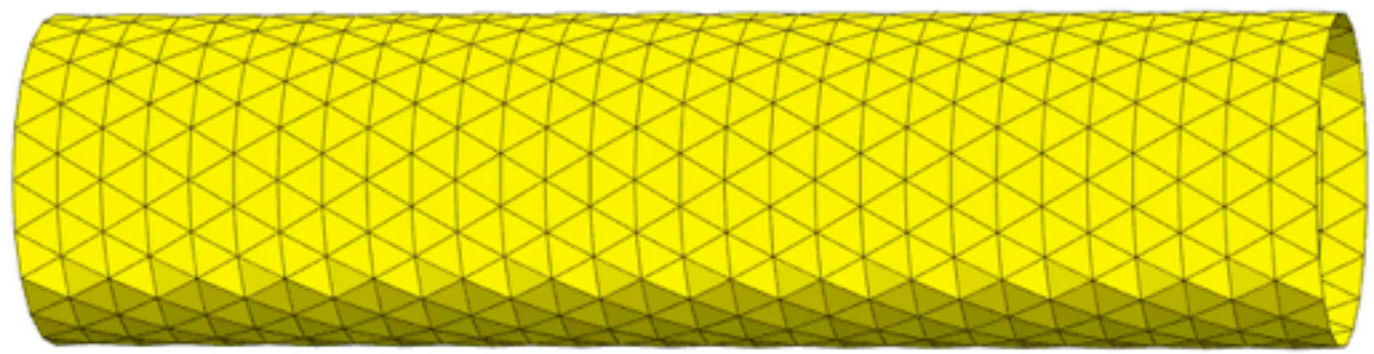
Plastic deformation of tubular crystals by dislocation glide



Daniel Beller, David Nelson

HARVARD
John A. Paulson
School of Engineering
and Applied Sciences

Geometry, elasticity, fluctuations, and order in 2D soft matter
Kavli Institute for Theoretical Physics
January 14, 2016

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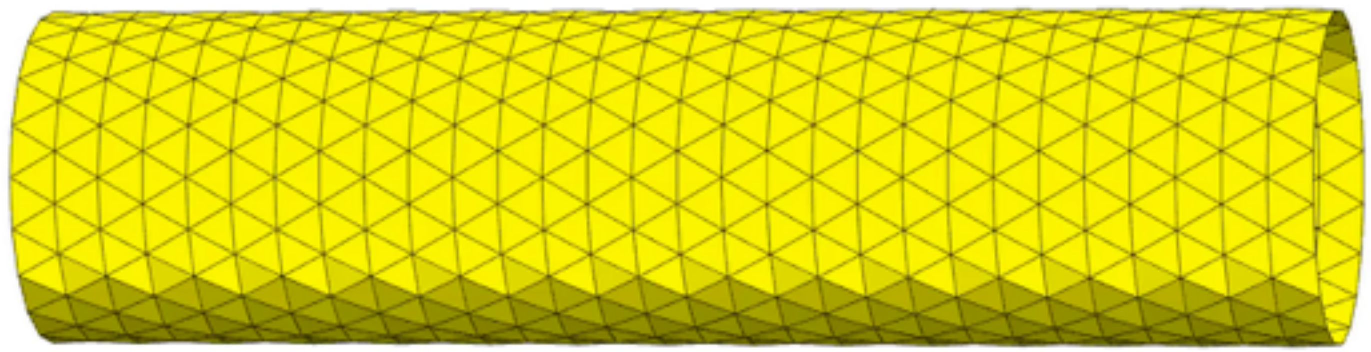

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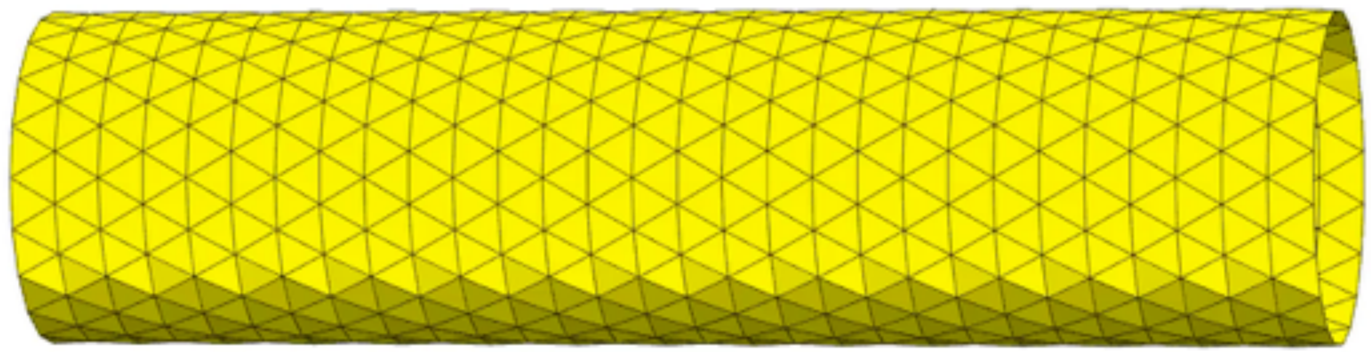

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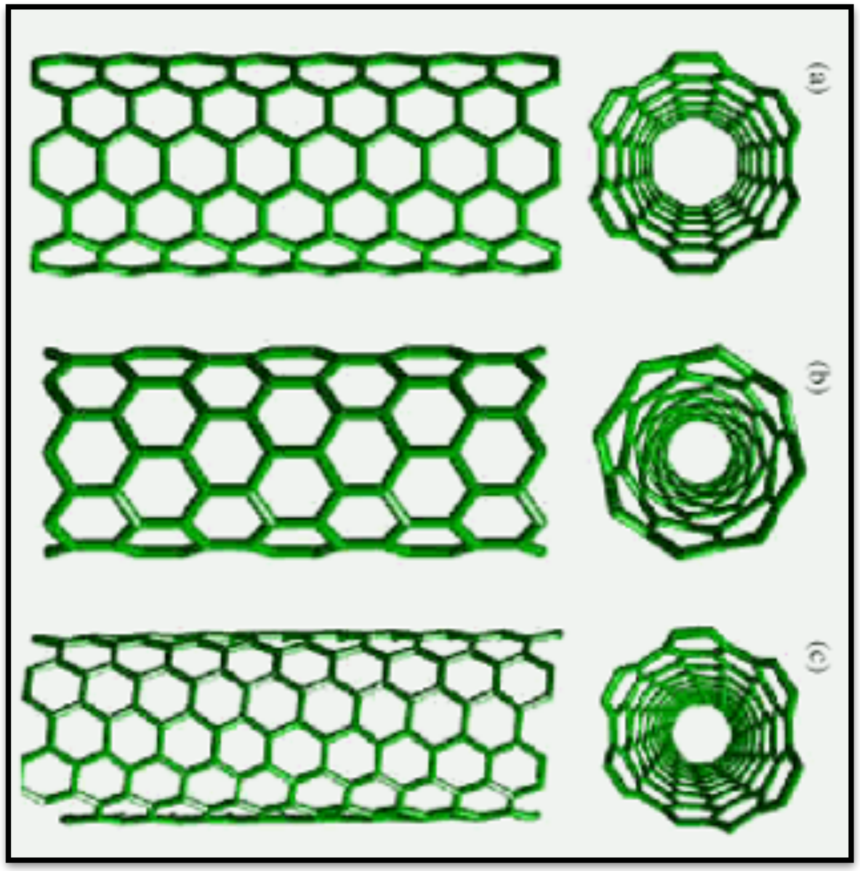
Examples of tubular crystals

Single-walled carbon nanotubes



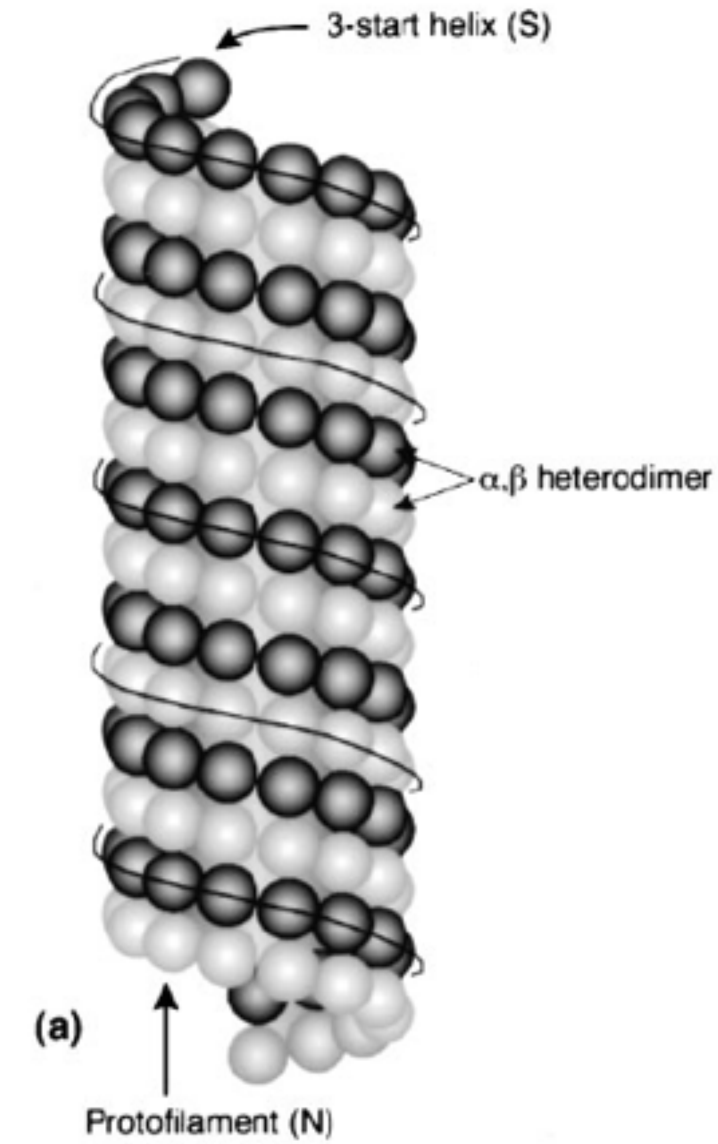
armchair

zigzag

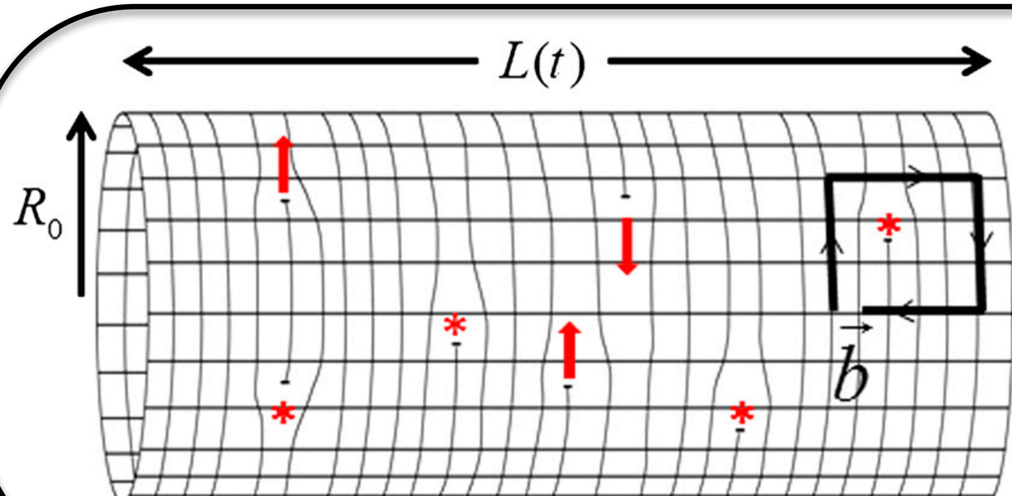
chiral

http://education.mrsec.wisc.edu/nanoquest/carbon/

Microtubules



Chrétien *et al.*, Eur Biophys J, 1998.

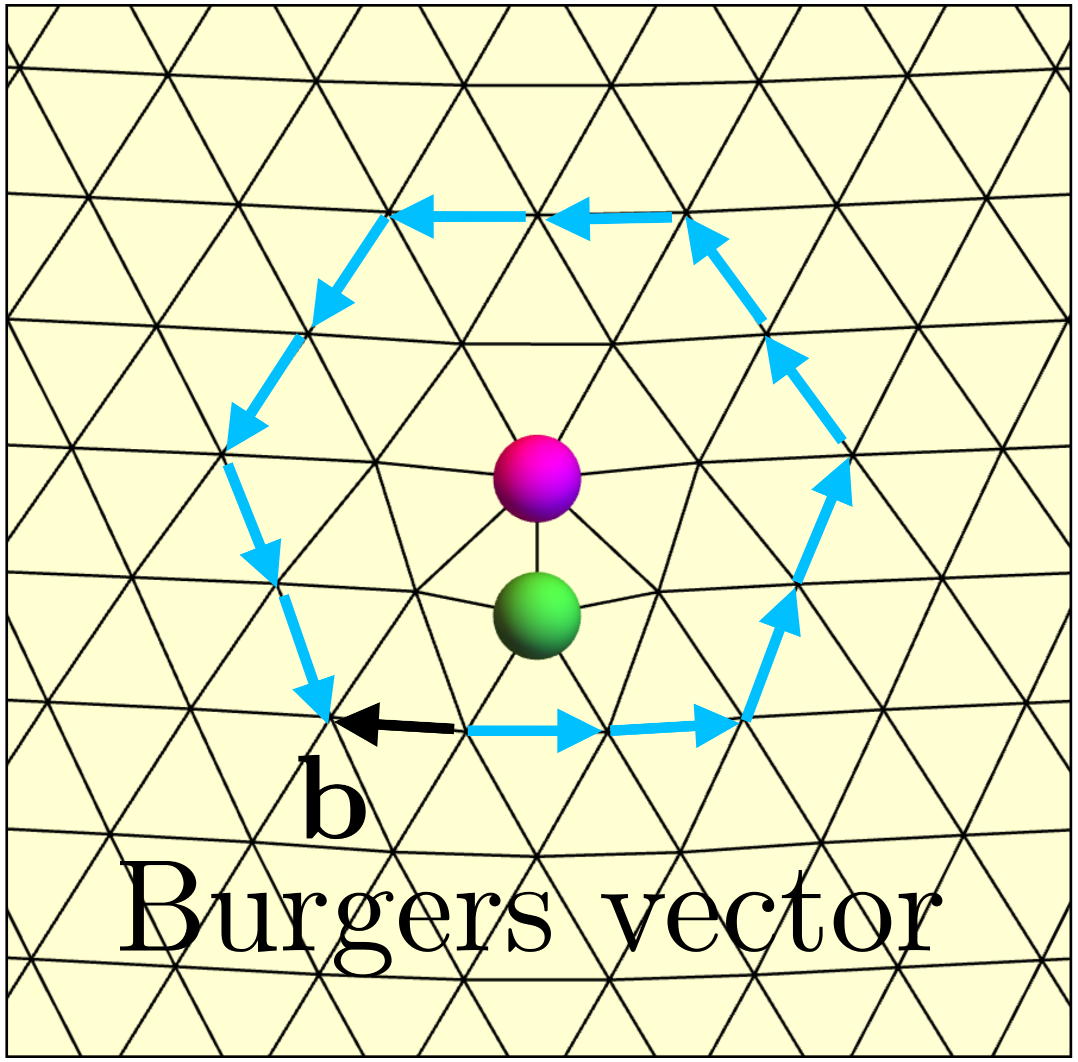


Bacterial cell wall (peptidoglycan mesh)

Amir & Nelson, "Dislocation-mediated growth of bacterial cell walls", PNAS 109:9833 (2012)

Each of these systems may contain dislocations…

Dislocation



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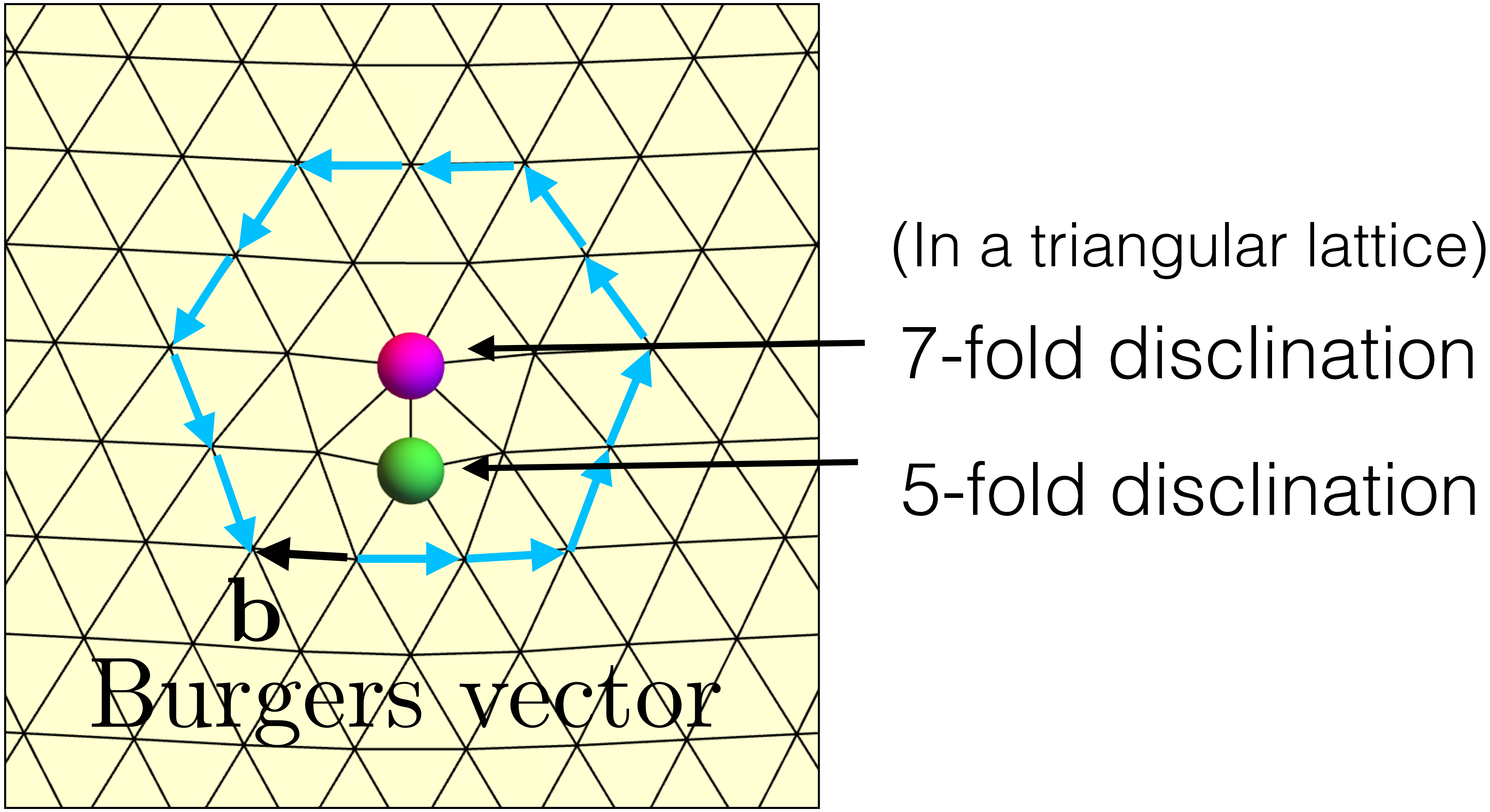
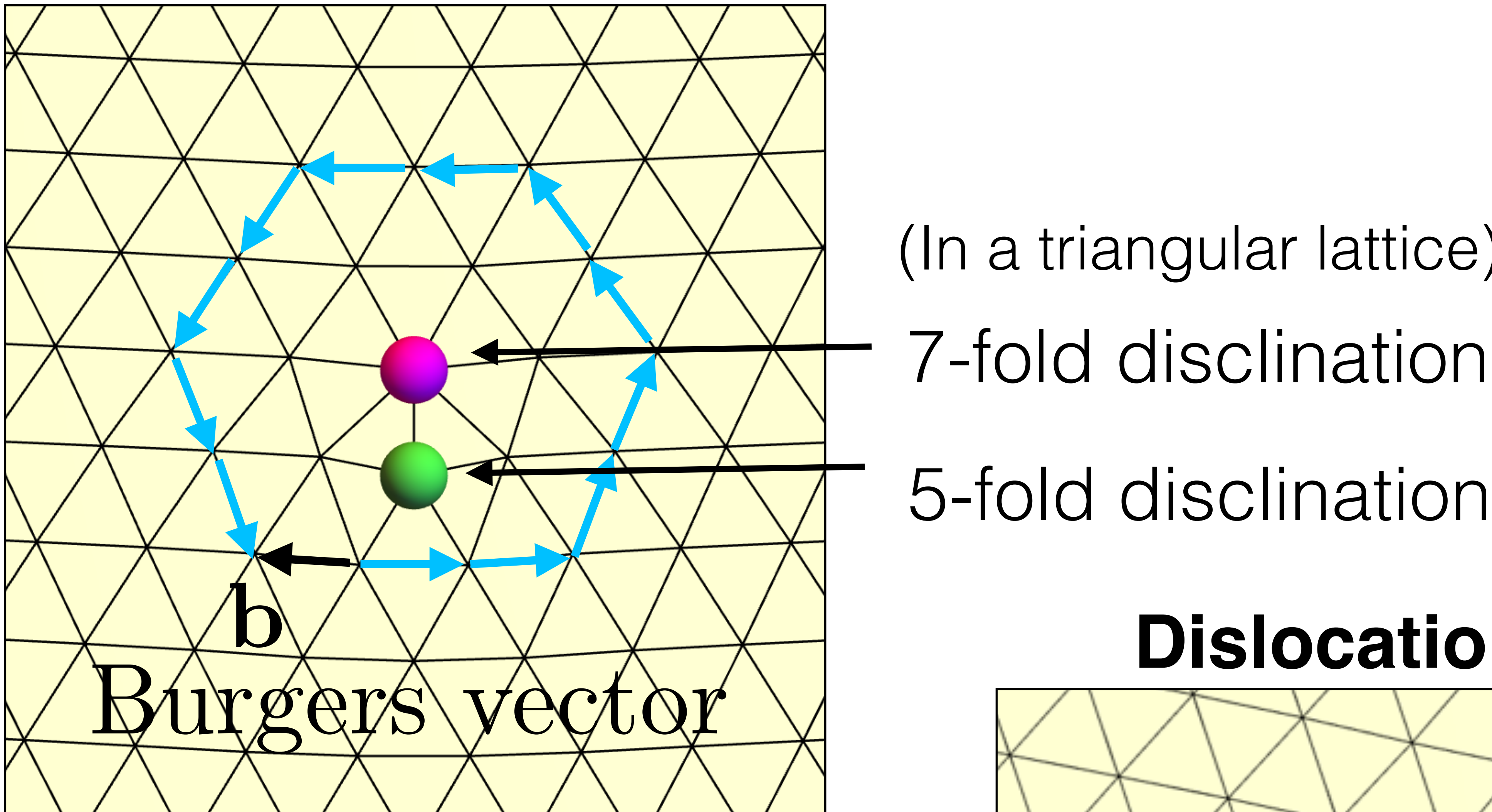
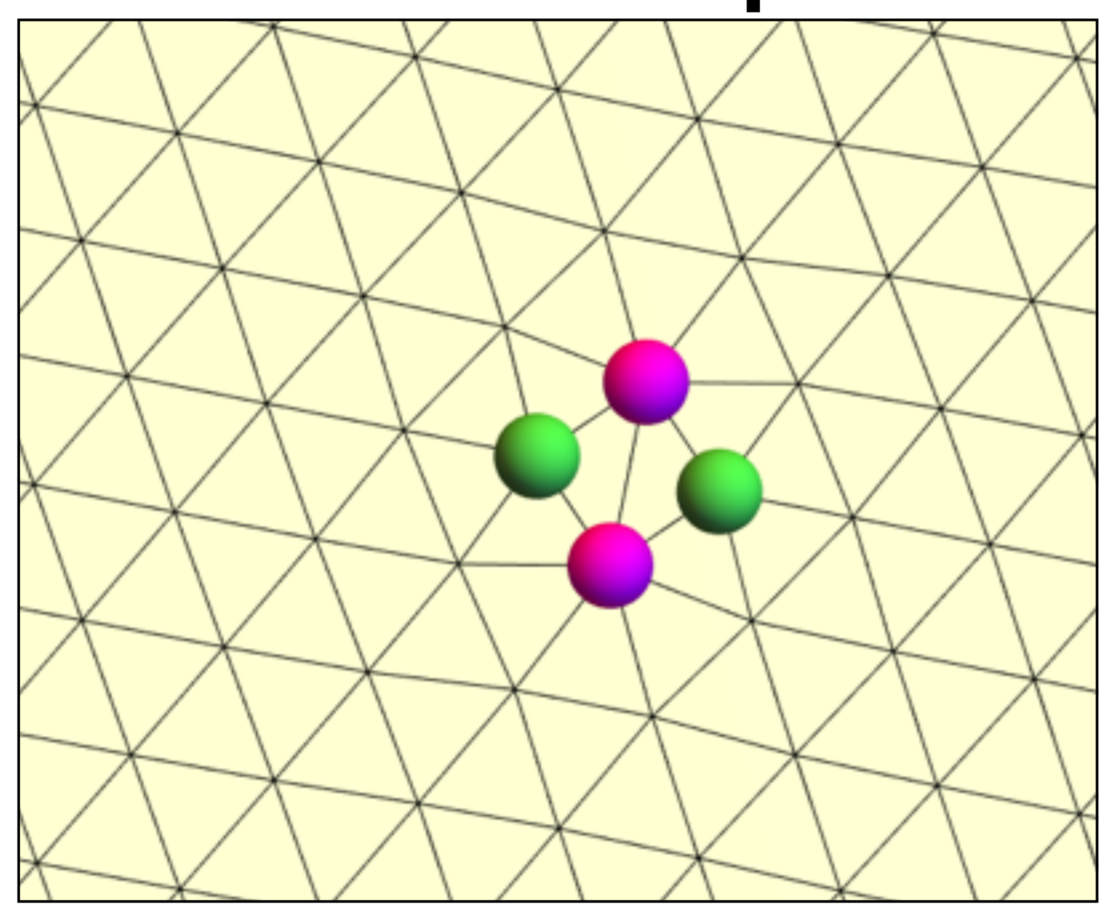


