The full angle-dependence of the four-loop cusp anomalous dimension in QED

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Based on JHEP 05 (2020) 025 and 2007.04851 [hep-th]

KITP 2020 Scattering Amplitudes and Beyond
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Max-Planck-Institut für Physik
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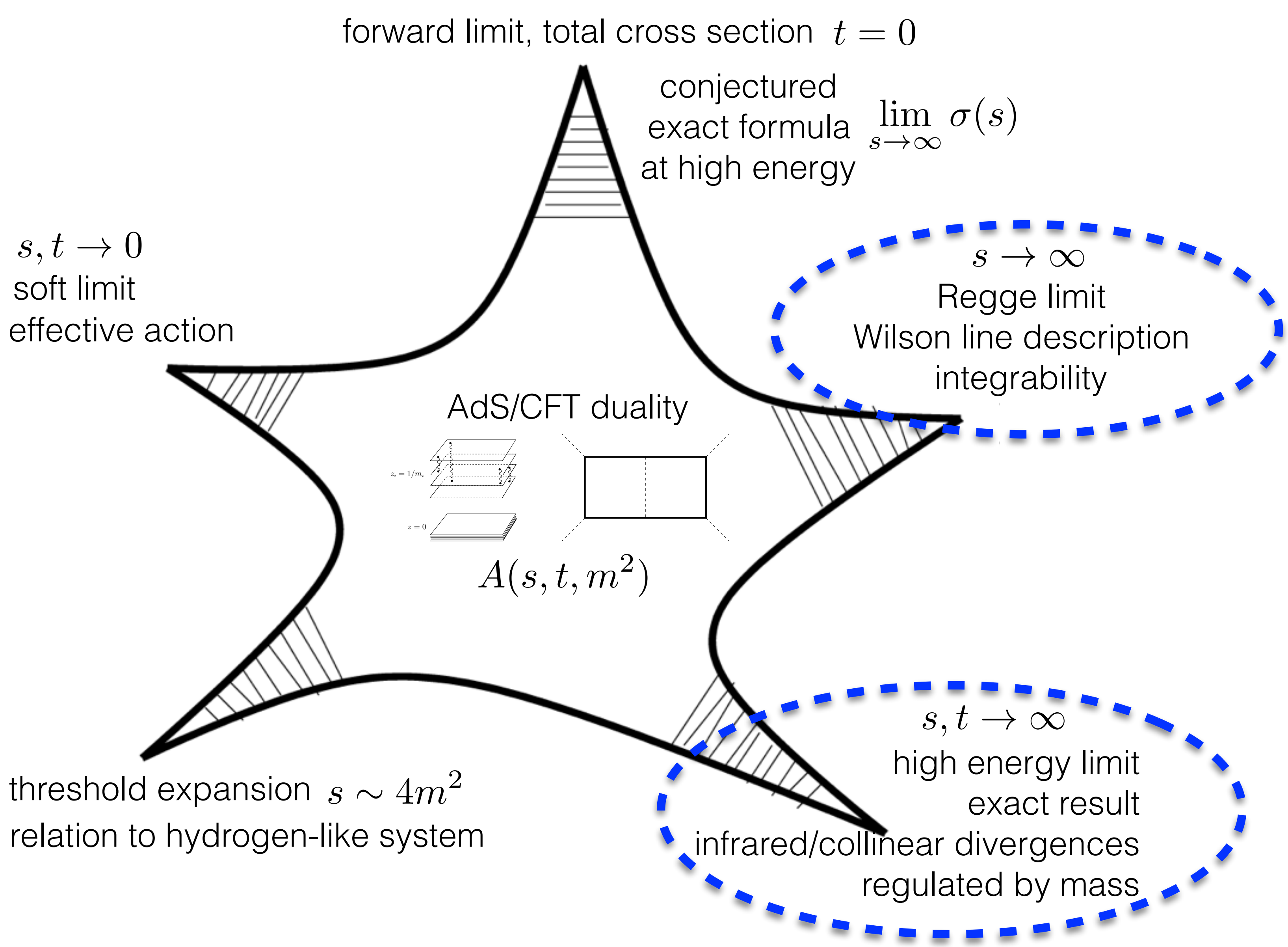
KITP 2017: Scattering amplitudes and beyond

