGREEMENT WITH EXPLICIT STRING THEORY LOOP CALCULATIONS

Higher-Rank Duality Groups

Duality Group $G(\mathbb{Z})$	space-time dimension
1	10A
$SL(2,\mathbb{Z})$	10B
$SL(2,\mathbb{Z})$	9
$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$	8
$SL(5,\mathbb{Z})$	7
$SO(5,5,\mathbb{Z})$	6
$E_{6(6)}(\mathbb{Z})$	5
$E_{7(7)}(\mathbb{Z})$	4
$E_{8(8)}(\mathbb{Z})$	3

Moduli space $G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$

Coefficients of R^4, D^4R^4 turn out to be specific automorphic functions for higher-rank groups that generalise the SL(2) Eisenstein series. These are

Langlands Eisenstein series $E^G_{\beta;s}$

associated with specific maximal parabolic subgroups of G, labelled by a simple root, β.

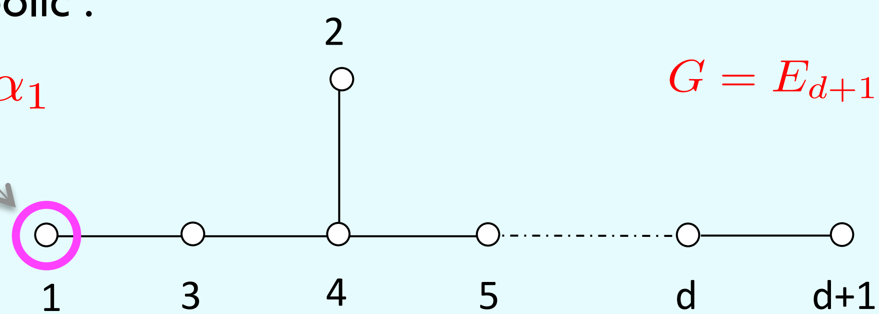
Choice of maximal parabolic :

B.c..'s require $\beta = \alpha_1$

With Levi sub-group

$L_{\alpha_1} = Spin(d,d)$

$G = E_{d+1}$ Dynkin diagram



Solutions

- ½-BPS and ¼-BPS Solutions: $E^{G}_{\alpha_1;\frac{3}{2}}\, R^4$, $E^{G}_{\alpha_1;\frac{5}{2}}\, D^4 R^4$

  Very striking simplifications when $s=\frac{3}{2}\,, \quad s=\frac{5}{2}$

- The coefficient of $\mathcal{E}^{G}_{(0,1)}\, D^6 R^4$ satisfies appropriate inhomogeneous Laplace eigenvalue equation – more difficult to analyse.

- Constant terms encode string perturbation results in compactified theories.

- Non-zero Fourier modes encode effects of D-instantons.

  A charge $k=d\times q$ instanton in D dimensions (almost always) identified with euclidean world-line of a charge q BPS black hole in $D+1$ dimensions wrapped d times around the circle of radius r_d.

  The instantons fill out orbits under the action of the Levi factor in the maximal parabolic sub-group:

  $\frac{1}{2}-$ BPS orbits are "Minimal Orbits" contained in $E^{E_{d+1}}_{\alpha_1;\frac{3}{2}}$

  $\frac{1}{4}-$ BPS orbits are "Next-to-Minimal Orbits" contained in $E^{E_{d+1}}_{\alpha_1;\frac{5}{2}}$

- Extended to $E_9\,,\ E_{10}\,,\ E_{11}$ (Fleig, Kleinschmidt, Persson)

- Extension to ½-maximal supersymmetry in D=3. Instantons in D=3 counts ½- and ¼-BPS dyons (c.f. DVV, …. formula). (Bossard,Cosnier-Horeau,Pioline)

Connection Between String Duality and SUGRA UV divergences

Maximal SUGRA has $\log\,\Lambda$ UV divergences in "Critical" dimensions $D = D_c$

These are reflected by $\log g$ terms in the corresponding automorphic functions

"Protected" terms

One loop in D=8 $\mathcal{R}^4$

Two loops in D=7 $d^4\,\mathcal{R}^4$

Three loops in D=6 $d^6\,\mathcal{R}^4$

Four loops in D=11/2 $d^8\,\mathcal{R}^4$

$$D_c = 4 + \frac{6}{L}$$

Five loops in D=??? $d^{??}\,\mathcal{R}^4$

Is $d^8\,\mathcal{R}^4$ protected ???

UV properties of supergravity

The coefficients of the UV divergences in maximal supergravity up to 3 loops in various dimensions > 4 are precisely reproduced by log terms in modular coefficients.

More Generally:
To what extent do string theory dualities constrain the structure of perturbative supergravity? – ultraviolet divergences??

Superstring perturbation theory is free of UV divergences. Can we understand the UV properties of supergravity by careful consideration of the low energy limit of string theory?