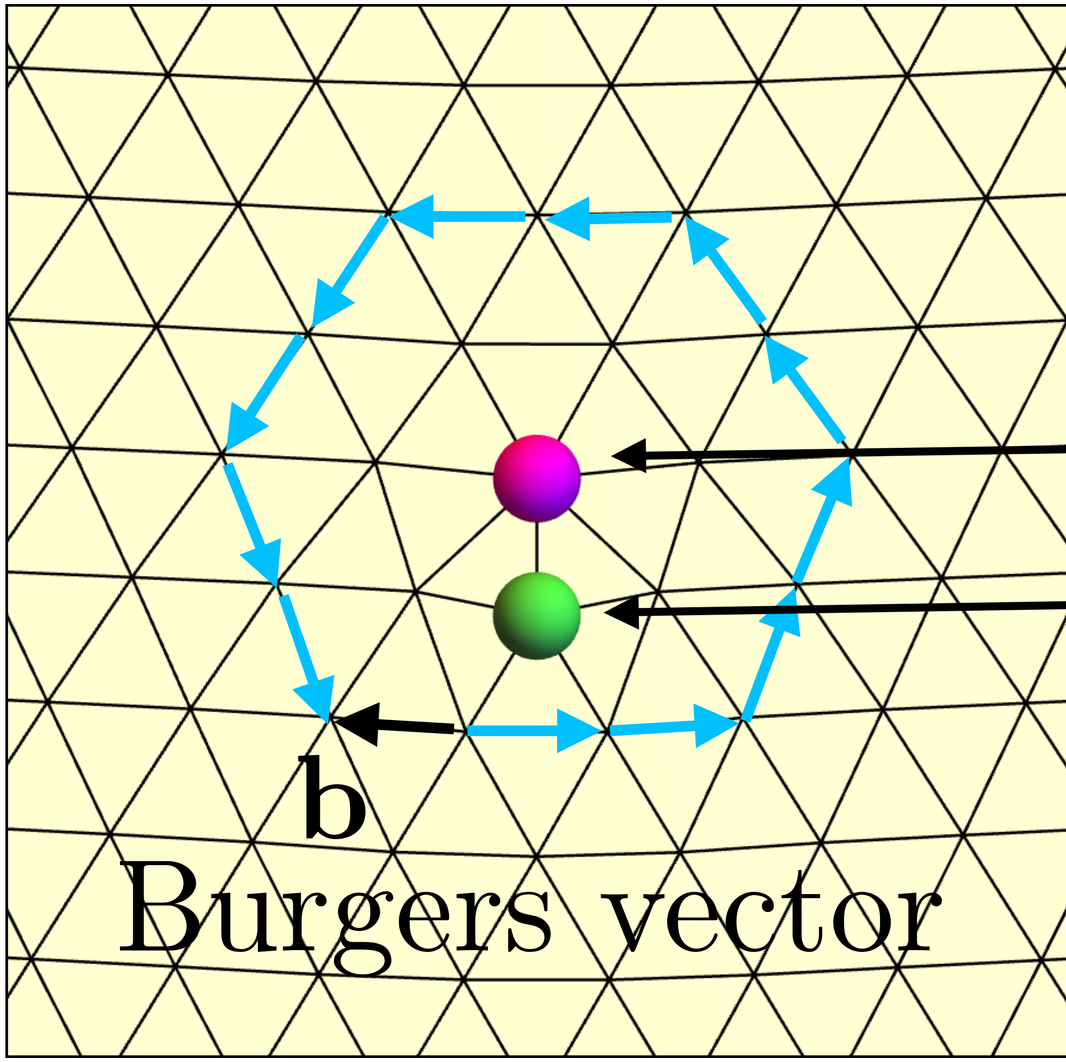
Dislocation



Dislocation pair



Dislocation

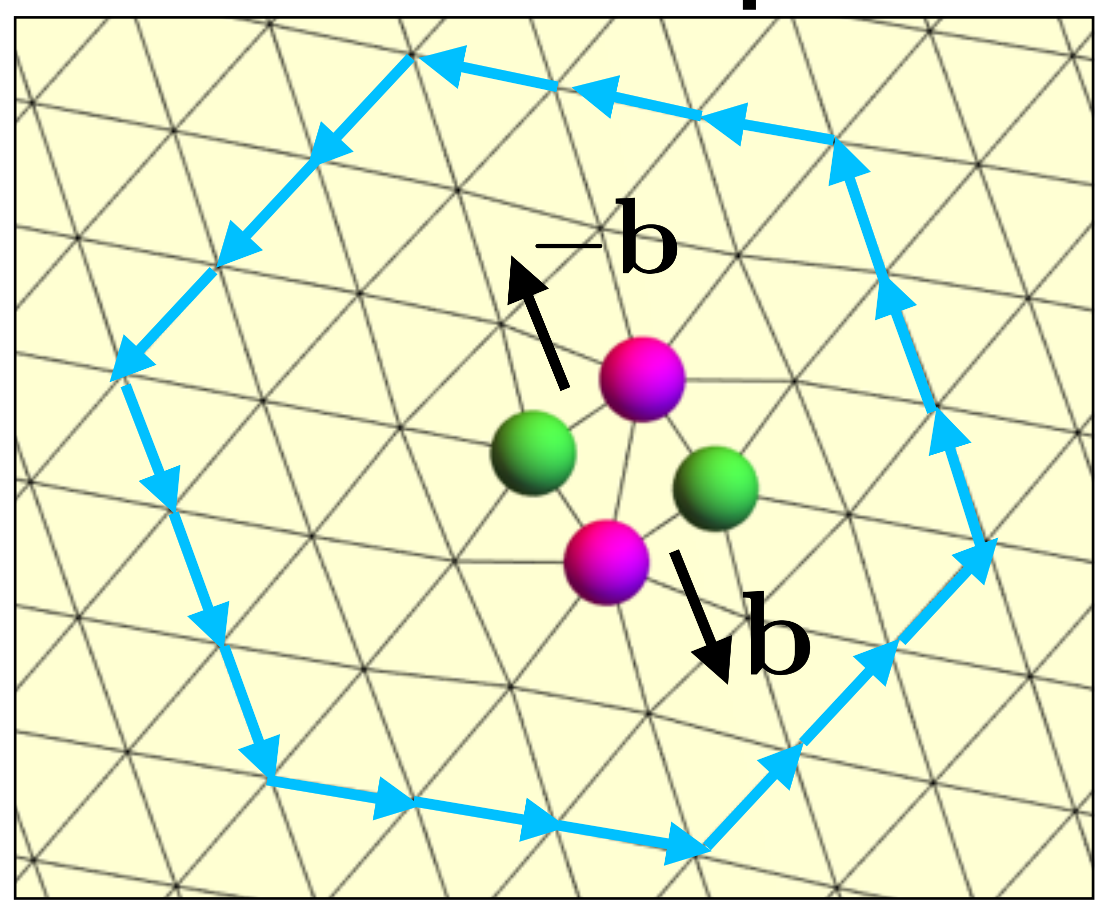


(In a triangular lattice)

7-fold disclination

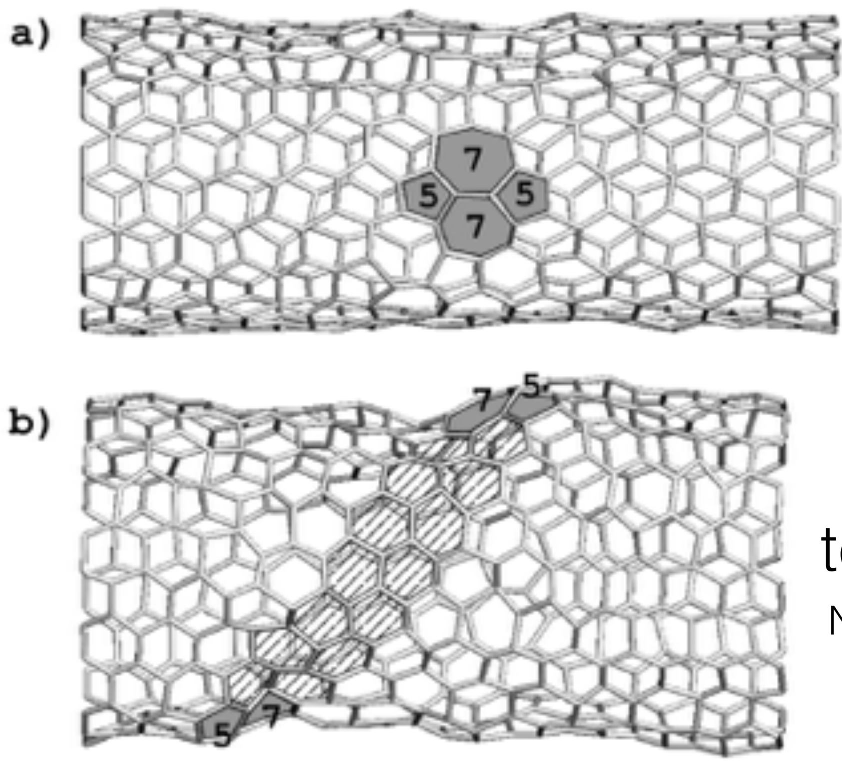
5-fold disclination

Dislocation pair

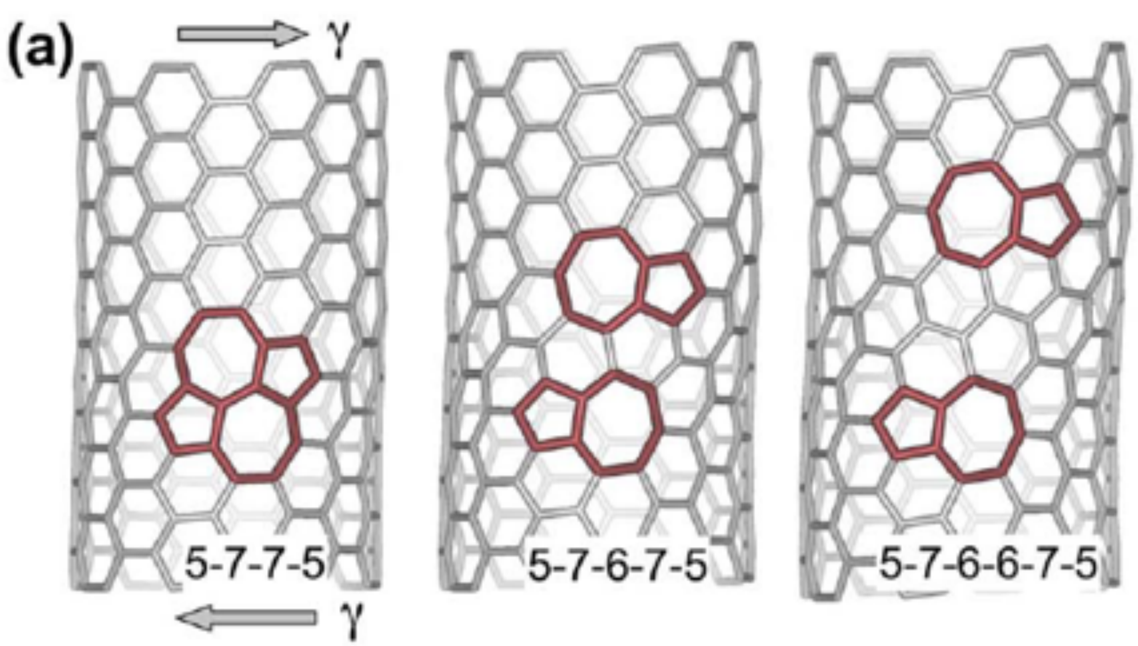


Single-walled carbon nanotubes plastically deform by dislocation motion at high temp

Simulation



Axial strain ~ 10%
temperature = 3000 K
Nardelli et al., PRL 81:4656 (1998)

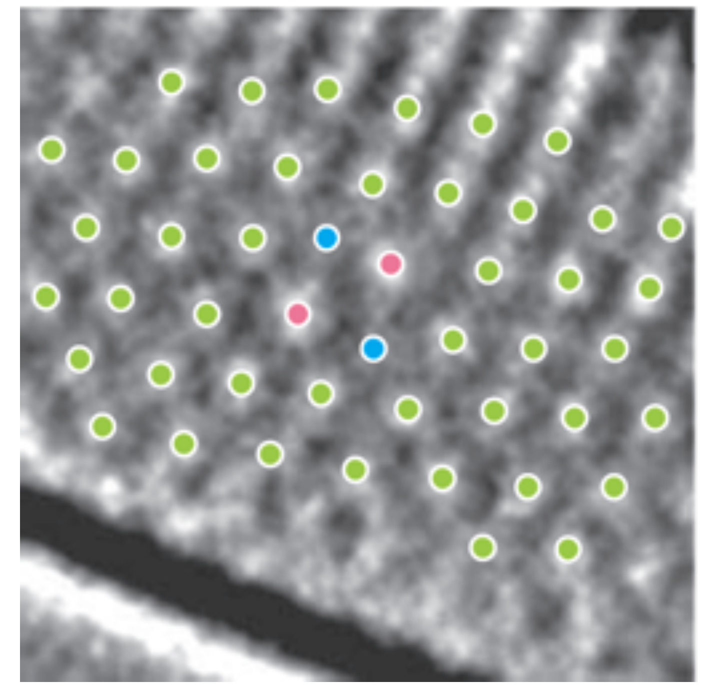


Torsional strain ~ 10%

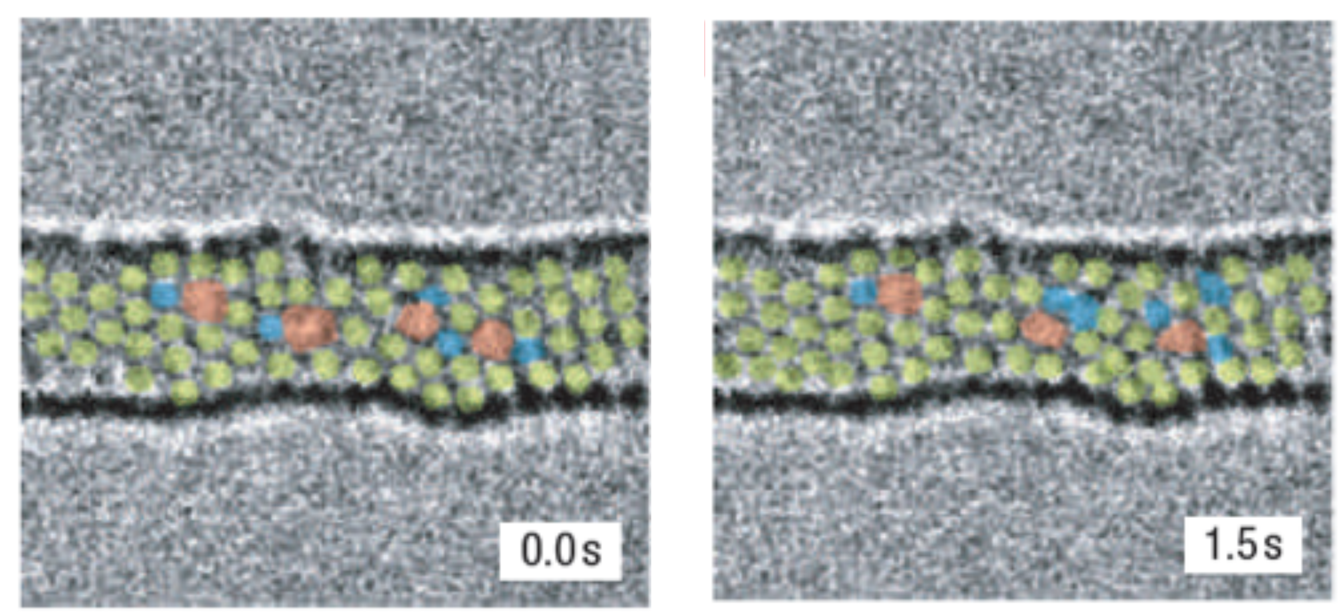
Zhang et al., J. Chem. Phys. 130:071101 (2009)

