Superstring Amplitudes beyond Tree-level

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Outline

- **QFT: proliferation of the number of Feynman diagrams**
  - with increasing number of external states
  - with the number of loops

- **Closed strings: a single diagram per loop order**
  - compact (super) Riemann surface Σ
  - genus = number of handles = number of loops

    (open-closed strings: cfr talks by Schlotterer and Stieberger)

- **Main topics**

  I prototypes of closed superstring loop amplitudes
  II construction of amplitudes in the RNS formulation
  III supermoduli
  IV low energy effective interactions
  V speculation on ambi-twistors

I. Prototypes

- **Type IIB four-graviton amplitude to one-loop order** (Green, Schwarz 1982)

$$\mathcal{A}^{(1)}(\varepsilon_i, k_i) = \mathcal{K}\tilde{\mathcal{K}} \int_{\mathcal{M}_1} \frac{d^2\tau}{(\mathrm{Im}\tau)^2} \mathcal{B}^{(1)}(s_{ij}|\tau)$$

  - Factorized basis of polarization tensors $\varepsilon_i^{\mu\nu} = \varepsilon_i^\mu \tilde{\varepsilon}_i^\nu$

$$s_{ij} = -\alpha'(k_i + k_j)^2/4 \qquad\qquad \mathcal{K} = t_8\, \varepsilon_1 \cdots \varepsilon_4\, k_1 \cdots k_4$$

  - Partial amplitude $\mathcal{B}$ is a modular function in $\tau \in \mathcal{M}_1 = \mathbb{H}/SL(2,\mathbb{Z})$

$$\mathcal{B}^{(1)}(s_{ij}|\tau) = \int_{\Sigma^4} \prod_{i=1}^{4} \frac{d^2 z_i}{\mathrm{Im}\tau} \exp\Big(\sum_{i<j} s_{ij}\, G(z_i - z_j|\tau) \Big)$$

  - $G(z|\tau)$ is the scalar Green function on the torus Σ of modulus τ.
  - Analogous formulas for Heterotic strings and more external states.

- **Singularity structure**
  - For fixed τ integrations over Σ produce poles in $\mathcal{B}$ at positive integers s_{ij}.
  - The integral over τ converges absolutely only for $\mathrm{Re}(s_{ij}) = 0$.
  - Analytic continuation requires decomposition of $\mathcal{M}_1$.
  - Massless singularities are produced by $\tau \to i\infty$ (cfr ambi-twistor string).

Loop momenta

- **Loop momenta were hidden but may be exposed**
  - Choose a canonical basis of cycles A, B of $H_1(\Sigma, \mathbb{Z})$.
  - Choose loop momentum p_μ flowing through the cycle A,

$$\int_{\mathcal{M}_1} \frac{d^2\tau}{(\mathrm{Im}\tau)^2} \mathcal{B}^{(1)}(s_{ij}|\tau) = \int_{\mathbb{R}^{10}} d^{10}p \int_{\mathcal{M}_1} \int_{\Sigma^4} \Big| \mathcal{F}(z_i, k_i, p|\tau) \Big|^2$$

  - The partial amplitude $\mathcal{F}$ is **locally holomorphic** in τ and z_i

$$\mathcal{F}(z_i, k_i, p|\tau) = d\tau \prod_{i=1}^{4} dz_i \, e^{i\pi\tau p^2 + 2\pi i p \sum_i k_i z_i} \prod_{i<j} \vartheta_1(z_i - z_j|\tau)^{-s_{ij}}$$

  - at the cost of non-trivial monodromy

$$\begin{aligned} \mathcal{F}(z_i + \delta_{i,\ell} A, k_i, p|\tau) &= e^{2\pi i k_\ell \cdot p} \mathcal{F}(z_i, k_i, p|\tau) \\ \mathcal{F}(z_i + \delta_{i,\ell} B, k_i, p|\tau) &= \mathcal{F}(z_i, k_i, p + k_\ell|\tau) \end{aligned}$$

  - Modular invariance of $\mathcal{A}^{(1)}$ guarantees independence of choices.
  - Hermitian pairing of $\mathcal{F}$ and $\bar{\mathcal{F}}$ is familiar from 2-d CFT where loop momenta p_μ label conformal blocks of 10 copies of $c = 1$.

Genus two

- **Siegel Upper half space $\mathcal{S}_2$**

$$\mathcal{S}_2 = \{\Omega_{IJ} = \Omega_{JI} \in \mathbb{C} \text{ with } I,J=1,2 \text{ and } Y = \mathrm{Im}\Omega > 0\}$$

  – $Sp(4,\mathbb{R})$ acts by $\Omega \to (A\Omega+B)(C\Omega+D)^{-1}$

$$M^t J M = J \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

  – $\mathcal{S}_2$ has $Sp(4,\mathbb{R})$-invariant metric ds_2^2 and volume form $d\mu_2$

$$ds_2^2 = \sum_{I,J,K,L=1,2} Y^{-1}_{IJ}\, d\bar\Omega_{JK}\, Y^{-1}_{KL}\, d\Omega_{LI}$$

- **Compact Riemann surfaces Σ**
  – Choose canonical homology basis of A_I, B_I cycles for $H_1(\Sigma,\mathbb{Z})$.
  – ω_I dual holomorphic (1,0) forms,

$$\oint_{A_I} \omega_J = \delta_{IJ} \qquad \oint_{B_I} \omega_J = \Omega_{IJ}$$

  – Riemann relations imply $\Omega \in \mathcal{S}_2$;
  – Modular group $Sp(4,\mathbb{Z})$; moduli space $\mathcal{M}_2 = \mathcal{S}_2 / Sp(4,\mathbb{Z})$.

Two-loop Type II superstring amplitudes

- **Type II four-graviton amplitude at genus 2** (ED, Phong 2001 – 2005)

$$\mathcal{A}^{(2)}(\varepsilon_i, k_i) = \mathcal{K}\tilde{\mathcal{K}} \int_{\mathcal{M}_2} d\mu_2 \, \mathcal{B}^{(2)}(s_{ij}|\Omega)$$

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \bar{\mathcal{Y}}}{(\det \operatorname{Im} \Omega)^2} \exp\Big(\sum_{i<j} s_{ij} \, G(z_i, z_j|\Omega) \Big)$$

  – $G(z_i, z_j)$ is the genus-two scalar Green function;

$$\mathcal{Y} = (t-u)\Delta(z_1,z_2) \wedge \Delta(z_3,z_4) + (s-t)\Delta(z_1,z_3) \wedge \Delta(z_4,z_2) + (u-s)\Delta(z_1,z_4) \wedge \Delta(z_2,z_3)$$

  – $\Delta(z_i, z_j)$ is a bi-holomorphic form independent of s, t, u.

$$\Delta(z,w) = \omega_1(z) \wedge \omega_2(w) - \omega_2(z) \wedge \omega_1(w)$$

  – Representation with loop momenta and holomorphic product as at 1-loop
  – reproduced (with fermions) in pure spinor formulation (Berkovits, Mafra 2005)

- **Singularity structure**
  – For fixed Ω integrations over Σ produce poles in $\mathcal{B}$ at positive integers s_{ij}.
  – The integral over Ω requires analytic continuation beyond $\operatorname{Re}(s_{ij}) = 0$.
  – Leading massless singularities from $\Omega_{11}, \Omega_{22} \to i\infty$ (cfr ambi-twistor string).

