Comments on conformal higher spin theory

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Based on:

"On triviality of S-matrix in conformal higher spin theory"
with M. Beccaria and S. Nakach arXiv:1607.06379

"On conformal higher spins in curved background"
with M. Grigoriev arXiv:1609.09381

"Induced action for conformal higher spins in curved background"
with M. Beccaria arXiv:1702.00222

• free complex scalar: $\Box\Phi = 0$

conserved $J_\mu = i(\Phi^* \partial_\mu \Phi - \partial_\mu \Phi^* \Phi)$ and stress $T_{\mu\nu}$

couple to external sources

$L = \partial^\mu \Phi^* \partial_\mu \Phi + A^\mu(x) J_m + h^{\mu\nu}(x) T_{\mu\nu} + ...$

integrate out Φ: local ($\log \infty$) part of 1-loop effective action

induced Maxwell + Weyl theory

$S = \int d^4x (-F^2_{\mu\nu} + C^2_{\mu\nu\lambda\rho})$

• free scalar equation admits also higher conserved currents:

$J_{\mu_1 ... \mu_s} = \Phi^* \partial_{\mu_1} ... \partial_{\mu_s} \Phi + ..., \quad s = 1, 2, 3, ...$

charges $\rightarrow$ infinite dim global symmetry

corresponding sources $h_{\mu_1 ... \mu_s}$ – symmetric traceless tensors:

conformal higher spins (CHS)

• induced action for infinite tower of fields

generalizes Maxwell and Weyl: $S = \int d^4x \sum_s h_s \partial^{2s} h_s + ...$

• local action with symmetry $\delta h_s = \partial \epsilon_{s-1} + \eta_2 \alpha_{s-2} + ...$

Motivation to study:

• unusual properties and simplifications due to underlying infinite-dimensional conformal HS symmetry (sums over infinite set of HS contributions, regularization consistent with symmetry)

• close connection to massless HS fields in AdS

CHS as toy model to study implications of HS symmetry:

- trivial partition function on a sphere
- trivial near-flat-space S-matrix (cf. Coleman-Mandula)
- non-trivial cancellation of conformal anomalies

• fundamental role of local conformal invariance? existence of consistent (UV finite, anomaly free) theories with local conformal symmetry? unitary issue?

Plan:

• flat space background:
action for CHS as induced one
corresponding S-matrix

• curved space background:
curved space CHS operators
partition function on S^4
a and c Weyl anomaly coefficients

Consistent HS theories:

• massless HS theory in AdS$_{d+1}$:

2-derivative kin term (unitary) but non-flat vacuum

dual to free CFT$_d$: e.g. scalar in vector rep of $U(N)$

S-matrix is “simple”:

reproduces correlators of currents in free CFT

• conformal HS theory:

has flat vacuum but higher derivative kin term (non-unitary)

S-matrix is “trivial” after summation over all spin exchanges

consistent with HS symmetry

Conformal higher spin theory ($d = 4$)

• generalization of Maxwell and Weyl:

$F^2_{\mu\nu} \sim h_1 \partial^2 h_1, \quad C^2_{\mu\nu\kappa\lambda} \sim h_2 \partial^4 h_2 + \partial^4 h_2 h_2 h_2 + ...$

• differential + algebraic ("Weyl") gauge symmetry

$\delta h_s = \partial \epsilon_{s-1} + \eta_2 \, \alpha_{s-2}$

can gauge-fix h_s to be transverse and traceless off-shell

• totally symmetric $h_{\mu_1 ... \mu_s}$ describes "pure" spin s:

maximal gauge symm consistent with locality at expense of higher-derivative kin terms [Fradkin, AT 85]

$$S^{(0)}_s = \int d^4 x \, h_s \, P_s \, \partial^{2s} \, h_s$$

$P_s \ \sim (\delta^\mu_\nu - \frac{\partial^\mu \partial_\nu}{\partial^2})^s$ – transv. traceless projector

• $\Delta(h_s) = 2 - s$: dimensionless coupling const

• interacting action consistent with symmetries can be defined as local induced action from scalar loop

• conformally invariant in flat space
number of derivatives in vertices fixed by dimensions

$$S_s = \frac{1}{g^2} \sum_s \int d^4x \Big(h_s \partial^{2s} h_s + \partial^{s_1+s_2+s_3-2}\, h_{s_1} h_{s_2} h_{s_3}$$

$$+ \partial^{s_1+s_2+s_3+s_4-4}\, h_{s_1} h_{s_2} h_{s_3} h_{s_4} + ... \Big)$$

• conformal symmetry: can be consistently defined
on any conformally flat background
• admits a background-independent formulation
and in general consistently defined near any
curved Bach-flat (e.g. Ricci-flat) background

Properties of free CHS theory

• regularized total number of d.o.f. =0:

$$\nu_{\rm tot} = \sum_{s=0}^{\infty} \nu_s = 0 \ , \qquad\qquad \nu_s = s(s+1) = 2, 6, ...$$

regularization: $\sum_{s=0}^{\infty} f(s) \ \to \ \sum_{s=0}^{\infty} f(s)\, e^{-\epsilon(s+\frac{1}{2})}\Big|_{\rm fin.}$

• equivalently, flat-space partition function is trivial:

$$Z_s = \left[\frac{(\det \Box_{s-1})^{s+1}}{(\det \Box_s)^s}\right]^{1/2} = (Z_0)^{\nu_s} \ , \qquad Z_0 = (\det \Box)^{-1/2}$$

$$Z = \prod_{s=0}^{\infty} (Z_0)^{\nu_s} = (Z_0)^{\nu_{\rm tot}} = 1$$

• with same regularization: $Z_{\rm CHS}(S^4)$=1, $\sum_{s=1}^{\infty} {\rm a}_s = 0$

consistent with relation between

1-loop Z of massless HS in AdS_5 and Z of CHS on S^4

[Giombi, Klebanov, Pufu, Safdi, Tarnopolsky 13; AT 13; Beccaria, Bekaert, AT 14]

• this definition of $\sum_s$ should be consistent

with underlying HS symmetry of CHS theory

"Quantized particle" approach: symmetries

start with quantized particle in external fields [Segal 02]

general phase space Hamiltonian $H(x,p)$

$$H(x,p) = \sum_{s=0}^{\infty} \mathrm{h}^{\mu_1\dots\mu_s}(x)\, p_{\mu_1}\dots p_{\mu_s} = \mathrm{h}_0(x) + \mathrm{h}^{\mu\nu}(x) p_\mu p_\nu + \dots$$

*-product of Weyl symbols $\rightarrow$ product of operators

$$* = \exp\left[\frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial x^\mu}\frac{\partial}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p_\mu}\frac{\partial}{\partial x^\mu}\right)\right]$$

Symmetries: canonical transfs of constraint $H(x,p) = 0$

$$\delta H = [H, \epsilon(x,p)]_* + \{H, \alpha(x,p)\}_*$$

gradient ϵ and algebraic α

• Quantum theory in x representation: $\widehat{H}\Phi(x) = 0$
action for scalar field in non-trivial background $H = \{\mathrm{h}_s\}$:

$$\mathcal{S}[\Phi, H] = \int d^4x\ \Phi^*(x)\ \widehat{H}(x, \partial_x)\ \Phi(x)$$

• Invariant under the gauge transformations of Φ and h_s

$$\delta\Phi = -(\widehat{\epsilon} + \widehat{\alpha})\Phi\,, \qquad \delta H = [H, \epsilon(x,p)]_* + \{H, \alpha(x,p)\}_*$$

• Choice of vacuum expansion point:
$H = H_{\mathrm{vac}} + h(x,p), \qquad h(x,p) = \sum_{s=0}^{\infty} h^{\mu_1 \ldots \mu_s}(x)\ p_{\mu_1} \ldots p_{\mu_s}$

$$\mathcal{S} = \int d^4x \Big[\Phi^*(x)\ \widehat{H}_{\mathrm{vac}}\,\Phi(x) + \sum_s h^{\mu_1 \ldots \mu_s}(x) J_{\mu_1 \ldots \mu_s}(\Phi)\Big]$$

• $$H_{\mathrm{vac}} = \tfrac{1}{2}\eta^{\mu\nu} p_\mu p_\nu$$

ϵ-gauge inv $\quad\rightarrow\qquad\qquad \partial^{\mu_1} J_{\mu_1\ldots\mu_s} = 0 \quad \text{if} \quad \Box\Phi = 0$

α-gauge inv $\quad\rightarrow\quad \eta^{\mu_1\mu_2} J_{\mu_1\mu_2\ldots\mu_s} - \frac{1}{2}\Box J_{\mu_3\ldots\mu_s} = 0$