I spoke about massive amplitudes in N=4 sYM



Wilson lines crucial to describe physical limits

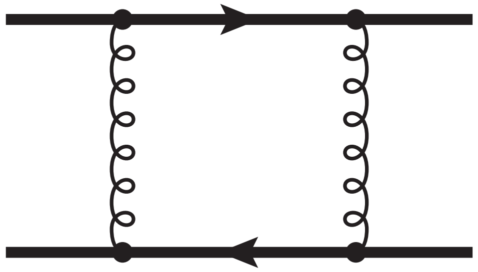
Wilson lines important in gauge theories

$$W = \frac{1}{N_R} \langle 0 | \, \mathrm{tr}_R \, P \exp \left(ig \oint_C dx^\mu A_\mu(x) \right) | 0 \rangle$$

Contour C

Anti-parallel lines:

Quark antiquark potential

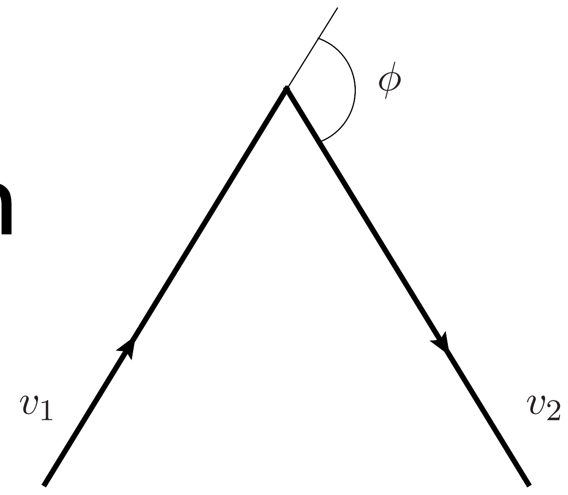


Multiple lines emanating from one point:

Soft anomalous dimension matrix, describes soft gluon effects in scattering processes

Two-line case: cusp anomalous dimension

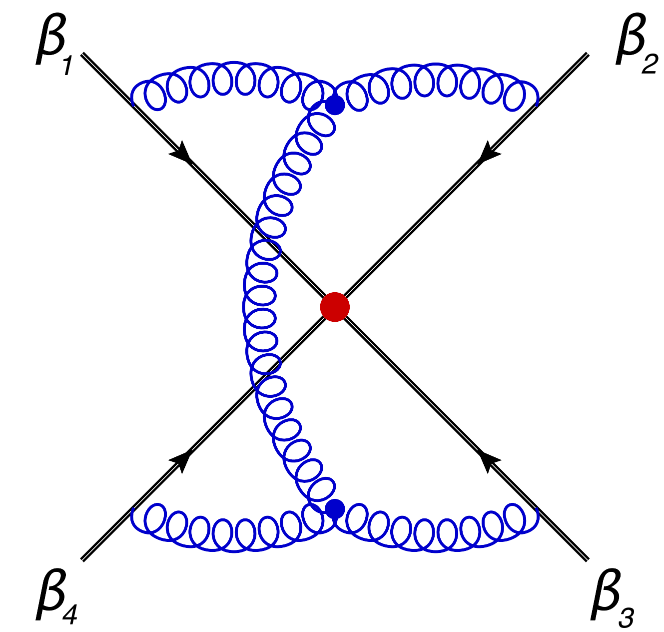
$$\cos\phi = \frac{v_1 \cdot v_2}{\sqrt{v_1^2 v_2^2}}$$



Simplicity in soft anomalous dimension

Massless case:

Corrections to dipole formula starting from three loops. Formula has relatively simple functional dependence, heavily constrained (bootstrap ideas).

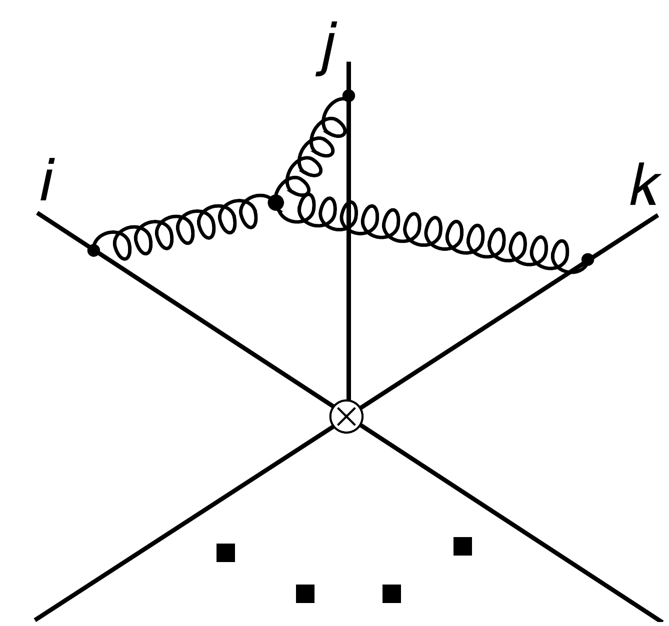


[Almelid, Duhr, Gardi 2015]
[Almelid Duhr, Gardi, McLeod, White, 2017]

Massive case:

Two-loop result very simple, despite complicated intermediate steps.