Two-loop Heterotic string amplitudes

- **Heterotic four NS boson amplitude at genus 2** (ED, Phong 2005)

$$\mathcal{A}_{\mathcal{O}}^{(2)}(\varepsilon_i, \tilde\varepsilon_i, k_i) = \mathcal{K} \int_{\mathcal{M}_2} d\mu_2 \, \mathcal{B}_{\mathcal{O}}^{(2)}(\tilde\varepsilon_i, k_i|\Omega)$$

$$\mathcal{B}_{\mathcal{O}}^{(2)}(s_{ij}|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \overline{\mathcal{W}_{\mathcal{O}}(\tilde\varepsilon_i, k_i)}}{(\det \mathrm{Im}\Omega)^2 \, \overline{\Psi_{10}(\Omega)}} \exp\Big(\sum_{i<j} s_{ij} \, G(z_i, z_j) \Big)$$

  - $\Psi_{10}(\Omega)$ is the Igusa cusp form.
  - Dependence on $\mathcal{O}$ according to channel:
    - ⋆ 4 gravitons $\mathcal{R}^4$
    - ⋆ 2 gravitons + 2 gauge bosons $\mathcal{R}^2 \mathrm{tr}(\mathcal{F}^2)$;
    - ⋆ or 4 gauge with 2 channels $(\mathrm{tr}\mathcal{F}^2)(\mathrm{tr}\mathcal{F}^2)$ and $\mathrm{tr}(\mathcal{F}^4)$
  - For example,

$$\mathcal{W}_{\mathcal{R}^4}(\tilde\varepsilon_i, k_i) = \frac{\langle \prod_{i=1}^4 \tilde\varepsilon_i \cdot \partial \tilde x(z_i) \, e^{ik_i \cdot \tilde x(z_i)} \rangle}{\langle \prod_{i=1}^4 e^{ik_i \cdot \tilde x(z_i)} \rangle}$$

$$\langle \tilde x^\mu(z) \tilde x^\nu(w) \rangle = -\eta^{\mu\nu} \, \ln E(z,w)$$

  - Gauge parts are obtained by the correlators of the current $(0,1)$-forms.

II. Construction of amplitudes in RNS formulation

- **Spinors ψ^μ on the worldsheet**
  - With Minkowski worldsheet signature
    - ⋆ opposite chirality spinors $\psi^\mu, \tilde\psi^\mu$ are *independent* Majorana-Weyl
    - ⋆ Dirac eq solved by $\psi^\mu(\tau - \sigma)$ and $\tilde\psi^\mu(\tau + \sigma)$
  - With Euclidean worldsheet signature
    - ⋆ right (-movers) $\tau - \sigma \to z$
    - ⋆ left (-movers) $\tau + \sigma \to \tilde z$
    - ⋆ worldsheet spinors $\psi^\mu, \tilde\psi^\mu$ become complex conjugate Weyl spinors
  - On a compact Riemann surface of genus g
    - ⋆ In local complex coordinates $(z, \tilde z)$ solved by $\psi^\mu(z)$ and $\tilde\psi^\mu(\tilde z)$
    - ⋆ Globally, ψ^μ are sections of spin bundle S (resp. $\tilde\psi^\mu$ of $\tilde S$)
    - ⋆ same bundle S (resp. $\tilde S$) for all μ by Lorentz invariance in $\mathbb{R}^{10}$
    - ⋆ $S^{\otimes 2} \approx \tilde S^{\otimes 2} \approx T^*_{(1,0)}(\Sigma)$
    - ⋆ 2^{2g} distinct spin bundles labelled by their spin structure.
    - ⋆ S and $\tilde S$ must be independent of one another.

Worldsheet fields

- **Type II**
  - $x^\mu, \psi^\mu, \tilde\psi^\mu, \ \mu = 0, 1, \cdots, 9$
  - $g_{mn} \longrightarrow b, c, \tilde b, \tilde c$ ghosts $+ g^0_{mn}$ (parametrizing even moduli)
    $\chi_m, \tilde\chi_m \longrightarrow \beta, \gamma, \tilde\beta, \tilde\gamma$ ghosts $+ \chi^0_m, \tilde\chi^0_m$ (parametrizing odd moduli)
  - $\psi^\mu, \beta, \gamma, \chi_m$ all have the same spin structure δ
    $\tilde\psi^\mu, \tilde\beta, \tilde\gamma, \tilde\chi_m$ all have the same spin structure $\tilde\delta$

- **Heterotic**
  - $x^\mu, \psi^\mu, \tilde\lambda^\alpha, \ \alpha = 1 \cdots 32$
  - $g_{mn} \longrightarrow b, c, \tilde b, \tilde c$ ghosts $+ g^0_{mn}$ (parametrizing even moduli)
    $\chi_m \longrightarrow \beta, \gamma$ ghosts $+ \chi^0_m$ (parametrizing odd moduli)
  - $\psi^\mu, \beta, \gamma, \chi_m$ all have the same spin structure δ
    $\tilde\lambda^\alpha$ grouped into $16_{\tilde\delta_1} \oplus 16_{\tilde\delta_2}$ for $E_8 \times E_8$ or into $32_{\tilde\delta}$ for $Spin(32)/\mathbb{Z}_2$

Chiral amplitudes

- **Both Type II and Heterotic require chiral amplitudes**
  - Link chirality with holomorphicity on (super) moduli space
  - Chiral amplitude from non-chiral by *chiral splitting*
    (ED, Phong 1988-91)
  - Approach via holomorphic super-Riemann surfaces
    (Witten, arXiv:1209.2459, 1209.5461, 1304.2832, 1306.3621)
- **Full amplitudes will be constructed by pairing left and right**
  - cfr "double copy construction"

Conformal structures and deformations

- **An orientable 2-dim manifold with Riemannian metric is a Riemann surface**
  - complex manifold (holomorphic transition functions)
  - complex structure $J : T(\Sigma) \to T(\Sigma), \quad J^2 = -I$ is integrable
  - in terms of the metric and local coordinates, $J^m{}_n = \sqrt{g}\, g^{mp} \varepsilon_{pn}$
  - Moduli space $\mathcal{M}_h = \{J\}/\mathrm{Diff}(\Sigma)$ of genus h Riemann surfaces

- **Moduli space itself is a complex "manifold" (actually orbifold)**
  - Its tangent space splits $T(\mathcal{M}_g) = T_{(1,0)}(\mathcal{M}_g) \oplus T_{(0,1)}(\mathcal{M}_g)$
  - associated Beltrami differentials $\delta J = \delta J_{\bar z}{}^z \oplus \delta J_z{}^{\bar z}$

$$\dim_{\mathbb{C}} \mathcal{M}_h = \begin{cases} 0 & h = 0 \\ 1 & h = 1 \\ 3h-3 & h \geq 2 \end{cases}$$

- **Beltrami differentials provide a parametrization of finite deformations**
  - Fix complex structure J and associated local complex coordinates $(z, \tilde z)$
  - metric is conformally flat $ds^2 = |dz|^2$
  - finite deformations to $ds^2 = |dz + \mu_{\bar z}{}^z d\bar z|^2$ with $\delta \mu_{\bar z}{}^z = \delta J_{\bar z}{}^z$