• after redefinition $J_s \rightarrow$ conserved traceless Noether currents corresponding to symmetries of $\Box\Phi = 0$ in R^d [Eastwood 02]

• their algebra = HS algebra of conformal spins in R^d = HS algebra of massless spins in AdS_{d+1} [Vasiliev, Fradkin, Linetsky]

Action for h_s: [AT 02; Segal 02; Bekaert, Mourad, Joung 10]

• $\log \Lambda_{\rm UV}$ term of scalar 1-loop action $\log\det \widehat{H}$

$$S[h] = \text{“Seeley coeff”} = t^0 \text{ term in } \operatorname{Tr} e^{-t\widehat{H}}\Big|_{t\to 0}, \quad H = H_{\rm vac} + h$$

• inherits CHS symm: $\quad \delta h = [H, \epsilon(x,p)]_* + \{H, \alpha(x,p)\}_*$

$\rightarrow \quad \delta h_{\mu_1\ldots\mu_s} = \partial_{(\mu_1}\epsilon_{\mu_2\ldots\mu_s)} + \eta_{(\mu_1\mu_2}\alpha_{\mu_3\ldots\mu_s)} + O(h)$

• this construction can be generalized [Grigoriev, AT 16] to curved vacuum expansion point: $\quad H_{\rm vac} = \frac{1}{2} g^{\mu\nu}(x) p_\mu p_\nu$

CHS as induced theory: AdS/CFT

start with free $U(N)$ scalar CFT $\quad \int d^4x\, \Phi_i^* \partial^2 \Phi_i$

- tower of on-shell conserved traceless currents

$J_s = \Phi_i^* \,\mathcal{J}_s\, \Phi_i \sim \Phi_i^* \partial_{(\mu_1} ... \partial_{\mu_s)} \Phi_i + ...$

- implies infinite tower of conserved charges:

symmetries of $\Box\Phi = 0 \to$ HS symmetry [Eastwood, Vasiliev]

- generating functional for correlators of currents:

add $h_s J_s$ and integrate out Φ_i

$$\Gamma[h] = N\ \log\det\Big(-\partial^2 + \textstyle\sum_s h_s\, \mathcal{J}_s\Big)\ , \qquad \mathcal{J}_s \sim \partial^s$$

- source fields = CHS fields h_s: $\quad \Delta(h_s) = 2 - s$
- CHS theory: gauge theory for

HS symmetries (conf Killing tensors) of $\Box\Phi = 0$

$\delta h_{\mu_1\cdots\mu_s} = \partial_{(\mu_1} \varepsilon_{\mu_2\cdots\mu_s)} + \eta_{(\mu_1\mu_2}\, \alpha_{\mu_3\cdots\mu_s)} + O(h)$

(cf. Weyl gravity as "gauge theory of conformal group")

• vectorial AdS/CFT: [Klebanov, Polyakov 02]

J_s dual to massless HS fields in AdS_{d+1}

$\Gamma[h]$ should follow from Vasiliev-type theory in AdS$_{d+1}$

upon integrating over AdS$_{d+1}$ fields ϕ_s with Dirichlet b.c.

$$e^{-\Gamma[h]} = \int_{\phi_s\big|_{\partial\mathrm{AdS}}=h_s} [d\phi_s] \exp\big(- N S_{\mathrm{HS}}[\phi]\big)$$

• full $\Gamma[h]$ is non-local and does not have CHS symmetry

but its log divergent part is local and invariant:

$$\Gamma[h] \quad \to \quad N S_{\mathrm{CHS}}[h] \log \Lambda_{\mathrm{UV}} + \dots$$

$$N S_{\mathrm{HS}}[\phi]\Big|_{\mathrm{on-shell}} \to \quad N S_{\mathrm{CHS}}[h] \log \Lambda_{\mathrm{IR}} + \dots$$

• CHS action as induced action:

$$S_{\rm CHS} \sim \log\det\Delta(h)\Big|_{\log\Lambda_{\rm UV}}, \qquad \Delta(h) = -\partial^2 + \sum_s \mathcal{J}_s h_s$$

• familiar low-spin cases ($s=0,1,2$) in covariant form

$$L = \sqrt{g}\Big[\, g^{\mu\nu} D_\mu\Phi^* D_\nu\Phi + (\tfrac{1}{6}R + h'_0)\Phi^*\Phi\Big]\,, \qquad D_\mu = \partial_\mu + \tfrac{i}{2}h'_\mu$$

$$L = \partial_\mu\Phi^*\partial^\mu\Phi + \textstyle\sum_s h_s\Phi^*\mathcal{J}_s\Phi$$
$$= \partial_\mu\Phi^*\partial^\mu\Phi + h_0\Phi^*\Phi + ih^\mu\Phi^*\partial_\mu\Phi + \tfrac{1}{2}h^{\mu\nu}\partial_\mu\Phi^*\partial_\nu\Phi + \ldots$$

by local field redefinition ($h'_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$)

$$h'_0 = h_0 + \tfrac{1}{4}h_\mu h^\mu + \tfrac{1}{96}(\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \ldots) + \ldots$$

$$h'_\mu = h_\mu + \tfrac{1}{2}h_{\mu\nu}h^\nu + \tfrac{1}{4}h_{\mu\nu}h^{\nu\lambda}h_\lambda + \ldots, \quad h'_{\mu\nu} = \tfrac{1}{2}h_{\mu\nu} + \tfrac{1}{4}h_{\mu\lambda}h^\lambda_\nu + \ldots$$

log divergent part of scalar log det

$$S[h'_0, h'_1, h'_2] = \int d^4x\sqrt{g}\Big(h'^2_0 - \tfrac{1}{24}F'^2_{\mu\nu} + \tfrac{1}{60}C^2_{\mu\nu\lambda\rho}\Big)$$

Computing CHS action as induced action

[Beccaria, Nakach, AT 16]

• 2-, 3- and 4-point vertices in CHS action from UV pole part of scalar loop integrals with J_s insertions

• same as local limit of correlators of currents $< J_{s_1}(x_1)....J_{s_n}(x_n) > \Big|_{x_i \to x}$

• coupling of external CHS fields to complex scalar

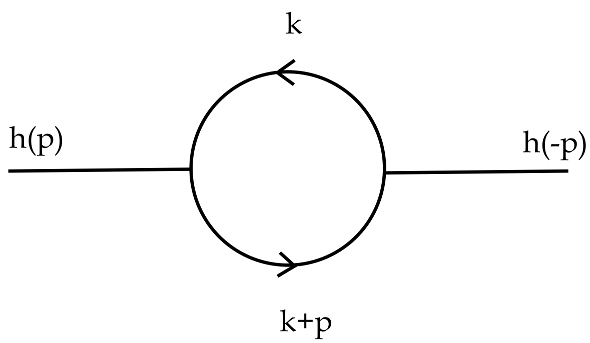
$$L = -\partial_\mu \Phi^* \, \partial^\mu \Phi + \sum_{s=0}^{\infty} J_{\mu(s)} \, h^{\mu(s)} \, , \qquad J_{\mu(s)} \equiv J_{\mu_1 \ldots \mu_s}$$

$$J_{\mu(s)}(x) = \tfrac{i^s \, 2^s}{(2s)!} \sum_{k=0}^{s} \binom{s}{k} \binom{\frac{s+k-1}{2}}{s} \, G^{(k)}_{\mu(s)}(x)$$

$$G^{(k)}_{\mu(s)}(x) = (\partial - \partial')^{\mu(k)} (\partial + \partial')^{\mu(s-k)} \Phi(x) \, \Phi^*(x') \Big|_{x=x'}$$

$$S = \int d^4x \Big(\sum_s h_s \partial^{2s} h_s + \sum_{s_i} \partial^{s_1+s_2+s_3-2} h_{s_1} h_{s_2} h_{s_3} + \sum_{s_i} \partial^{s_1+s_2+s_3+s_4-4} h_{s_1} h_{s_2} h_{s_3} h_{s_4} + ... \Big)$$

• kinetic term:

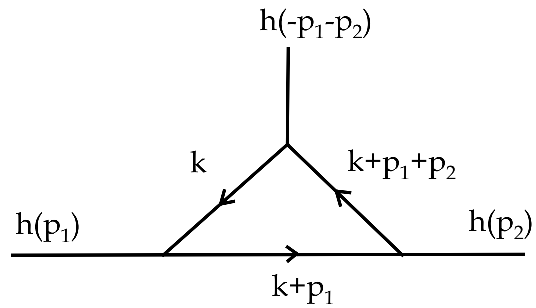


$$= \int \frac{d^d k}{(2\pi)^d} \; \frac{N(k,p)}{k^2\,(k+p)^2}$$

$\frac{1}{\varepsilon} = \frac{1}{d-4}$ UV pole part (for TT field h_s):

$$S_2 = \frac{1}{2^s\,(2s+1)!} \int d^4x \, h_{\mu(s)} \, \Box^s \, h^{\mu(s)}$$

• cubic vertex: pole part of



example: 1-1-s

$$V_{\mu\nu\rho(s)} = \int \frac{d^dk}{(2\pi)^d}\; \frac{k_\mu (k+p_1)_\nu (k+p_1+p_2)_{\rho(s)}}{k^2 (k+p_1)^2 (k+p_1+p_2)^2}\Bigg|_{\frac{1}{\varepsilon}\text{ part}}$$