[Mitov, Sterman, Sung; Ferroglia, Neubert Pecjak, Yang, 2009]
[Chien, Schwartz, Simmons-Duffin, Stewart, 2011]



Properties cusp anomalous dimension

Matter-dependent terms to three loops follow simple recursive pattern. Holds for some but not all four-loop color structures. [Grozin, Henn, Korchemsky, Marquard, 2014]

[Grozin, Henn, Stahlhofen 2017; Brüser, Grozin, Henn, Stahlhofen,2019]

To three loops, color dependence only via C_F, C_A .

At four loops, quartic Casimir terms: $d_R \sim Tr_R(T^a T^b T^c T^d)$

We wish to determine the matter-dependent quartic Casimir terms at four loops:

$$\Gamma_{\text{cusp}}|_{\alpha_s^4} = \left(\frac{\alpha_s}{\pi}\right)^4 \frac{d_R d_F}{N_R} \left[n_f B(\phi) + n_s C(\phi)\right]$$

The research team

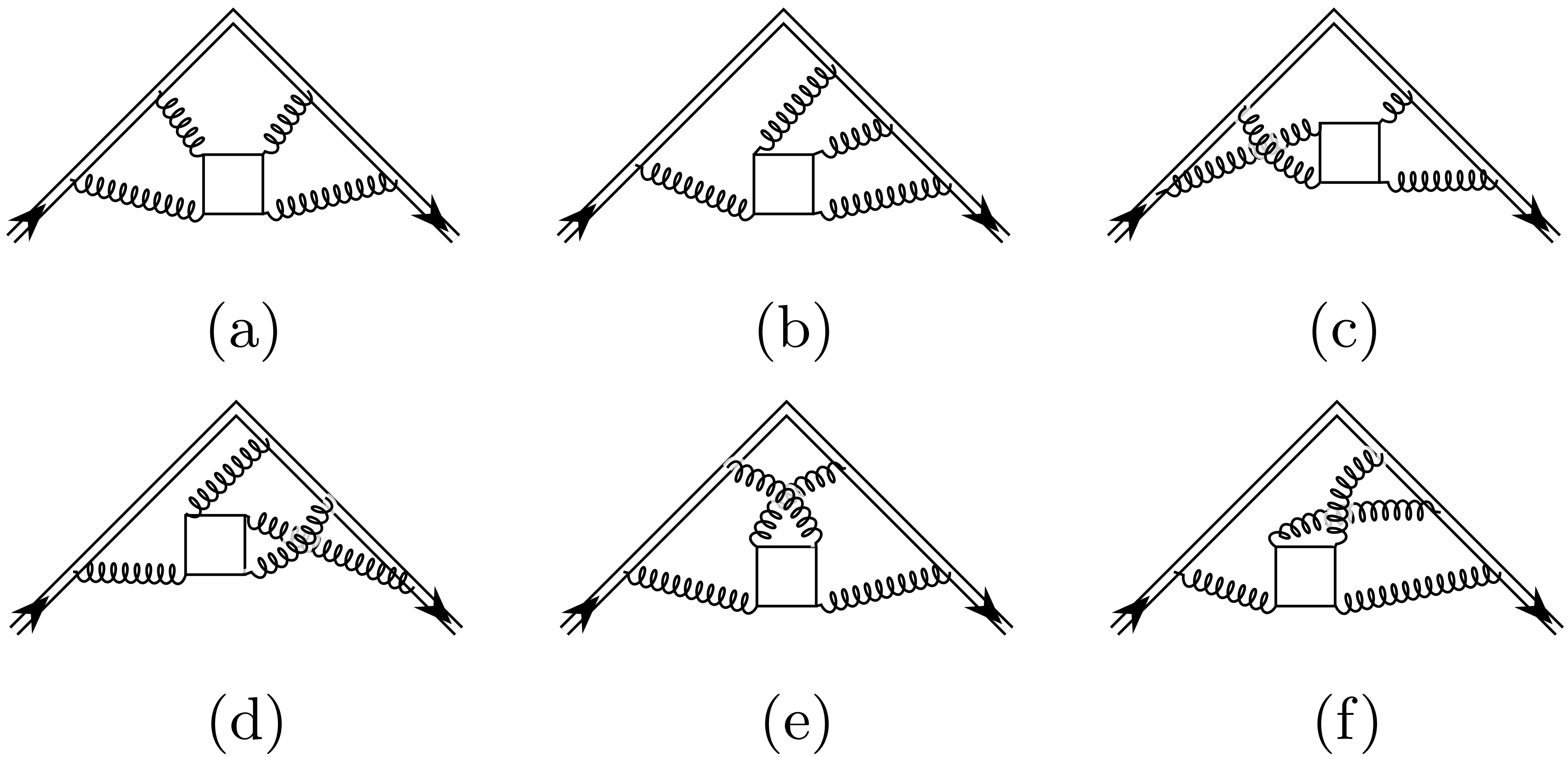
Robin Brüser (Siegen)