Experiment

Dislocation pairs found at T = 2273 K

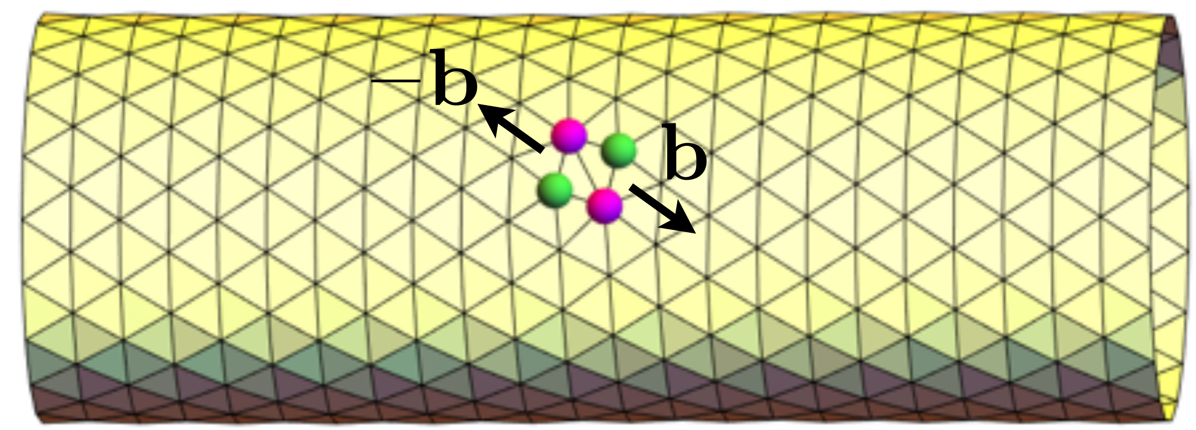


Dislocations migrate in presence of kink



Suenaga et al., Nature Nanotech. 2:358 (2007)

In this talk:
Dislocations in triangular crystals on tubes



Plastic deformation of tubular crystals

- <u>Background: Phyllotactic geometry of tubular crystals</u>
- Mechanics of plastic deformation: Analytic predictions
- Numerical modeling
- Necks in tubes: Radius profiles near dislocations

Phyllotaxis (“leaf-arrangement”) in Botany

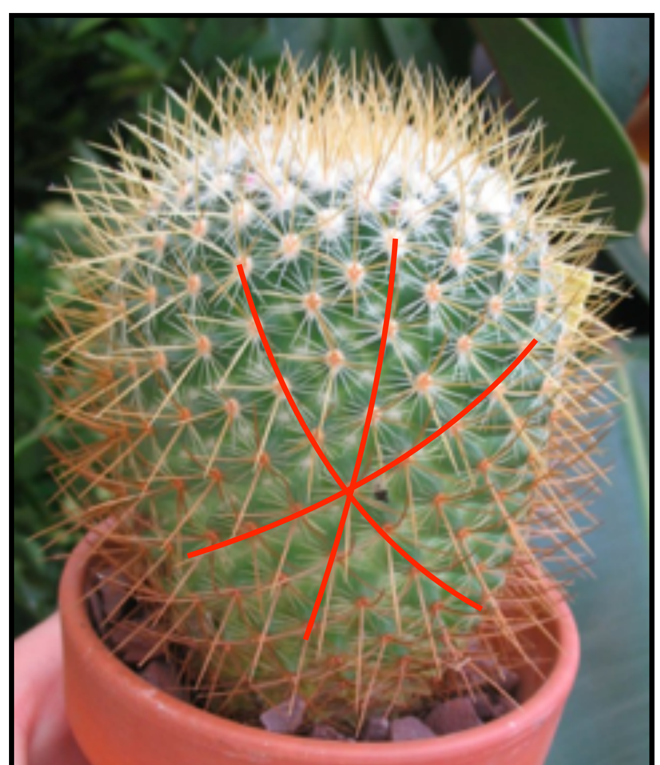
(Not the subject of this talk, but fascinating!)

sunflower
Pennybacker et al.,
Physica D 306:48 (2015)
^^^ *a great review article!*

aloe
Wikipedia

Romanesco broccoli
www.fourmilab.ch

Pineapple (D.A.B./Whole Foods)



Pincushion cactus
www.cactuslovers.com

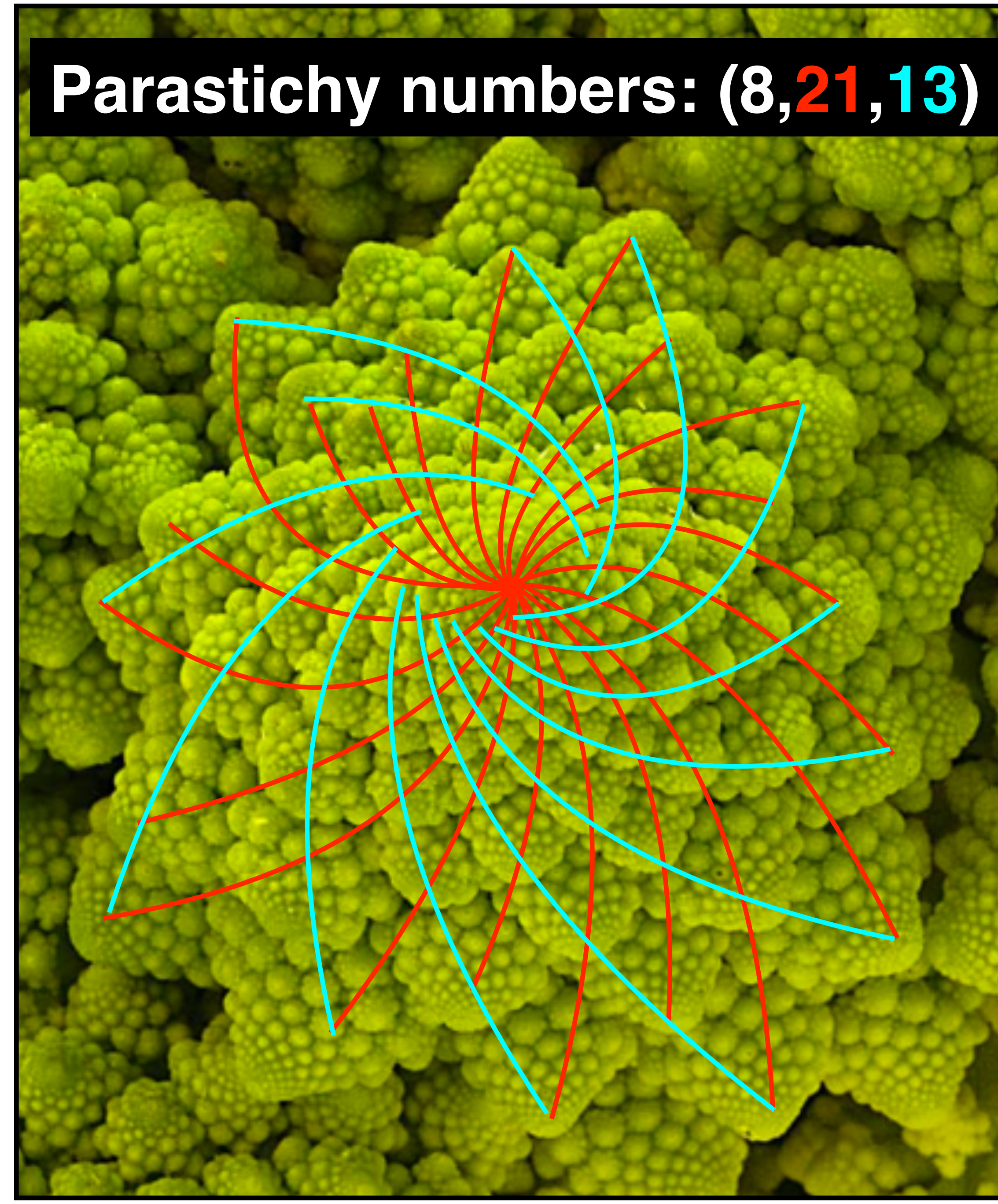
Pine cone
Warren Photographic

Parastichies
Lattice lines →
Spirals or helices

Phyllotaxis ("leaf-arrangement") in Botany

(Not the subject of this talk, but fascinating!)

Phyllotactic packing is described by **parastichy numbers** = number of distinct parastichies in a *parastichy family*



Romanesco broccoli

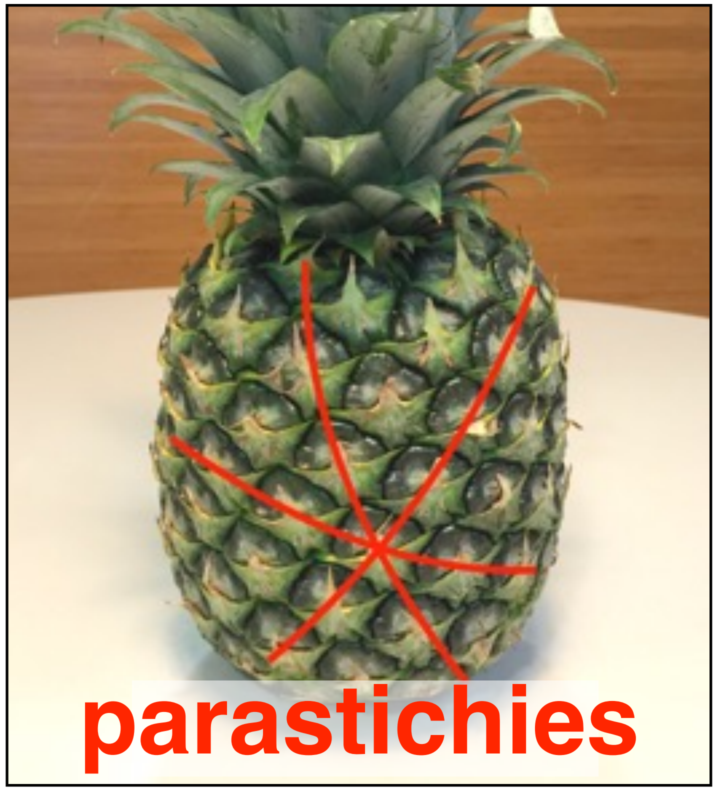
www.fourmilab.ch

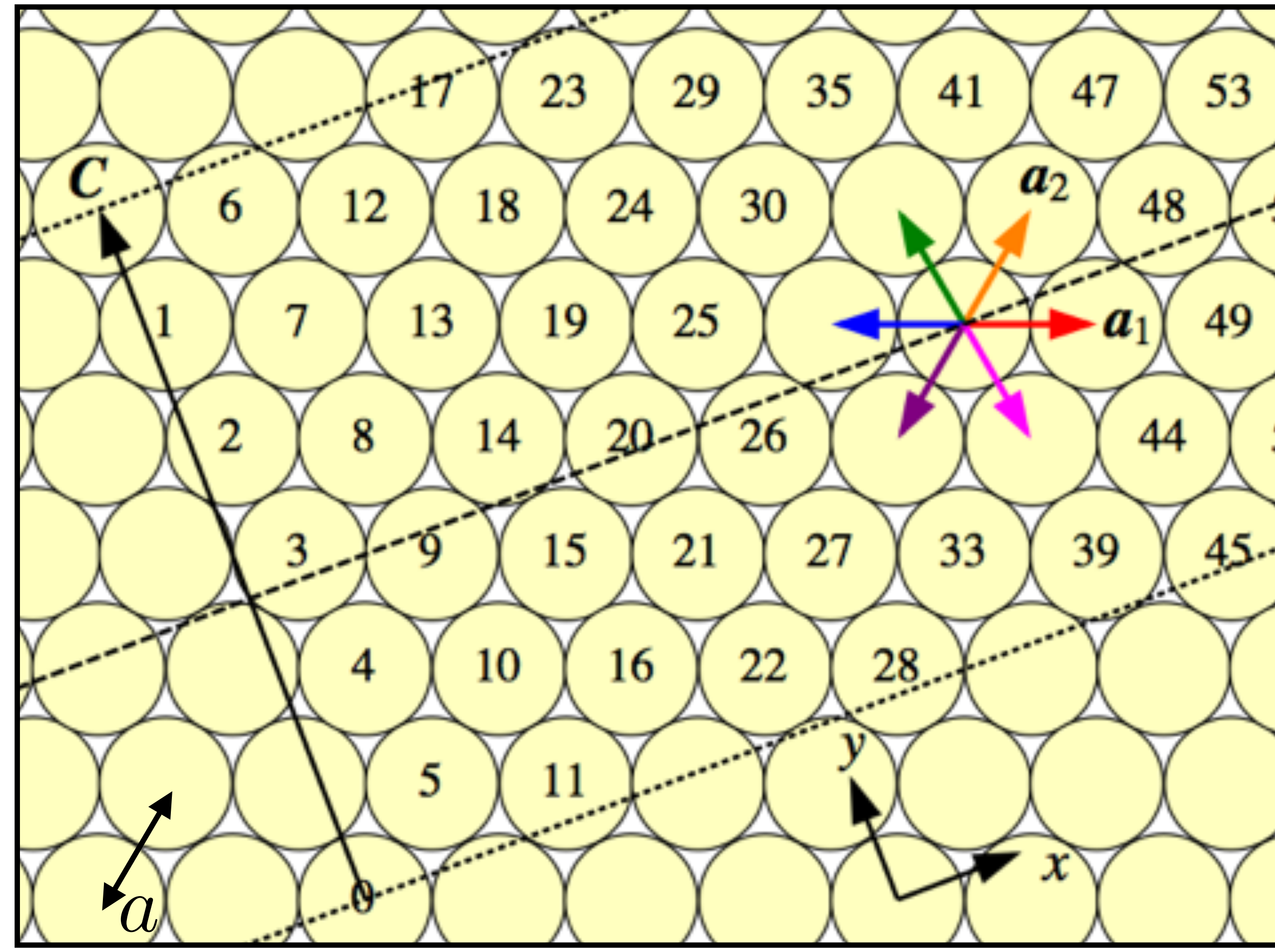
Phyllotaxis as the geometry of tubular crystals

Erickson, Science 181:705 (1973)

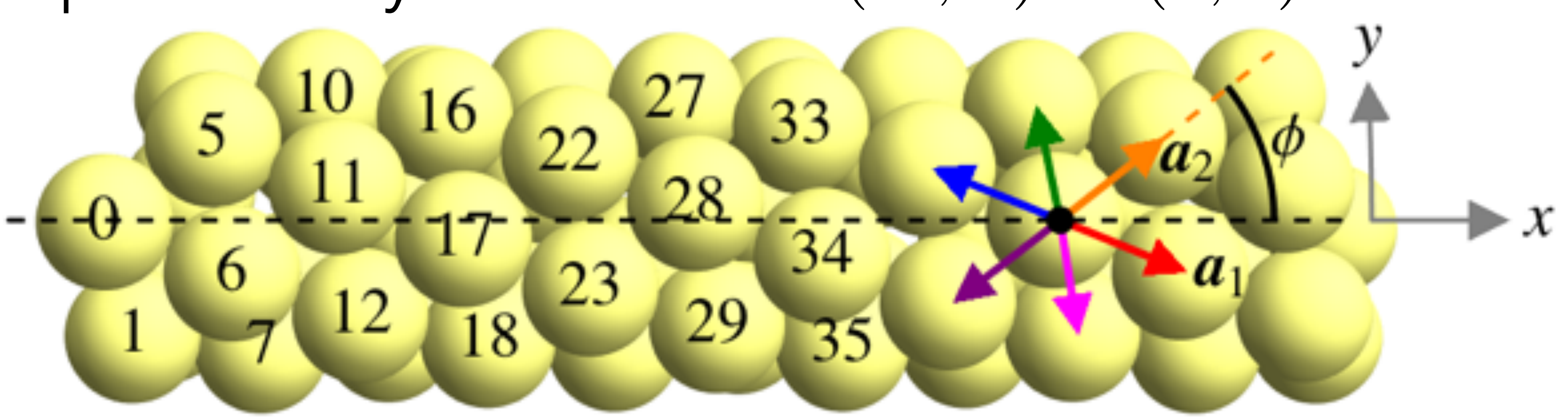
Circumference

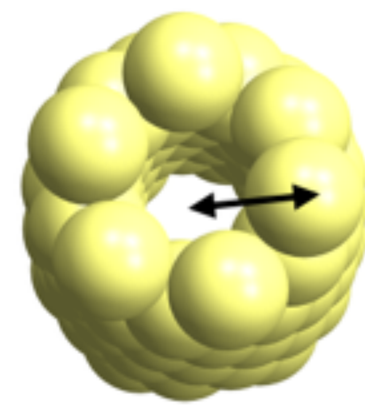
$$\mathbf{C} = m\mathbf{a}_2 - n\mathbf{a}_1$$





parastichy numbers $(m, n) = (6, 5)$





$R = |\mathbf{C}|/2\pi$

$$\tan\phi \approx \frac{2}{\sqrt{3}}\left(\frac{m}{n} - \frac{1}{2}\right) \qquad R \approx \frac{1}{2\pi}|\mathbf{C}| = \frac{a}{2\pi}\sqrt{m^2 + n^2 - mn}$$

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the geometry of tubular crystals

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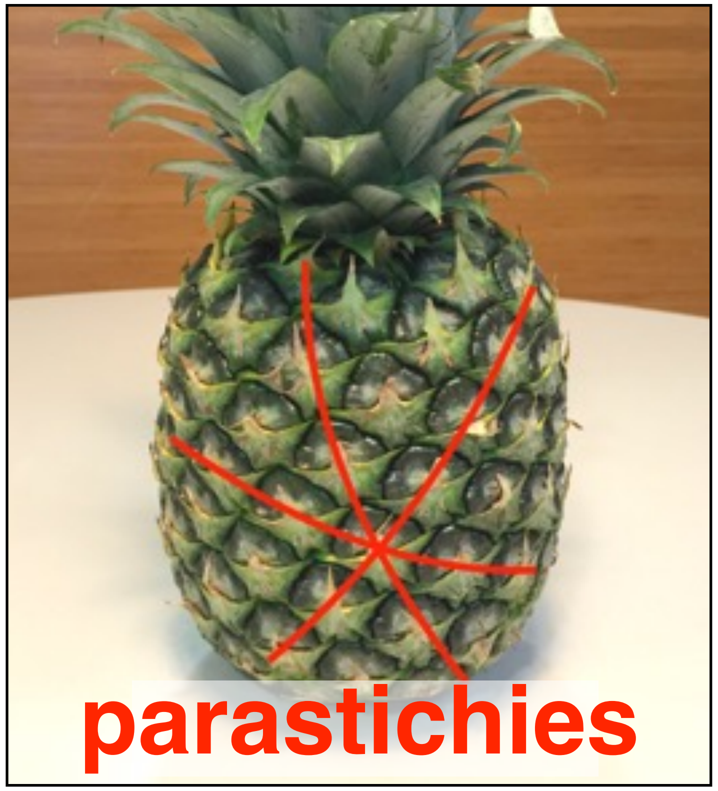

