Scattering Amplitudes:

From Gauge Theory to Gravity

Loop-level relations for gauge-theory and gravity amplitudes

Oliver Schlotterer (AEI Potsdam)

based on 1612.00417 with S. He (part I & II)

and 1705.abcde with S. He, C. Mafra, Y. Zhang (part III)

18.04.2017

Different formulations of double copy

color ordered amplitudes

manifest gauge-/diffeo'invariance

cubic diagrams $\in \Gamma_{g\text{-loop}}$

manifest locality

KLT relations '86

$$M^{\text{tree}}_{\text{grav}} = A^{\text{tree}}_{\text{YM}} \otimes \tilde{A}^{\text{tree}}_{\text{YM}}$$

$\xrightarrow[\text{kin. Jacobis}]{\text{establish}}$

BCJ '08: YM-kinematics n_i

$$M^{\text{tree}}_{\text{grav}} = \sum_{i \in \Gamma_{\text{tree}}} \frac{n_i \, \tilde{n}_i}{\prod_{\text{edge } \alpha_i} k^2_{\alpha_i}}$$

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BCJ' 10: natural g-loop version

$$M_{\rm grav}^{(g)} = \int {\rm d}^{g \cdot d} \ell \sum_{i \in \Gamma_g} \frac{n_i(\ell) \, \tilde{n}_i(\ell)}{\prod_{\alpha_i} k_{\alpha_i}^2(\ell)}$$

need Jacobi-satisfying $n_i(\ell)$

or **[see Zvi's talk & 1701.02519]**

Universal to any number d of spacetime dimensions & #(SUSY)!

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this talk **[He, OS 1612.00417]**

$$M_{\rm grav}^{(1)} = \int \frac{d^d \ell}{\ell^2} \, a_{\rm YM}^{(1)}(\ell) \otimes \tilde{a}_{\rm YM}^{(1)}(\ell)$$

rearrange $\longleftarrow$ propagators

BCJ' 10: natural g-loop version

$$M_{\rm grav}^{(g)} = \int d^{g \cdot d} \ell \sum_{i \in \Gamma_g} \frac{n_i(\ell) \, \tilde{n}_i(\ell)}{\prod_{\alpha_i} k_{\alpha_i}^2(\ell)}$$

any representation of $a_{\rm YM}^{(1)}$

? ?

need Jacobi-satisfying $n_i(\ell)$

but ?? higher-loop KLT ??

or **[see Zvi's talk & 1701.02519]**

Universal to any number d of spacetime dimensions & #(SUSY)!

KLT relations in gravity $\leftrightarrow$ BCJ relations in YM

Consistency condition of KLT formula $M^{\text{tree}}_{\text{grav}} = A^{\text{tree}}_{\text{YM}} \otimes \tilde{A}^{\text{tree}}_{\text{YM}}$:

BCJ amplitude relations among color-ordered $A^{\text{tree}}_{\text{YM}}(\rho(1,2,\ldots,n))$

$\Longrightarrow$ $(n-3)!$ linearly independent choices of $\rho \in S_n$

[Stephan's & Emil's talk; Bern, Carrasco, Johansson 2008]

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Loop-level KLT $M_{\text{grav}}^{(1)} = \int \frac{\mathrm{d}^d\ell}{\ell^2}\, a_{\text{YM}}^{(1)}(\ell) \otimes \tilde{a}_{\text{YM}}^{(1)}(\ell)$ requires ...

... one-loop BCJ relations among "partial integrands" $a_{\text{YM}}^{(1)}(\rho(\ldots))$

$\Longrightarrow$ at most $(n-1)!$ linearly independent choices of ρ

(will see extra degeneracy that depends on supersymmetry)

Einstein–Yang–Mills (EYM): coupling gauge theory to gravity

$\exists$ numerous tree-level amplitude relations with the flavour of

$$A^{\text{tree}}_{\text{EYM}}\left(\begin{smallmatrix}\text{gauge}\\ \text{gravity}\end{smallmatrix}\right) = A^{\text{tree}}_{\text{YM}}\begin{pmatrix}\text{gauge}\\ \text{only!}\end{pmatrix} \otimes (\text{polarizations for extra graviton})$$

[Stieberger, Taylor; Nandan, Plefka, OS, Wen; de la Cruz, Kniss, Weinzierl; OS; Du, Teng, Wu; Du, Feng, Fu, Huang; Chiodaroli, Günaydin, Johansson, Roiban; Feng, Teng]

Consequence of double-copy formulation EYM $=$ YM $\otimes$ (YM $\oplus$ scalar)

[Chiodaroli, Günaydin, Johansson, Roiban 1408.0764, 1703.00421]

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This talk: one-loop amplitude relations for gauge multiplets in the loop

$$A^{(1)}_{\text{EYM}} = \int \frac{\mathrm{d}^d\ell}{\ell^2}\left[a^{(1)}_{\text{YM}}(\ell) \otimes (\text{extra polarizations})\right] ,$$

also see recent all-loop relations for 1 external graviton

[Chiodaroli, Günaydin, Johansson, Roiban 1703.00421]

Our method: CHY formalism / ambitwistor strings

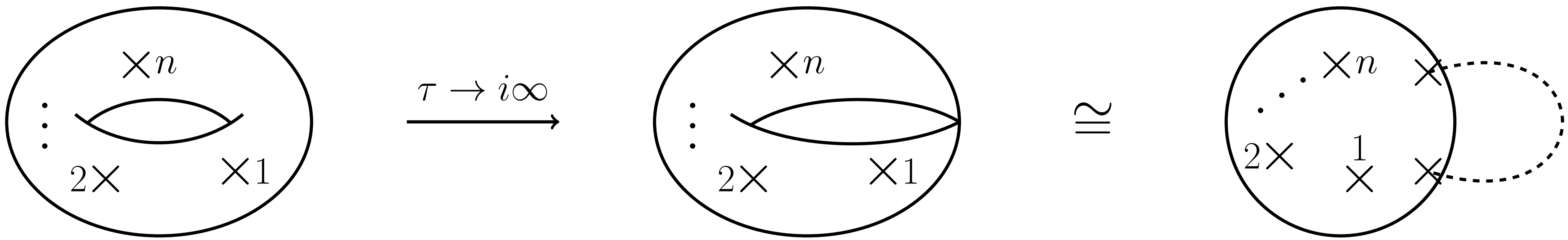
$\longrightarrow$ integrands share double-copy structure with superstrings

& new perspectives on the field-theory limit $\alpha' \to 0$

[Cachazo, He, Yuan 1306.6575, 1307.2199, 1309.0885]

[Mason, Skinner 1311.2564; Adamo, Casali, Skinner 1312.3828]