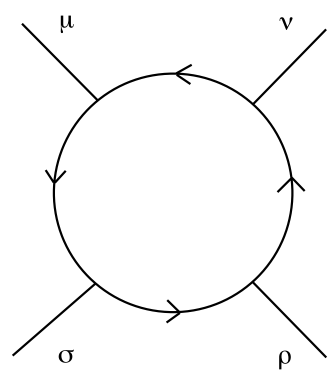
$$\begin{aligned} S_3(1,1,s) = \frac{1}{(s+2)!}\int d^4x\Big[&\partial^{\rho(s)} h_\mu h^\mu h_{\rho(s)} - 2h_\mu\, \partial^\mu\, \partial_{\rho(s-1)}\, h_\nu\, h^{\nu\rho(s-1)} \\ &- \tfrac{s}{2}\partial^{\rho(s-2)}\Box h^\mu h^\nu h_{\mu\nu\rho(s-2)} - \tfrac{s}{2}\partial^{\rho(s-2)} h^\mu \Box h^\nu h_{\mu\nu\rho(s-2)} \\ &- \partial_\lambda \partial^{\rho(s-2)} h^\mu \partial^\lambda h^\nu h_{\mu\nu\rho(s-2)}\Big] \end{aligned}$$

e.g. 1-1-2 is like in Maxwell $\int d^4x\, \sqrt{g}\, g^{\mu\nu} g^{\lambda\rho} F_{\mu\lambda} F_{\mu\rho}$

$$S_3(1,1,2) = \frac{1}{24} \int d^4x \Big[\partial_\rho h_\mu \, \partial_\sigma h^\mu h^{\rho\sigma} - 2\partial_\rho h_\mu \, \partial^\mu \, h_\nu \, h^{\nu\rho} + 2\, h^\mu \, \Box h^\nu h_{\mu\nu} + \partial_\lambda h^\mu \partial^\lambda h^\nu h_{\mu\nu} \Big]$$

• quartic vertex:

e.g. 4-vector contact term from pole part of diagram



$\frac{1}{16} \int d^4x (h_\mu h^\mu)^2$ combining into $\int d^4x (h_0 + \frac{1}{4} h_\mu h^\mu)^2$:
contribution to 1-1-1-1 scattering cancels against h_0 exchange

• similarly for 2-2-s and 2-2-2-2 vertices, etc.

S-matrix of CHS theory in flat vacuum

[Beccaria, Nakach, AT 16]

• compute tree-level CHS 4-point amplitudes A_4

for external states = massless ($\Box h_s = 0$) modes in flat space

• A_4 turns out to be zero after summation

over all spin s intermediate states

• this appears to be a consequence of CHS global symmetry

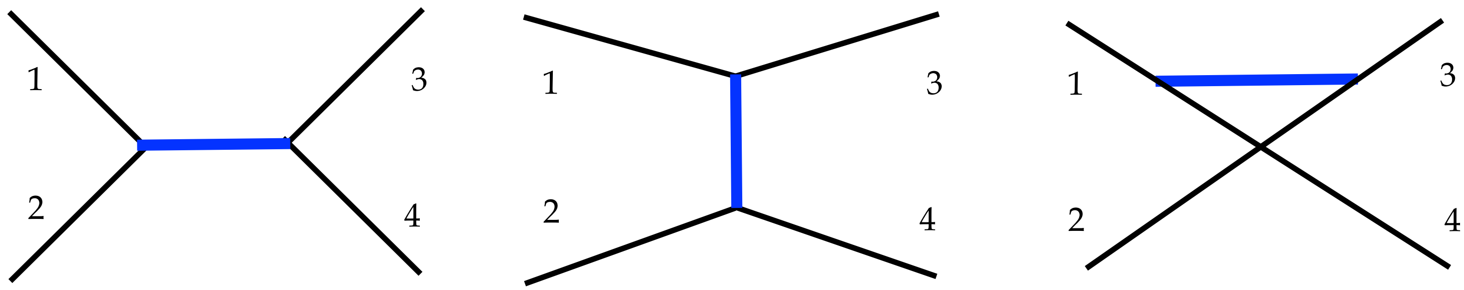
first illustrate this on simplest example:

scattering of external scalars via exchange of

infinite tower of CHS fields

Scalar scattering via conformal HS exchange

[Joung, Nakach, AT 15]



$$S[\Phi, h] = \int d^4x \big[\Phi^* \partial^2 \Phi + \sum_{s=0}^{\infty} h_s \, J_s(\Phi)\big] + S[h]$$

$$S[h] = \frac{1}{g^2} \sum_{s=0}^{\infty} \int h_s \, P_s \partial^{2s} \, h_s + \mathcal{O}(h^3)$$

- h_0 coupled to $\Phi^*\Phi$; h_μ to $i\Phi^*\partial_\mu\Phi + c.c.$; $h_{\mu\nu}$ to $T_{\mu\nu}$, etc.
- h_s exchange with propagator $\sim \frac{1}{p^{2s}}$ and p^s in the vertices:

scale invariance, no dimensional parameters

Four-scalar tree-level scattering amplitude

t-channel amplitude

$$A^{(\mathsf{t})}(\mathsf{s},\mathsf{t},\mathsf{u}) = g^2 F(\tfrac{\mathsf{s}-\mathsf{u}}{\mathsf{s}+\mathsf{u}}) \ , \qquad F(z) \equiv \sum_{s=0}^{\infty} (s+\tfrac{1}{2}) P_s(z)$$

$\mathsf{s},\mathsf{t},\mathsf{u}$ are Mandelstam variables: $\mathsf{s}+\mathsf{t}+\mathsf{u}=0$

$P_s(z)$ – Legendre polynomial

- amplitude is scale-invariant: depends on ratios of $\mathsf{s},\mathsf{t},\mathsf{u}$
- summing over spins:

$$\textstyle\sum_{s=0}^{\infty} f(s) \to \sum_{s=0}^{\infty} f(s)\, e^{-\varepsilon(s+\frac{1}{2})}\Big|_{\epsilon\to 0,\ \mathrm{fin}}$$

one finds that amplitude is δ-function in phase space

$$F(z) = \delta(z-1)$$

Total amplitude: sum of channels

$$A_{\Phi\Phi\to\Phi\Phi} = g^2 \left[\delta(\tfrac{\mathsf{s}}{\mathsf{t}}) + \delta(\tfrac{\mathsf{s}}{\mathsf{u}}) \right]$$

in c.o.m. frame $\vec{p}_1 + \vec{p}_2 = 0 = \vec{p}_3 + \vec{p}_4$
scattering angle: $\frac{\mathsf{s}}{\mathsf{t}} = -(\sin^2\frac{\theta}{2})^{-1}, \qquad \frac{\mathsf{s}}{\mathsf{u}} = -(\cos^2\frac{\theta}{2})^{-1}$
arguments of delta-functions never vanish for real θ

$$A_{\Phi\Phi\to\Phi\Phi} = 0$$

$$A_{\Phi\Phi^*\to\Phi\Phi^*} = \frac{g^2}{2}\left[\delta(\tfrac{\mathsf{u}}{\mathsf{t}}) + \delta(\tfrac{\mathsf{u}}{\mathsf{s}}) \right] = \frac{g^2}{2}\left[\delta(\cot^2\tfrac{\theta}{2}) - \delta(\cos^2\tfrac{\theta}{2}) \right]$$

t-channel and s-channel contributions cancel each other

$$A_{\Phi\Phi^*\to\Phi\Phi^*} = 0$$

thus individual spin s exchange contributions are nontrivial
but total amplitude =0

• underlying HS symmetry constrains the S-matrix

$A_4 = 0$ is implied by global part of CHS gauge symmetry:

conformal group generators plus higher spin generators

• in particular: "hyper-translations"

$$\delta\Phi = \epsilon^{m_1....m_r}\partial_{m_1}...\partial_{m_r}\Phi$$

fix $A_4(\mathsf{s},\mathsf{t},\mathsf{u}) = k_1(\mathsf{t},\mathsf{u})\,\delta(\mathsf{s}) + k_2(\mathsf{s},\mathsf{u})\,\delta(\mathsf{t}) + k_3(\mathsf{t},\mathsf{s})\,\delta(\mathsf{u})$