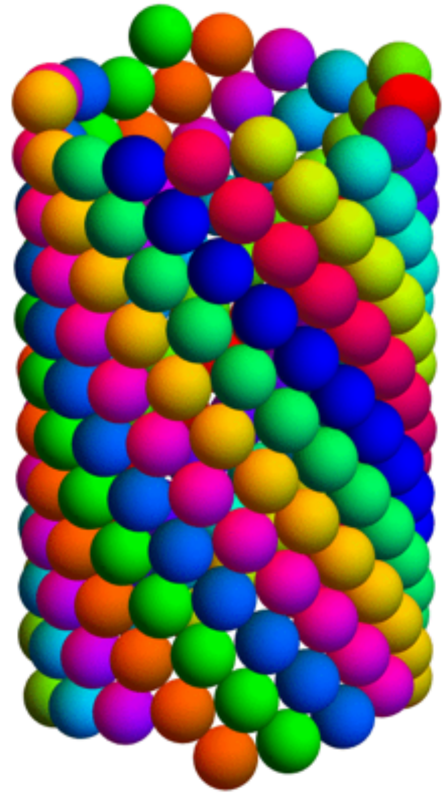

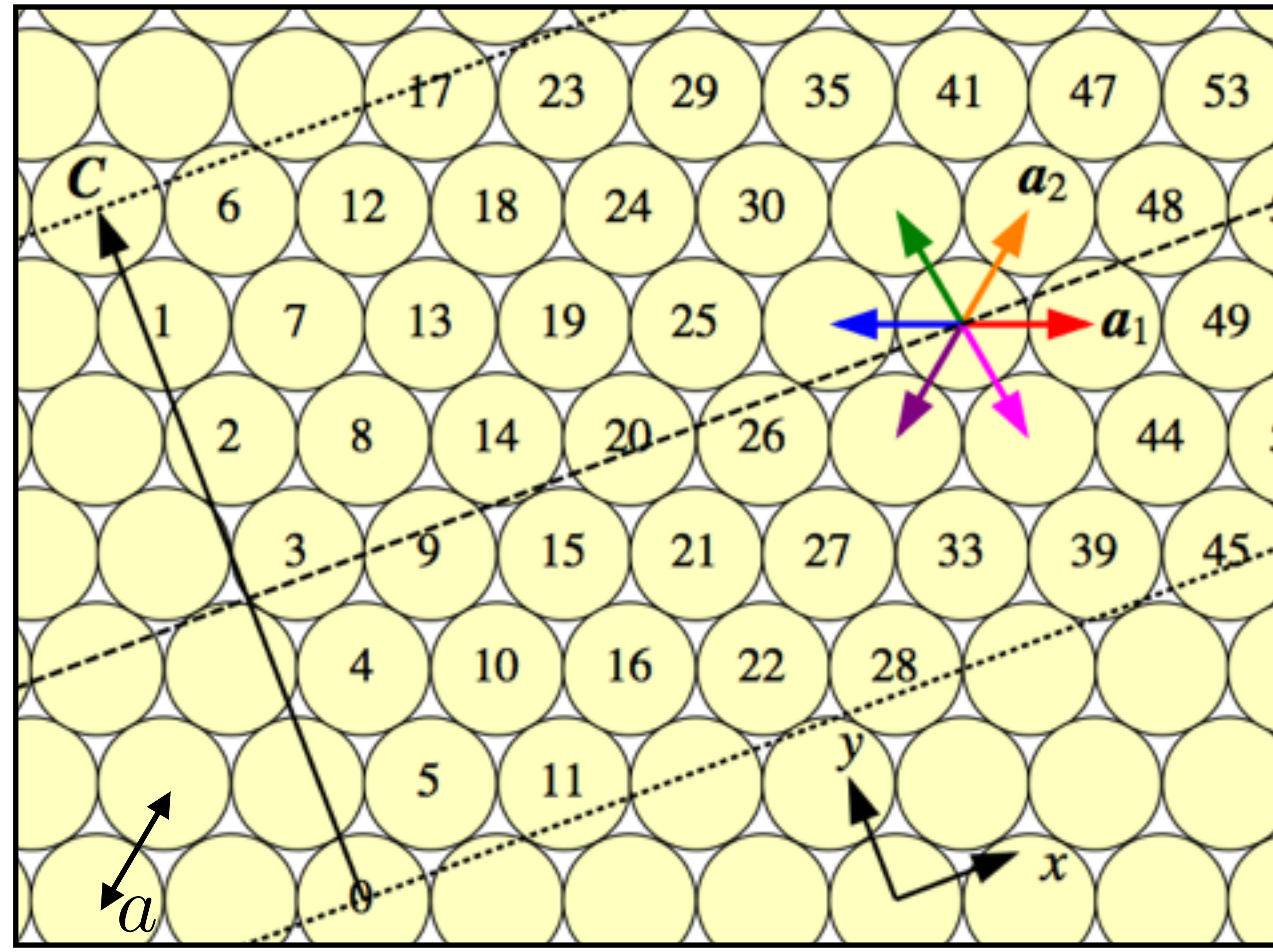

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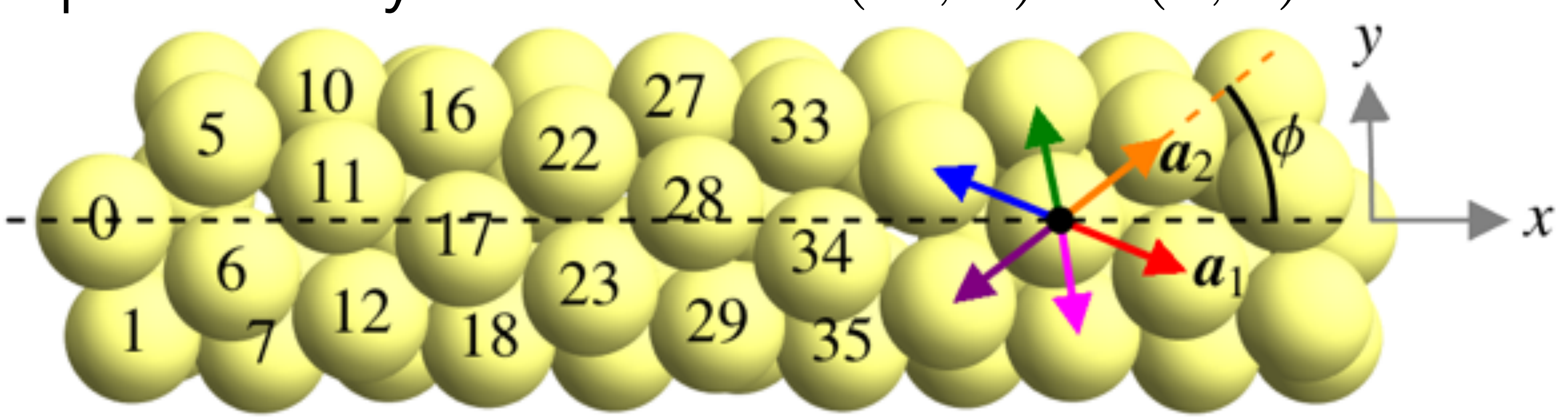

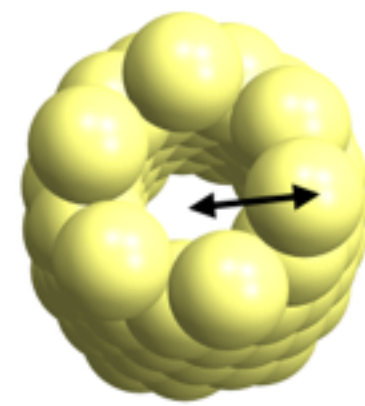

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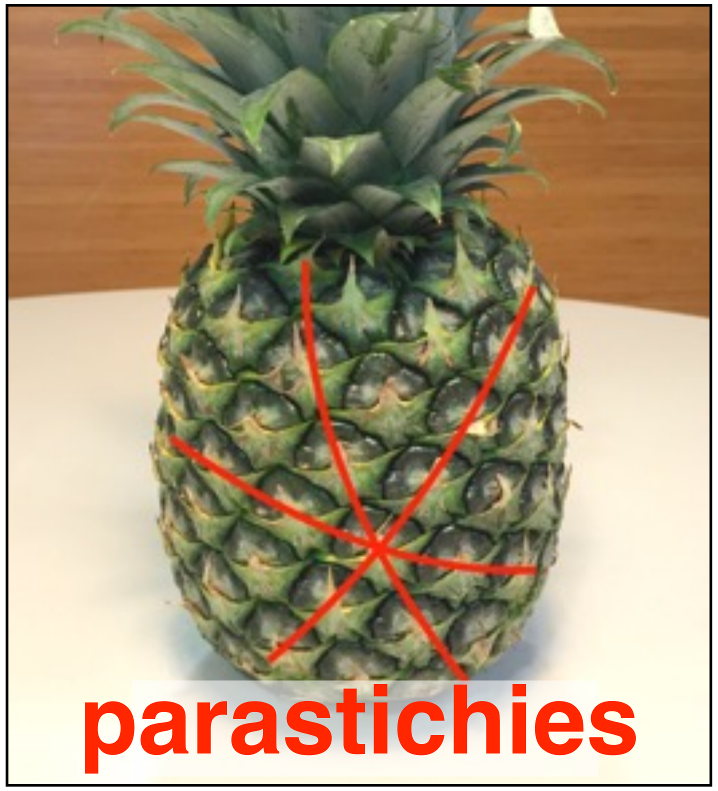

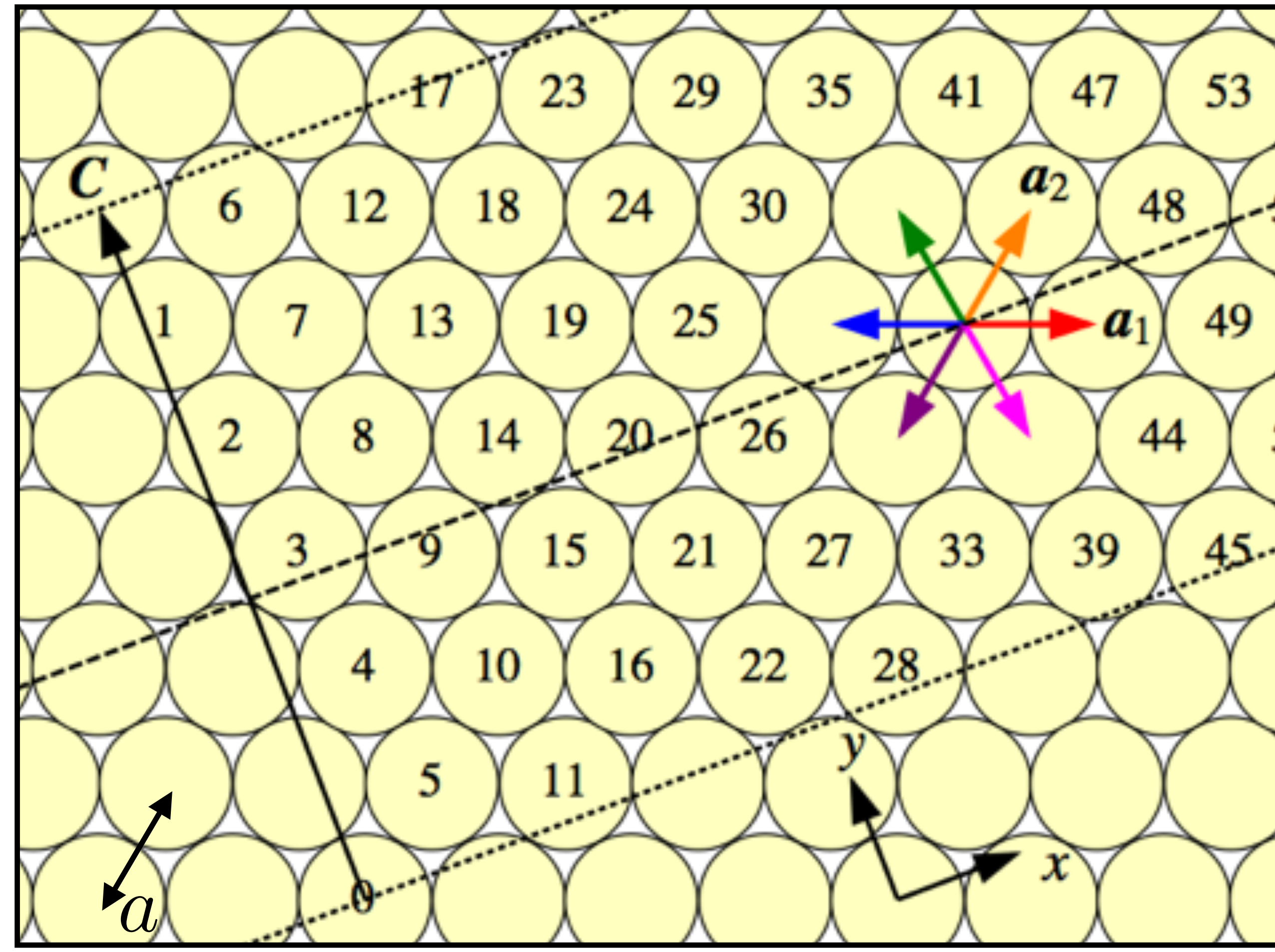

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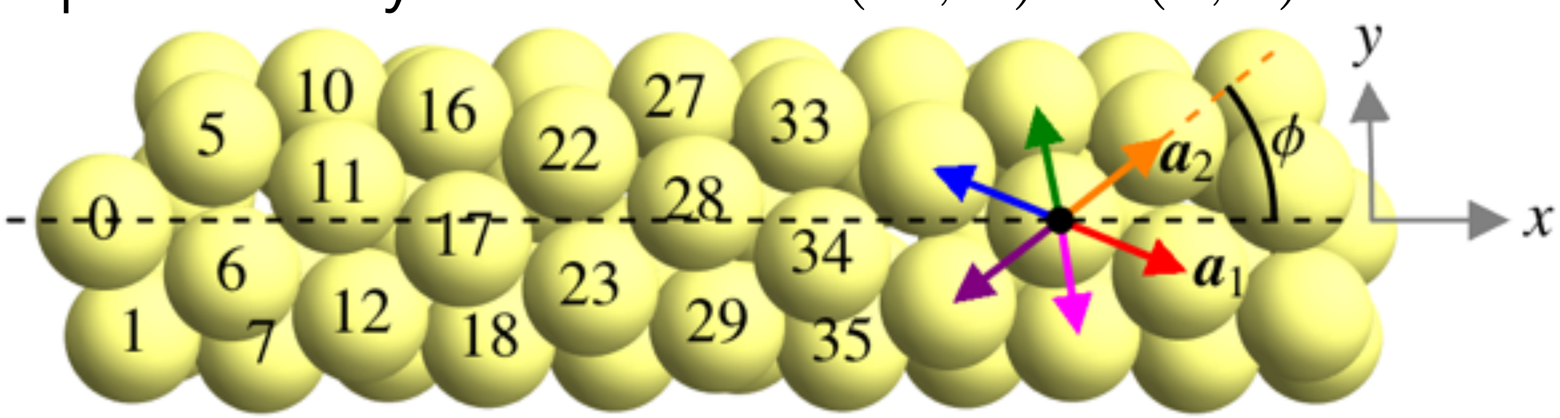

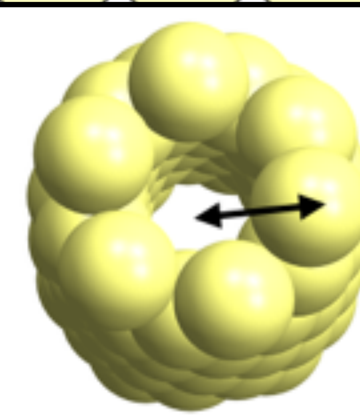

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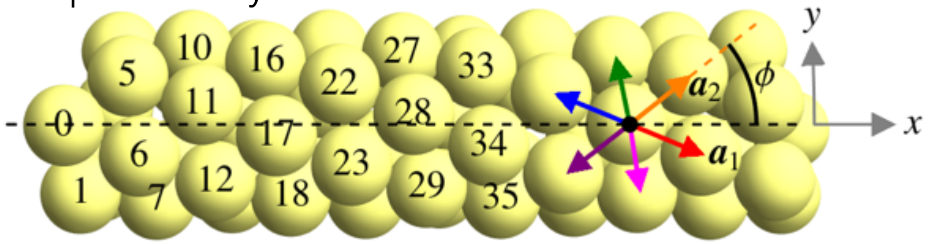
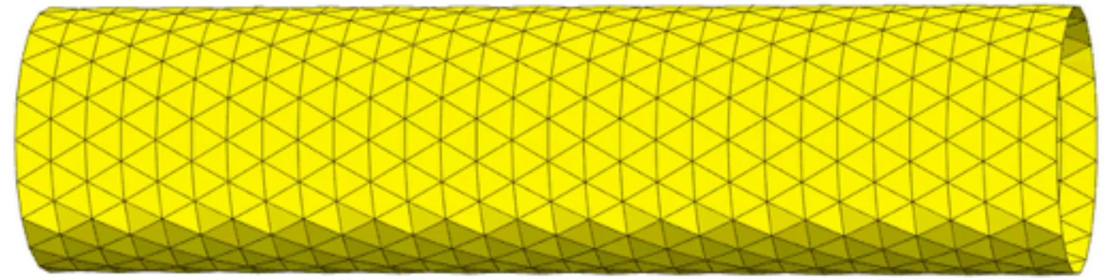


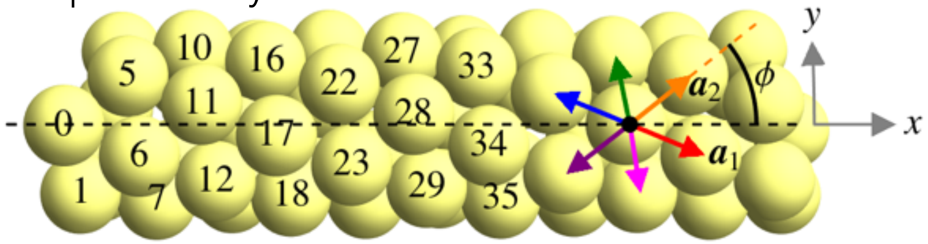


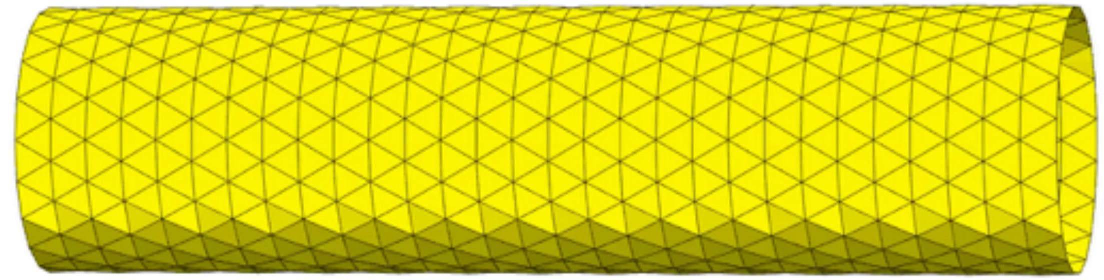
$(m,n)=(20,20)\to(20,19)$

<u>This talk is about</u>
Parastichy transitions, i.e., changes in (m,n),
as plastic deformations
accomplished by unbinding and separation of
pairs of **dislocation defects**

Key questions

- How much stress is required to plastically deform a tubular crystal via dislocation motion?
- How do the softest plastic modes change the tube geometry?
- How well does continuum elasticity theory predict deformations in very small tubes?
- How does a crystals' bending modulus change the plastic deformation mechanics as compared with the plane?





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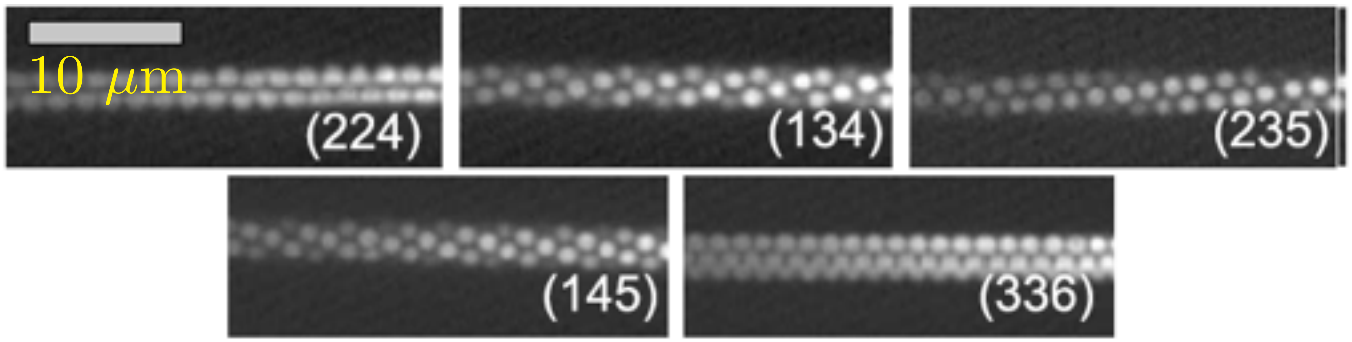
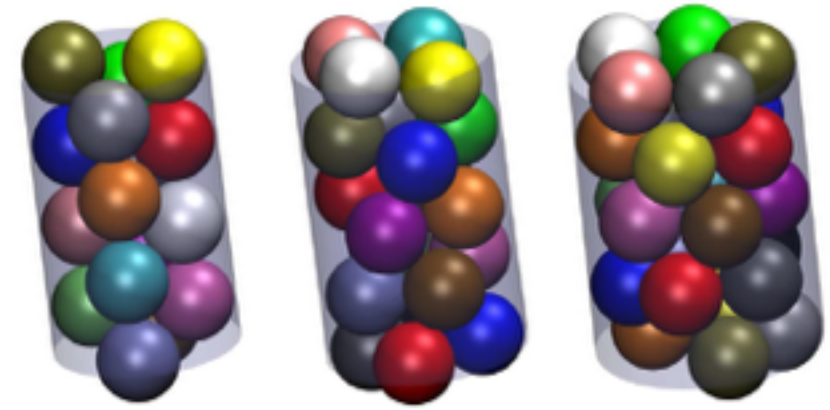
Phyllotactic packings on cylinders of *fixed radius*: Recent work

Colloidal spheres in capillaries



Lohr et al., PRE 81:040401 (2010)



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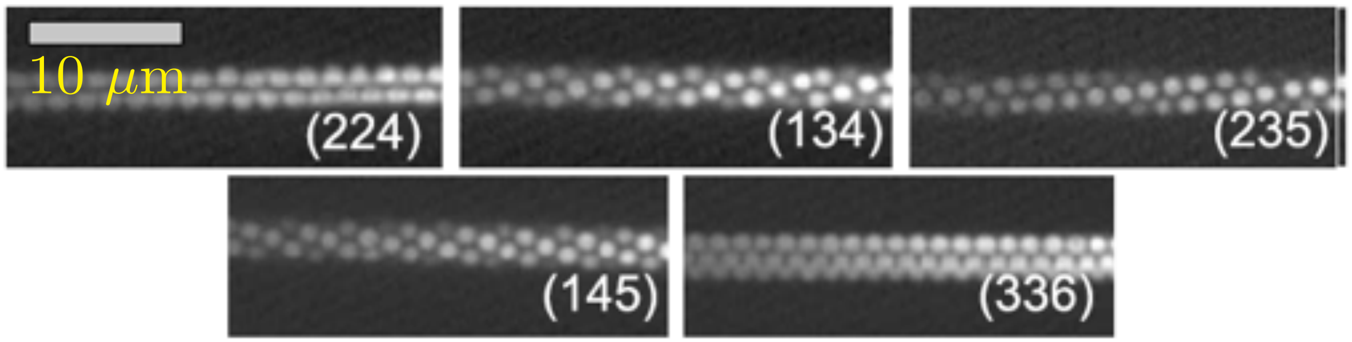

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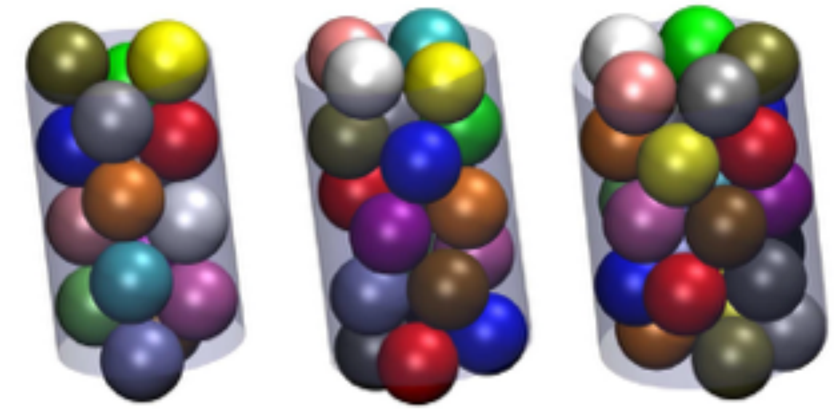

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What happens if the cylinder radius is incommensurate with any perfect phyllotactic packing?