Super conformal structures

- **Super Riemann surfaces** (Friedan, Martinec, Shenker 1986)
  - complex dimension $1|1$ locally isomorphic to $\mathbb{C}^{1|1}$
  - transition functions are superconformal in $z|\theta$
  - transforming the superderivative $D_\theta = \partial_\theta + \theta\partial_z$ by scaling

$$f : (z|\theta) \to (\hat z|\hat\theta) \qquad\qquad D_\theta = F(z,\theta) D_{\hat\theta}$$

  - Locally: generates the holomorphic $\mathcal{N}=1$ superconformal algebra
  - Globally: $T\Sigma$ has a completely non-integrable subbundle $\mathcal{D}$ of rank $0|1$
  - Moduli space $\mathfrak{M}_h = \{\mathcal{J}\}/\mathrm{Diff}(\Sigma)$
    $=$ equivalence classes of superconformal structures $\mathcal{J}$

$$\dim_{\mathbb{C}} \mathfrak{M}_h = \begin{cases} 0|0 & h=0 \\ 1|0 \text{ or } 1|1 & h=1 \quad \text{even or odd spin structure} \\ 3h-3|2h-2 & h \geq 2 \end{cases}$$

  - odd modulus at $h=1$ odd spin structure is a book keeping device
  - odd moduli really first appear at genus 2

  - Global extension to SRS with NS and R punctures (Witten 2011)

Superstring worldsheet and moduli spaces (Witten 2011)

- **Heterotic**
  - Left : Riemann surface Σ_L, moduli space $\mathcal{M}_L$
  - Right : super Riemann surface Σ_R, moduli space $\mathfrak{M}_R$
  - Worldsheet is a cycle $\Sigma \subset \Sigma_L \times \Sigma_R$ of dimension $1|1$
    subject to $\Sigma_{\rm red} = {\rm diag}(\Sigma_{L\,{\rm red}} \times \Sigma_{R\,{\rm red}})$: $\tilde z = \bar z + {\rm nilpotent}$
    (reduced space obtained by setting all nilpotent variables to zero)
  - Moduli space is a cycle $\Gamma \subset \mathcal{M}_L \times \mathfrak{M}_R$ of dim $3h-3|2h-2$ for $h \geq 2$

- **Type II**
  - Left : super Riemann surface Σ_L, moduli space $\mathfrak{M}_L$
  - Right : super Riemann surface Σ_R, moduli space $\mathfrak{M}_R$
  - Worldsheet is a cycle $\Sigma \subset \Sigma_L \times \Sigma_R$ of dimension $1|2$
    subject to $\Sigma_{\rm red} = {\rm diag}(\Sigma_{L\,{\rm red}} \times \Sigma_{R\,{\rm red}})$
  - Moduli space is a cycle $\Gamma \subset \mathfrak{M}_L \times \mathfrak{M}_R$ of dim $3h-3|4h-4$ for $h \geq 2$

- **Super-Stokes theorem ensures independence of the choice of cycles**
  - in amplitudes with BRST invariant vertex operators

Worldsheet action for Heterotic strings

- **Worldsheet is** $\Sigma \subset \Sigma_L \times \Sigma_R$
  - superconformal structure $\mathcal{J}$ for Σ_R with local coordinates $z|\theta$
  - conformal structure $\tilde{J}$ for Σ_L, with local coordinates $\tilde{z}$

$$\begin{aligned} X^\mu(\tilde{z}, z, \theta) &= x^\mu(\tilde{z}, z) + \theta \psi^\mu(\tilde{z}, z) \\ \Lambda^\alpha(\tilde{z}, z, \theta) &= \lambda_-^\alpha(\tilde{z}, z) + \theta \ell^\alpha(\tilde{z}, z) \end{aligned}$$

- **Superconformal invariant matter action**

$$I_M[X^\mu, \Lambda^\alpha, \mathcal{J}, \tilde{J}] = \int_\Sigma [d\tilde{z} dz|d\theta] \Big[\partial_{\tilde{z}} X^\mu D_\theta X_\mu + \sum_\alpha \Lambda^\alpha D_\theta \Lambda^\alpha \Big]$$

  - Integrating out θ, we recover familiar action

$$I_M = \int d\tilde{z} dz \Big[\partial_{\tilde{z}} x^\mu \partial_z x_\mu + \psi^\mu \partial_{\tilde{z}} \psi_\mu + \sum_\alpha (\lambda^\alpha \partial_z \lambda^\alpha + \ell^\alpha \ell^\alpha) \Big]$$

  - Superconformal algebra on fields generated by

$$\begin{aligned} T_{zz} &= -\frac{1}{2} \partial_z x^\mu \partial_z x_\mu + \frac{1}{2} \psi^\mu \partial_z \psi_\mu \\ S_{z\theta} &= \psi^\mu \partial_z x_\mu \end{aligned}$$

Deformations of superconformal structures

- **Under deformation of the conformal structure $\tilde J$ on Σ_L**

$$\delta I = \int d\tilde z dz \, \delta \tilde J_z{}^{\tilde z} \, T_{\tilde z \tilde z}$$

- **Under deformation of superconformal structure $\mathcal{J}$ on Σ_R**

$$\delta I = \int d\tilde z dz \, \left[\delta J_{\tilde z}{}^z \, T_{zz} + \delta \chi_{\tilde z}{}^\theta \, S_{z\theta} \right]$$

  – $\chi_{\tilde z}{}^\theta$ is the "worldsheet gravitino" field
  – δJ and $\delta \tilde J$ are even Beltrami differentials

- **Assemble deformations into super-conformal invariant action**

$$\begin{aligned}
\mathcal{S}_{z\theta} &= S_{z\theta} + \theta T_{zz} \\
\delta H_{\tilde z}{}^z &= \delta J_{\tilde z}{}^z + \theta \delta \chi_{\tilde z}{}^\theta \\
\delta H_\theta{}^{\tilde z} &= \theta \delta \tilde J_z{}^{\tilde z} + \text{auxiliary} \\
\delta I &= \int_\Sigma [d\tilde z dz | d\theta] \left(\delta H_{\tilde z}{}^z \mathcal{S}_{z\theta} + \delta H_\theta{}^{\tilde z} T_{\tilde z \tilde z} \right)
\end{aligned}$$

Full Heterotic string amplitude

- **Parametrization of the deformations $\delta H_{\bar z}{}^z$ and $\delta H_\theta{}^{\tilde z}$ by slice in $\{\mathcal{J}, \tilde J\}$**

$$\delta H_{\bar z}{}^z = \partial_{\bar z} V^z + \sum_A H_A dm_A \qquad H_A = \partial \mathcal{J}_{\bar z}{}^z / \partial m_A$$

$$\delta H_\theta{}^{\tilde z} = D_\theta V^{\tilde z} + \sum_a \tilde H_a d\tilde m_a \qquad \tilde H_a = \partial(\theta J_z{}^{\tilde z}) / \partial \tilde m_a$$