Christoph Dlapa (MPP)

Johannes Henn (MPP)

Kai Yan (MPP)

Few Feynman diagrams contribute to the quartic Casimir color structure



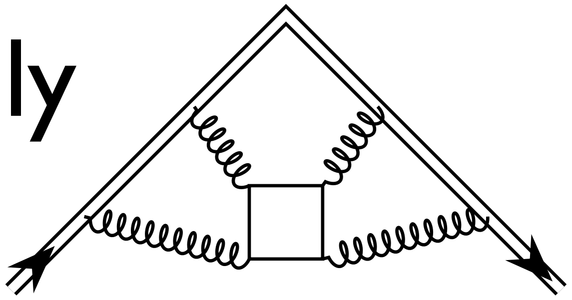
We write them in covariant gauge, and use state-of-the art IBP programs for integral reduction.

[Smirnov 2019 (FIRE6)][Lee 2013 (LiteRed)][Peraro 2019 (FiniteFlow)]

We improve the canonical differential equations method for the calculation of the Feynman integrals

Canonical differential equations method. [Henn, 2013]

Automation needed, since each integral family involves hundreds of integrals.

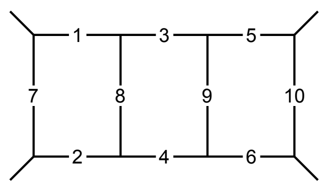
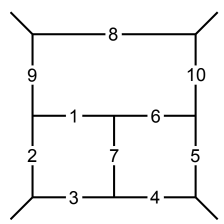
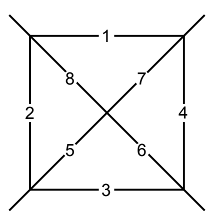
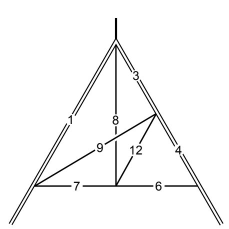
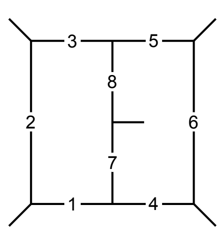


Equations are canonical if all integrals are UT (uniform transcendental weight). The new method requires only one UT integral! [Höschele, Hoff, Ueda 2014] [Dlapa, Henn, Yan 2019]

Public algorithm:

https://github.com/UT-team/INITIAL

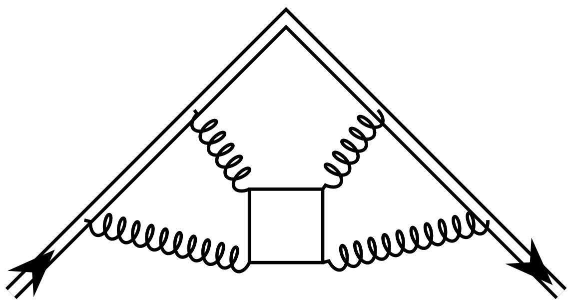
Algorithm is efficient for many coupled integrals, and in multi-variable case

Type of problem		#MI	#vars	#letters	time [min.]	Memory [MB]
Full three-loop DE		26 | 3	1	2	2	330
		41 | 3	1	2	34	1710
Full four-loop DE		19 | 12	1	2	1	240
HQET DE on cut		17 | 17	1	3	2	390
Five-point integrals DE on cut		9 | 9	4	17	5	510

[Dlapa, Henn, Yan 2019]