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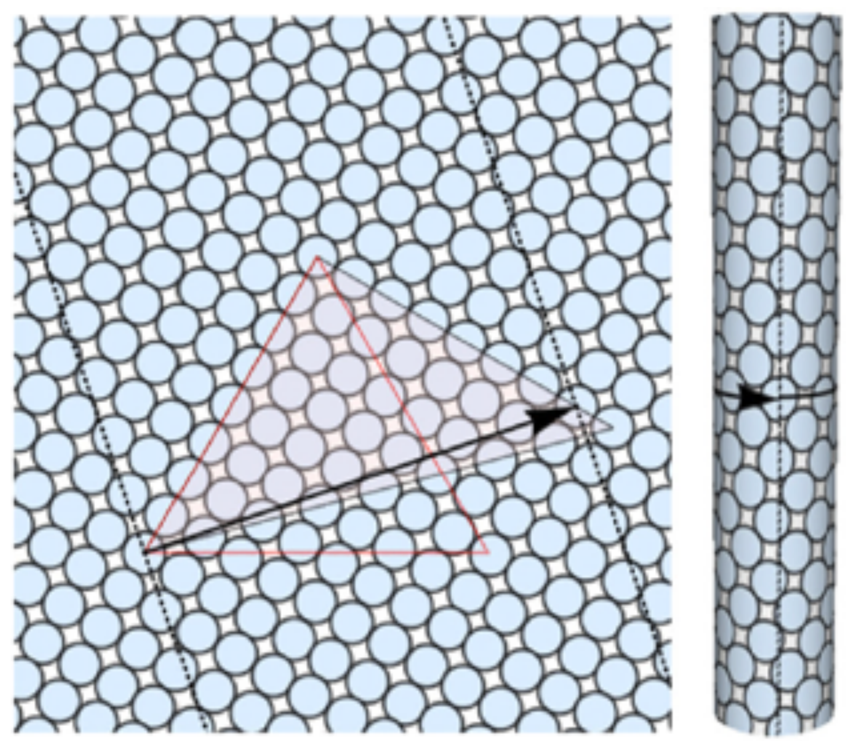
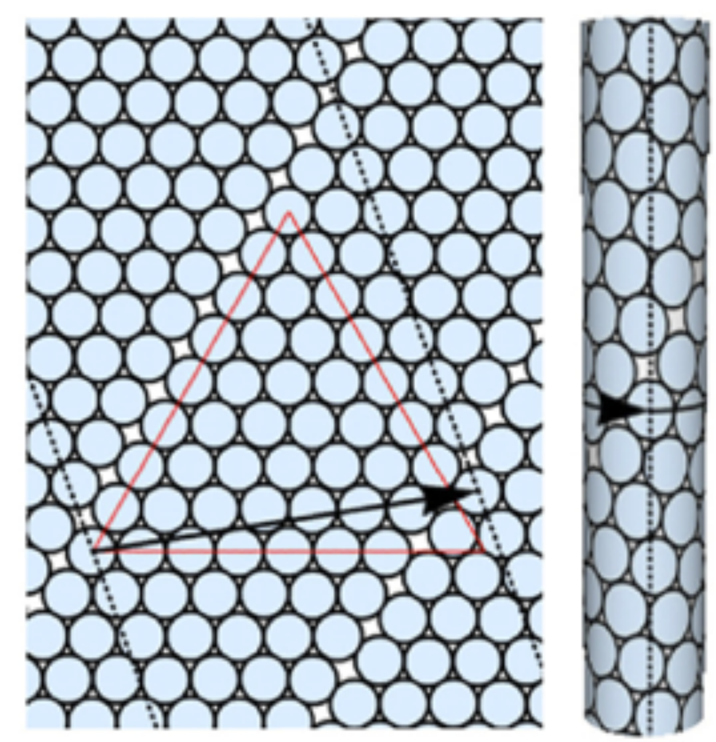


Rhombic (or “oblique”) lattice



Helical “line-slip” defects in a triangular lattice

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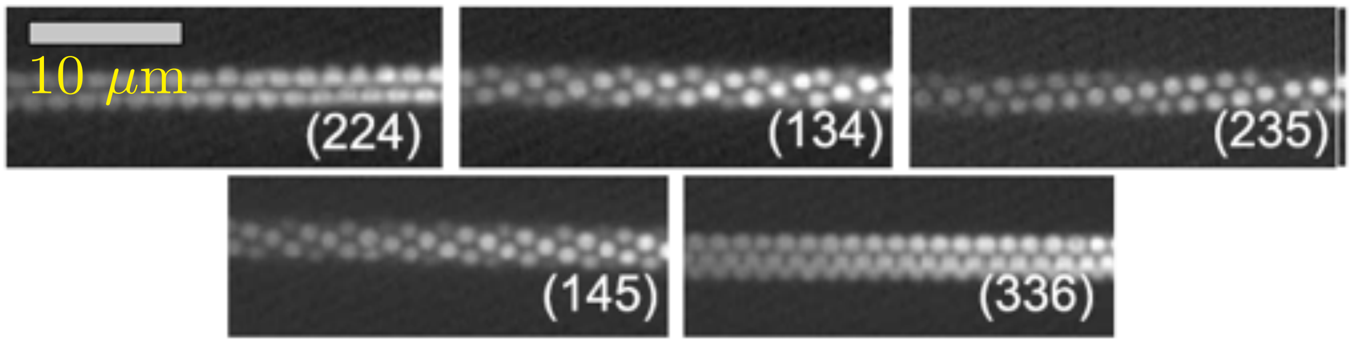

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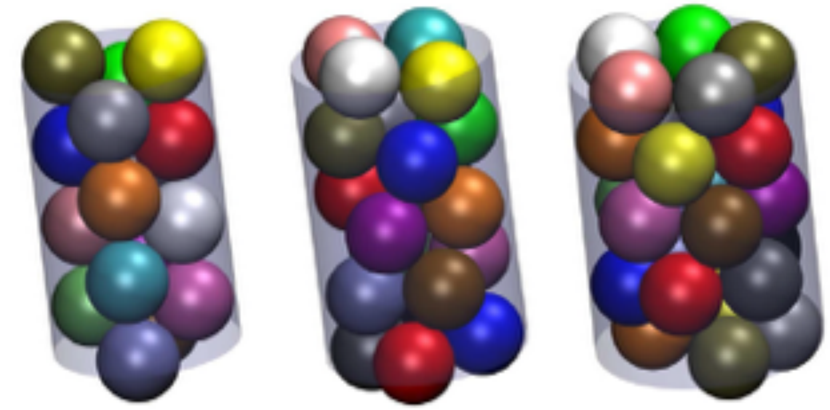

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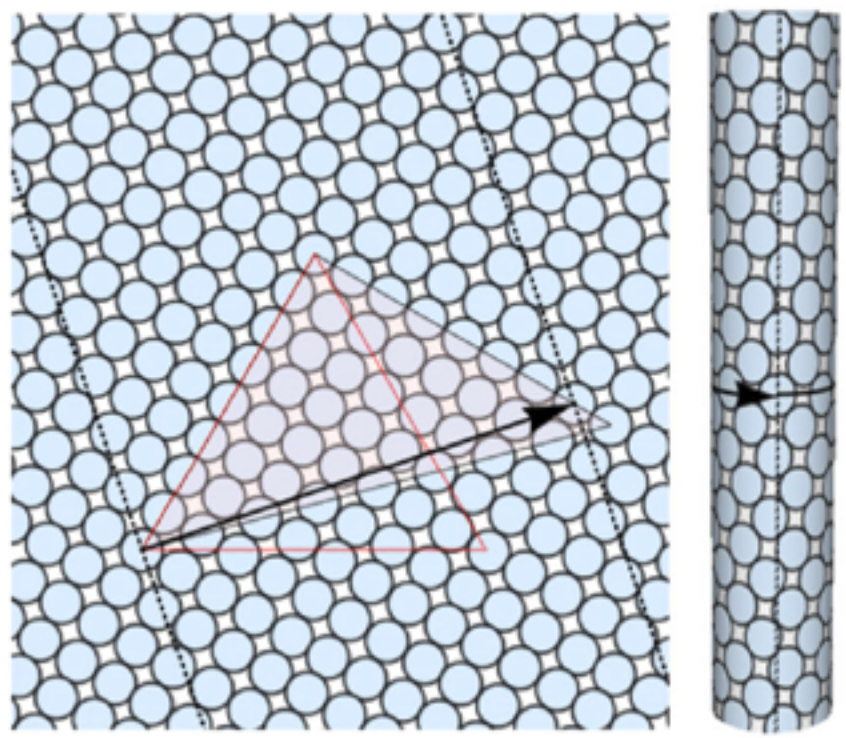

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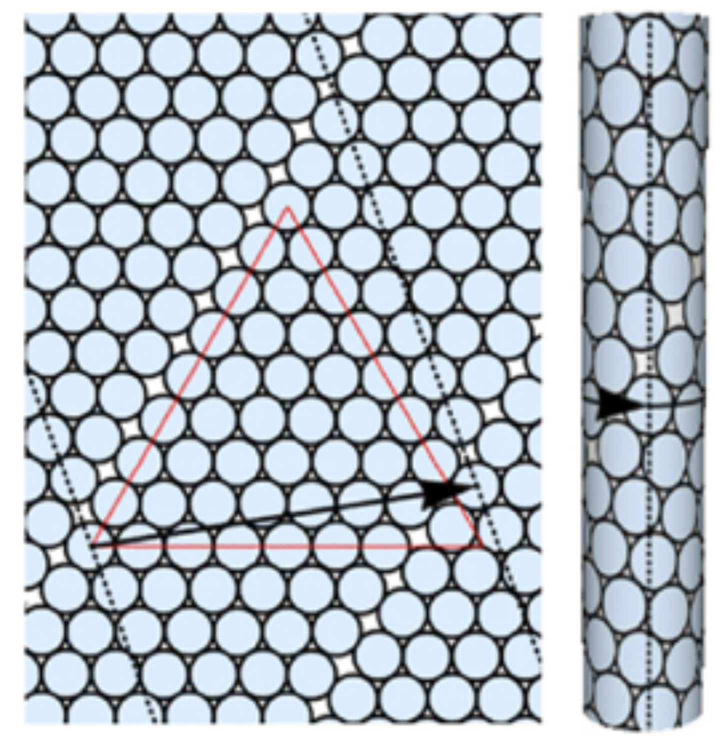

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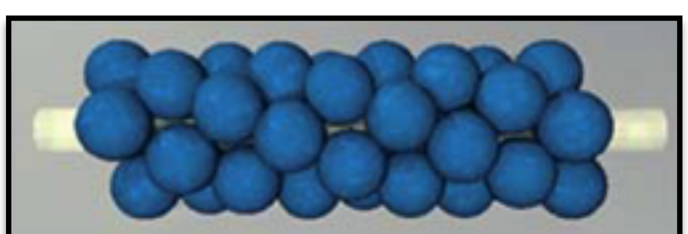

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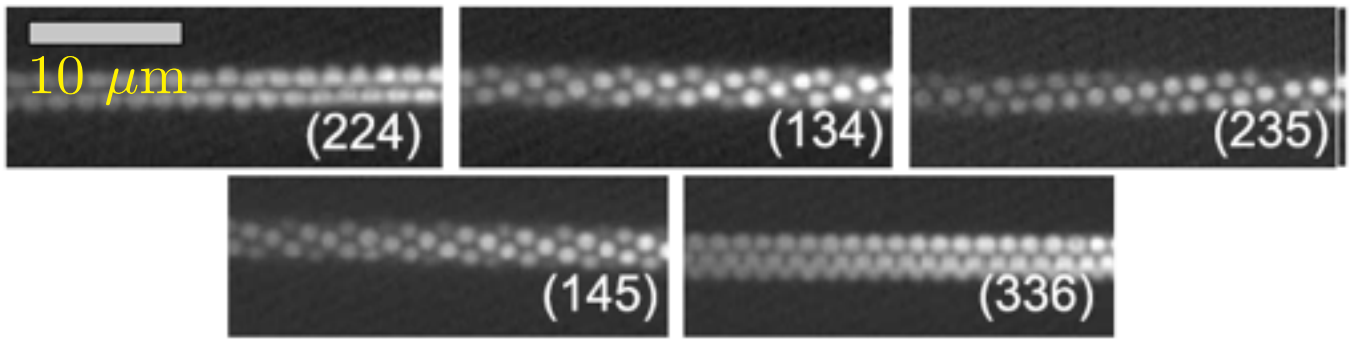

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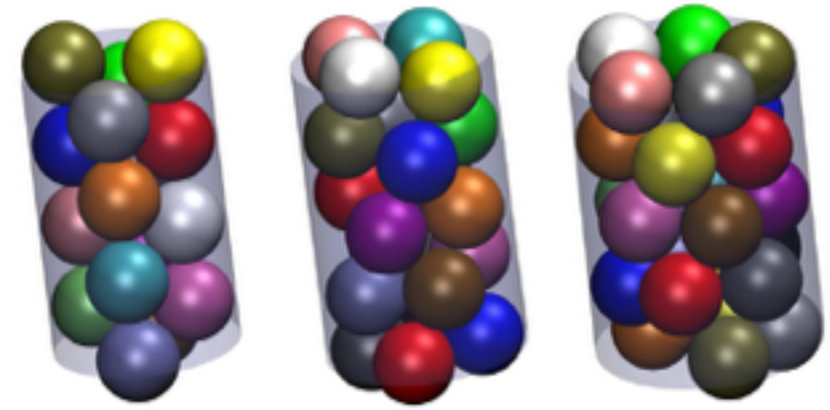

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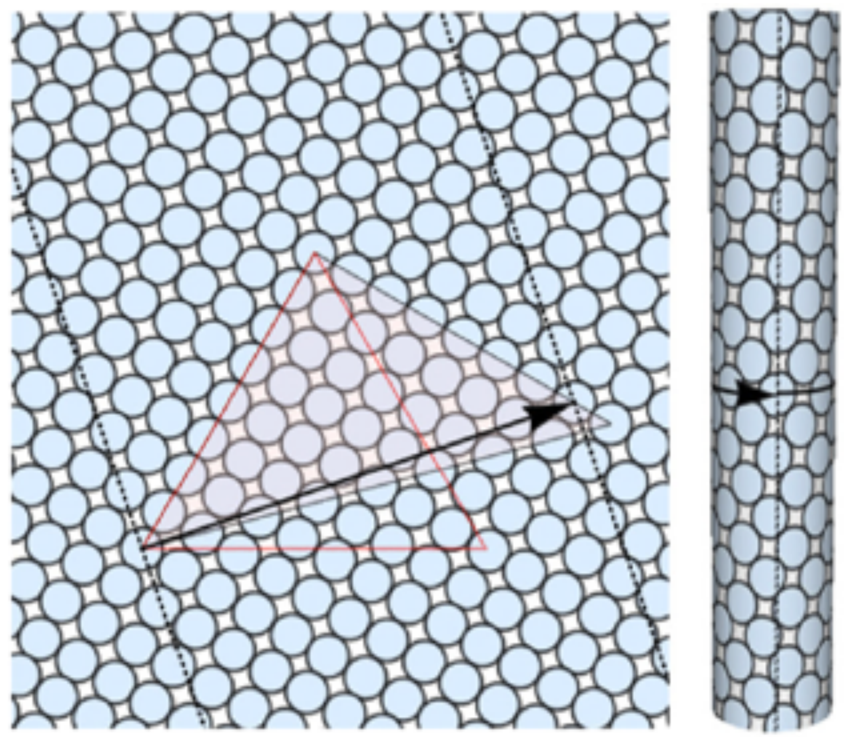

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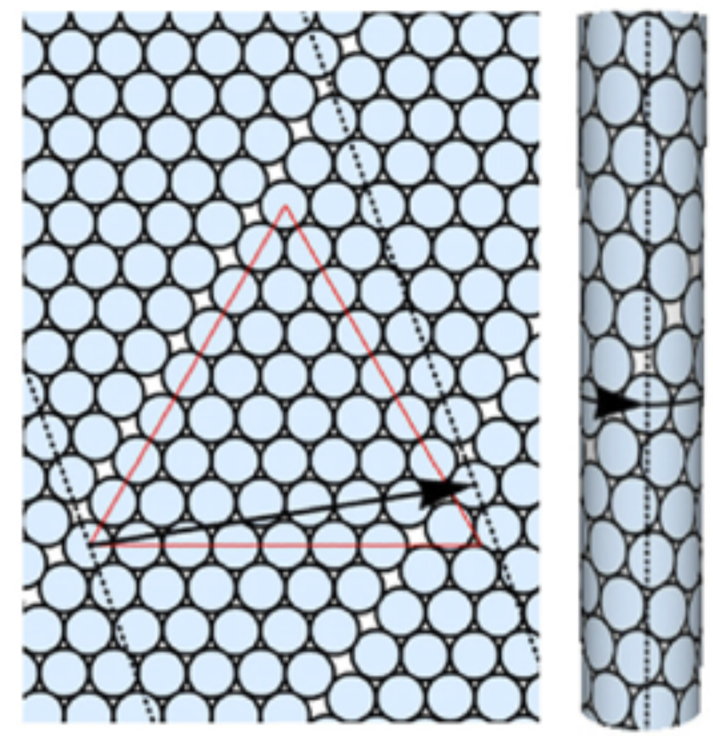

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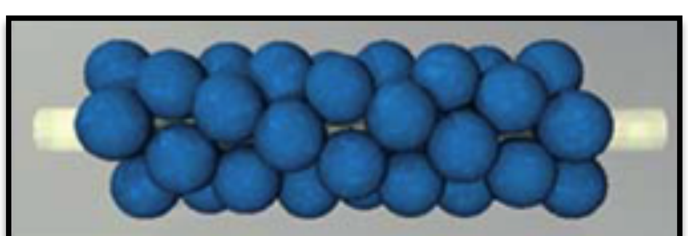

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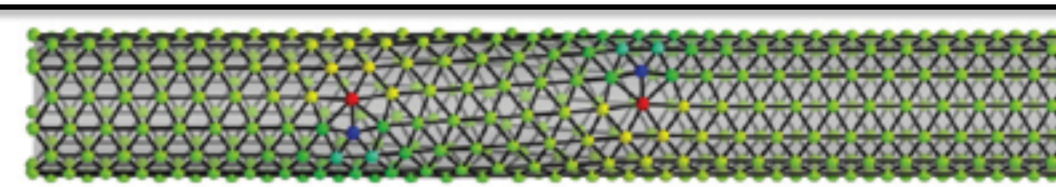
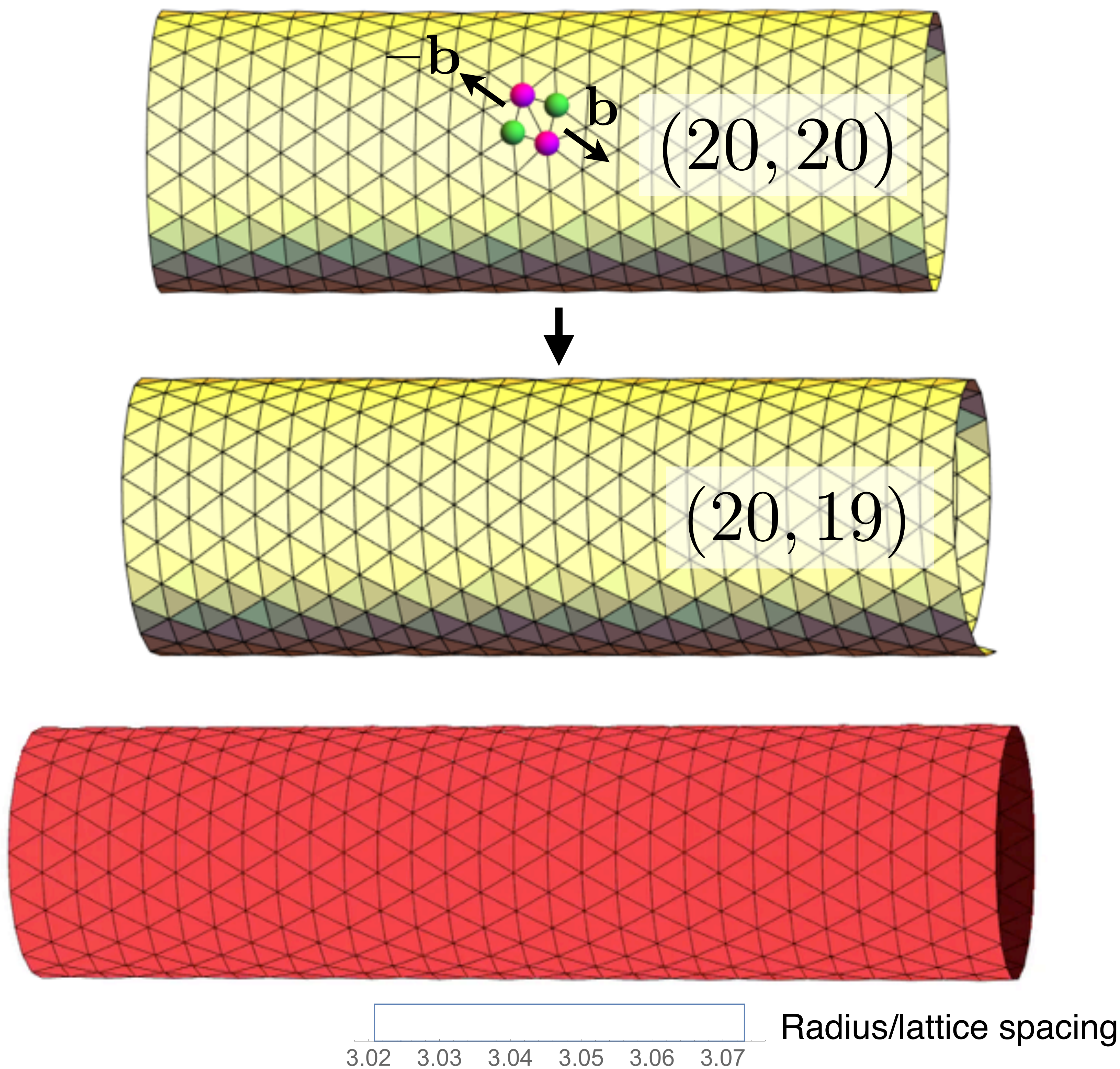
Dislocation interaction energetics on a cylinder