  – Introducing ghost fields

$$V^z \to C^z = c^z + \theta \gamma^\theta \qquad \delta H_{\bar z}{}^z \to B_{z\theta} = \beta_{z\theta} + \theta b_{zz}$$

$$V^{\tilde z} \to \tilde C^{\tilde z} = c^{\tilde z} + \text{aux} \qquad \delta H_\theta{}^{\tilde z} \to \tilde B_{\tilde z \tilde z} = b_{\tilde z \tilde z} + \text{aux}$$

- **Super conformal invariant ghost action**

$$I_{\rm gh} = \int_\Sigma [d\tilde z dz | d\theta] \left[B_{z\theta} \partial_{\bar z} C^z + \tilde B_{\tilde z \tilde z} D_\theta C^{\tilde z} + B_{z\theta} \sum_A H_A dm_A + \tilde B_{\tilde z \tilde z} \sum_a \tilde H_a d\tilde m_a \right]$$

- **Assembling all factors, we obtain the integrand on $\mathcal{M}_L \times \mathfrak{M}_R$**

$$\int D(XB\tilde B C \tilde C)\, \mathcal{V}_1 \cdots \mathcal{V}_n \prod_{a,A} [d\tilde m_a dm_A]\, \delta\left(\langle \tilde B, \tilde H_a \rangle \right) \delta\left(\langle B, H_A \rangle \right) e^{-I_M - I_{\rm gh}}$$

  – $\mathcal{V}_1 \cdots \mathcal{V}_n$ are BRST-invariant vertex operators.

Chiral amplitudes via Chiral splitting

- **In Type II superstrings, both Σ_L and Σ_R are super Riemann surfaces**
  - Deformation must involve independent ${\chi_{\bar z}}^\theta$ and ${\chi_z}^{\tilde\theta}$
  - ghosts B, C couple only to ${\chi_{\bar z}}^\theta$ and $\tilde B, \tilde C$ only to ${\chi_z}^{\tilde\theta}$
  - 2-d supergravity action for X^μ couples left to right chiralities

$$I_m = \int_\Sigma d^2z \Big[\partial_z x^\mu \partial_{\bar z} x_\mu + \psi^\mu \partial_{\bar z}\psi_\mu - {\chi_{\bar z}}^\theta \psi^\mu \partial_z x_\mu + \cdots + {\chi_{\bar z}}^\theta {\chi_z}^{\tilde\theta} \psi^\mu \tilde\psi^\nu\Big]$$

- **Chiral splitting is established at fixed internal loop momenta**
  - Fix canonical homology basis $A_I, B_I, I = 1, \cdots, h$ on Σ of genus h
  - h independent internal loop momenta p_I^μ are defined across the cycles A_I

    (Verlinde, Verlinde; ED, Phong 1988)

$$p_I^\mu = \oint_{A_I} dz\, \partial_z x^\mu + \oint_{A_I} d\tilde z\, \partial_{\tilde z} x^\mu$$

  - Amplitude is an integral over p_I^μ of a product of chiral amplitudes

$$\int_{\mathbb{R}^{10}} dp_I^\mu\, \mathcal{F}_L({\mu_z}^{\tilde z}, {\chi_z}^{\tilde\theta}, \varepsilon_i, k_i, p_I^\mu)\, \mathcal{F}_R({\mu_{\tilde z}}^z, {\chi_{\tilde z}}^\theta, \tilde\varepsilon_i, k_i, p_I^\mu)$$

  - with k_i external momenta, and polarization tensors $\varepsilon_i^{\mu\nu} = \varepsilon_i^\mu \tilde\varepsilon_i^\nu$

Chiral amplitudes

- **Chiral amplitude $\mathcal{F}_R$ has supermoduli deformations of only Σ_R**

  (similarly $\mathcal{F}_L$ has supermoduli deformations of only Σ_L)
  - $\mathcal{F}_R$ computed with effective rules for chiral fields x_+, ψ_+^μ
  - and chiral vertex operators $\mathcal{V}_1^+ \cdots \mathcal{V}_N^+$

$$\mathcal{F}_R = \left\langle \mathcal{V}_1^+ \cdots \mathcal{V}_N^+ \, e^{p_I^\mu \oint_{B_I} dz \partial_z x_+^\mu} \exp \int_{\Sigma_{\rm red}} d\tilde z dz (\mu_{\tilde z}{}^z T_{zz} + \chi_{\tilde z}{}^\theta S_{z\theta}) \right\rangle_{x_+, \psi_+}$$

  - with stress tensor and supercurrent evaluated on chiral fields x_+, ψ_+^μ

$$\begin{aligned} T_{zz} &= -\frac{1}{2} \partial_z x_+^\mu \partial_z x_+^\nu \eta_{\mu\nu} + \frac{1}{2} \psi_+^\mu \partial_z \psi_+^\nu \eta_{\mu\nu} + T_{zz}^{\rm gh}(b, c, \beta, \gamma) \\ S_{z\theta} &= -\frac{1}{2} \psi_+^\mu \partial_z x_+^\nu \eta_{\mu\nu} + S_{z\theta}^{\rm gh}(b, c, \beta, \gamma) \end{aligned}$$

  - $\langle \cdots \rangle_{x_+, \psi_+}$ indicates Wick contractions of x_+^μ, ψ_+^μ with
  - effective chiral Green functions

$$\begin{aligned} \langle \psi_+^\mu(z) \psi_+^\nu(w) \rangle &= -\eta^{\mu\nu} S(z,w) && \text{Szegö kernel} \\ \langle x_+^\mu(z) x_+^\nu(w) \rangle &= -\eta^{\mu\nu} \ln E(z,w) && \text{prime form} \end{aligned}$$

III. Parametrization of supermoduli

- **Superconformal structure $\mathcal{J} \in \mathfrak{M}_h$ specifies transition functions**
  - Practical calculations mostly use parametrization by gravitino field ${\chi_{\tilde z}}^\theta$
- **Local parametrization of bosonic moduli (in conformal-invariant theory)**
  - Complex structure J with metric $g = |dz|^2$ in local coordinates $(z, \tilde z)$
  - deformation of complex structure by Beltrami differential to $g' = |dz + \mu d\tilde z|^2$
  - realized in CFT by insertion of $\int_\Sigma d\tilde z dz \, {\mu_{\tilde z}}^z \, T_{zz}$
- **Local parametrization of supermoduli (in superconformal-invariant theory)**
  - Start with $\Sigma_{\rm red}$ with complex structure given by $J \in \mathfrak{M}_{\rm red}$
  - Deformation of super conformal structure realized by insertion of T and S

$$\int_{\Sigma_{\rm red}} d\tilde z dz \, \left({\mu_{\tilde z}}^z T_{zz} + {\chi_{\tilde z}}^\theta S_{z\theta} \right)$$

  - χ and μ parametrized by local odd coordinates on $\mathfrak{M}_h$
  - nilpotency guarantees that these deformations are exact
- **Globally, there is no holomorphic projection $\mathfrak{M}_h \to \mathcal{M}_h$ for $h \geq 3$**
  - proven for $h \geq 5$ (Donagi, Witten 2013)

The super period matrix

- **For $h=2$ and even spin structure there is a natural projection $\mathfrak{M}_2 \to \mathcal{M}_2$ via the super period matrix** (E.D. & Phong 1988)
  - Fix canonical homology basis A_I, B_I for $H^1(\Sigma, \mathbf{Z})$
  - There exist $h=2$ superholomorphic forms $\hat\omega_I$ with super periods $\hat\Omega$

$$\oint_{A_I} \hat\omega_J = \delta_{IJ} \qquad \qquad \oint_{B_I} \hat\omega_J = \hat\Omega_{IJ}$$