• scale invariance: $A_4(\lambda^2\,\mathsf{s}, \lambda^2\,\mathsf{t}, \lambda^2\,\mathsf{u}) = A_4(\mathsf{s},\mathsf{t},\mathsf{u})$

• solution consistent with crossing and scaling symmetry

$$A_4(\mathsf{s},\mathsf{t},\mathsf{u}) = 0$$

$\star$ special prescription for summation over s

with which tree-level amplitude vanishes

is thus consistent with underlying global CHS symmetry

Scattering of conformal higher spin fields

[Beccaria, Nakach, AT 16]

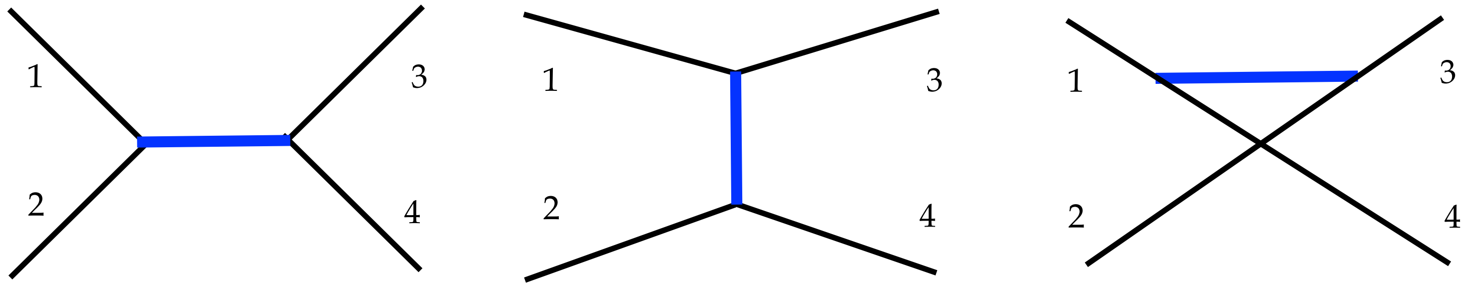
• $s = 1$ case is standard vector but for $s \geqslant 2$
higher-derivative $\Box^s$ kinetic term: non-unitary theory

• definition of S-matrix: amputated Green's functions
computed with full CHS vertices and internal propagators
but with particular – massless spin s – asymptotic states
[e.g. for $s = 2$: Weyl graviton with 6 d.o.f. – but choose
only standard helicity ± 2 gravitons as asymptotic states]

CHS 4-particle tree level amplitude

helicities $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and s, t, u $\quad (p_i^2 = 0$ for legs)

exchange diagrams



$s = 1$ scattering: $11 \to 11$

spin s exchange: two 1-1-s vertices
and TT spin s propagator $\quad (p_{\rho(s)} \equiv p_{\rho_1} ... p_{\rho_s})$

$$V_{\alpha\beta\rho(s)}(p,q) = \frac{1}{(s+2)!}\Big\{\eta_{\alpha\beta}\left[\, p_{\rho(s)} + \, q_{\rho(s)}\right]$$

$$- \, \eta_{\alpha\rho_1} p_\beta p_{\rho_2} \ldots p_{\rho_s} + \eta_{\beta\rho_1} q_\alpha p_{\rho_2} \ldots p_{\rho_s} - \eta_{\beta\rho_1} q_\alpha q_{\rho_2} \ldots q_{\rho_s} + \eta_{\alpha\rho_1} p_\beta q_{\rho_2} \ldots q_{\rho_s}$$

$$- \, \eta_{\alpha\rho_1}\eta_{\beta\rho_2}\, p_{\rho_3} \ldots p_{\rho_s}\, p \cdot q - \eta_{\alpha\rho_1}\eta_{\beta\rho_2}\, q_{\rho_3} \ldots q_{\rho_s}\, p \cdot q \Big\}$$

• $s = 2$ exchange ($\Box^{-2}$):
same as in conformal supergravity ($L = -F^2 + C^2 + \ldots$)
only MHV are non-zero (++++, +++-,... =0)

λ	$A_{\mathrm{s}}^{(2)}$	$A_{\mathrm{t}}^{(2)}$	$A_{\mathrm{u}}^{(2)}$
$\pm\pm\mp\mp$	0	$\frac{5}{48}\,\frac{\mathrm{s}^2}{\mathrm{t}^2}$	$\frac{5}{48}\,\frac{\mathrm{s}^2}{\mathrm{u}^2}$
$\pm\mp\mp\pm$	$\frac{5}{48}\,\frac{\mathrm{u}^2}{\mathrm{s}^2}$	$\frac{5}{48}\,\frac{\mathrm{u}^2}{\mathrm{t}^2}$	0

• $s = 4$ exchange ($\Box^{-4}$):
again only MHV are non-zero:

λ	$A_{\mathrm{s}}^{(4)}$	$A_{\mathrm{t}}^{(4)}$	$A_{\mathrm{u}}^{(4)}$
$\pm\pm\mp\mp$	0	$\frac{\mathrm{s}^2\,(28\,\mathrm{s}^2+42\,\mathrm{s}\,\mathrm{t}+15\,\mathrm{t}^2)}{80\,\mathrm{t}^4}$	$\frac{\mathrm{s}^2\,(28\,\mathrm{s}^2+42\,\mathrm{s}\,\mathrm{u}+15\,\mathrm{u}^2)}{80\,\mathrm{u}^4}$
$\pm\mp\mp\pm$	$\frac{\mathrm{u}^2\,(28\,\mathrm{u}^2+42\,\mathrm{s}\,\mathrm{u}+15\,\mathrm{s}^2)}{80\,\mathrm{s}^4}$	$\frac{\mathrm{u}^2\,(28\,\mathrm{u}^2+42\,\mathrm{t}\,\mathrm{u}+15\,\mathrm{u}^2)}{80\,\mathrm{t}^4}$	0

• General spin s exchange 11→11 amplitudes ($\neq 0$)

$$A_{\mathrm{t}}^{(s)}(\pm\pm\mp\mp) = \mathrm{c}_s \Big(\frac{\mathrm{s}}{\mathrm{t}}\Big)^s \mathrm{P}_s\Big(\frac{\mathrm{t}}{\mathrm{s}}\Big), \qquad A_{\mathrm{u}}^{(s)}(\pm\pm\mp\mp) = \mathrm{c}_s \Big(\frac{\mathrm{s}}{\mathrm{u}}\Big)^s \mathrm{P}_s\Big(\frac{\mathrm{u}}{\mathrm{s}}\Big),$$

$$A_{\mathrm{s}}^{(s)}(\pm\mp\mp\pm) = \mathrm{c}_s \Big(\frac{\mathrm{u}}{\mathrm{s}}\Big)^s \mathrm{P}_s\Big(\frac{\mathrm{s}}{\mathrm{u}}\Big), \qquad A_{\mathrm{t}}^{(s)}(\pm\mp\mp\pm) = \mathrm{c}_s \Big(\frac{\mathrm{u}}{\mathrm{t}}\Big)^s \mathrm{P}_s\Big(\frac{\mathrm{t}}{\mathrm{u}}\Big)$$

$$\mathrm{c}_s = \frac{2\,s+1}{2\,(s-1)\,s\,(s+1)\,(s+2)}$$

$$\mathrm{P}_s(x) = x^{s-2}\,P_{s-2}^{(4,0)}\Big(\frac{x+2}{x}\Big), \qquad \text{order } s-2\,, \qquad\qquad s = 2, 4, 6, \ldots$$

$P_n^{(a,b)}(x)$ = Jacobi polynomials

$$\mathrm{P}_s(x) = \sum_{j=2}^{s} \frac{1}{(j-2)!\,(j+2)!}\,\frac{(s+j)!}{(s-j)!}\,x^{s-j} \sim x^{s-2}\,{}_2F_1\Big(2-s, s+3, 5; -\frac{1}{x}\Big)$$

Sum over spins

total $++--$ amplitude: t + u-channel

$$A^{(s)} = \mathrm{c}_s \left[\left(\frac{\mathrm{s}}{\mathrm{t}}\right)^s \mathrm{P}_s\left(\frac{\mathrm{t}}{\mathrm{s}}\right) + \left(\frac{\mathrm{s}}{\mathrm{u}}\right)^s \mathrm{P}_s\left(\frac{\mathrm{u}}{\mathrm{s}}\right) \right]$$

$$A^{(s)} = \sigma_s(x) + \sigma_s(-1-x), \quad \sigma_s(x) = \mathrm{c}_s\, x^{-s}\, \mathrm{P}_s(x)\ , \quad x = \frac{\mathrm{t}}{\mathrm{s}}$$