Amir, Paulose, Nelson, PRE 87:042314 (2013)

This work:

Dislocation-mediated plastic deformation of tubular crystals where the tube radius is *not fixed*: R varies in space and time



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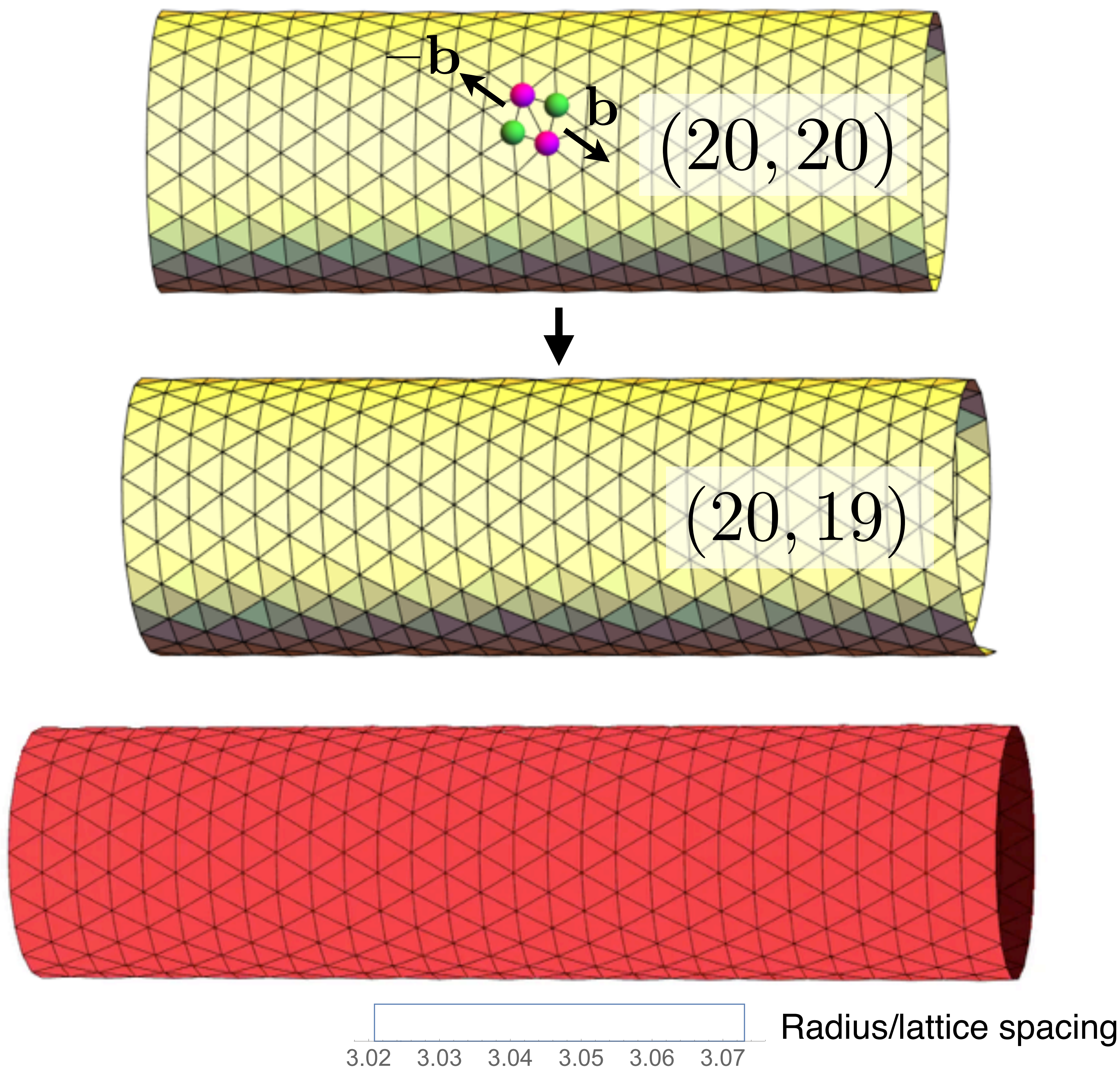
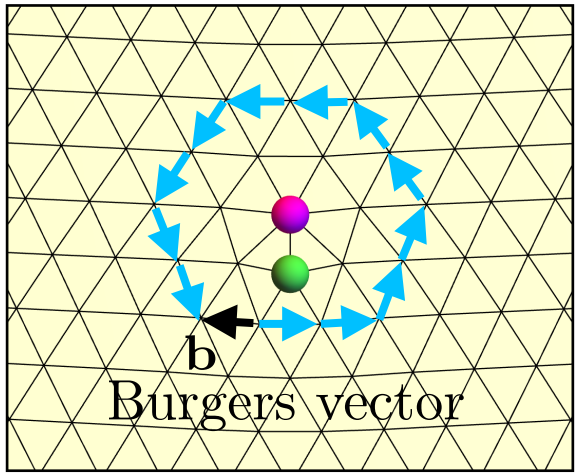
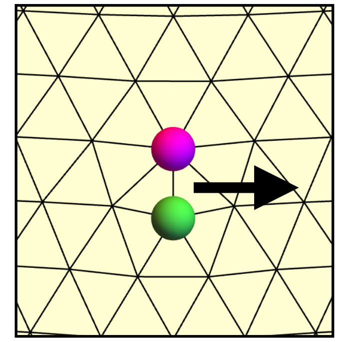


Dislocation motion: Glide and climb

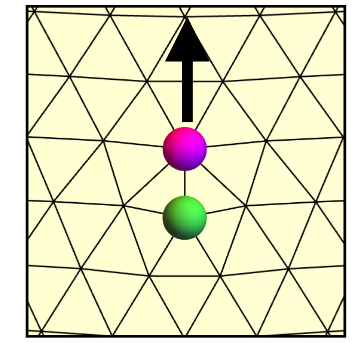


Glide: $\mathrm{d}\mathbf{x}/\mathrm{d}t \parallel \mathbf{b}$

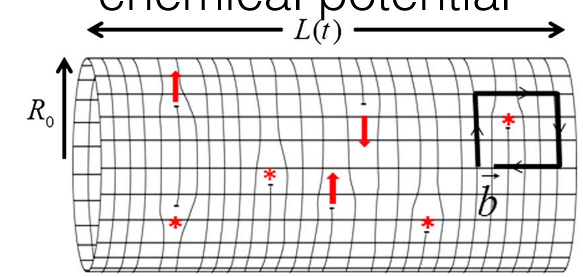


usually easier for mass-conserving systems

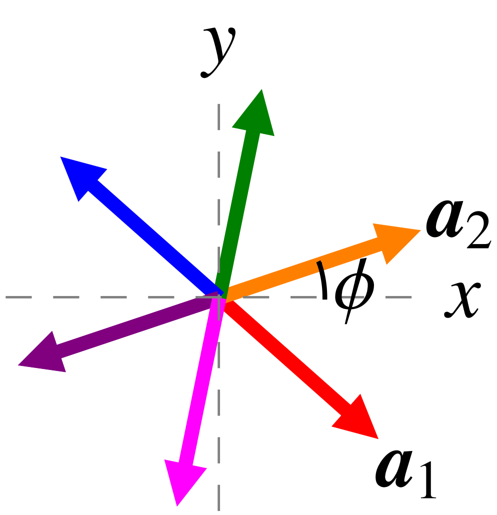
Climb: $\mathrm{d}\mathbf{x}/\mathrm{d}t \perp \mathbf{b}$



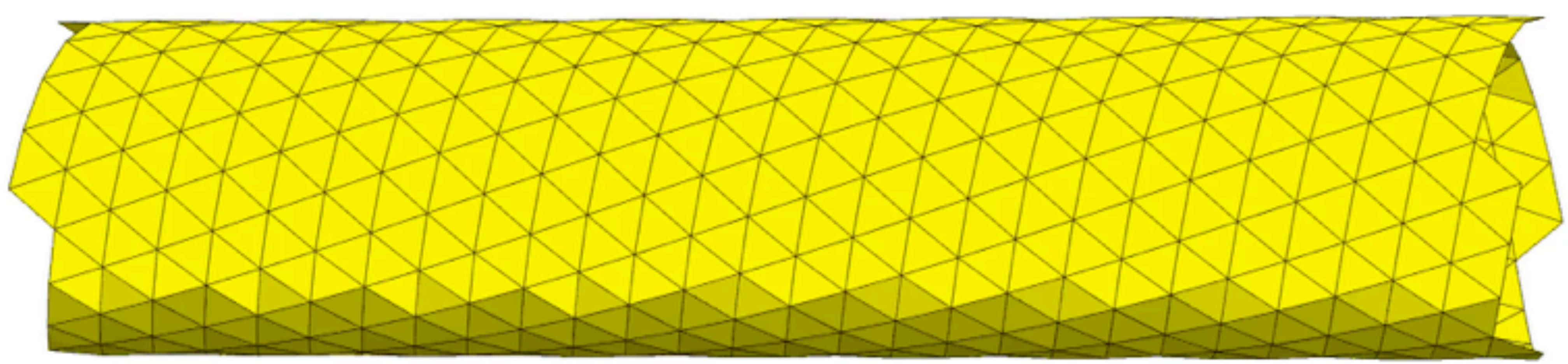
relevant to systems with growth/ chemical potential



e.g., Bacterial cell wall growth
Amir & Nelson, PNAS 109:9833 (2012)

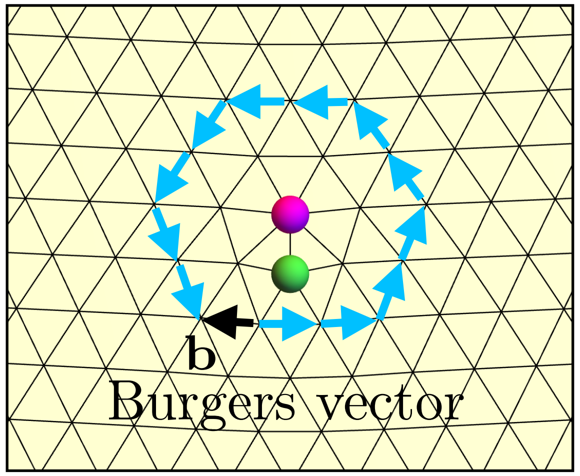


$\mathbf{b}$ of the right-moving dislocation

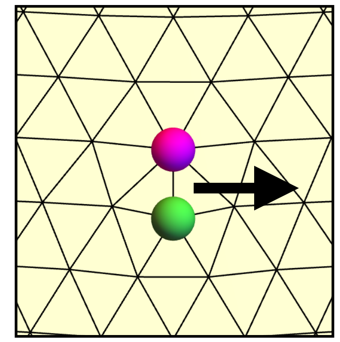


The six elementary Burgers vector pairs on a triangular lattice

Dislocation motion: Glide and climb



Glide: $\mathrm{d}\mathbf{x}/\mathrm{d}t \parallel \mathbf{b}$



usually easier for mass-conserving systems

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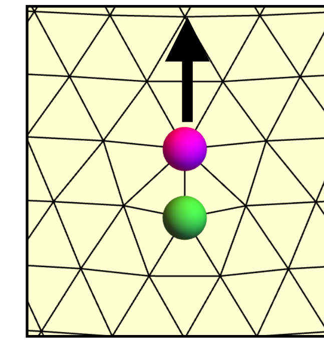
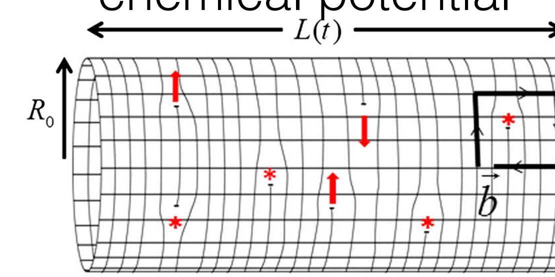


relevant to systems with growth/ chemical potential



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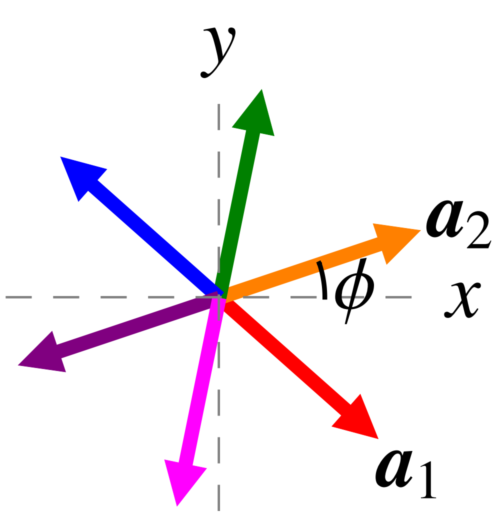

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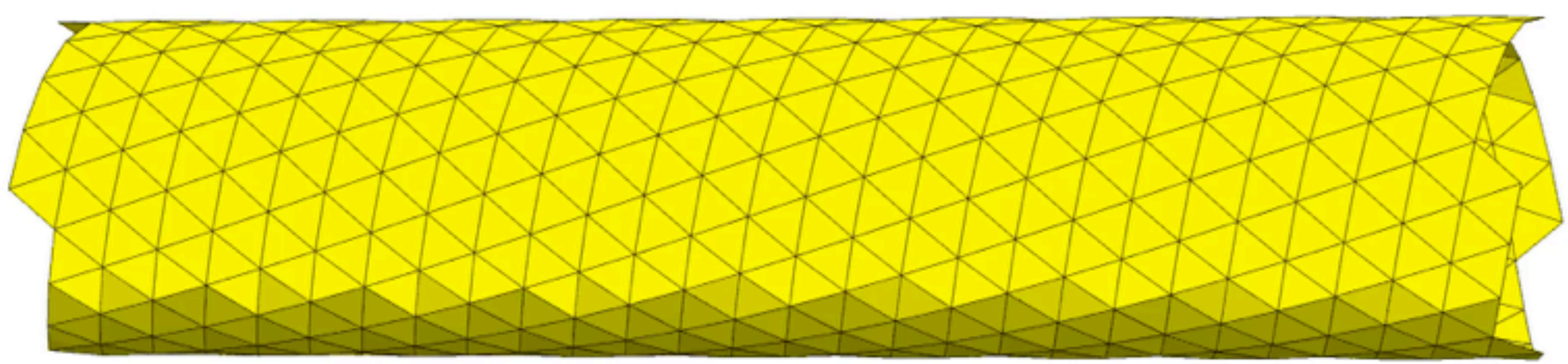
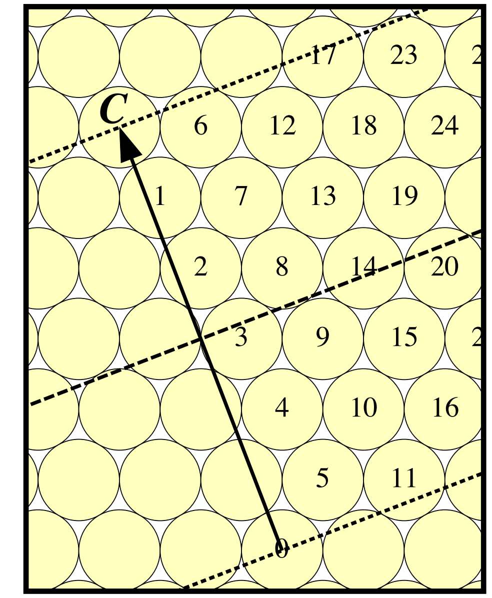


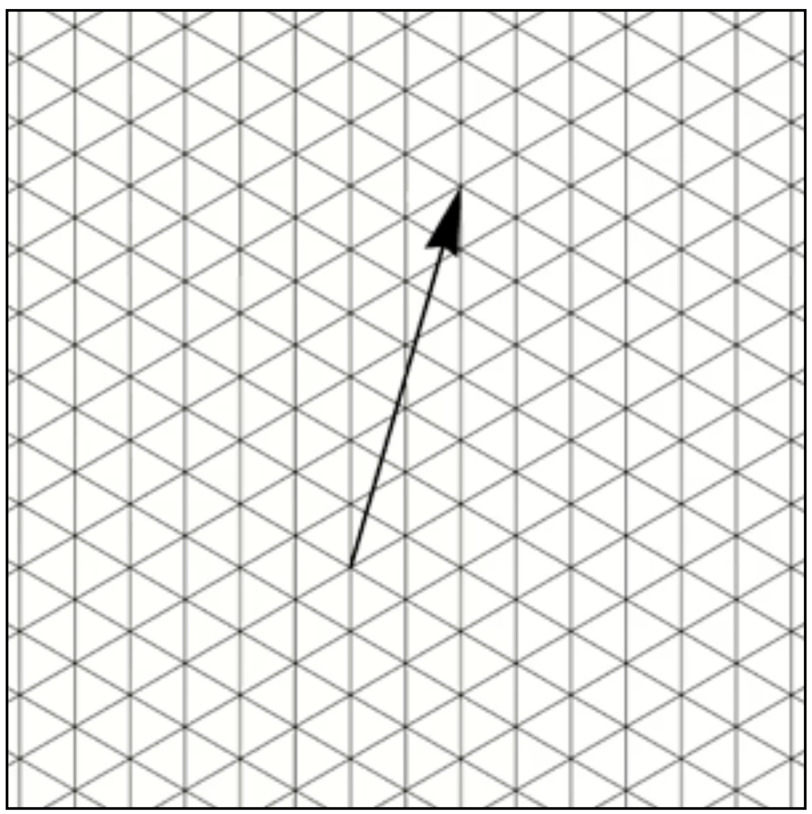
The six elementary Burgers vector pairs on a triangular lattice

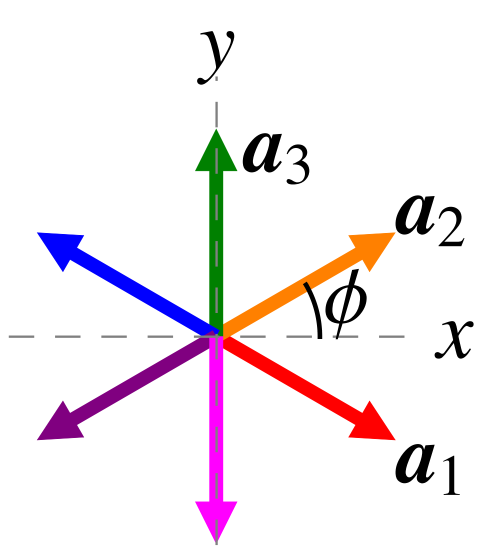
A dislocation passing through the system changes (m, n)



Altered circumference vector: $\mathbf{C}' = \mathbf{C} + \mathbf{b}$

$\mathbf{C} = m\mathbf{a}_2 - n\mathbf{a}_1$





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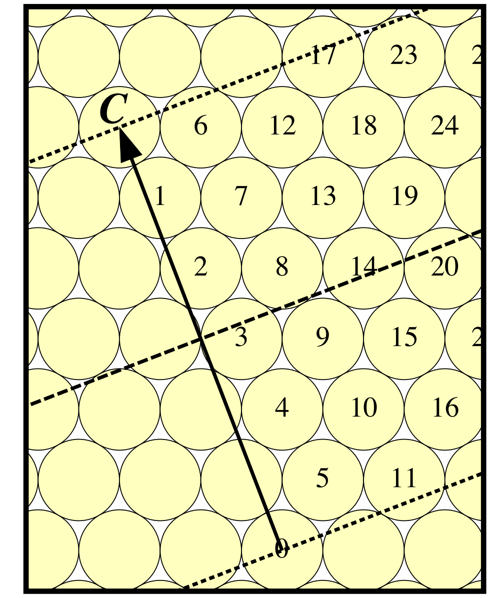