  - Explicit formula in terms of (g, χ), and Szego kernel S_δ

$$\hat\Omega_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int\int \omega_I(z)\chi(z) S_\delta(z,w)\chi(w)\omega_J(w)$$

  - $\hat\Omega_{IJ}$ is locally supersymmetric with $\hat\Omega_{IJ} = \hat\Omega_{JI}$ and $\mathrm{Im}\,\hat\Omega > 0$
  - Every $\hat\Omega$ corresponds to an ordinary Riemann surface
  - Szegö kernel $S_\delta(z,w|\Omega)$ is non-singular on $\mathcal{M}_2$

$\Rightarrow$ **Projection using $\hat\Omega$ is holomorphic and natural for genus 2**

Projecting and pairing Chiral Amplitudes

- **Chiral Amplitudes on $\mathfrak{M}_2$**
  - Natural supersymmetric parametrization of $\mathfrak{M}_2$ by $(\hat\Omega_{IJ}, \zeta^\alpha)$
  - involve measure $d\mu[\delta](\hat\Omega, \zeta)$ and correlation functions $\mathcal{C}[\delta](\varepsilon_i, k_i, p_I|\hat\Omega, \zeta)$

- **Projection to chiral amplitudes on $\mathcal{M}_2$**
  - by integrating over ζ and summing over δ at fixed $\hat\Omega$

$$\mathcal{R}(\varepsilon_i, k_i, p_I|\hat\Omega) = \sum_\delta \int_\zeta d\mu[\delta](\hat\Omega, \zeta)\, \mathcal{C}[\delta](\varepsilon_i, k_i, p_I|\hat\Omega, \zeta)$$

$$\mathcal{L}(\tilde\varepsilon_i, k_i, p_I|\hat\Omega) = \sum_{\tilde\delta} \int_{\tilde\zeta} d\mu[\tilde\delta](\hat\Omega, \tilde\zeta)\, \mathcal{C}[\tilde\delta](\tilde\varepsilon_i, k_i, p_I|\hat\Omega, \tilde\zeta)$$

  - for heterotic, $\mathcal{L}$ is chiral half of bosonic string, has no integral in $\tilde\zeta$
  - phase factors determined by $Sp(4,\mathbb{Z})$ modular invariance

- **Pairing left and right chiral amplitudes, integrating over p_I and Ω**

$$\mathcal{A}^{(2)}(\varepsilon_i, \tilde\varepsilon_i, k_i) = \int_{\mathcal{M}_2} d\Omega \int dp_I^\mu\, \mathcal{R}(\varepsilon_i, k_i, p_I|\Omega)\, \overline{\mathcal{L}(\tilde\varepsilon_i, k_i, p_I|\Omega)}$$

  - Integral over p_I can be carried out explicitly.

Singularities in the projection $\mathfrak{M}_2 \to \mathcal{M}_2$

- **Projection $\mathfrak{M}_2 \to \mathcal{M}_2$ is holomorphic, but integration extends to boundary**
  - are there singularities on the boundary of $\mathfrak{M}_2$?

$$\Omega = \begin{pmatrix} \tau & u \\ u & \sigma \end{pmatrix} \qquad \begin{matrix} u \to 0 & \text{separating node} \\ \sigma \to i\infty & \text{non-separating node} \end{matrix}$$

  - Key ingredient in $\hat{\Omega}$ is the Szegö kernel

$$S_\delta(z, w|\Omega) = \frac{\vartheta[\delta](z - w|\Omega)}{\vartheta[\delta](0|\Omega) E(z, w)}$$

  - As $u \to 0$ we have $\vartheta[\delta](0|\Omega) \to \vartheta[\delta_1](0|\tau)\,\vartheta[\delta_2](0|\tau)$
  - Even $\delta = [\delta_1, \delta_2]$ with δ_1, δ_2 odd produces a singularity in S_δ and $\hat{\Omega}$

- **Physical effects**
  - singularity killed by ψ-zero modes in $\mathbb{R}^{10}$ (space-time susy)
  - contribution when susy is broken by radiative corrections (Witten 2013)
  - Two-loop vacuum energy in Heterotic strings on CY orbifold $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$
    - ⋆ zero for $E_8 \times E_8 \to E_6 \times E_8$ with unbroken susy
    - ⋆ non-zero for $Spin(32)/\mathbb{Z}_2 \to SO(26) \times U(3)$ with broken susy

      (ED, Phong 2013; Berkovits, Witten 2014)
  - cfr one-loop susy breaking (Atick, Dixon, Sen; Dine Seiberg, Witten 1987)

Singularities in the projection $\mathfrak{M}_3 \to \mathcal{M}_3$

- **Some basic structure theorems**
  - A hyper-elliptic surface is a branched double cover of the sphere S^2
  - All genus 1 and all genus 2 surfaces are hyper-elliptic
  - Hyper-elliptic surfaces form a subvariety of $\mathcal{M}_3$ of complex codimension 1 (referred to as the hyper-elliptic divisor)

- **The period matrix (for even spin structure) for genus 3 is given by**

$$\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int\!\!\int \omega_I(z)\chi(z)S_\delta(z,w)\chi(w)\omega_J(w) + \mathcal{O}(\chi^4)$$

  - For genus 3, $\vartheta[\delta](0|\Omega) = 0$ for some δ on the hyper-elliptic divisor of $\mathfrak{M}_3$ (which crosses the interior of $\mathfrak{M}_3$)
  - the presence of additional Dirac zero modes kills the effect of this singularity

  - But it is another δ that produces a singularity in $\hat{\Omega}$ (subtle) (Witten 2015)
  - Rules out earlier proposals for the genus 3 superstring measure

IV. Low Energy Effective Interactions

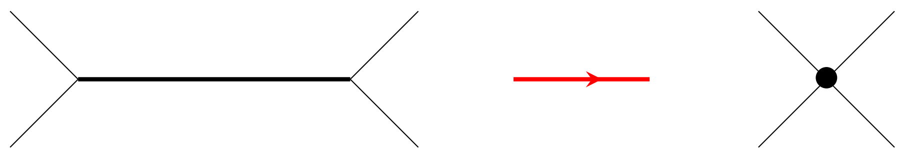
- **Four-graviton amplitude in Type II at genus 0**

$$\mathcal{A}^{(0)} = \mathcal{K}\tilde{\mathcal{K}}\,\frac{1}{stu}\,\frac{\Gamma(1-s)\,\Gamma(1-t)\,\Gamma(1-u)}{\Gamma(1+s)\,\Gamma(1+t)\,\Gamma(1+u)}$$

– Low energy expansion corresponds to $|s|,\ |t|,\ |u| \ll 1$

$$\underbrace{\frac{1}{stu}}_{\text{massless}} + \underbrace{2\zeta(3)}_{\mathcal{R}^4} + \underbrace{\zeta(5)(s^2+t^2+u^2)}_{D^4\mathcal{R}^4} + \underbrace{2\zeta(3)^2 stu}_{D^6\mathcal{R}^4} + \cdots$$

– Exchange of massive string states produces local effective interactions.