@ 1-loop: torus worldsheet $\longrightarrow$ nodal Riemann sphere



[Yvonne's talk; Geyer, Mason, Monteiro, Tourkine 1507.00321 & 1511.06315]

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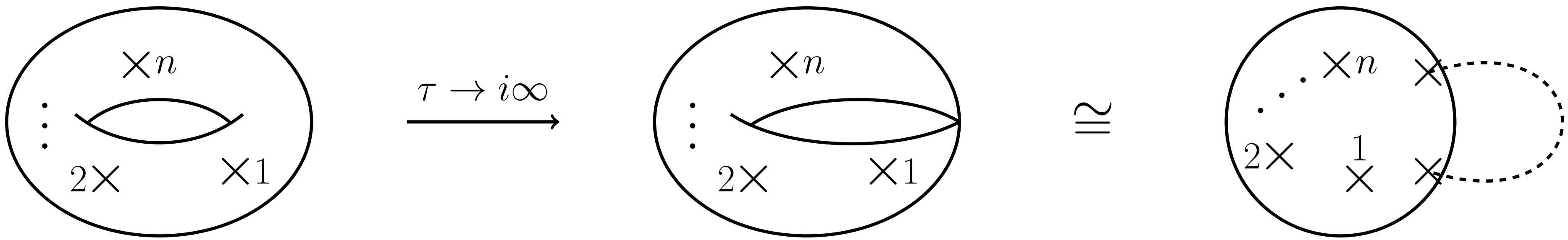


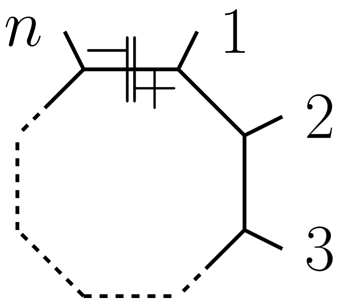
[Yvonne's talk; Geyer, Mason, Monteiro, Tourkine 1507.00321 & 1511.06315]

- natural framework to define partial integrand $a^{(1)}_{\rm YM}(\ldots)$
- one-loop BCJ relations from scattering equations
- one-loop KLT relations from double-copy structure of the integrand

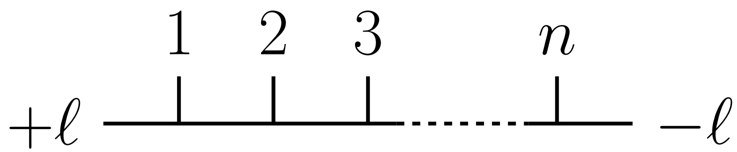
[He, Mafra, OS, Zhang 1705.abcde]

Outline

I. Partial integrands and one-loop BCJ relations



$\Longrightarrow$



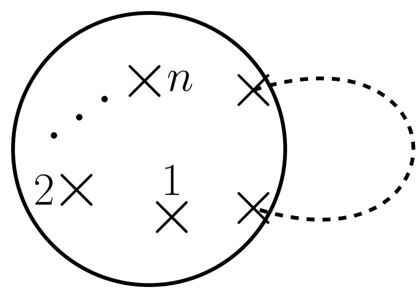
$+ \ \mathrm{cyclic}(1,2,\ldots,n)$

II. One-loop KLT formula and EYM relations

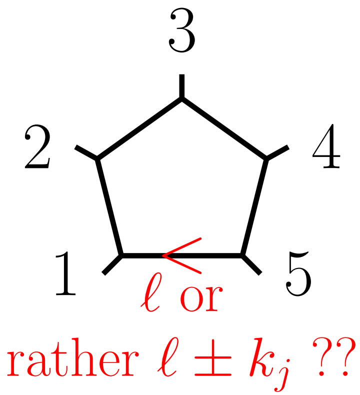
$$\sum_{\rho,\tau\in S_{n-1}} a(+,\rho(1,2,\ldots,n{-}1),n,-)\, S[\rho\,|\,\tau]_\ell\, \tilde{a}(+,\tau(1,2,\ldots,n{-}1),-,n)$$

III. CHY derivation of one-loop amplitude relations

$$a(\tau(1,2,\ldots,n,+,-)) \;=\; \int \mathrm{d}\mu_{n+2}^{\mathrm{tree}}$$



I. 1 New representations of Feynman integrals

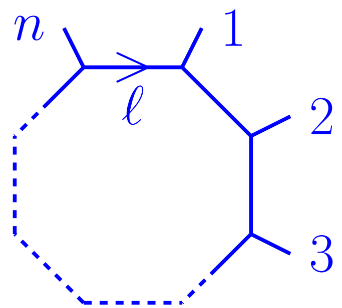


Ambiguity: where to put the loop momentum?

$\rightarrow$ confusing when numerator $n_i(\ell)$ depends on ℓ

Resolution: Canonicalize ℓ-dependence via

partial fraction (PF) $\Longrightarrow$ democratic sum over positions of ℓ



$$= \int \frac{\mathrm{d}^d\ell}{\ell^2\,(\ell+k_1)^2\,(\ell+k_{12})^2\,\ldots\,(\ell+k_{12\ldots n-1})^2}$$

$$= \sum_{i=0}^{n-1} \int \frac{\mathrm{d}^d\ell}{(\ell+k_{12\ldots i})^2} \prod_{\substack{j=0\\ j\neq i}}^{n-1} \frac{1}{(\ell+k_{12\ldots j})^2-(\ell+k_{12\ldots i})^2}$$

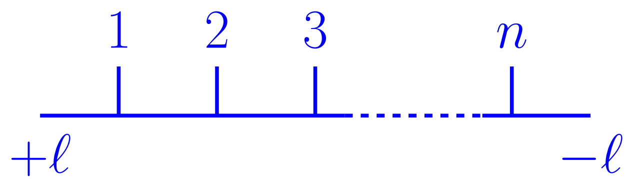
linear in ℓ

PF

$$= \int \frac{\mathrm{d}^d\ell}{\ell^2} \sum_{i=0}^{n-1} \prod_{j=0}^{i-1} \frac{1}{s_{j+1,j+2,\ldots,i,-\ell}} \prod_{j=i+1}^{n-1} \frac{1}{s_{i+1,i+2,\ldots,j,+\ell}}$$

shift ℓ

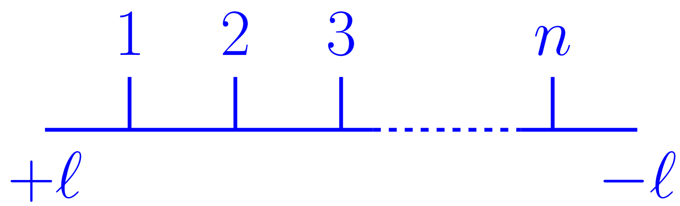
$$= \int \frac{\mathrm{d}^d\ell}{\ell^2}$$



$+ \text{cyclic}(1,2,\ldots,n)$

[Yvonne's talk; Q-cuts: BBBCDF 1509.02169]

I. 2 Partial integrands



Each term in partial-fractionized n-gon

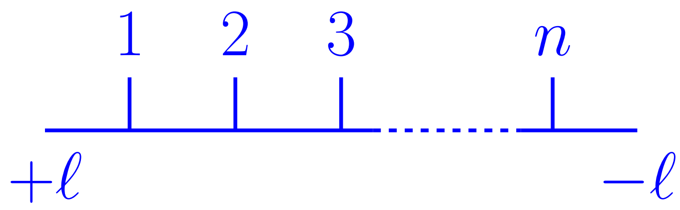
has a notion of cyclic ordering in $n{+}2$ legs $(1, 2, \ldots, n, -\ell, +\ell)$.