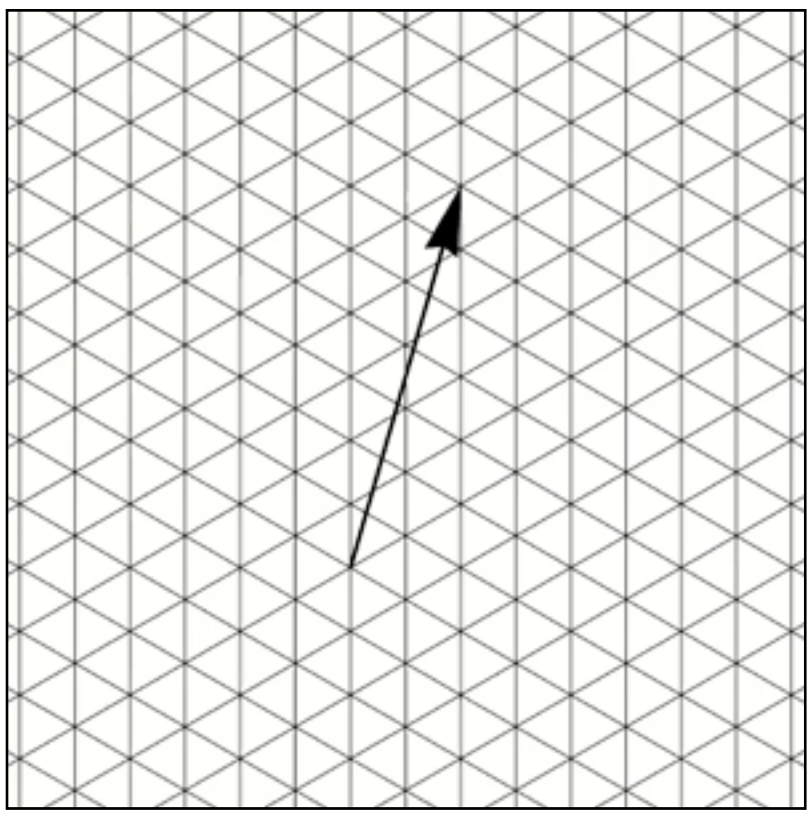

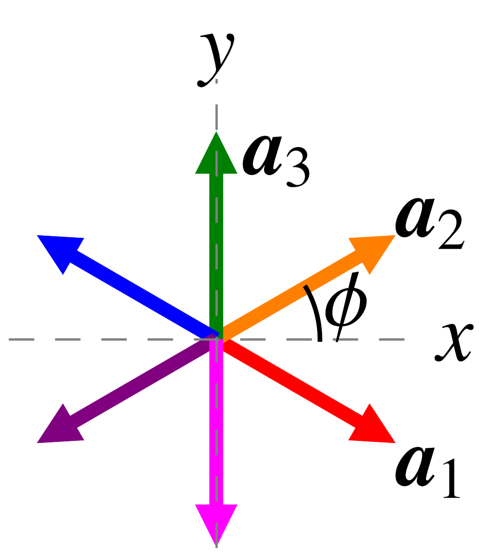

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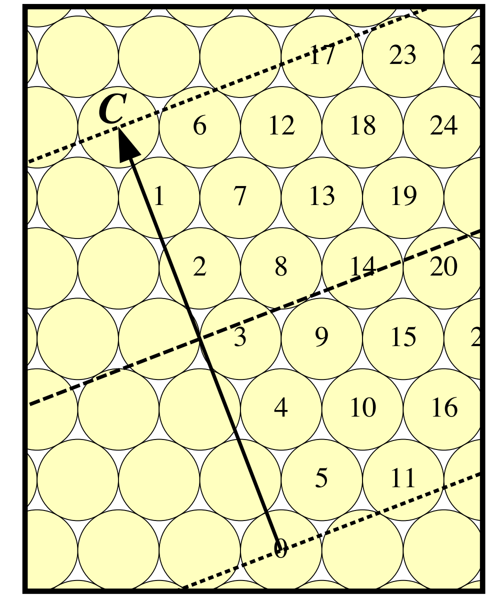

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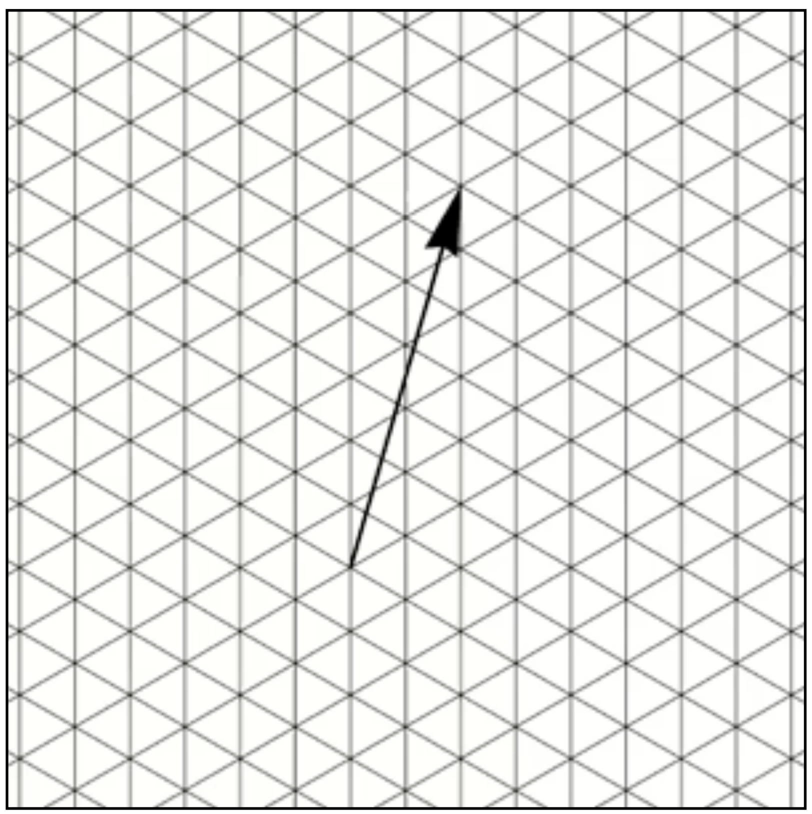
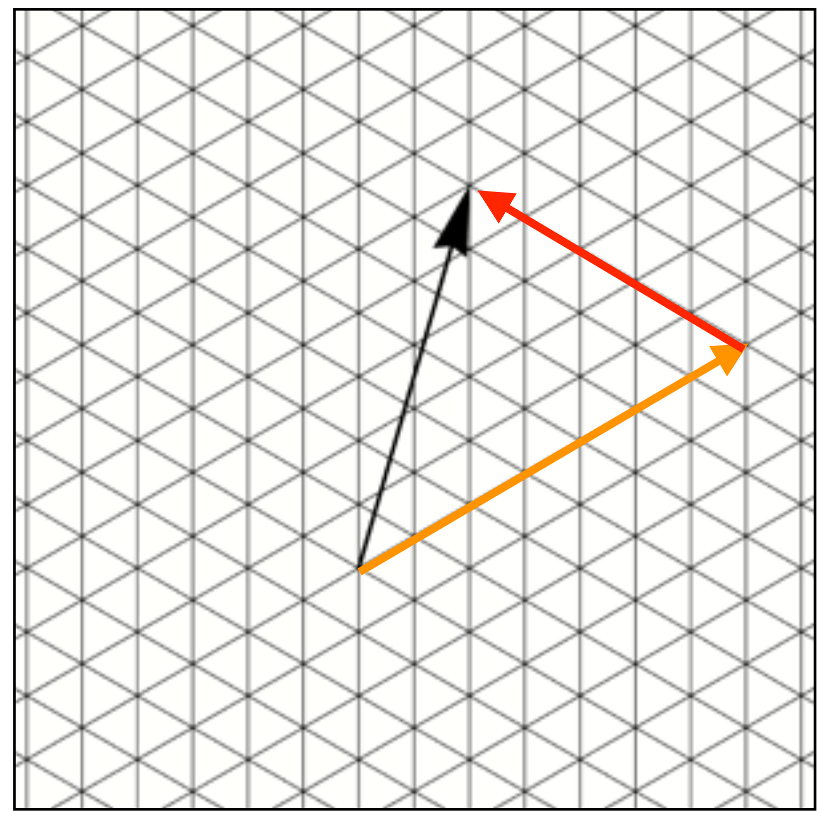
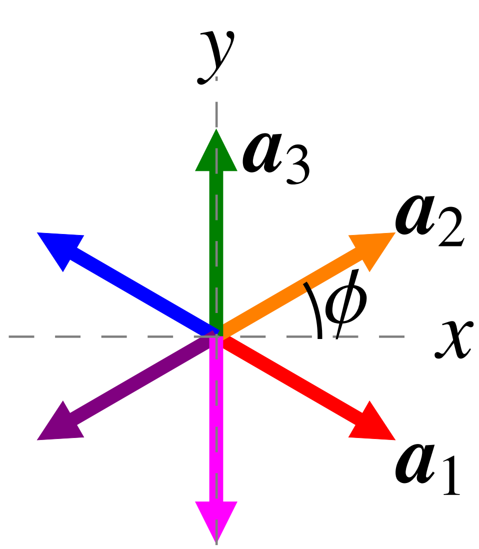




$$\mathbf{C} = 7\mathbf{a}_2 - 5\mathbf{a}_1$$



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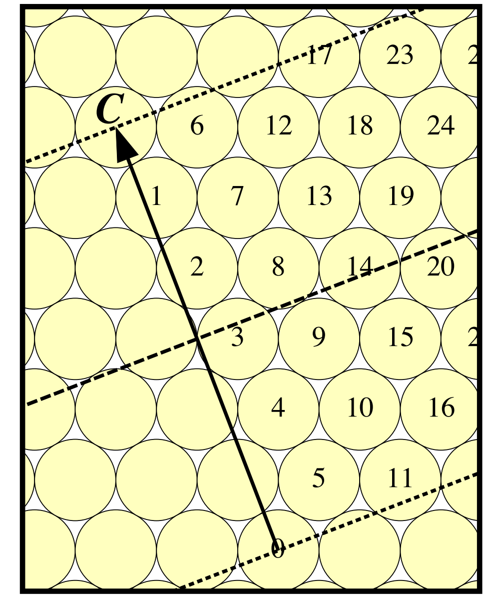

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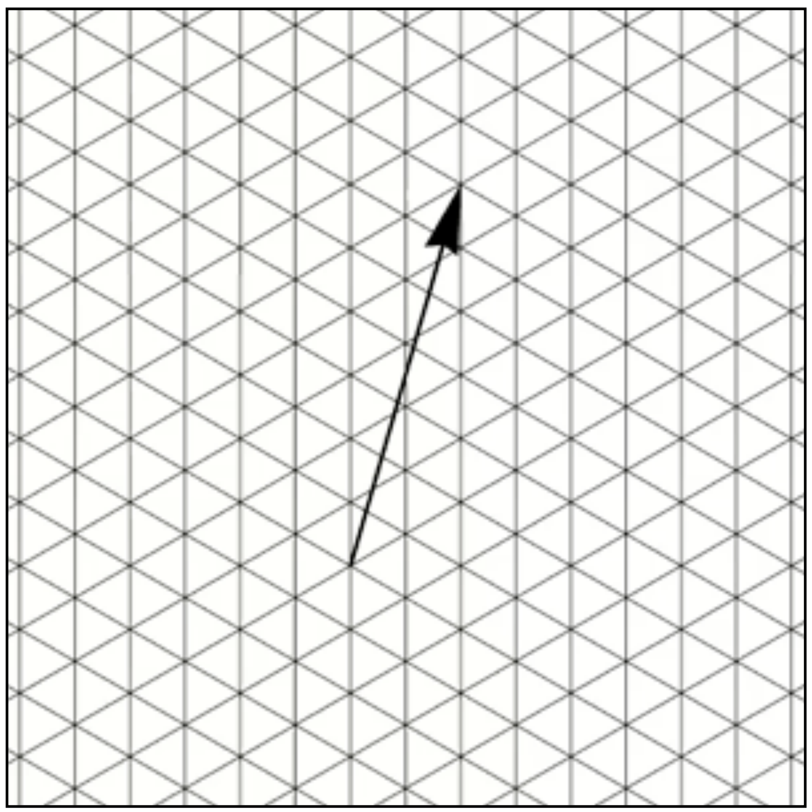

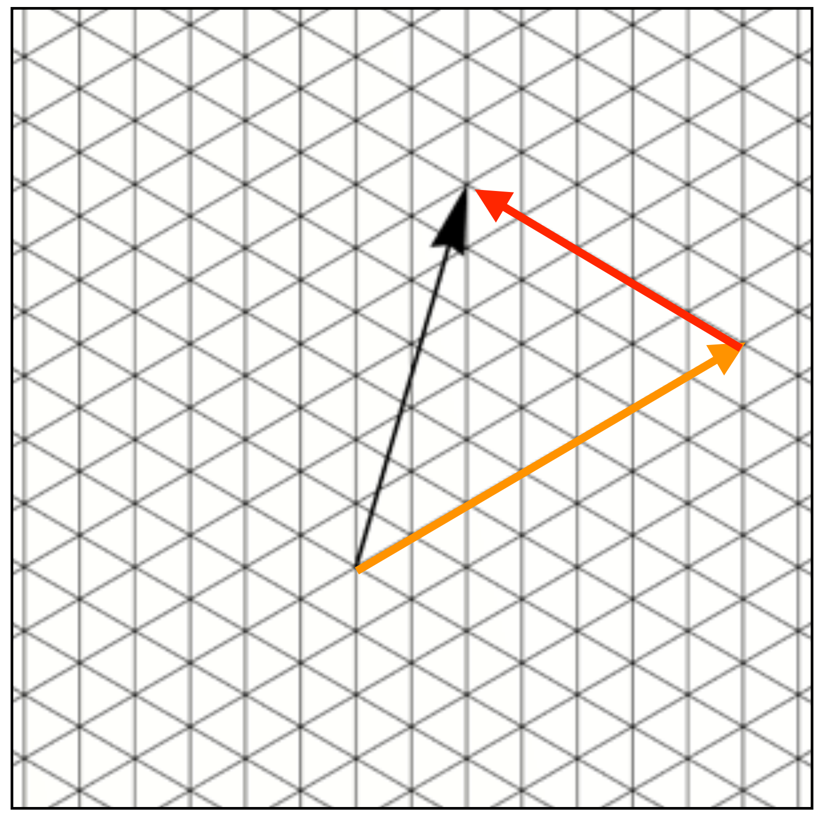

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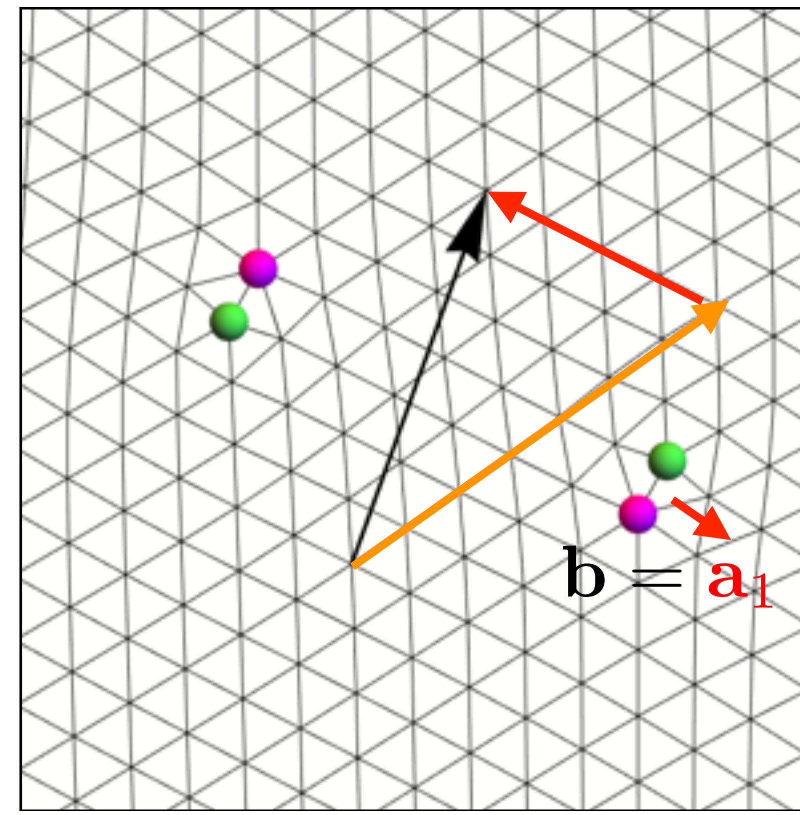

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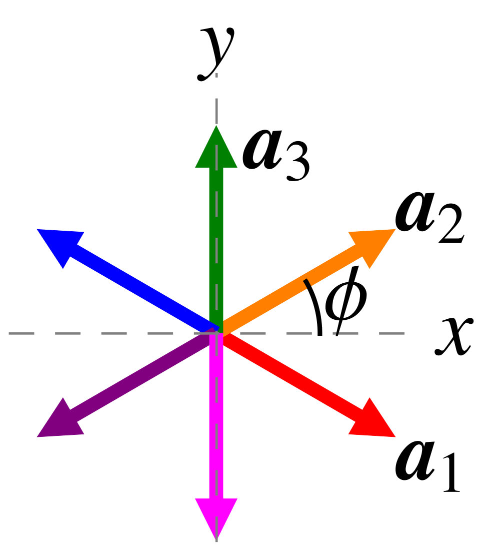
$(m, n) \to (m, n-1)$



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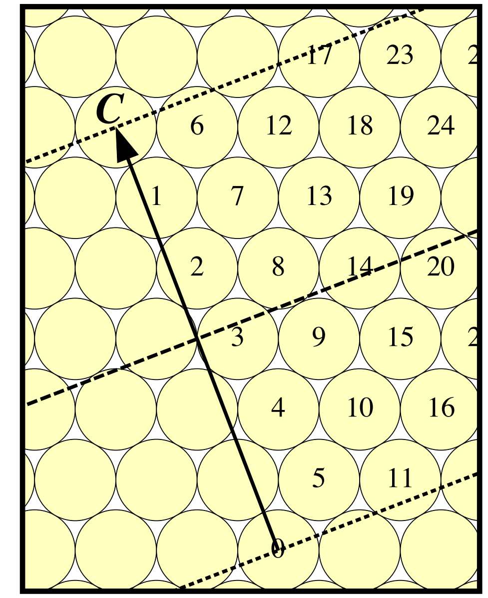

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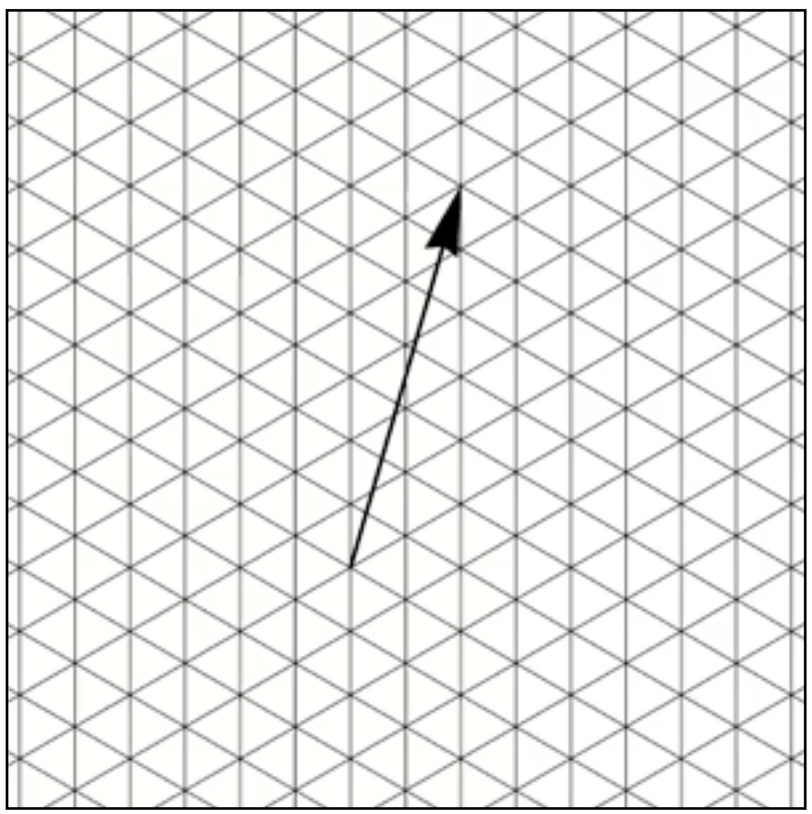