- **Four-graviton amplitudes in Type II at higher genus**
  – provide effective interactions of the type $D^{2w}\mathcal{R}^4$ for $w \geq 0$
  – interplay with predictions from supersymmetry and S-duality in Type IIB

(cfr talk by Michael Green)

Genus-one effective interactions in Type II

- **Recall the Type II genus-one four-graviton amplitude,**

$$\mathcal{A}^{(1)}(\varepsilon_i, k_i) = \mathcal{K}\tilde{\mathcal{K}} \int_{\mathcal{M}_1} \frac{d^2\tau}{\tau_2^2} \mathcal{B}^{(1)}(s_{ij}|\tau)$$

$$\mathcal{B}^{(1)}(s_{ij}|\tau) = \prod_{i=1}^{4} \int_\Sigma \frac{d^2 z_i}{\tau_2} \exp\Big\{ \sum_{i<j} s_{ij}\, G(z_i - z_j|\tau) \Big\}$$

- **For fixed τ, expansion of $\mathcal{B}^{(1)}$ in powers of s_{ij}**
  - finite radius of convergence $|s_{ij}| < 1$
  - integrals $\int_{\Sigma^4} G^w$ are convergent modular functions, contribute to $D^{2w}\mathcal{R}^4$
  - G is given by a Fourier sum over torus momenta $(m,n) \in \mathbb{Z}^2$

$$G(\alpha + \beta\tau|\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2}{\pi|m+\tau n|^2} e^{2\pi i(m\beta - n\alpha)}$$

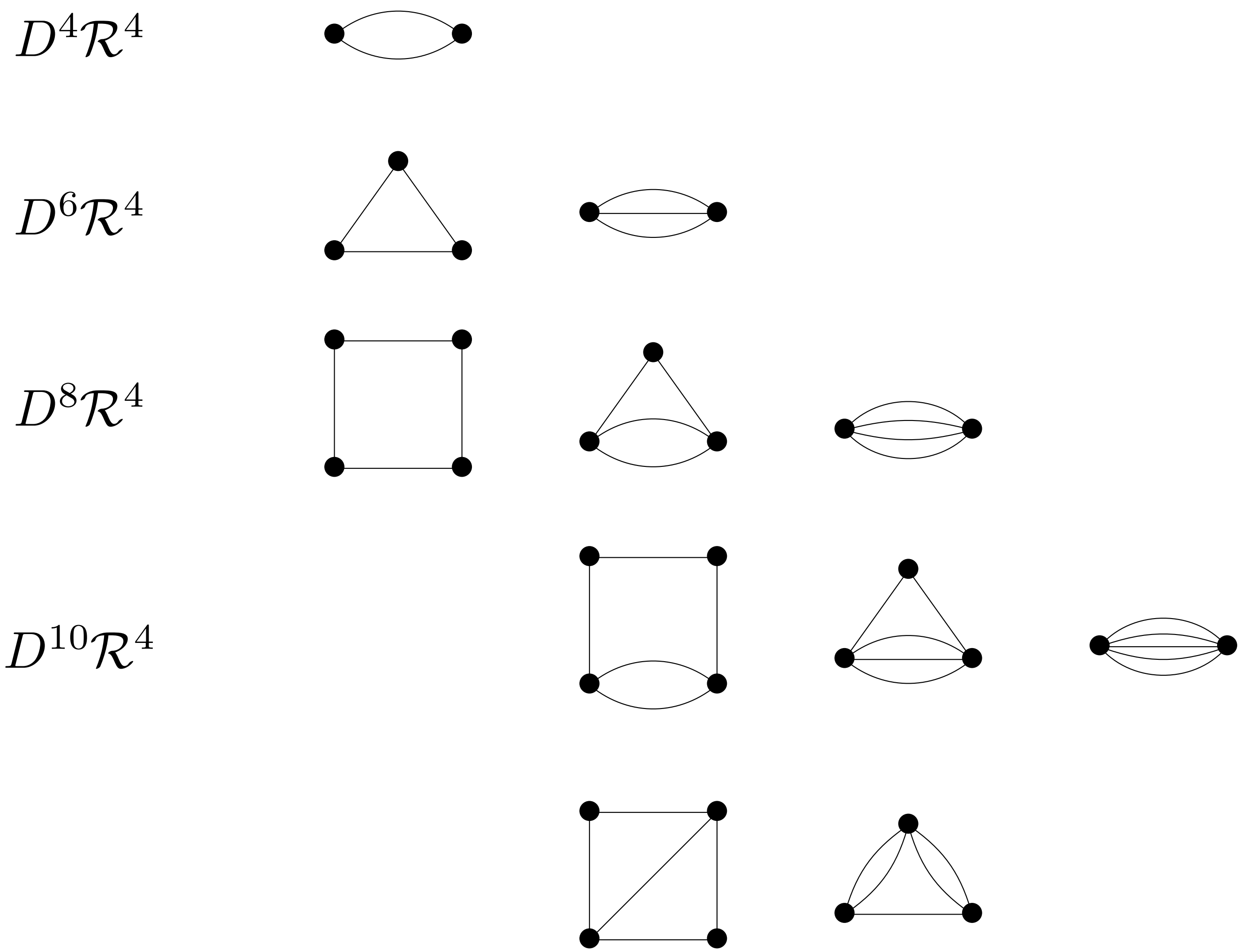
- **Integrating $\mathcal{B}^{(1)}$ over τ produces branch cuts in s_{ij} starting at 0**
  - extract non-analytic part in s_{ij} prior to extracting effective interactions

    (Green, Vanhove 2000; Green, Russo Vanhove 2008)