Decompose single-trace subamplitudes into n “partial integrands”

$$A^{(1)}(1,2,\ldots,n) \;=\; \int \frac{\mathrm{d}^d\ell}{\ell^2} \sum_{i=1}^{n} \underbrace{a(1,2,\ldots,i,-,+,i{+}1,\ldots,n)}$$

- all propagators linear in ℓ
- only those diagrams compatible with cycle $(1, 2, \ldots, i, -, +, i{+}1, \ldots, n)$

I. 2 Partial integrands



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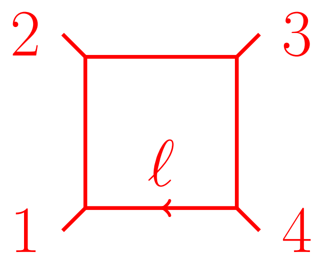
- all propagators linear in ℓ
- only those diagrams compatible with cycle $(1,2,\ldots,i,-,+,i{+}1,\ldots,n)$

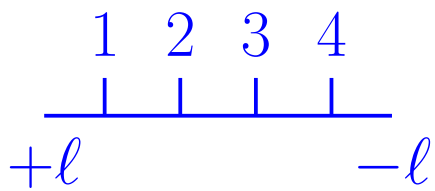
e.g. 4pt @ max. SUSY:

$$A^{(1)}_{\rm max}(1,2,3,4) \;=\; n_{\rm box} \times$$



cyc. orbit of

$$\Longrightarrow \;\; a_{\rm max}(1,2,3,4,-,+) \;=\; \frac{n_{\rm box}}{s_{1,\ell}\, s_{12,\ell}\, s_{123,\ell}} \quad \longleftrightarrow$$



Decompose single-trace subamplitudes into n "partial integrands"

$$A^{(1)}(1,2,\ldots,n) \;=\; \int \frac{\mathrm{d}^d\ell}{\ell^2} \sum_{i=1}^{n} a(1,2,\ldots,i,-,+,i{+}1,\ldots,n)$$

Properties of partial integrand $a(1,2,\ldots,i,-,+,i{+}1,\ldots,n)$

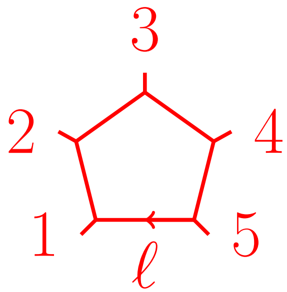
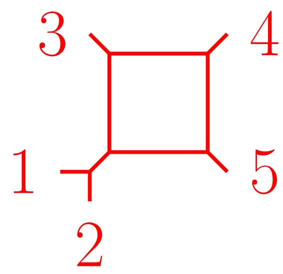
- all propagators made linear in ℓ via partial fraction
- only those diagrams compatible with cycle $(1,2,\ldots,i,-,+,i{+}1,\ldots,n)$
- universal to any number of spacetime dim's d and supercharges
- can be viewed as forward limit of color-ordered $(n{+}2)$-point tree

[Cachazo, He, Yuan 1512.05001]

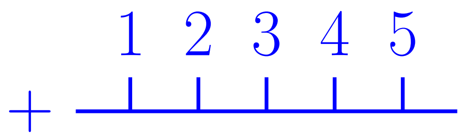
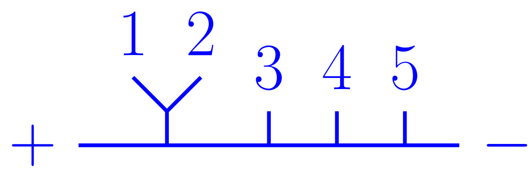
- each partial integrand is separately **gauge invariant**

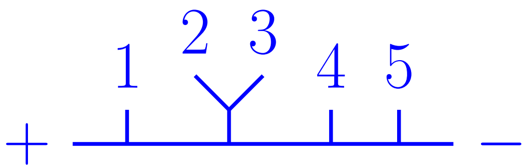
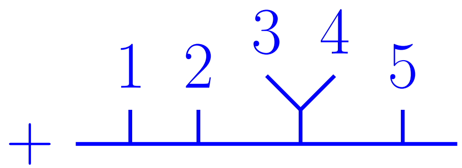
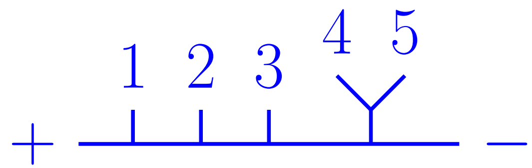
I. 3 Five-point example @ maximal SUSY

By the no-triangle property, one pentagon & five boxes per single-trace

$A^{(1)}_{\rm max}(1,2,3,4,5) =$  $+ \Big\{$  $+ \,{\rm cyc}(1,2,3,4,5)\Big\}$

Partial integrands only have 4 out of 5 boxes (no massive corner with k_{51}):

$a_{\rm max}(1,2,3,4,5,-,+) =$  & 

&  &  & 

Individual numerators are gauge dependent,

e.g. they change under linearized gauge trf. $e_i \to k_i$.

cf. 5pt BCJ numerators in pure-spinor superspace [Mafra, OS 1410.0668]

Explicit five-point numerators for external gluons via generalized t_8-tensor

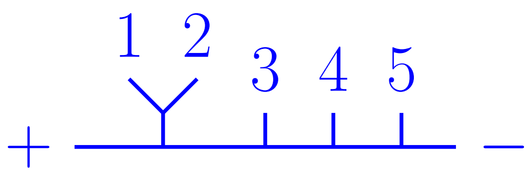
$$t_8(A,B,C,D) \;=\; \mathrm{tr}(f_A f_B f_C f_D) \;-\; \frac{1}{4}\,\mathrm{tr}(f_A f_B)\,\mathrm{tr}(f_C f_D) \;+\; \mathrm{cyc}(B,C,D)$$

with linearized field-strength $f_1^{mn} = k_1^m e_1^n - k_1^n e_1^m$ & two-particle current