- use generating function for Jacobi polynomials $P_{s-2}^{(4,0)}$

$$\sum_{s=2}^{\infty} x^{-s}\, \mathrm{P}_s(x)\, z^{s-2} = \frac{1}{x^2} \frac{16}{\sqrt{z^2 - \frac{2z(x+2)}{x} + 1}\, \left(\sqrt{z^2 - \frac{2z(z+2)}{z} + 1} - z + 1\right)^4}$$

$$\sigma(x) = \sum_{s=2,4,6,\ldots}^{\infty} \sigma_s(x) = \lim_{z\to 1} \sum_{s=2,4,6,\ldots}^{\infty} \mathrm{c}_s\, x^{-s}\, \mathrm{P}_s(x)\, z^{s-2}$$
$$= \frac{1}{8} \left[-2\,x + 2\,(x+1)\,x\, \log\left(\frac{1}{x} + 1\right) - 1 \right].$$

- total amplitude is then zero as in the scalar scattering case

$$A(x) = \sum_{s=2,4,6,\ldots}^{\infty} A^{(s)}(x) = \sigma(x) + \sigma(-1-x) = 0$$

Generalization to $s > 1$ external states

Why Jacobi polynomials appear?

compare to partial wave expansion in terms

of intermediate angular momentum J states [Jacob, Wick 1959]

$$A_{\{\lambda_i\}} = R_{\{\lambda_i\}}(\theta) \sum_J (J+\tfrac{1}{2}) \mathrm{F}^{(J)}_{\{\lambda_i\}}(\mathrm{s})\ P^{(|\lambda+\mu|,|\lambda-\mu|)}_{J-M}(\cos\theta)$$

$$\lambda = \lambda_1 - \lambda_2, \quad \mu = \lambda_3 - \lambda_4, \qquad M = \max(|\lambda|, |\mu|)$$

$$R_{\{\lambda_i\}}(\theta) = \left(\cos\tfrac{\theta}{2}\right)^{|\lambda+\mu|} \left(\sin\tfrac{\theta}{2}\right)^{|\lambda-\mu|} = \left(-\tfrac{\mathrm{u}}{\mathrm{s}}\right)^{\frac{1}{2}|\lambda+\mu|} \left(-\tfrac{\mathrm{t}}{\mathrm{s}}\right)^{\frac{1}{2}|\lambda-\mu|}$$

- J-th partial wave as exchange of TT spin J CHS field:

for massive field ($m^2 \sim \mathrm{s}$)

$$(\Box + m^2)\psi_{m_1\ldots m_J} = 0, \qquad \partial^{m_1}\psi_{m_1\ldots m_J} = \psi^{m_1}{}_{m_1\ldots m_J} = 0$$

• scale invariance controls how F depends on s

e.g., for dim 1 external particles $\mathrm{F}^{(J)}_{\{\lambda_i\}}(\mathrm{s}) = \mathrm{const}$

• general prediction for Jacob-Wick coefficient for scattering of CHS fields of dim $\Delta_i = 2 - |\lambda_i|$ (no dim$\neq 0$ parameters!)

$$\mathrm{F}^{(J)}_{\{\lambda_i\}}(\mathrm{s}) = k_{\lambda,\mu}\, \frac{[J-\max(|\lambda|,|\mu|)]!}{[J+\min(|\lambda|,|\mu|)]!}\, \mathrm{s}^r \ , \qquad r = 2 - \tfrac{1}{2}\sum_{i=1}^{4} \Delta_i$$

Special cases $(J = s)$:

• $00 \to 00$

$$A_{0,0;0,0}(\mathrm{s},\theta) = \sum_{s=0,2,\ldots} (s+\tfrac{1}{2})\mathrm{F}^{(s)}_0\, P^{(0,0)}_s(\cos\theta)$$

• $+1+1 \to +1+1$

t-channel $\quad (\cos\theta = -1 - 2\,\frac{\mathrm{s}}{\mathrm{t}})$

$$A_{++;++}(\theta) = (\sin\tfrac{\theta}{2})^{-4} \sum_{s=2,4,\ldots} (s+\tfrac{1}{2})\mathrm{F}^{(s)}_+\, P^{(4,0)}_{s-2}(\cos\theta)$$

Comments on Weyl gravity

$L = C^2_{mknl} \sim (\partial_k \partial_l h_{mn} + ...)^2$

• can choose TT gauge: $h^m_m = 0, \ \partial^m h_{mn} = 0$

free eq: $\Box^2 h_{mn} = 0$ solved by [Stelle 78; Riegert 84]

$$h_{mn} = h^{(1)}_{mn} + h^{(2)}_{mn} = (a_{mn} + b_{mn} u_k x^k) e^{ip\cdot x} + c.c.$$

$$p^2 = 0 \ , \quad u^2 = -1 \ , \quad u \cdot p \neq 0 \ , \quad a^m_m = b^m_m = 0$$

• $h^{(1)}_{mn}$ – spin 2 and spin 1 massless modes;

$h^{(2)}_{mn}$ – spin 2 ghost mode – grows in time, negative energy

residual gauge freedom: $p^m = (p, 0, 0, p), \quad u^m = (1, 0, 0, 0)$

$$a_{11} + a_{22} = b_{11} + b_{22} = 0 \ , \quad a_{m3} = b_{m3} = b_{m0} = 0$$

• modes: 2+2+2 =6 dynamical d.o.f.

$(a_{11} \pm i a_{12}) e^{ip\cdot x}$:	physical $\lambda = \pm 2$ massless tensor
$(a_{01} \pm i a_{02}) e^{ip\cdot x}$:	$\lambda = \pm 1$ massless vector
$(b_{11} \pm i b_{12}) x^0 e^{ip\cdot x}$:	ghost $\lambda = \pm 2$ massless tensor

• Higher-derivative actions admit 2-derivative forms

$\phi \Box^2 \phi \ \to \ \psi \Box \phi - \psi^2$

$R^2_{mn} - \frac{1}{3} R^2 \to u^{mn} R_{mn} - u^{mn} u_{mn} + \ldots$

Weyl gravity: in terms of $h_{mn}, \ \widetilde{h}_{mn}, h_m$ [Metsaev 07]

• can define standard scattering S-matrix (with usual LSZ rules)

if asymptotic states are physical massless spin 2 gravitons

• intermediate states – all modes – effective $\frac{1}{p^4}$ propagator:

non-unitary theory

• 4-graviton amplitude in Weyl theory: found to be 0

e.g. from 4-graviton amplitude in $L = \epsilon R + C^2$

in the limit $\epsilon \to 0$ (propagator $\frac{1}{\epsilon p^2 + p^4} \to \frac{1}{p^4}$)

similar result from other approaches:

- start with Weyl gravity in dS_4 or AdS_4 space ($\Lambda \neq 0$)

Neumann boundary condition: selects the Einstein graviton mode

then S-matrix is $\Lambda \times$ (Einstein S-matrix) [Maldacena 11]

$\Lambda \to 0$ gives trivial S-matrix of Weyl theory in flat 4d space

[Adamo, Mason 13]

- start with twistor superstring theory [Berkowits, Witten 04]

and compute 4-graviton S-matrix [Dolan, Ihry 08]

result is non-zero but only due to presence of extra

non-minimal scalar coupling in $\alpha' \to 0$ limit of twistor string:

$(1 + \phi + ...)C^2_{mnkl} + \phi\Box^2\phi \quad \to \quad C^2_{mnkl}\Box^{-2}C^2_{abcd}$

$s = 2$ scattering via CHS exchange

- $+2{+}2 \to +2{+}2$: contribution from from $s > 2$ exchanges:

t-channel $++ \to ++$ or $+ + --$ MHV

$$A_{++;++}(\mathrm{t},\theta) = \frac{\mathrm{s}^4}{\mathrm{t}^4} \sum_{s=4,6,\dots} (s+\tfrac{1}{2})\mathrm{F}^{(s)}\mathrm{t}^2\, P^{(8,0)}_{s-4}(\cos\theta)$$

explicit computation gives for full (t + u- channel) amplitude

$$A^{(s)} = \mathrm{c}_s\,\mathrm{s}^2 \left[\left(\tfrac{\mathrm{s}}{\mathrm{t}}\right)^{s-2} \mathrm{P}_s\!\left(\tfrac{\mathrm{t}}{\mathrm{s}}\right) + \left(\tfrac{\mathrm{s}}{\mathrm{u}}\right)^{s-2} \mathrm{P}_s\!\left(\tfrac{\mathrm{u}}{\mathrm{s}}\right)\right]$$

$$\mathrm{P}_s(x) = x^{s-2}\,\mathrm{P}^{(8,0)}_{s-4}\left(\tfrac{x+2}{x}\right), \qquad \mathrm{c}_s = \frac{9}{32}\frac{2s+1}{(s-3)\dots(s+4)}$$