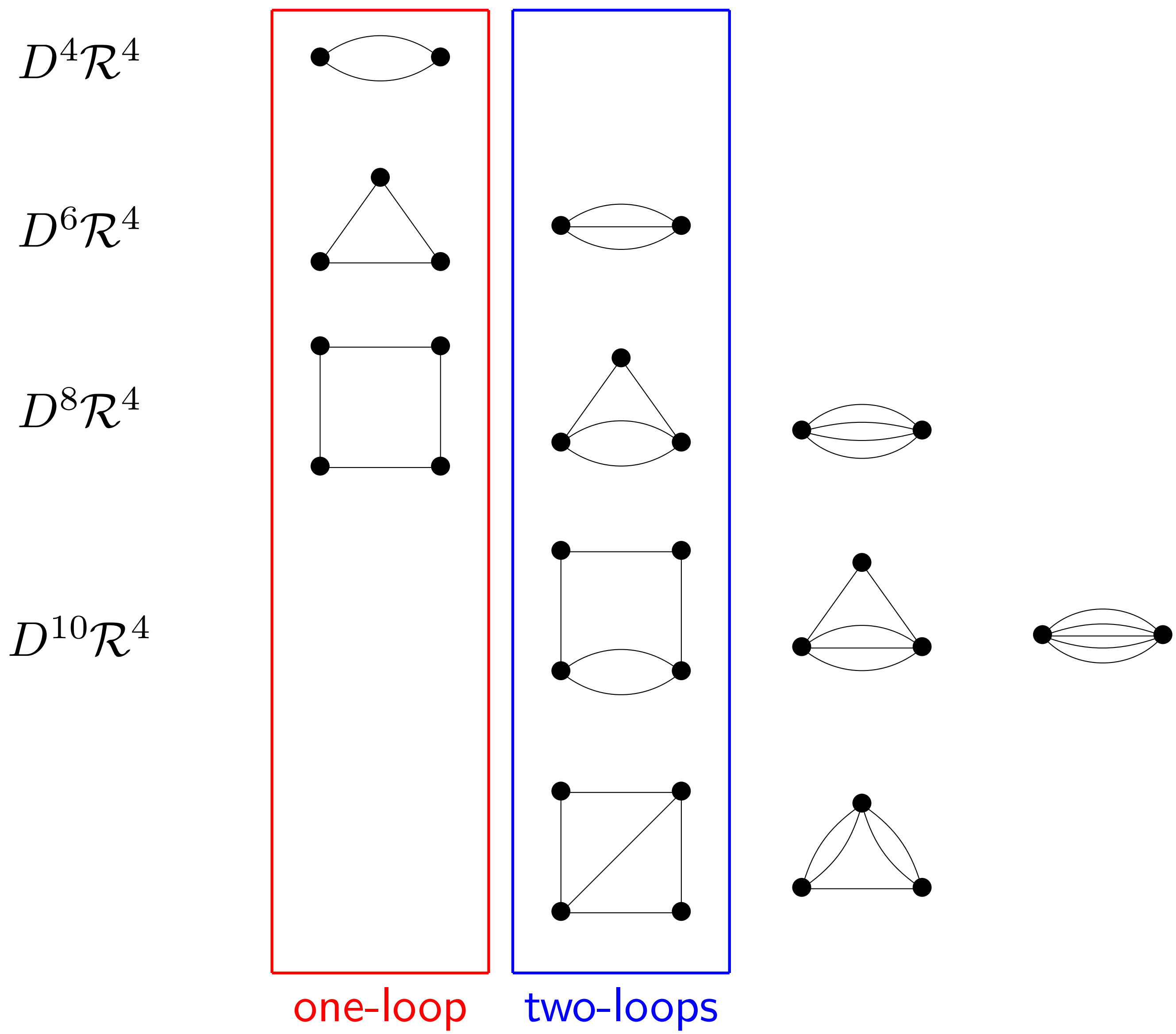
Modular graph functions

- **Graph in the expansion of $D^{2w}\mathcal{R}^4 \Longrightarrow$ Modular Function**

  (ED, Green, Gurdogan, Vanhove 2015; ED, Green 2016; ED, Kaidi 2016)



Modular graph functions



One-loop : Eisenstein series

- **One-loop worldsheet Feynman diagram with k bivalent vertices**

$$\prod_{i=1}^{k} \int_\Sigma \frac{d^2 z_i}{\tau_2} G(z_i - z_{i+1}|\tau) = \sum_{\substack{p=m+\tau n \neq 0 \\ m,n \in \mathbb{Z}}} \frac{\tau_2^k}{\pi^k |p|^{2s}}$$

- **Non-holomorphic Eisenstein series** are defined by,

$$E_s(\tau) = \sum_{\substack{p=m+\tau n \neq 0 \\ m,n \in \mathbb{Z}}} \frac{\tau_2^s}{\pi^s |p|^{2s}}$$

- **Properties**
  - absolutely convergent for $\mathrm{Re}(s) > 1$; analytically continue to $s \in \mathbb{C}$
  - reflection relation $\Gamma(s) E_s(\tau) = \Gamma(1-s) E_{1-s}(\tau)$
  - modular invariant under $SL(2,\mathbb{Z})$, $E_s(\tau') = E_s(\tau)$ with $\tau' = \frac{a\tau+b}{c\tau+d}$
  - Laplace-eigenvalue equation,

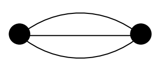
$$\Big(\Delta - s(s-1)\Big) E_s(\tau) = 0 \qquad\qquad \Delta = 4\tau_2^2 \partial_\tau \partial_{\bar\tau}$$

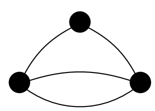
Two-loops : modular graph functions

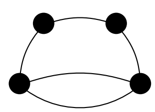
- **Feynman diagrams evaluate to the modular functions**

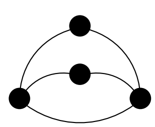
$$C_{a_1,a_2,a_3}(\tau) = \sum_{\substack{p_r = m_r + \tau n_r \neq 0 \\ m_r, n_r \in \mathbb{Z},\, r=1,2,3}} \delta\left(\sum_{r=1}^{3} p_r\right) \prod_{r=1}^{3} \left(\frac{\tau_2}{\pi |p_r|^2}\right)^{a_r}$$

– contribute to $D^{2w}\mathcal{R}^4$ with the *weight* given by $w = a_1 + a_2 + a_3$
– satisfy (inhomogeneous) Laplace-eigenvalue equations

$w = 3$ $\quad C_{1,1,1} =$  $\quad (\Delta - 0)C_{1,1,1} = 6E_3$

$w = 4$ $\quad C_{2,1,1} =$  $\quad (\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2$

$w = 5$ $\quad C_{3,1,1} =$  $\quad (\Delta - 6)C_{3,1,1} = 3C_{2,2,1} + 16E_5 - 4E_2E_3$

$w = 5$ $\quad C_{2,2,1} =$  $\quad (\Delta - 0)C_{2,2,1} = 8E_5$

– Note that eigenvalues are of the form $s(s-1)$ for $s = 1, 2, 3$

Structure Theorem

- $C_{a,b,c}(\tau)$ **are linear combinations of modular functions** $\mathfrak{C}_{w;s;\mathfrak{p}}(\tau)$ **satisfying**

$$\big(\Delta - s(s-1)\big)\mathfrak{C}_{w;s;\mathfrak{p}} = \mathfrak{F}_{w;s;\mathfrak{p}}\big(E_{s'}\big)$$

  – $\mathfrak{C}_{w;s;\mathfrak{p}}$ and $\mathfrak{F}_{w;s;\mathfrak{p}}$ of weight $w = a + b + c$ (with $E_{s'}$ assigned weight s');
  – $\mathfrak{F}$ is a polynomial of total degree 2 in $E_{s'}$ with $2 \leq s' \leq w$

$$s = w - 2\mathfrak{m} \qquad \mathfrak{m} = 1, \cdots, \left[\frac{w-1}{2}\right] \qquad \mathfrak{p} = 0, \cdots, \left[\frac{s-1}{3}\right]$$

- **Examples at low weight**

$w = 3$	$s = 1$	$0^{(1)}$
$w = 4$	$s = 2$	$2^{(1)}$
$w = 5$	$s = 1, 3$	$0^{(1)} \oplus 6^{(1)}$
$w = 6$	$s = 2, 4$	$2^{(1)} \oplus 12^{(2)}$
$w = 7$	$s = 1, 3, 5$	$0^{(1)} \oplus 6^{(1)} \oplus 20^{(2)}$
$w = 8$	$s = 2, 4, 6$	$2^{(1)} \oplus 12^{(2)} \oplus 30^{(2)}$
$w = 9$	$s = 1, 3, 5, 7$	$0^{(1)} \oplus 6^{(1)} \oplus 20^{(2)} \oplus 42^{(3)}$