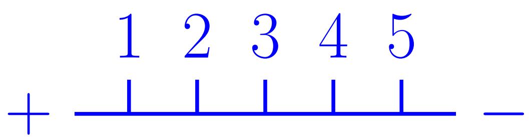
$$e_{12}^m \;=\; e_2^m\,(k_2\cdot e_1) \;+\; \tfrac{1}{2}\,k_1^m\,(e_1\cdot e_2) \;-\; (1\leftrightarrow 2)$$

$$f_{12}^{mn} \;=\; k_{12}^m\,e_{12}^n \;-\; s_{12}\,e_1^m\,e_2^n \;-\; (m\leftrightarrow n)$$

Gauge trf. $t_8(12,3,4,5)\big|_{e_1\to k_1} = s_{12}\,t_8(2,3,4,5)$ cancels between diag's



$$\longleftrightarrow \quad \frac{t_8(21,3,4,5)}{s_{12}\;\; s_{12,\ell}\, s_{123,\ell}\, s_{1234,\ell}}$$



$$\longleftrightarrow \quad \frac{\ell_m\Big[e_1^m\,t_8(2,3,4,5) \;+\; (1\leftrightarrow 2,3,4,5)\Big]}{s_{1,\ell}\,s_{12,\ell}\,s_{123,\ell}\,s_{1234,\ell}}$$

$$-\frac{1}{2}\;\frac{\Big[t_8(12,3,4,5) \;+\; (12\leftrightarrow 13,14,\ldots,45)\Big]}{s_{1,\ell}\,s_{12,\ell}\,s_{123,\ell}\,s_{1234,\ell}}$$

I. 4 Amplitude relations of partial integrands

From intuition partial integrands $\leftrightarrow$ forward limit of $(n+2)$-point trees

$$a(1,2,\ldots,n,-,+) \;=\; \sum_{\text{states } \phi^*,\phi} A^{\text{tree}}(1,2,\ldots,n,\phi^*_{-\ell},\phi_\ell)$$

can export tree-level amplitude relations.

- KK-relations $\Rightarrow$ $(n-2)!$- "basis" $\{A^{\text{tree}}(1,\rho(2,\ldots,n-1),n),\ \rho\in S_{n-2}\}$

$$\Longrightarrow \text{ non-planar } a(\ldots,+,\ldots,-) \text{ determined by } a(\ldots,-,+)$$

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- from BCJ-relations $0=\sum_{j=2}^{n-1}(k_1\cdot k_{23\ldots j})A^{\text{tree}}(2,3,\ldots,j,1,j+1,\ldots,n)$

$$\boxed{0 \;=\; \sum_{i=1}^{n-1}(\ell\cdot k_{12\ldots i})\; a(1,2,\ldots,i,+,i+1,\ldots,n,-)}$$

at most $(n-1)!$ independent $a(\ldots)$, with SUSY-dependent degeneracy

II. 1 KLT amplitude relations

1st double-copy formula: tree-level KLT relations **[Kawai, Lewellen, Tye 1986]**

$$M^{\text{tree}}_{\text{grav}} = \sum_{\rho,\tau\in S_{n-3}} A^{\text{tree}}(1,\rho(2,3,\ldots,n{-}2),n,n{-}1) \times S[\rho\,|\,\tau]_1\, \tilde{A}^{\text{tree}}(1,\tau(2,3,\ldots,n{-}2),n{-}1,n)$$

kernel $S[\rho|\tau]_1 \sim s_{ij}^{n-3}$ recursively determined by $S[j|j]_i = k_i \cdot k_j$ and

$$S[A,j\,|\,B,j,C]_i = k_j\cdot(k_i{+}k_B)\, S[A\,|\,\underbrace{B,C}]_i$$

$A \uparrow a_1, a_2, \ldots, a_x$ $\qquad$ $\underbrace{B,C} \uparrow b_1, b_2, \ldots, b_y, c_1, \ldots, c_z$

[Bern, Dixon, Perelstein, Rozowsky 1998]

[Bjerrum-Bohr, Damgaard, Feng, Sondergaard, Vanhove 2010]

"Secretly" permutation invariant by BCJ relations among $A^{\text{tree}}(\ldots)$.

Identify $\sum_{\text{states}} A^{\text{tree}}(\ldots, \phi^*_{-\ell}, \ldots, \phi_\ell) \leftrightarrow a(\ldots, -, \ldots, +)$

in both gauge-theory copies of $(n{+}2)$-point tree-level KLT formula

$$M^{\text{1-loop}}_{\text{grav}} = \int \frac{\mathrm{d}^d\ell}{\ell^2} \sum_{\rho,\tau \in S_{n-1}} a(+, \rho, n, -)\, S[\rho\,|\,\tau]_\ell\, \tilde{a}(+, \tau, -, n)$$

with e.g. $\rho = \rho(1, 2\ldots, n{-}1)$, same for τ and tree-level kernel $S[\rho\,|\,\tau]_\ell$.

Identify $\sum_{\text{states}} A^{\text{tree}}(\ldots,\phi^*_{-\ell},\ldots,\phi_\ell) \leftrightarrow a(\ldots,-,\ldots,+)$

in both gauge-theory copies of $(n+2)$-point tree-level KLT formula

$$M^{\text{1-loop}}_{\text{grav}} = \int \frac{\mathrm{d}^d\ell}{\ell^2} \sum_{\rho,\tau\in S_{n-1}} a(+,\rho,n,-)\, S[\rho\,|\,\tau]_\ell\, \tilde{a}(+,\tau,-,n)$$

with e.g. $\rho = \rho(1,2\ldots,n-1)$, same for τ and tree-level kernel $S[\rho\,|\,\tau]_\ell$.

e.g. 4pt supergravity amplitude @ maximal SUSY

$$M^{\text{1-loop}}_{\text{max}} = \left|t_8(1,2,3,4)\right|^2 \int \frac{\mathrm{d}^d\ell}{\ell^2} \left\{ \frac{1}{s_{1,\ell}s_{12,\ell}s_{123,\ell}} + \text{perm}(1,2,3,4) \right\}$$

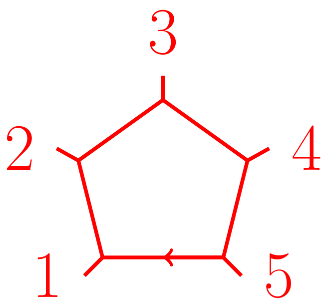
is reproduced from KLT formula

$$\int \frac{\mathrm{d}^d\ell}{\ell^2} \sum_{\rho,\tau\in S_3} a(+,\rho(1,2,3),4,-)\, S[\rho\,|\,\tau]_\ell\, \tilde{a}(+,\tau(1,2,3),-,4)$$

with $a(+,1,2,3,4,-) = \frac{t_8(1,2,3,4)}{s_{1,\ell}s_{12,\ell}s_{123,\ell}}$ & 4-term expression for $a(+,1,2,3,-,4)$.