Complicated functions in intermediate steps, but not needed for final result

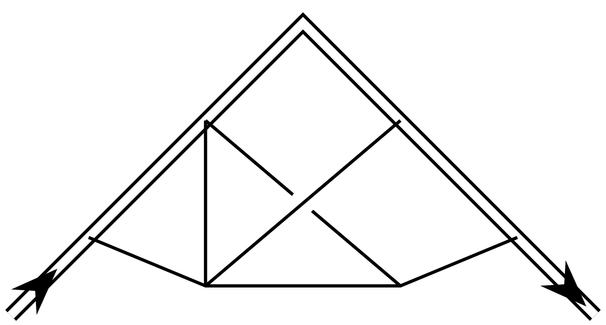
Canonical differential equations:



$$d\vec{f}(x,\epsilon) = \epsilon \sum_k \mathbf{m}_k \left[d \log \alpha_k(x)\right] \vec{f}(x,\epsilon)\,,$$

$$\vec{\alpha} = \{x, 1+x, 1-x, 1+x^2, 1-x+x^2, \tfrac{1-\sqrt{-x}}{1+\sqrt{-x}}, \tfrac{1-\sqrt{-x}+x}{1+\sqrt{-x}+x}\}$$

Possibly non-polylogarithmic integral sector:



$$d\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = d\begin{pmatrix} -\frac{i}{2\sqrt{3}} \ln\frac{y-y_+}{y-y_-} & \frac{2}{3}\ln y + \frac{1}{6}\ln(y-y_+)(y-y_-) \\ \frac{1}{2}\ln(y-y_+)(y-y_-) & \frac{i}{2\sqrt{3}}\ln\frac{y-y_+}{y-y_-} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix},$$

$$y = (1-x^2)/x$$

We used our algorithm plus other ideas to obtain this. [Lee, 2014]

But final answer is polylogarithmic, only alphabet $\alpha = \{x, 1\pm x, 1+x^2\}$ needed!

Four-loop result and checks

We find $(x = e^{i\phi})$

$$B = \frac{1+x^2}{1-x^2} B_1 + \frac{x}{1-x^2} B_2 + \frac{1-x^2}{x} B_3 + B_4$$

Polylogarithms of weight three to seven.

- Gauge invariance check
- Small angle limit $\phi \to 0, x \to 1$ agrees [Grozin, Henn, Stahlhofen 2017] [Brüser, Grozin, Henn, Stahlhofen,2019]
- Massless limit $x \to 0$: light-like cusp anomalous dimension correctly reproduced [Lee, Smirnov^2, Steinhauser, 2019; Henn, Peraro, Stahlhofen, Wasser, 2019]
- Anti-parallel lines limit $\phi \to \pi, x \to -1$: quark-antiquark potential checked [Lee, Smirnov^2, Steinhauser, 2016]

A surprising zero in the anti-parallel lines limit

Anti-parallel lines limit: $\phi = \pi - \delta, \delta \to 0$

[Kilian, Mannel, Ohl, 1993]
[Grozin, Henn, Korchemsky, Marquard, 2015]

$$\Gamma_{\text{cusp}} \overset{\delta \to 0}{\longrightarrow} -C_R \frac{\alpha_s}{\delta} V \,,$$

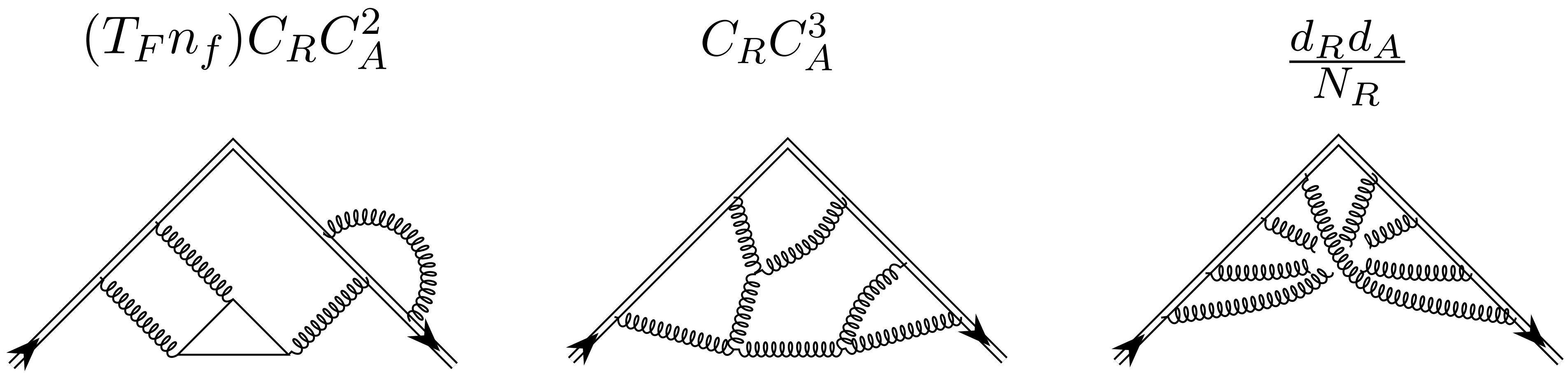
$$B \;=\; -\,\frac{\pi}{\delta}\Big(\frac{79\pi^2}{72} - \frac{23\pi^4}{48} + \frac{5\pi^6}{192} + \frac{l_2\pi^2}{2} + \frac{l_2\pi^4}{12} - \frac{l_2^2\pi^4}{4} - \frac{61\pi^2\zeta_3}{24} + \frac{21\pi^2\zeta_3 l_2}{4}\Big) + \mathcal{O}(\delta)\,,$$