Constructive Proof

- **A natural generating function is given by,**

$$\mathcal{W}(t_1,t_2,t_2|\tau) = \sum_{a,b,c=1}^{\infty} t_1^{a-1} t_2^{b-1} t_3^{c-1} C_{a,b,c}(\tau)$$

Summing gives sunset diagram for three scalars with masses $-t_r$,

$$\mathcal{W}(t_1,t_2,t_2|\tau) = \sum_{\substack{p_r=m_r+\tau n_r \neq 0 \\ m_r,n_r \in \mathbb{Z}}} \delta\Big(\sum_r p_r\Big) \prod_{r=1}^{3} \left(\frac{1}{\pi |p_r|^2/\tau_2 - t_r} \right)$$

- **Laplacian given by differential operator in the masses acting on $\mathcal{W}$,**

$$\Delta \mathcal{W} - \mathfrak{L}^2 \mathcal{W} = \text{quadratic polynomial in } E_s$$

  – Polynomials in t_r homogeneous of degree w on hyperbolic plane
  – $\mathfrak{L}^2$ is the quadratic Casimir of $SO(2,1)$ acting on this hyperbolic plane

- $SO(2,1)$ representation theory gives constructive proof of Structure Theorem.

Type IIB effective interactions at genus-two

- **Recall Type II four-graviton amplitude at genus 2,**

$$\begin{aligned}\mathcal{A}^{(2)}(\varepsilon_i, k_i) &= \mathcal{K}\tilde{\mathcal{K}} \int_{\mathcal{M}_2} d\mu_2 \, \mathcal{B}^{(2)}(s_{ij}|\Omega) \\ \mathcal{B}^{(2)}(s_{ij}|\Omega) &= \int_{\Sigma^4} \mathcal{Y} \wedge \bar{\mathcal{Y}} \exp \sum_{i<j} s_{ij} \, G(z_i, z_j)\end{aligned}$$

  - $\mathcal{Y} = (s-t)\Delta(z_1, z_3) \wedge \Delta(z_4, z_2) + 2$ permutations;
  - $\Delta(z_i, z_j)$ is a holomorphic form independent of s, t, u.

- **Contributions to local effective interactions**
  - $\mathcal{R}^4$: zero, since $\mathcal{Y}$ vanishes for $s = t = u = 0$
  - $D^4\mathcal{R}^4$: non-zero, $\mathcal{B}^{(2)}$ constant on $\mathcal{M}_2$;
  - $D^6\mathcal{R}^4$: non-zero, one power of G brought down in integral over Σ^4

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = 32(s^2 + t^2 + u^2) + 192 \, stu \, \varphi(\Omega) + \mathcal{O}(s^4, \cdots)$$

  - $\varphi(\Omega)$ coincides with the Zhang -Kawazumi invariant [ED, Green 2013]

The Zhang-Kawazumi invariant for genus-two

- **The ZK-invariant is given as follows**

$$8\varphi(\Omega) = \sum_{I,J,K,L} \Big(Y^{-1}_{IJ} Y^{-1}_{KL} - 2 Y^{-1}_{IL} Y^{-1}_{JK} \Big) \int_{\Sigma^2} G(x,y) \omega_I(x) \overline{\omega_J(x)} \omega_K(y) \overline{\omega_L(y)}$$

  – equivalent to definition via Arakelov geometry [Zhang 2007, Kawazumi 2008]

- **Coefficient of genus-two $D^6\mathcal{R}^4$ interaction involves** $\int_{\mathcal{M}_2} d\mu_2 \, \varphi(\Omega)$

  – Direct evaluation appeared completely out of reach ... until ...

- **Theorem** [ED, Green, Pioline, R. Russo 2014]

$$(\Delta - 5)\varphi = -2\pi \delta_{SN}$$

  – Δ is the Laplace-Beltrami operator on $\mathcal{M}_2$ with Siegel metric ds^2_2;
  – δ_{SN} has support on separating node (into two genus-one surfaces)
  – The integral over $\mathcal{M}_2$ reduces to an integral over $\partial\mathcal{M}_2$

$$\int_{\mathcal{M}_2} d\mu_2 \, \varphi = \frac{1}{5} \int_{\mathcal{M}_2} d\mu_2 \Big(\Delta\varphi + 2\pi\delta_{SN} \Big) = \frac{2\pi^3}{45}$$

Supersymmetry and S-duality

- **Laplace-eigenvalue eq from space-time supersymmetry** [Green, Sethi, 1998]
  - Eisenstein series = unique modular solution with polynomial growth at cusp

- **Predicts vanishing contributions for high enough loop order,**

$\mathcal{R}^4$	1/2 BPS	$h \geq 2$	$E_{\frac{3}{2}}$
$D^4\mathcal{R}^4$	1/4 BPS	$h \geq 3$	$E_{\frac{5}{2}}$
$D^6\mathcal{R}^4$	1/8 BPS	$h \geq 4$	$(\Delta - 12)\mathcal{E}_{D^6\mathcal{R}^4} = (E_{\frac{3}{2}})^2$

[Green, Gutperle, Vanhove 1997; Green, Vanhove 2005]

- **Predicts relations between non-vanishing contributions** (e.g. with tree-level),

$\mathcal{R}^4$	$h = 1$	[Green, Gutperle 1997]
$D^4\mathcal{R}^4$	$h = 2$	[ED, Gutperle, Phong 2005]
$D^6\mathcal{R}^4$	$h = 2 \quad \text{(ZK)}$	[ED, Green, Pioline, Russo 2014]
	$h = 3$	[Gomez, Mafra 2013]

V. Speculation on ambi-twistor strings

- **Can ambi-twistor prescription be obtained from chiral amplitudes ?**

1. – Moduli space compactified by Deligne-Mumford divisors $\Delta = \overline{\mathfrak{M}_h} - \mathfrak{M}_h$
   – Reducible to separating and non-separating divisors with normal crossings
   – Maximal intersection of non-separating divisors Δ_0 define cohomology class
   $\Longrightarrow$ Instead of integrating over $\mathfrak{M}_h$, integrate over Δ_0

2. Rescale loop momenta $p_I^\mu \, \mathrm{Im}(\Omega)_{IJ} \, p_J^\mu \to \ell_I^\mu \ell_I^\mu$

3. Superconformal structure deformations $\delta_{ww} \to$ insert T_{ww} and $\delta_{w\theta} \to$ insert $S_{w\theta}$

$$\mathcal{F} = \exp\left\{ i\pi p\Omega p + 2\pi i p \sum_i k_i \int^{z_i} \omega + \sum_{i<j} (k_i k_j \ln E(i,j) - i\varepsilon_i k_j \partial_i \ln E(i,j) + \cdots) \right\}$$

Variational equations $\delta_{ww}\mathcal{F} = \Pi(w)^2 \mathcal{F}$

$$\Pi^\mu(w) = \pi i p_I \omega_I(w) + \sum_i (k_i \partial_w \ln E(w,i) + \varepsilon_i \partial_w \partial_i \ln E(w,i) + \cdots)$$

– Setting $\Pi(w)^2 = 0$ condition for all $w \in \Sigma \approx$ produces a closed form on Δ_0