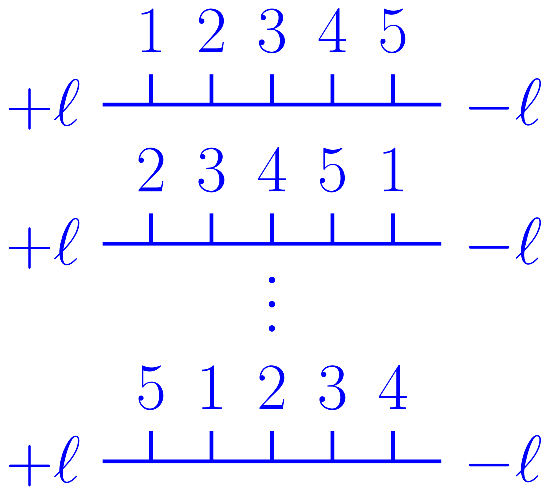
II. 2 Cubic-graph equivalent of one-loop KLT

Rearranged Feynman integrals $\Rightarrow$ more cubic diagrams, e.g.



propagators $(\ell + k_{12\ldots j})^2$
only **one** numerator

partial fraction $\xrightarrow{\hspace{2cm}}$ shift in ℓ



$\leftrightarrow n_{+|12345|-}(\ell)$

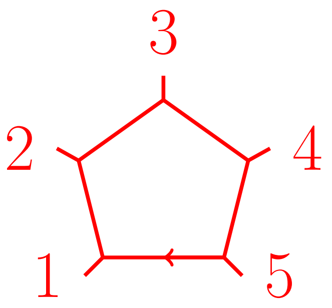
$\leftrightarrow n_{+|23451|-}(\ell)$

$\vdots$

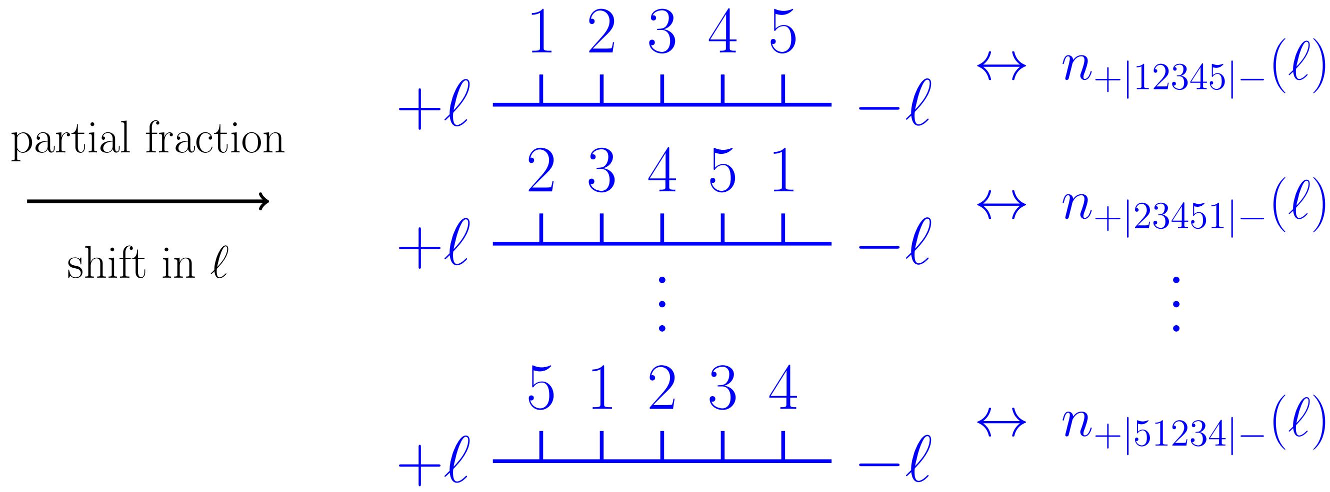
$\leftrightarrow n_{+|51234|-}(\ell)$

II. 2 Cubic-graph equivalent of one-loop KLT

Rearranged Feynman integrals $\Rightarrow$ more cubic diagrams, e.g.



propagators $(\ell + k_{12\ldots j})^2$
only **one** numerator



One-loop KLT $\Leftrightarrow$ separate squaring of the 5 numerators on the right

$$M^{\text{1-loop}}_{\text{grav}} = \int \frac{\mathrm{d}^d\ell}{\ell^2} \sum_{\rho,\tau\in S_{n-1}} a(+,\rho,n,-)\, S[\rho\,|\,\tau]_\ell\, \tilde{a}(+,\tau,-,n)$$

$$= \int \frac{\mathrm{d}^d\ell}{\ell^2} \sum_{i\in\Gamma^{n+2}_{\text{tree}}} \frac{n_i(\ell)\,\tilde{n}_i(\ell)}{\prod_{\text{edge }\alpha_i} s_{\alpha_i}(\ell)} \quad \begin{matrix}\longleftarrow \text{kin. Jacobi rel's} \\ \longleftarrow \text{linear in } \ell\end{matrix}$$

$\exists$ all-multiplicity method for BCJ numerators **[He, Mafra, OS, Zhang 1705.abcde]**

II. 3 EYM relations at one loop

E.g. n external gluons, one ext. graviton $\{e_p, p\}$ & internal gauge states:

notion of partial integrand carries over to EYM:

$$A^{(1)}_{\text{EYM}}(1,2,\ldots,n\,|\,p) = \int \frac{\mathrm{d}^d\ell}{\ell^2} \sum_{i=1}^{n} \underbrace{a_{\text{EYM}}(1,2,\ldots,i,-,+,i{+}1,\ldots,n\,|\,p)}$$

- all propagators linear in ℓ
- forward limit of $A^{\text{tree}}_{\text{EYM}}(\ldots\,|\,p)$

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- all propagators linear in ℓ
- forward limit of $A^{\rm tree}_{\rm EYM}(\ldots\,|\,p)$

Recycle tree-level relations such as

$$A^{\rm tree}_{\rm EYM}(1,2,\ldots,n\,|\,p) \ = \ \sum_{j=1}^{n-1} (e_p \cdot k_{12\ldots j})\, A^{\rm tree}(1,2,\ldots,j,p,j{+}1,\ldots,n)$$

[Stieberger, Taylor 1606.09616]

$\exists$ analogous relations for multiple ext. gravitons and color traces.