where $l_2 = \log(2)$. No $\mathcal{O}(1)$ term!

Similarly, we produce systematic expansions in small angle and light-like limits.

Few color structures missing for full QCD result

Sample diagrams:



1) We computed all matter-dependent quartic Casimir terms. This means that the gluon quartic Casimir term could be obtained from N=4 super Yang-Mills result!

2) In addition to this, only (simpler) planar calculation needed. The integrals we computed should be helpful.

Full four-loop QED result

$$\Gamma_{\rm cusp}(x,\alpha) = \gamma(\alpha) A(x) + \left(\frac{\alpha}{\pi}\right)^4 n_f B(x) + \mathcal{O}(\alpha^5)\,,$$

Light-like cusp:

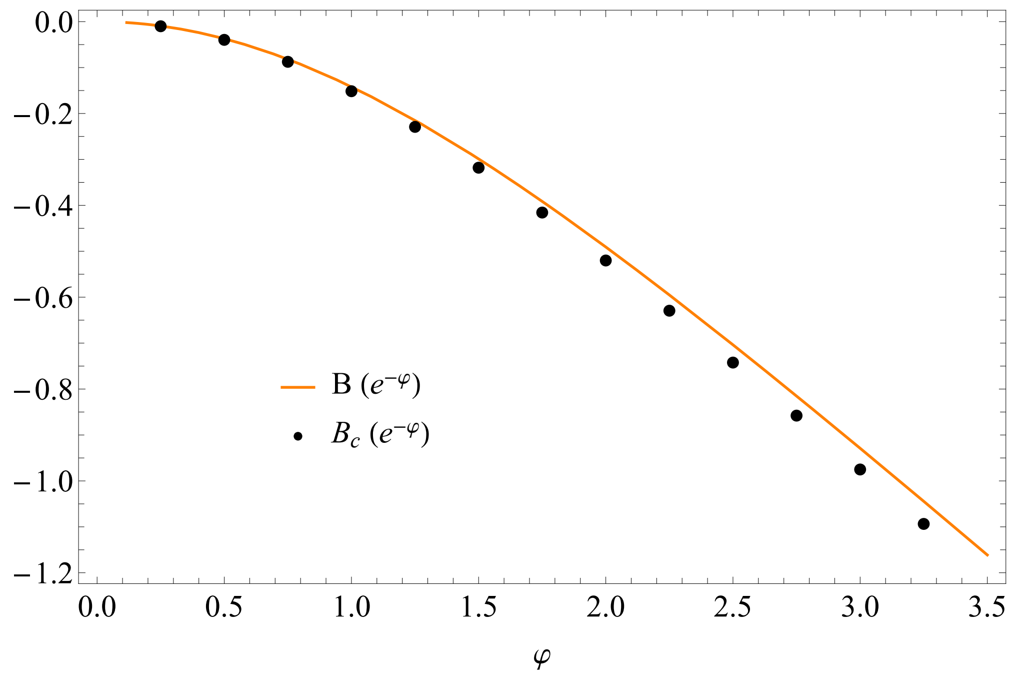
$$\gamma(\alpha) = \left(\frac{\alpha}{\pi}\right) - \frac{5n_f}{9}\left(\frac{\alpha}{\pi}\right)^2 + \left(-\frac{n_f^2}{27} - \frac{55n_f}{48} + n_f\zeta_3\right)\left(\frac{\alpha}{\pi}\right)^3$$
$$+ \left[n_f^3\left(-\frac{1}{81} + \frac{2\zeta_3}{27}\right) + n_f^2\left(\frac{299}{648} + \frac{\pi^4}{180} - \frac{10\zeta_3}{9}\right)\right.$$
$$\left. + n_f\left(\frac{143}{288} + \frac{37\zeta_3}{24} - \frac{5\zeta_5}{2}\right)\right]\left(\frac{\alpha}{\pi}\right)^4, \qquad (4)$$