- sum over spins:

$$\sigma(x) = \textstyle\sum_{s=4,6,\dots}^{\infty} \sigma_s(x) = \lim_{z\to 1} \sum_{s=4,6,\dots}^{\infty} \mathrm{c}_s\, x^{-(s-2)}\,\mathrm{P}_s(x)\, z^{s-4}$$

$$= \tfrac{1}{4320}\left[60\,(x+1)^3\,x^3\,\log\left(\tfrac{1}{x}+1\right) - 60\,x^5 - 150\,x^4 - 110\,x^3 - 15\,x^2 + 3\,x - 1\right]$$

- total $s > 2$ exchange vanishes: t- and u- channels cancel

  $\sigma(x) + \sigma(-1-x) = 0$

- contribution of $s = 0, 2$ exchanges + 2222 contact vertex

$$A^{0,\mathrm{s}}_{++;++} = \frac{\mathrm{s}^2}{18432}, \quad A^{0,\mathrm{t}}_{++;++} = \frac{\mathrm{t}^2\,\mathrm{u}^4}{2048\,\mathrm{s}^4}, \quad A^{0,\mathrm{u}}_{++;++} = \frac{\mathrm{t}^4\,\mathrm{u}^2}{2048\,\mathrm{s}^4},$$

$$A^{2,\mathrm{s}}_{++;++} = \frac{\mathrm{s}^2+6\,\mathrm{s}\,\mathrm{t}+6\,\mathrm{t}^2}{92160}, \quad A^{2,\mathrm{t}}_{++;++} = \frac{\mathrm{u}^2(2\,\mathrm{s}^4-10\,\mathrm{s}^3\,\mathrm{t}+33\,\mathrm{s}^2\,\mathrm{t}^2-24\,\mathrm{s}\,\mathrm{t}^3+3\,\mathrm{t}^4)}{30720\,\mathrm{s}^4}$$

$$A^{2,\mathrm{u}}_{++;++} = \frac{\mathrm{t}^2\,(2\,\mathrm{s}^4-10\,\mathrm{s}^3\,\mathrm{u}+33\,\mathrm{s}^2\,\mathrm{u}^2-24\,\mathrm{s}\,\mathrm{u}^3+3\,\mathrm{u}^4)}{30720\,\mathrm{s}^4}$$

$$A^{\mathrm{contact}}_{++;++} = -\frac{\mathrm{s}^6-\mathrm{s}^5\,\mathrm{t}+26\,\mathrm{s}^4\,\mathrm{t}^2+63\,\mathrm{s}^3\,\mathrm{t}^3+54\,\mathrm{s}^2\,\mathrm{t}^4+27\,\mathrm{s}\,\mathrm{t}^5+9\,\mathrm{t}^6}{7680\,\mathrm{s}^4}$$

non-trivial cancellation: total 2222 amplitude =0

$$A^{0,\mathrm{s}} + A^{0,\mathrm{t}} + A^{0,\mathrm{u}} + A^{2,\mathrm{s}} + A^{2,\mathrm{t}} + A^{2,\mathrm{u}} + A^{\mathrm{contact}} = 0$$

- similar cancellation checked for 1122 amplitude
- conjecture: full massless-state CHS S-matrix is trivial
- should follow again from underlying global CHS symmetry

HS charges $\to$ triviality of S-matrix (cf. Coleman-Mandula)

CHS symmetries

$$h(x,p) \equiv h_{\mu_1\ldots\mu_s}(x)\, p^{\mu_1}\ldots p^{\mu_s}$$

$$f(x,p) \star g(x,p) = f(x,p)\, e^{\frac{i}{2}(\overleftarrow{\partial}_x\cdot\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\cdot\overrightarrow{\partial}_x)}\, g(x,p)$$

- diff and algebraic symm of scalar-CHS system [Segal 02]

$$\delta_\epsilon h(x,p) = (p\cdot\partial_x)\epsilon(x,p) - \tfrac{i}{2}\big[h(x,p),\epsilon(x,p)\big]_\star$$

$$\delta_\alpha h(x,p) = (p^2 - \tfrac{1}{4}\partial_x^2)\alpha(x,p) - \tfrac{1}{2}\big\{h(x,p),\alpha(x,p)\big\}_\star$$

$$\delta_{\epsilon+i\alpha}\Phi(x) = e^{-\frac{i}{2}\partial_{x'}\cdot\partial_p}\big[\epsilon(x,p)+i\alpha(x,p)\big]\,\Phi(x')\Big|_{x=x',\ p=0}$$

$$\delta h = \delta_0 h + \delta_1 h\ , \qquad \delta_0 h_s \sim \partial\epsilon_{s-1} + \eta\alpha_{s-2}$$

- global symmetry from: $\delta_1 h \sim \epsilon\,\partial h + \partial\epsilon\; h + \ldots$ for special ϵ
- spin s field transforms in terms of $s' < s$ fields

$$\delta_1 h_0 \sim \textstyle\sum_k \epsilon^{\mu(k)}\partial_{\mu(k)}h_0\ , \quad \delta_1 h^\rho \sim \sum_k \Big[\epsilon^{\rho\mu(k)}\partial_{\mu(k)}h_0 + \epsilon^{\mu(k)}\partial_{\mu(k)}h^\rho\Big]$$

$$\delta_1 h^{\rho\sigma} \sim \textstyle\sum_k \Big[\epsilon^{\rho\sigma\mu(k)}\partial_{\mu(k)}h_0 + \epsilon^{\mu(k)(\rho}\partial_{\mu(k)}h^{\sigma)} + \frac{1}{2!k!}\epsilon^{\mu(k)}\partial_{\mu(k)}h^{\rho\sigma}\Big]$$

- constraints on amplitudes as in external scalar scattering case

CHS fields in curved background

Expansion near vacuum with non-trivial $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$:

• Weyl-invariant quadratic action known for $s = 1$ and $s = 2$

• $s > 2$: kinetic operator $\mathcal{O}_s = \nabla^{2s} + \ldots$ – diff and Weyl inv

but to be consistent with CHS gauge symm.:

$g_{\mu\nu}$ should solve Bach eqs $\quad (\nabla^\mu \nabla^\nu + \frac{1}{2} R^{\mu\nu}) C_{\lambda\mu\nu\rho} = 0$

• $\mathcal{O}_s$ simplifies (factorizes) on conf-flat background:

explicitly known on S^4 or AdS_4 [AT 13; Metsaev 14; Nutma,Taronna 14]

and $S^1 \times S^3$ [Bekaert, Beccaria, AT 14]

• quantum consistency? anomalies?

conformal → Weyl symmetry: $g'_{mn} = \lambda^2(x) g_{mn}$

Weyl anomaly: $\quad T^m_m = -\mathrm{a}\, R^* R^* + \mathrm{c}\, C^2$

Weyl gravity ($s = 2$) is anomalous: $\mathrm{a}_2 = \frac{87}{20}, \ \mathrm{c}_2 = \frac{199}{30}$

• one way to cancel anomaly – add fermions: supersymmetry

$N=4$ conformal supergravity + 4 $N=4$ Maxwell multiplets

is anomaly free: $\mathrm{a}=\mathrm{c}=0$ [Fradkin, AT 82]

• alternative: sum over infinite number of CHS contributions

• CHS fields with $s>2$:

to find a_s: enough to know partition function on S^4

to find c_s: need to know $\mathcal{O}_s$ on Ricci-flat background

CHS partition function on S^4

• Maxwell theory on S^4 $(R=12, \ r=1)$

$$Z_1=\left[\frac{\det\Delta_0(0)}{\det\Delta_{1\perp}(3)}\right]^{1/2}, \qquad \Delta_s(M^2)\equiv-\nabla_s^2+M^2$$

• Weyl graviton: $C^2 \to \frac{1}{2} h\, \Delta_{2\perp}(2)\, \Delta_{2\perp}(4)\, h$

$$Z_2 = \left[\frac{\det \Delta_{1\perp}(-3)}{\det \Delta_{2\perp}(2)}\right]^{1/2} \left[\frac{\det \Delta_{0}(-4)}{\det \Delta_{2\perp}(4)}\right]^{1/2}$$

• CHS operator: factorization into "partially-massless"

$$\mathcal{O}_s = \nabla^{2s} + ... = \prod_{k=0}^{s-1} \Delta_{s\perp}(M^2_{s,k}) \ , \qquad M^2_{s,k} = 2 + s - k - k^2$$