[Nandan, Plefka, OS, Wen; OS; Du, Feng, Fu, Huang; Chiodaroli, Günaydin, Johansson, Roiban; Feng, Teng]

One-loop amplitude relation among partial integrands via forward limit

$$A^{\text{tree}}_{\text{EYM}}(1,2,\ldots,n\,|\,p) \;=\; \sum_{j=1}^{n-1} (e_p\cdot k_{12\ldots j})\, A^{\text{tree}}(1,2,\ldots,j,p,j{+}1,\ldots,n)$$

$$\Downarrow$$

$$\begin{aligned} a_{\text{EYM}}(+,1,2,\ldots,n,-\,|\,p) \;=\; & -(e_p\cdot \ell)\, a(+,1,2,\ldots,n,-,p) \\ & + \sum_{j=1}^{n-1} (e_p\cdot k_{12\ldots j})\, a(+,1,2,\ldots,j,p,j{+}1,\ldots,n,-) \end{aligned}$$

Gauge invariance under $e_p \to p$ follows from BCJ relations among $a(\ldots)$.

One-loop amplitude relation among partial integrands via forward limit

$$A^{\text{tree}}_{\text{EYM}}(1,2,\ldots,n\,|\,p) \;=\; \sum_{j=1}^{n-1} (e_p \cdot k_{12\ldots j})\, A^{\text{tree}}(1,2,\ldots,j,p,j{+}1,\ldots,n)$$

$$\Downarrow$$

$$a_{\text{EYM}}(+,1,2,\ldots,n,-\,|\,p) \;=\; -\,(e_p\cdot \ell)\, a(+,1,2,\ldots,n,-,p) + \sum_{j=1}^{n-1} (e_p \cdot k_{12\ldots j})\, a(+,1,2,\ldots,j,p,j{+}1,\ldots,n,-)$$

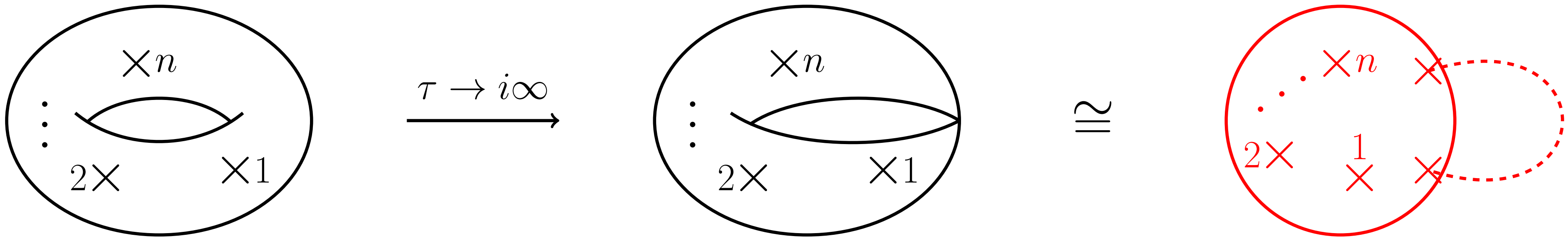
Gauge invariance under $e_p \to p$ follows from BCJ relations among $a(\ldots)$.

Example for recombination to Feynman integral: $n = 3$ @ max. SUSY:

$$A^{(1),\max}_{\text{EYM}}(1,2,3;p) \;=\; t_8(1,2,3,p) \times \int \frac{\mathrm{d}^d\ell \;\; (e_p\cdot\ell)}{\ell^2\,(\ell{+}k_{123})^2\,(\ell{+}k_{23})^2\,(\ell{+}k_3)^2} + \text{cyc}(1,2,3)$$

III. CHY derivation of one-loop amplitude relations

Ambitwistor 1-loop prescription can be localized on nodal Riemann spheres



[Yvonne's talk; Geyer, Mason, Monteiro, Tourkine 1507.00321 & 1511.06315]

Integrands factorize into halves for color or kinematics

$$M^{(1)}_{L\otimes R} = \int \frac{\mathrm{d}^d\ell}{\ell^2} \int \prod_{i=2}^{n} \mathrm{d}\sigma_i\, \delta\left(\frac{\ell\cdot k_i}{\sigma_i} + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{k_i\cdot k_j}{\sigma_{ij}}\right) \widehat{I}_L(\ell)\, \widehat{I}_R(\ell)$$

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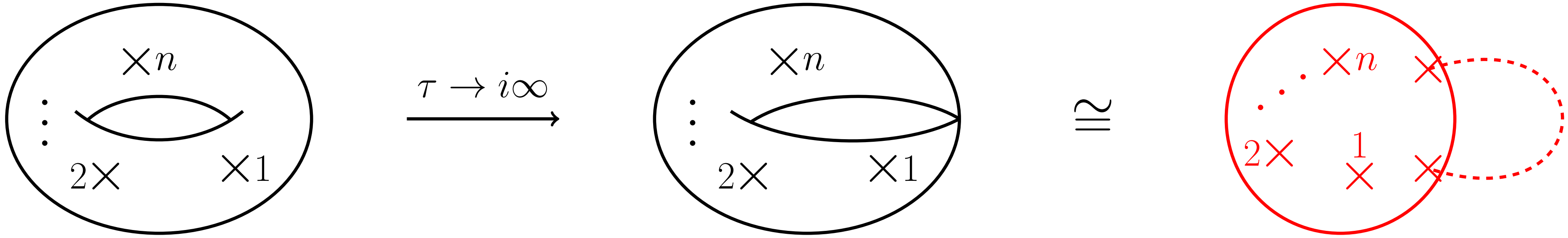

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Recognize as SL_2-fixed expression $(\sigma_+, \sigma_-, \sigma_1) = (0, \infty, 1)$ with $k_\pm = \pm\ell$

$$M^{(1)}_{L\otimes R} = \int \frac{\mathrm{d}^d\ell}{\ell^2} \lim_{k_\pm \to \pm\ell} \int \mathrm{d}\mu^{\text{tree}}_{n+2}\, I_L(\ell)\, I_R(\ell)$$

with tree-level measure $\mathrm{d}\mu^{\text{tree}}_{n+2}$ of the CHY formalism.