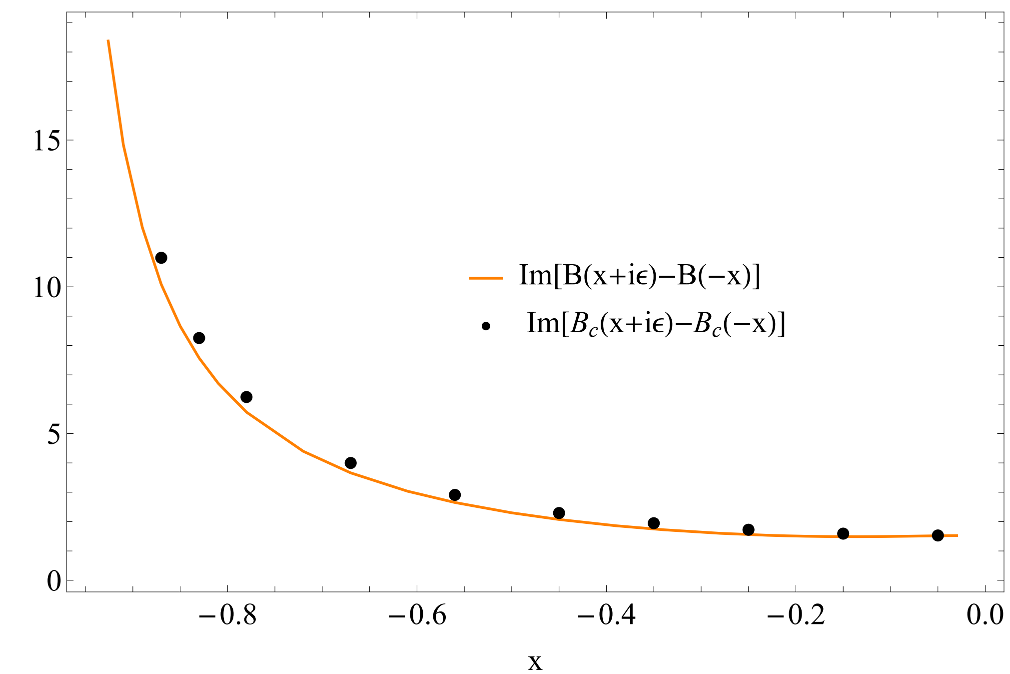
One loop function: $A = -\frac{1+x^2}{1-x^2}\log x - 1\,,$

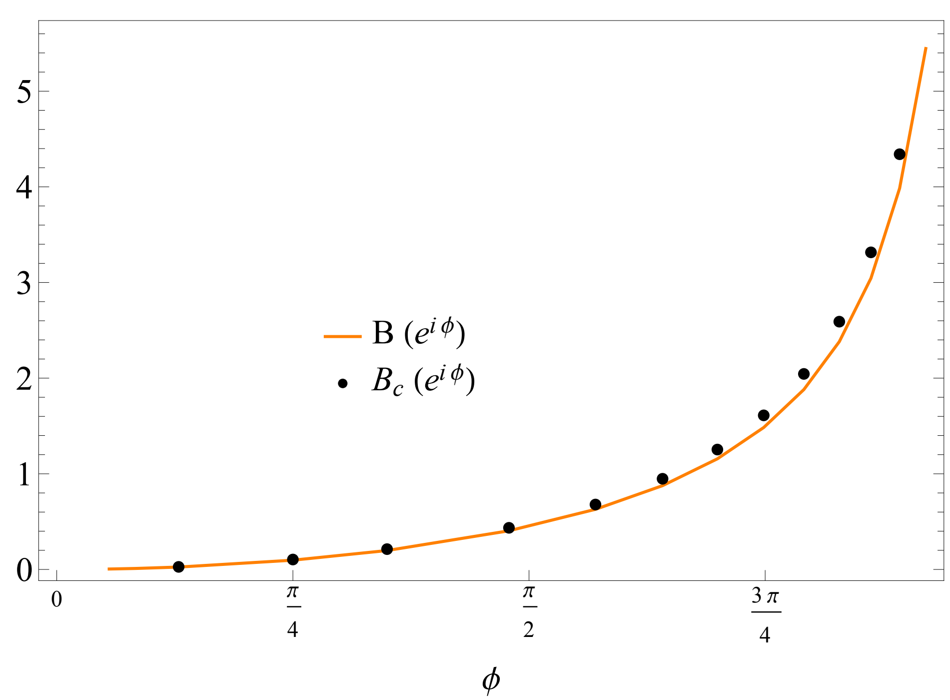
New four-loop function B. How different is B from A?