• get simple generalization of flat-space Z

$$Z(S^4) = \prod_{s=1}^{\infty} Z_s, \qquad Z_s = \prod_{k=0}^{s-1} Z_{s,k} \ , \quad Z_{s,k} = \left[\frac{\det \Delta_{k\perp}(M^2_{k,s})}{\det \Delta_{s\perp}(M^2_{s,k})}\right]^{1/2}$$

$$\ln Z = -B_4 \ln \Lambda_{\rm UV} + ..., \qquad B_4 = \int d^4x \sqrt{g}\, b_4 \Big|_{S^4} = -\mathrm{a}_s$$

• summing contributions of 2nd order operators [AT 13]

$$\begin{aligned} \mathrm{a}_s &= \textstyle\sum_{k=0}^{s-1} \Big(\mathrm{a}[\Delta_{s\perp}(2+s-k-k^2)] - \mathrm{a}[\Delta_{k\perp}(2+k-s-s^2)]\Big) \\ &= \tfrac{1}{180}\nu^2(14\nu+3)\ , \qquad\qquad\qquad \nu = s(s+1) \end{aligned}$$

• same coefficient found via massless HS AdS_5 relation

[Giombi, Klebanov, Pufu, Safdi, Tarnapolsky 13]

$$\ln \frac{Z_s^{(-)}}{Z_s^{(+)}} = \ln Z_s = \mathrm{a}_s \ln \Lambda_{\rm IR} + ..., \qquad \mathrm{vol}(\mathrm{AdS}_5) \sim \ln \Lambda_{\rm IR}$$

• with $e^{-\epsilon(s+\frac{1}{2})}$ regularization prescription for $\sum_s$ consistent with CHS symmetries get

$$\sum_{s=1}^{\infty} \mathrm{a}_s = 0$$

• finite parts cancel too: $\quad Z(S^4) = 1$

[Giombi, Klebanov, Safdi 14; Beccaria, AT 15]

Ricci-flat background

• Maxwell vector: $(\Delta_1)_{mn} = -(\nabla^2)_{mn} + R_{mn}, \quad \Delta_0 = -\nabla^2$

$$Z_1 = \Big[\frac{(\det \Delta_0)^2}{\det \Delta_1}\Big]^{1/2}$$

• Weyl graviton: 4-th order operator factorizes:
square of Einstein op. $\quad (\Delta_2)_{mn,kl} = -(\nabla^2)_{mn,kl} - 2C_{mknl}$

$$Z_2 = \Big[\frac{(\det \Delta_1)^3}{(\det \Delta_2)^2}\Big]^{1/2}$$

• if assume that factorization of $\mathcal{O}_s$ true also for $s > 2$:
s factors of "massless" spin s 2nd-order operator

$$Z_s = \Big[\frac{(\det \Delta_{s-1})^{s+1}}{(\det \Delta_s)^s}\Big]^{1/2}, \qquad \Delta_s = -\nabla^2 - s(s-1)C_{....}$$

same structure as in flat space but with covariant operators Δ_s

• from Seeley coefficients for Δ_s get [AT 13]

$$\mathrm{c}_s - \mathrm{a}_s = \tfrac{1}{720}\nu_s(15\nu_s^2 - 45\nu_s + 4)\ , \qquad \nu_s = s(s+1)$$

with same summation over spins prescription

$$\textstyle\sum_{s=1}^{\infty}(\mathrm{c}_s - \mathrm{a}_s) = 0$$

• then a- and c- anomalies or UV ∞ appear to vanish:
suggests novel mechanism of UV finiteness due to
summation of ∞ number of bosonic fields (cf. string theory)

• $\sum_s \mathrm{c}_s = 0$ remains a conjecture:

• $\mathcal{O}_{s>2}$ does not factorize on $R_{mn} = 0$ backgr [Nutma, Taronna 14]
but obstruction to factorization $\sim \nabla_{.} C_{....}$ should not change c_s

• CHS action does not diagonalize on $R_{mn} = 0$ backgr:
mixing terms [Grigoriev, AT 16] contribute to c_s [Beccaria, AT 17]

Curved space background: spin 1 – 3 mixing

[Beccaria, AT 17]

• flat space:

$$S_0 = \int d^4x \, \Phi^* \, \partial^2 \, \Phi, \qquad \partial^{a_1} J_{a_1 \cdots a_s} = 0, \qquad J^{a_1}{}_{a_1 \cdots a_s} = 0$$

$$J_a = i \, \Phi^* \, \partial_a \, \Phi + c.c.$$

$$J_{ab} = \Phi^* \, \partial_a \partial_b \, \Phi - 2 \, \partial_a \Phi^* \, \partial_b \Phi + \tfrac{1}{2} \eta_{ab} \, \partial^c \Phi^* \, \partial_c \Phi + c.c.$$

$$J_{abc} = i \Big[\Phi^* \partial_a \partial_b \partial_c \Phi - 9 \partial_{(a} \Phi^* \partial_b \partial_{c)} \Phi + 3 \eta_{(ab} \, \partial^p \Phi^* \partial_p \partial_{c)} \Phi \Big] + c.c.$$

adding interaction with background fields:

$$S_{int} = \sum_s \int d^4x \, h^{a_1 \cdots a_s}(x) \, J_{a_1 \cdots a_s}$$

inv under $\delta h_{a_1 \cdots a_s} = \partial_{(a_1} \, \epsilon_{a_2 \cdots a_s)} + \eta_{(a_1 a_2} \, \alpha_{a_3 \cdots a_s)}$ mod $\partial^2 \Phi$ terms

extended off shell if transform Φ and add terms linear in h_s

• curved space:

$$S_0 = \int d^4x \, \sqrt{g} \, \Phi^* \big(-\nabla^2 + \tfrac{1}{6} \, R \big) \, \Phi$$

$$S_{int} = \sum_s \int d^4x \, \sqrt{g} \, h^{a_1 \cdots a_s}(x) \, J_{a_1 \cdots a_s}$$

• require $\nabla^{a_1} J_{a_1 \cdots a_s} = 0, \qquad J^{a_1}{}_{a_1 \cdots a_s} = 0$

then will have inv under backgr-cov gauge transfs

$\delta h_{a_1 \cdots a_s} = \nabla_{(a_1} \epsilon_{a_2 \cdots a_s)} + g_{(a_1 a_2} \alpha_{a_3 \cdots a_s)}$

• require also Weyl inv w.r.t. backgr metric: $w = w(x)$

$\delta_w g_{ab} = 2\, w\, g_{ab}, \quad \delta_w \Phi = -w\, \Phi, \quad \delta_w h_{a_1 \cdots a_s} = 2\,(s-1)\, w\, h_{a_1 \cdots a_s}$

• such covariant currents exist for $s = 1$ and $s = 2$:

$$J_a = i \left(\Phi^* \nabla_a \Phi - \nabla_a \Phi^* \Phi \right), \qquad \nabla^a J_a = 0$$

$$J_{ab} = \frac{6}{\sqrt{g}} \frac{\delta S_0}{\delta g^{ab}} = \left(\Phi^* \nabla_a \nabla_b \Phi - 2\, \nabla_a \Phi^* \nabla_b \Phi + c.c \right)$$
$$+ g_{ab} \nabla_c \Phi^* \nabla^c \Phi - (R_{ab} - \tfrac{1}{6} g_{ab} R) \Phi^* \Phi$$

• but $s \geqslant 3$ cases are different:

$\nabla^{a_1} J_{a_1 \cdots a_s} \neq 0$ – given by terms with lower-rank J_s

• $s = 3$: unique traceless current with Weyl-inv $S_{int} = \int h_3 J_3$:

$$J_{abc} = i\Big[\Phi^* \nabla_{(a}\nabla_b\nabla_{c)}\,\Phi - 9\,\nabla_{(a}\,\Phi^*\,\nabla_b\nabla_{c)}\,\Phi + 3\,g_{(ab}\,\nabla^p\,\Phi^*\,\nabla_p\nabla_{c)}\,\Phi$$
$$+2g_{(ab}\,\Phi^*\,\nabla^2\,\nabla_{c)}\,\Phi + \tfrac{1}{2}\,g_{(ab}\,R\,\Phi^*\,\nabla_{c)}\,\Phi - 7\,R_{(ab}\,\Phi^*\,\nabla_{c)}\Phi\Big] + c.c.$$

• J_3 conserved only in conformally-flat background:

$$\nabla_a\,J^{abc} = 8C^{pbcq}\nabla_{(p}J_{q)} + 32\nabla_{(p}C^{pbcq}J_{q)}$$

• 1+3 action $S_{int} = \int d^4x\sqrt{g}(h^a J_a + h^{abc}J_{abc})$
is invariant under $\delta h_a = \partial_a\epsilon$ and combined transformations

$$\delta h_{abc} = \nabla_{(a}\,\epsilon_{bc)}\ , \qquad \delta h_a = -8\,C_{abcd}\,\nabla^d\,\epsilon^{bc} + 24\,\nabla^d C_{abcd}\,\epsilon^{bc}$$