III. 1 Partial integrands & BCJ relations from CHY

$$M^{(1)}_{L\otimes R} = \int \frac{\mathrm{d}^d\ell}{\ell^2} \lim_{k_\pm \to \pm\ell} \int \mathrm{d}\mu^{\text{tree}}_{n+2} \, I_L(\ell) \, I_R(\ell)$$

Half-integrand I_{color} is combination of $(n+2)$-point Parke–Taylor factors

$$P(i_1, i_2, \ldots, i_k) = \frac{1}{\sigma_{i_1 i_2} \sigma_{i_2 i_3} \ldots \sigma_{i_{k-1} i_k} \sigma_{i_k i_1}}$$

Half-integrand I_{kin} from correlator of gauge multiplet vertex operators U_j

$$\widehat{I}_{\text{kin}}(\ell) = \lim_{\tau \to i\infty} \frac{\langle U_1(\sigma_1) \, U_2(\sigma_2) \, \ldots \, U_n(\sigma_n) \rangle_\tau}{\sigma_1 \, \sigma_2 \, \ldots \, \sigma_n}$$

- essentially the correlators from RNS or pure-spinor superstring

- at 4pt with maximal SUSY: $\widehat{I}^{\text{4pt}}_{\text{kin}}(\ell) = \frac{t_8(1,2,3,4)}{\sigma_1\sigma_2\sigma_3\sigma_4}$ which descends from

$$I^{\text{4pt}}_{\text{kin}}(\ell) = t_8(1,2,3,4) \sum_{\rho \in S_4} P(+, \rho(1,2,3,4), -)$$

$$M^{(1)}_{L\otimes R} \;=\; \int \frac{\mathrm{d}^d\ell}{\ell^2} \lim_{k_\pm \to \pm\ell} \int \mathrm{d}\mu^{\text{tree}}_{n+2}\; I_L(\ell)\, I_R(\ell)$$

For partial integrand $a(\ldots)$, take a single Parke–Taylor factor as I_L:

$$a(\tau(1,2,\ldots,n,+,-)) \;=\; \lim_{k_\pm \to \pm\ell} \int \mathrm{d}\mu^{\text{tree}}_{n+2}\; P(\tau(1,2,\ldots,n,+,-))\, I_{\text{kin}}(\ell)$$

$\Rightarrow$ inherit gauge invariance from $I_{\text{kin}}(\ell)$ @ support of scattering equations

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$\Rightarrow$ inherit gauge invariance from $I_{\text{kin}}(\ell)$ (@ support of scattering equations)

BCJ relations among $a(\ldots)$ follow from scattering equations

$$0 = \int \mathrm{d}\mu^{\text{tree}}_{n+2} \sum_{i=1}^{n-1} (\ell\cdot k_{12\ldots i})\, P(1,2,\ldots,i,+,i{+}1,\ldots,n,-)\, I_R(\ell)$$

valid before $k_\pm \to \pm\ell$. For supersymmetric correlators $I_R(\ell) \to I_{\text{kin}}(\ell)$,

no forward-limit divergences arise and one can take $k_\pm \to \pm\ell$

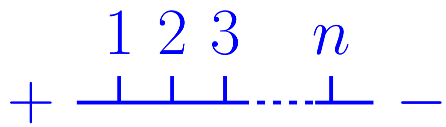
$$0 = \sum_{i=1}^{n-1} (\ell\cdot k_{12\ldots i})\, a(1,2,\ldots,i,+,i{+}1,\ldots,n,-)$$

III. 2 KLT relations from CHY

Also $I_{\rm kin}(\ell)$ can be expanded in Parke–Taylor factors

$$\widehat{I}_{\rm kin}(\ell) = \lim_{\tau \to i\infty} \frac{\langle\, U_1(\sigma_1)\, U_2(\sigma_2)\ \ldots\ U_n(\sigma_n)\,\rangle_\tau}{\sigma_1\, \sigma_2\ \ldots\ \sigma_n}$$

$$I_{\rm kin}(\ell) = \sum_{\rho \in S_n} P(+, \rho(12\ldots n), -)\, N_{+|\rho(1,2,\ldots,n)|-}(\ell)$$

$\Rightarrow$ BCJ master numerators $N_{+|123\ldots n|-}(\ell)$ for n-gons 

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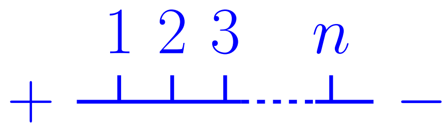

Scattering equations yield BCJ basis of $(n{-}1)!$ Parke–Taylor factors,

$$I_{\text{kin}}(\ell) \ = \sum_{\rho,\tau \in S_{n-1}} P(+,\rho,-,n)\, S[\rho\,|\,\tau]_\ell\, a(+,\tau,n,-)\ ,$$

master numerators combine to gauge-invariant combinations $S[\cdot|\cdot]_\ell\, a(\ldots)$.

cf. tree-level [Cachazo, He, Yuan 1309.0885] and cf. string correlator

[Mafra, OS, Stieberger 1106.2645; Brödel, OS, Stieberger 1304.7267]

Next, combine BCJ-basis representation of $I_{\mathrm{kin}}(\ell)$ for left-movers

with the definition of partial integrands for right-movers

$$\tilde{a}(\tau(1,2,\ldots,n,+,-)) \;=\; \lim_{k_\pm\to\pm\ell}\int \mathrm{d}\mu_{n+2}^{\mathrm{tree}}\, P(\tau(1,2,\ldots,n,+,-))\, \tilde{I}_{\mathrm{kin}}(\ell)\,.$$

$\Longrightarrow$ short proof of one-loop KLT (valid for SUSY $I_{\mathrm{kin}}(\ell)$ or $\tilde{I}_{\mathrm{kin}}(\ell)$):

$$M_{\mathrm{grav}}^{(1)} \;=\; \int \frac{\mathrm{d}^d\ell}{\ell^2} \lim_{k_\pm\to\pm\ell}\int \mathrm{d}\mu_{n+2}^{\mathrm{tree}}\, \underbrace{I_{\mathrm{kin}}(\ell)}_{\sum_{\rho,\tau\in S_{n-1}} P(+,\rho,-,n)\, S[\rho\,|\,\tau]_\ell\, a(+,\tau,n,-)}\, \tilde{I}_{\mathrm{kin}}(\ell)$$

$$= \int \frac{\mathrm{d}^d\ell}{\ell^2} \sum_{\rho,\tau\in S_{n-1}} \tilde{a}(+,\rho,-,n)\, S[\rho\,|\,\tau]_\ell\, a(+,\tau,n,-)$$

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$$\sum_{\rho,\tau\in S_{n-1}} P(+,\rho,-,n)\,S[\rho\,|\,\tau]_\ell\,a(+,\tau,n,-)$$

$$=\; \int \frac{\mathrm{d}^d\ell}{\ell^2}\sum_{\rho,\tau\in S_{n-1}} \tilde{a}(+,\rho,-,n)\,S[\rho\,|\,\tau]_\ell\,a(+,\tau,n,-)$$

Note that the expansion of $I_{\rm kin}(\ell)$ in terms of $P(+,\rho,-,n)$

introduces singular coefficients in absence of SUSY.

III. 3 Explicit representations & string correlators

Explicit results for ambitwistor correlators $I_{\rm kin}^{(g)} \sim \langle U_1(\sigma_1)\ldots U_n(\sigma_n)\rangle^{(g)}$

at genus $g \leq 3$ can be imported from the superstring literature.