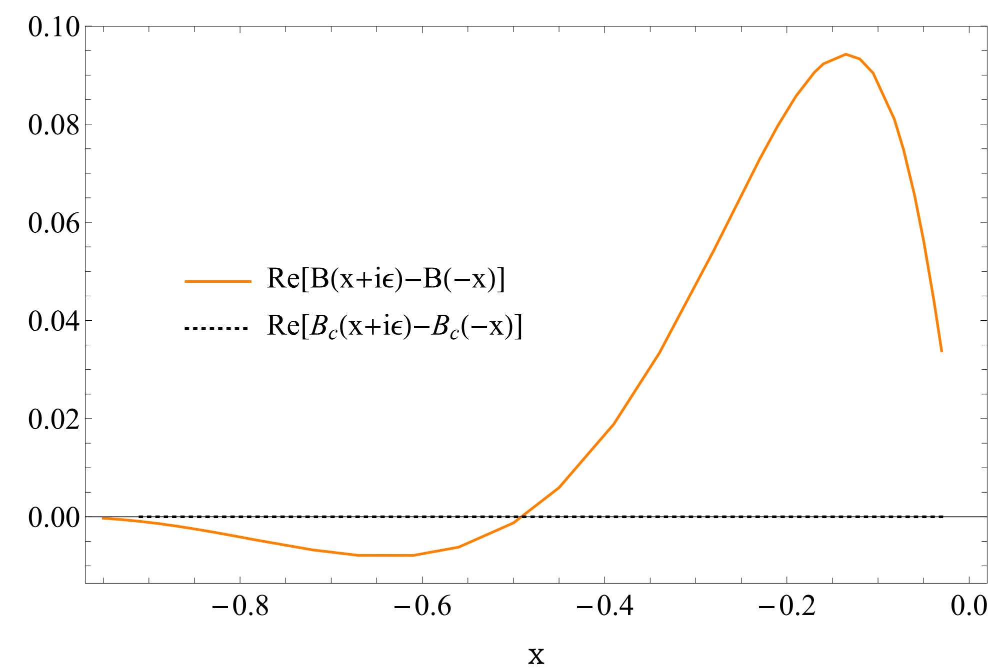
Surprisingly good agreement between rescaled one-loop formula and full four-loop result!

$$B_c(x) = \left(\frac{\pi^2}{6} - \frac{\zeta_3}{3} - \frac{5\zeta_5}{3}\right) A(x) \approx -0.484 \times \left[-\frac{1+x^2}{1-x^2}\log x - 1\right]$$









First non-planar terms in N=4 sYM quark-antiquark potential

$$V_{\rm sYM}|_{\alpha_s^3} = \left(\frac{\alpha_s}{\pi}\right)^3 \left[C_A^3 V_1 + d_R d_A/(N_R C_R) V_2\right]$$

We find (using supersymmetric decomposition, gluon terms from [Lee, Smirnov^2, Steinhauser, 2016]):

$$V_2 = 7\pi^2 - \frac{47\pi^4}{24} + \frac{413\pi^6}{1440} + \frac{116\pi^2 l_2}{3} + \frac{3\pi^4 l_2}{3} + \frac{2}{3}\pi^4 l_2^2 - \frac{17}{12}\pi^2 l_2^4 - 34\pi^2 \mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{89}{4}\pi^2\zeta_3 - 14\pi^2 l_2 \zeta_3 \,. \quad (11)$$

Here $l_2 = \log(2)$

This is for the bosonic Wilson loop. Can it be obtained from integrability? [Correa, Maldacena, Sever 2012; Drukker 2012; Gromov, Levkovich-Maslyuk, 2016; Correa, Leoni, Luque, 2018]

Conclusions and discussion

- Obtained full four-loop QED angle-dependent cusp anomalous dimension
- Result is qualitatively well described by rescaled one-loop function
- Analytic result depends on relatively simple function alphabet. Are there better methods for obtaining this? Gives valuable input for bootstrap of soft anomalous dimension.