• to make invariance manifest (off-shell):
need also to transform Φ and add $h_1 h_3 + ...$ terms in S_{int}
(manifest spin 1 invariance: $\nabla_a\Phi \to D_a\Phi = \nabla_a\Phi + ih_a\Phi$)

• induced action inv under h-gauge transf
$e^{-\Gamma(h)} = \int d\Phi\, e^{-S(\Phi,h;g)}$, $\qquad \Gamma(h) = \mathrm{S}(h)\log\Lambda_{\mathrm{UV}} + \ldots$

• non-linear $h_s h_{s'} + ...$ terms produce contact terms
in generating functional for correlators of currents:
absent in correlators $\langle J(x_1)...J(x_n)\rangle$ at separated points
but contributing to local UV singular part – to induced action
• need contact terms to get e.g. covariant spin 1 + 2 action

$$\mathrm{S} = \int d^4x \,\sqrt{g}\,\big(- \tfrac{1}{12}\, F^2_{ab} + \tfrac{1}{120}\, C^2_{abcd}\big)$$

• expansion near g_{ab}:

$$\int d^4x\sqrt{g}\, C^2_{abcd} \to \int d^4x\sqrt{g}\big[B_{ab}(g)h^{ab} + h^{ab}\mathcal{O}_{abcd}(g)h^{cd} + ...\big]$$

$\mathcal{O}_4 = \nabla^4 + ...$ is gauge-inv $\delta h_{ab} = \nabla_{(a}\epsilon_{b)}$ if

$$B_{ab} = (\nabla^p\nabla^q + \tfrac{1}{2}R^{pq})C_{apqb} = 0$$

• expansion of S in h_s: manifest reparam and Weyl inv

$$\mathrm{S}(g,h) = \mathrm{S}^{(0)}(g) + \mathrm{S}^{(1)}(g,h) + \mathrm{S}^{(2)}(g,h) + \ldots$$

$$\mathrm{S}^{(1)} = \int\, B_{(s)}(g)\, h^{(s)}\,, \qquad \mathrm{S}^{(2)} = \int\, h^{(s)}\,\mathcal{O}_{s,s'}(g)\, h^{(s')}$$

• gauge invariance if $\langle J_{(s)}\rangle_{\rm UV} \sim B_{(s)}(g)$=0
Weyl-inv + $\nabla^a B_{a...} = 0 \to$ true for Bach-flat g [Grigoriev, AT]

Quadratic part of spin 1 + 3 induced action

$\mathrm{S}^{(2)} = \mathrm{S}_{11} + \mathrm{S}_{13} + \mathrm{S}_{33}$

$L_{11} = h^a \langle J_a\, J_b \rangle_{\mathrm{UV}} h^b = -\frac{1}{6}\, F_{ab}^2$

$L_{33} = h_3 \mathcal{O}_6\, h_3$: $\quad \mathcal{O}_6$ from $\langle J_{abc}\, J_{pqr} \rangle_{\mathrm{UV}}$ + contact term

$L_{13} = h^a\, \langle J_a\, J_{bcd} \rangle_{\mathrm{UV}}\, h^{bcd}$ + contact term

final result for the mixing term:

$L_{13} = 8\, F^{ab} \Big[C_a{}^{cdp}\, \nabla_p h_{bcd} + \big(\nabla_a R^{cd} - \nabla^c R_a^d \big) h_{bcd} \Big]$

Weyl-inv; vanishes for conformally-flat Einstein space g_{ab}

• in Bach-flat case: e.g. Einstein background $R_{ab} = \frac{1}{4} R g_{ab}$

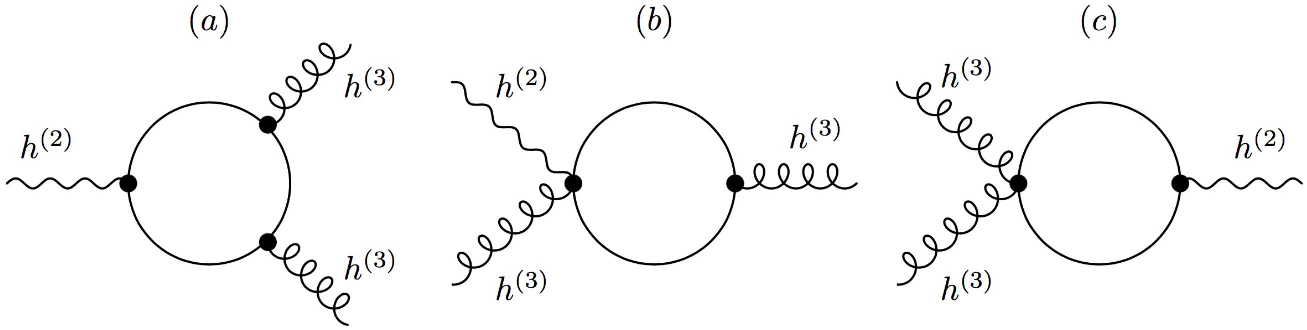
$$\mathrm{S}^{(2)} = \int d^4x \sqrt{g} \big[-\tfrac{1}{12} F_{ab}^2 + 8 C^{abcd} F_{ap} \nabla_d h^p{}_{bc} + h_3\, \mathcal{O}_6\, h_3 \big]$$

• invariant under spin 3 gauge transformations

$$\delta h_{abc} = \nabla_{(a}\, \epsilon_{bc)}\ , \qquad \delta h_a = -8\, C_{abcp}\, \nabla^p\, \epsilon^{bc}$$

• ϵh_3 term in variation of S_{13} is order CC:

$h_3\, \mathcal{O}_6\, h_3$ inv by itself only to 1st order in C [Nutma, Taronna 14]



linear in curvature terms in $\mathcal{O}_6$ can be found from UV part of $h_2 h_3 h_3$ 1-loop scalar diagrams: $\langle J_{abc}\, J_{pqr}\, J_{mn}\rangle_{\rm UV}$ + contact terms

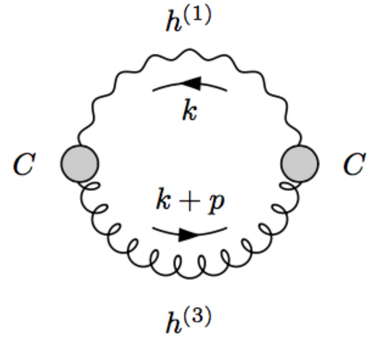
Spin 1–3 mixing term contribution to UV divergences

$$\Gamma = -\log Z = -\log \Lambda_{\rm UV} \int d^4x \sqrt{g}\, b_4(x) \quad + \quad \text{finite}$$

$$b_4 = -\mathrm{a}\, R^* R^* + \mathrm{c}\, C^2$$

• conf flat background: no mixing terms, $\mathcal{O}_s$ factorize and get

$$\mathrm{a}_s = \frac{1}{720}\, \nu_s \left(3\, \nu_s + 14\, \nu_s^2\right), \qquad \nu_s \equiv s(s+1)$$



• ignoring mixings and assuming that factorization holds also in Ricci-flat case [AT 13]

$\mathrm{c}_s \equiv \mathrm{c}_{ss} = \frac{1}{720}\,\nu_s\,(29\,\nu_s^2 - 42\,\nu_s + 4)$

• need to add mixing terms contribution to C^2 div:

example of 1–3 sector: $\mathrm{c}_1 = \frac{1}{10}$, $\mathrm{c}_3 = \frac{919}{15}$

$L = h_1(\nabla^2 + ...)h_1 + C\nabla h_1 \nabla h_3 + h_3(\nabla^6 + ...)h_3$

1-loop diagram gives non-trivial contribution:

$L_{\rm UV} = \mathrm{c}_{13} C_{abcd} C^{abcd} \log \Lambda_{\rm UV}\ , \qquad \mathrm{c}_{13} = \frac{392}{5}$

• need to find all mixing terms to decide if $\sum_{s,s'} \mathrm{c}_{ss'} = 0$

Conclusions

• theories with infinite number of massless higher spin fields:
importance of definition of quantum theory
consistent with underlying symmetries

• remarkable simplifications due to large HS symmetry:
• 1-loop $Z = 1$ on R^4 ($\sum_s \nu_s = 0$) and S^4 ($\sum_s \mathrm{a}_s = 0$)
• vanishing of scattering amplitudes with CHS exchange:
triviality of S-matrix implied by conformal HS symmetry

• intricate structure of interacting induced CHS action
mixing terms in non-trivial background to be understood $\rightarrow$
cancellation of c-anomalies $\sum_s \mathrm{c}_s = 0$ remains to be proved