[Emil's talk; Adamo, Casali, Skinner 2013; Adamo, Casali 2015]

- tree level: supersymmetrized Pfaffian $\sim$ n-point open-string correlator

[Gomez, Yuan 1312.5485]

known in terms of BCJ numerators in pure-spinor superspace & cpt's

[Mafra, OS, Stieberger 1104.5224; Mafra, OS 1510.08846]

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- one loop: bosonic correlators for $I^{(g)}_{\rm kin}$ studied in RNS superstring

[Green, Schwarz 1981; Tsuchiya 1988; Stieberger, Taylor 2002]

[Bjerrum-Bohr, Vanhove 2008; Brödel, Mafra, Matthes, OS 2014]

simplification in the $\tau \rightarrow i\infty$ limit $\Rightarrow$ prescription for BCJ numerators

[He, Mafra, OS, Zhang 1705.abcde]

- one loop: SUSY correlators for $I^{(g)}_{\text{kin}}$ studied in pure-spinor superstring

  [Berkovits 2004; Mafra, Stahn 2009; Mafra, OS 2012; Mafra, OS 2016]

- two loops: 4pt superstring correlators in RNS and pure-spinor variables

  [D'Hoker, Phong 2005; Berkovits 2005]

  $\Rightarrow$ 2-loop 4pt field-theory amplitude from CHY/ambitwistor setup

  [Geyer, Mason, Monteiro, Tourkine 1607.08887]

- partial results for 4pt 3-loop & 5pt 2-loop string correlator

  [Gomez, Mafra 1308.6567; Gomez, Mafra, OS 1504.02759]

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- one-loop string correlators with half- & quarter-maximal SUSY

[Bianchi, Santini hep-th/0607224; Bianchi, Consoli 1508.00421]

[Berg, Buchberger, OS 1603.05262 & 1611.03459]

all-multiplicity method for BCJ numerators with reduced SUSY

[He, Mafra, OS, Zhang 1705.abcde]

Conclusions & Outlook

- partial-fraction representation of Feynman integrals identifies

  partial integrands $a(\ldots)$ as gauge-invariant seeds @ one loop

- $a(\ldots) \cong$ forward limit of trees $\Rightarrow$ natural one-loop uplift

  of tree-level BCJ, KLT and EYM amplitude relations

- CHY formalism $\Rightarrow$ concise proofs of one-loop amplitude relations

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- partial-fraction representation of Feynman integrals identifies partial integrands $a(\ldots)$ as gauge-invariant seeds @ one loop

- $a(\ldots) \cong$ forward limit of trees $\Rightarrow$ natural one-loop uplift of tree-level BCJ, KLT and EYM amplitude relations

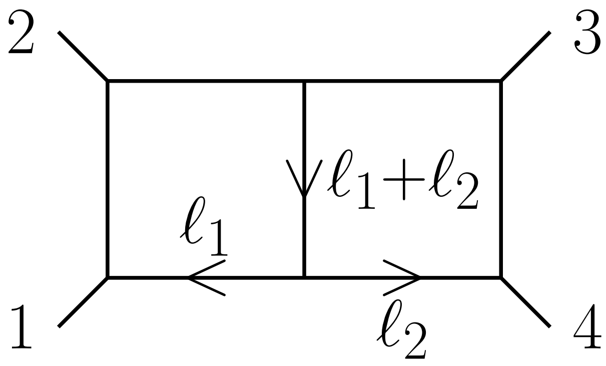
- CHY formalism $\Rightarrow$ concise proofs of one-loop amplitude relations

- How to systematically undo partial-fractioning of Feynman integrals?

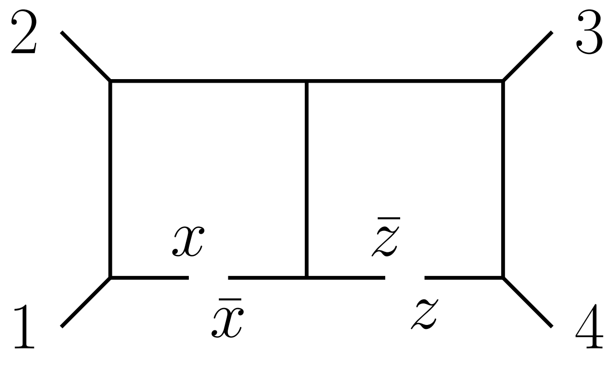
- Desirable to develop integration techniques for propagators linear in ℓ.

[Arkani-Hamed, Yuan: in progress]

• need suitable higher-loop partial integrands $a^{(g)}$ for g-loop KLT rel's



$A^{(2)}(1,2,3,4)$



$a^{(2)}(1,2,3,4,z,\bar{z},\bar{x},x)$

$$\overset{?}{\Longrightarrow} \quad M_{\rm grav}^{(g)} = \prod_{j=1}^{g} \int \frac{\mathrm{d}^d \ell_j}{\ell_j^2} \sum_{\rho,\tau \in \ldots} a^{(g)}(\ldots) S[\ldots|\ldots] \tilde{a}^{(g)}(\ldots)$$

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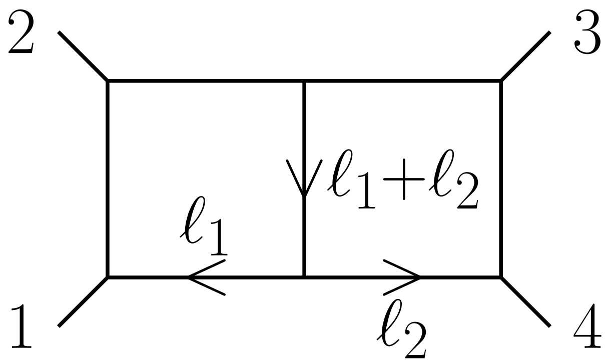

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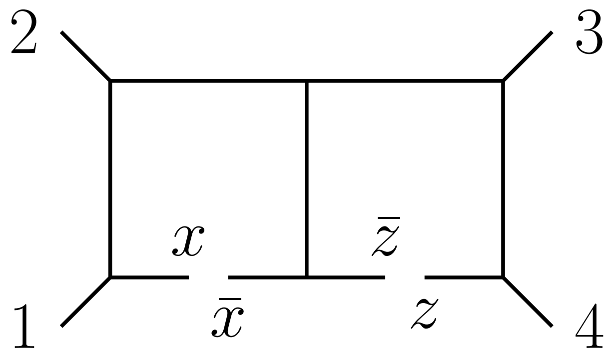

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- loop-level KLT in string theory? connections with monodromy rel's?

[Stephan's talk; Vanhove, Tourkine 1608.01665; Hohenegger, Stieberger 1702.04963]

Thank you for your attention !