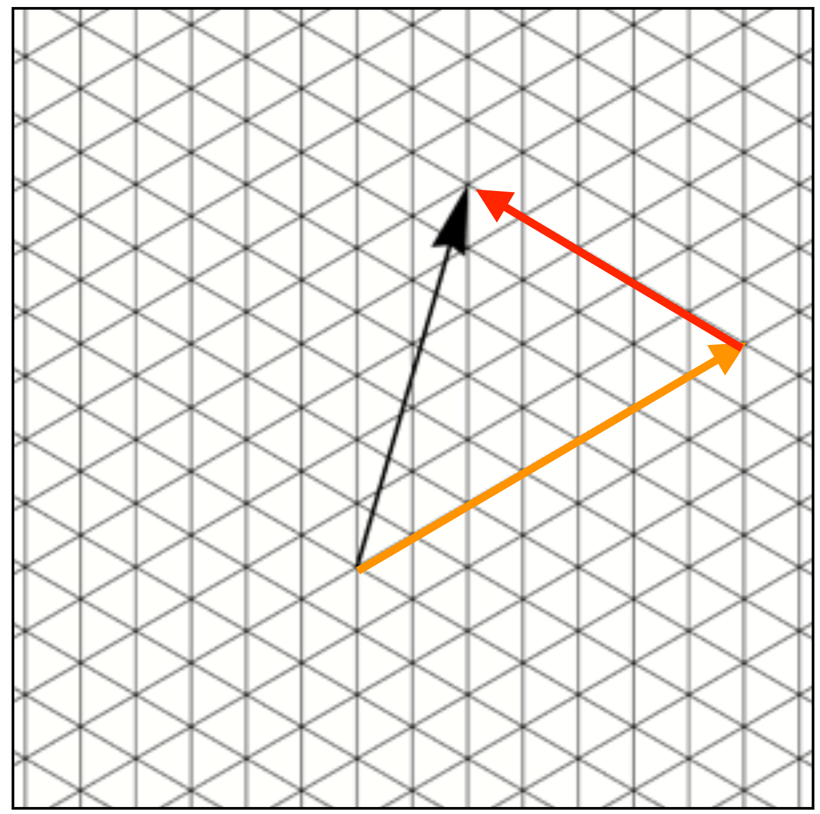

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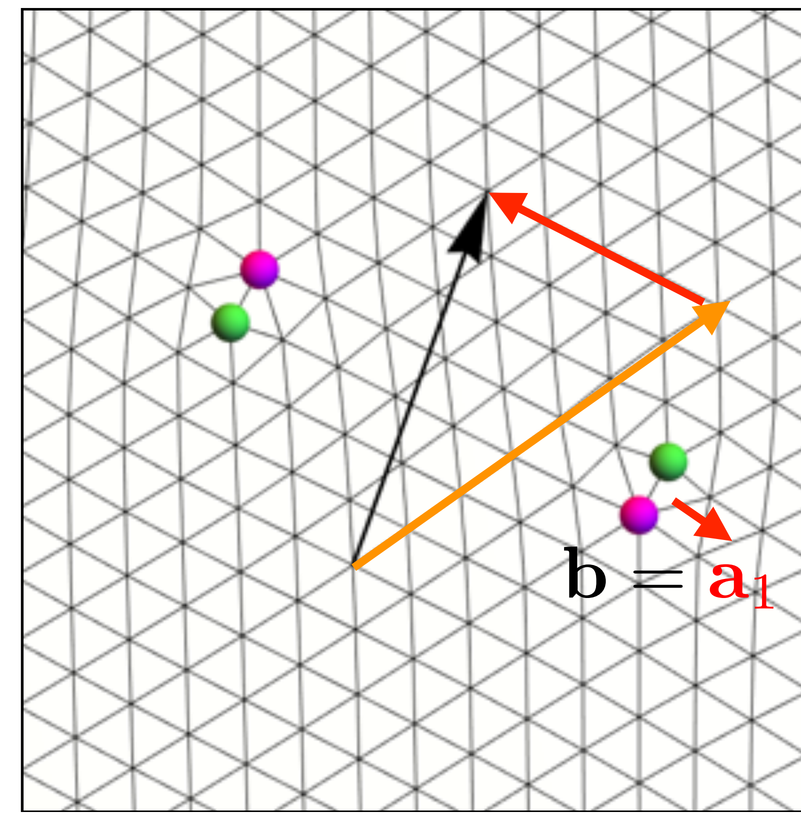

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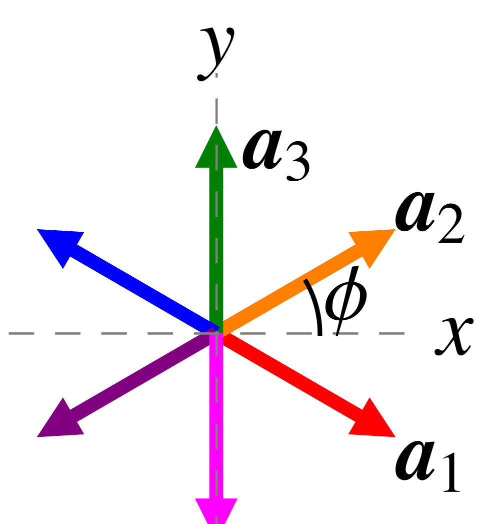

$\mathbf{b}$	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$-\mathbf{a}_1$	$-\mathbf{a}_2$	$-\mathbf{a}_3$
Δm	0	+1	+1	0	−1	−1
Δn	−1	0	+1	+1	0	−1

A dislocation passing through the system changes (m,n)

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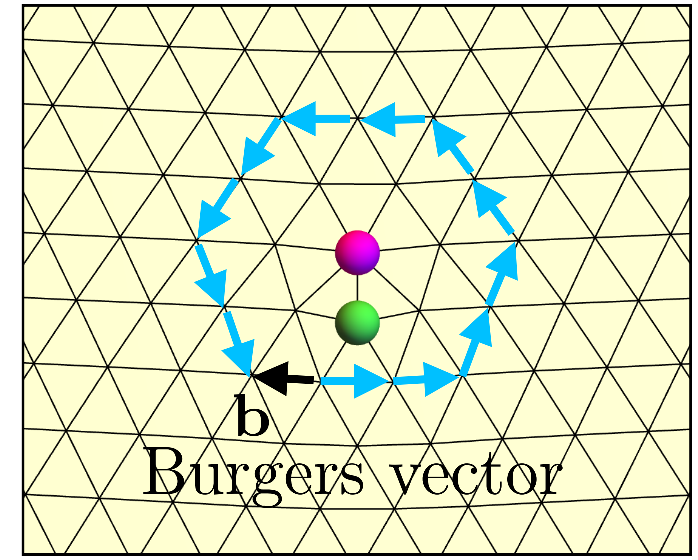
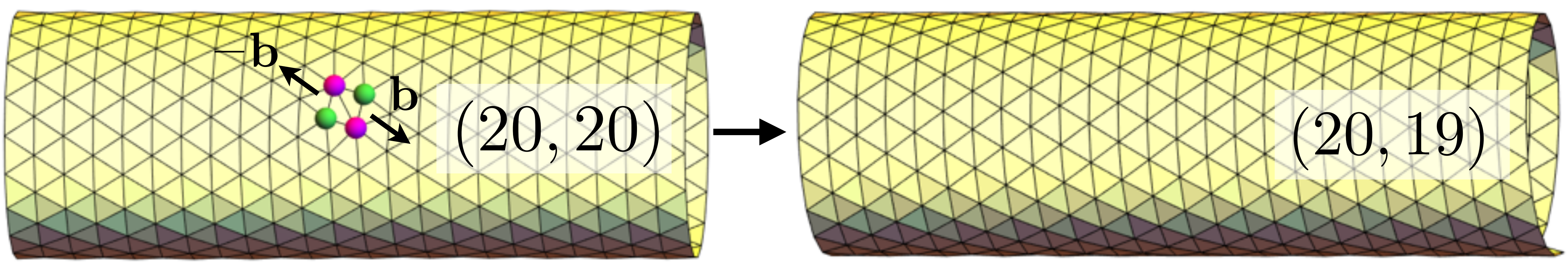
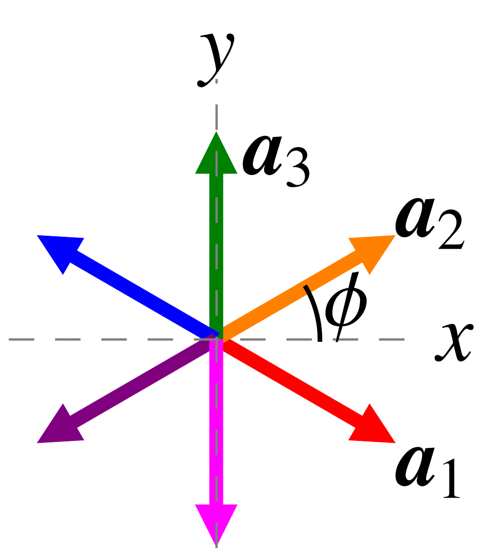




The right-moving dislocation has $\mathbf{b} = \mathbf{a}_1$



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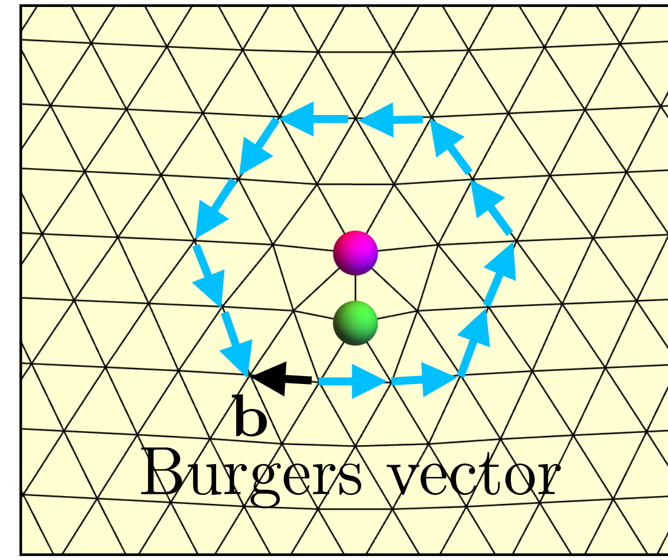
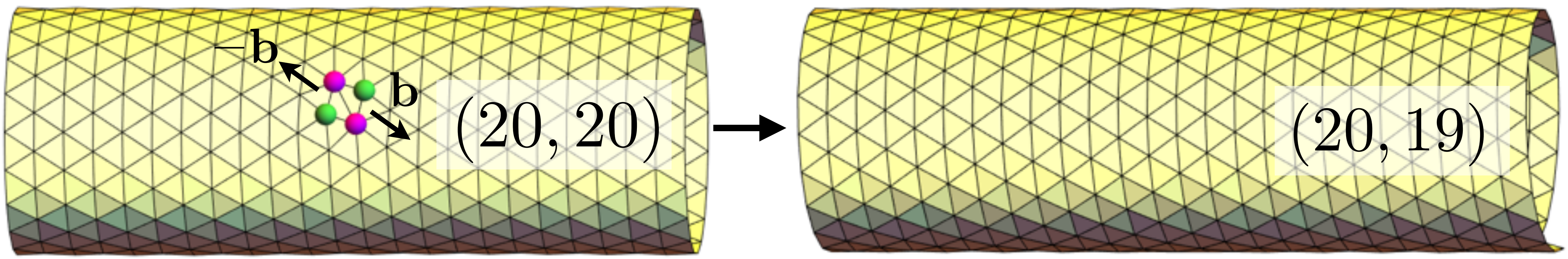
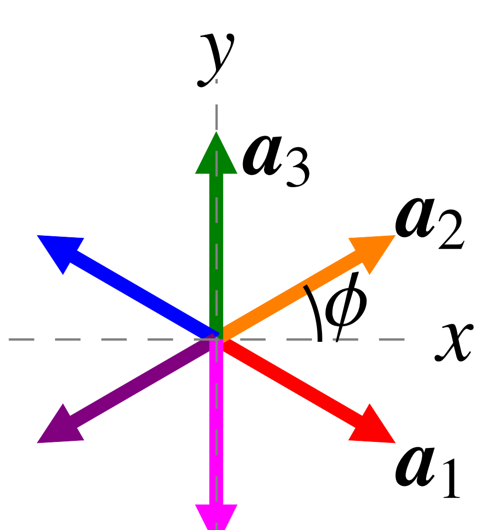




The right-moving dislocation has $\mathbf{b} = \mathbf{a}_1$

Dislocation motion $\Rightarrow$ Parastichy transition!

$(\Delta m, \Delta n) \Rightarrow \Delta R$, $\Delta\phi \Rightarrow$ Plastic deformation!



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Plastic deformation of tubular crystals

- Background: Phyllotactic geometry of tubular crystals
- <u>Mechanics of plastic deformation: Analytic predictions</u>
- Numerical modeling
- Necks in tubes: Radius profiles near dislocations

Energetics of dislocations on the plane

Stretching energy

$$\boxed{\begin{aligned} E_s &= \frac{1}{2}\int d\mathbf{x}\left(2\mu u_{ij}u_{ij} + \lambda u_{kk}^2\right) \\ &= \frac{1}{2}\cdot\frac{3}{8}Y\int d\mathbf{x}\left(2u_{ij}u_{ij} + u_{kk}^2\right) \end{aligned}}$$

- μ, λ = Lamé coefficients
- "Harmonic springs" assumption:

$$\mu = \lambda = \frac{3}{8}Y$$

where $\boxed{Y = 4\pi A = \text{Young's modulus}}$

- strain tensor $u_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)$
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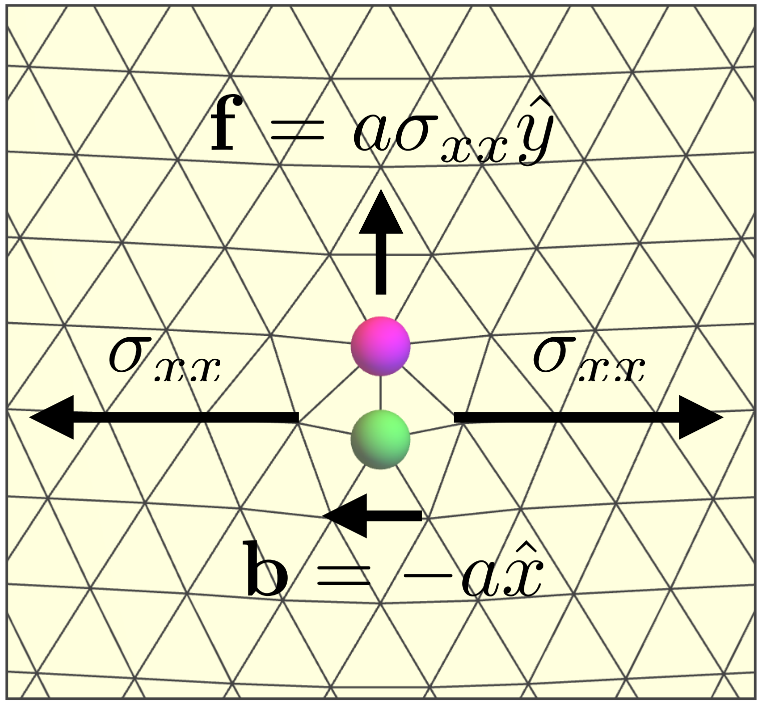
Peach-Kohler force:
Force on a dislocation $\mathbf{b}$ in a stress field σ

$$f_i = \epsilon_{ijz} b_k \sigma_{jk}$$



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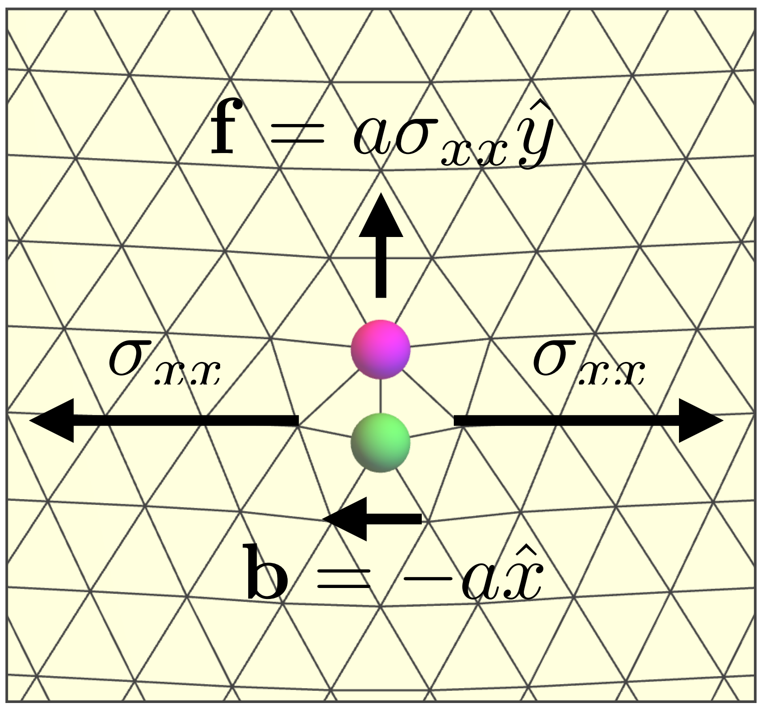

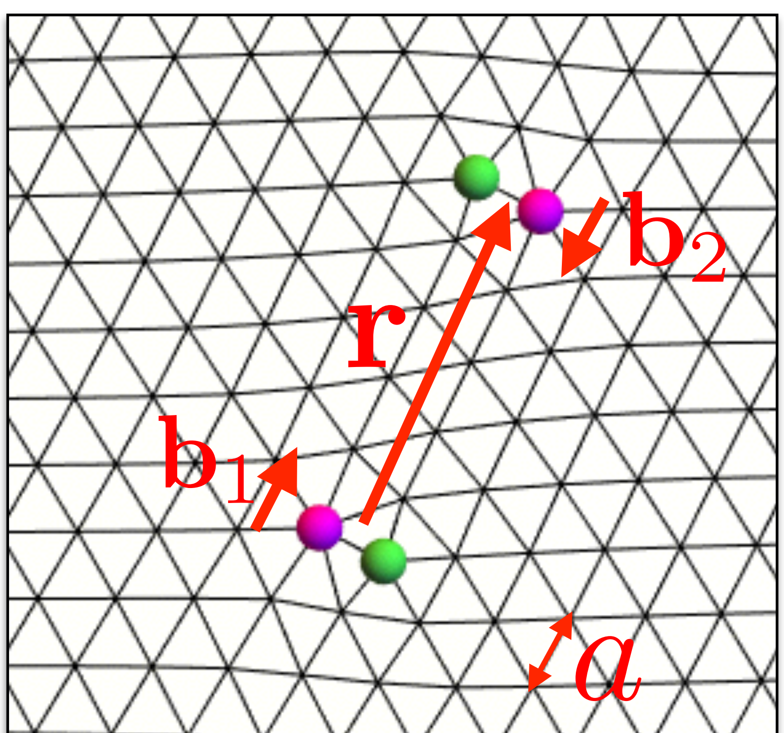

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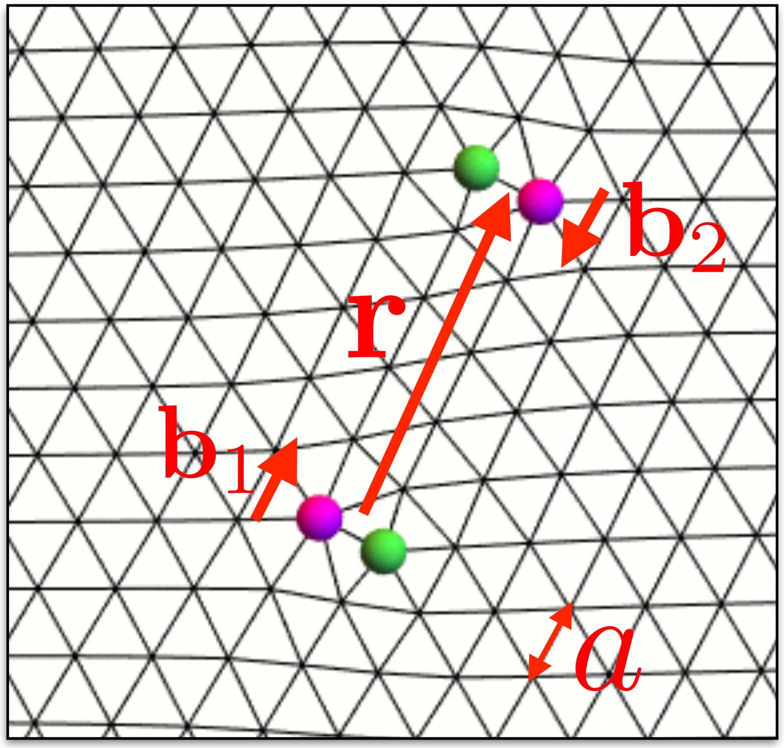

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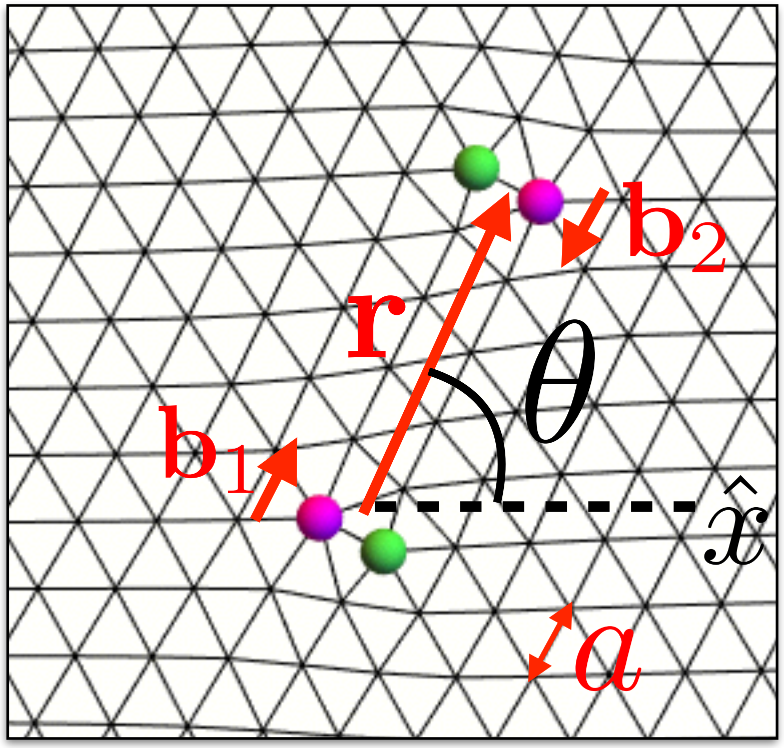
$$\mathbf{b}_1 = -\mathbf{b}_2 \equiv \mathbf{b} = a(\cos\theta, \sin\theta)$$



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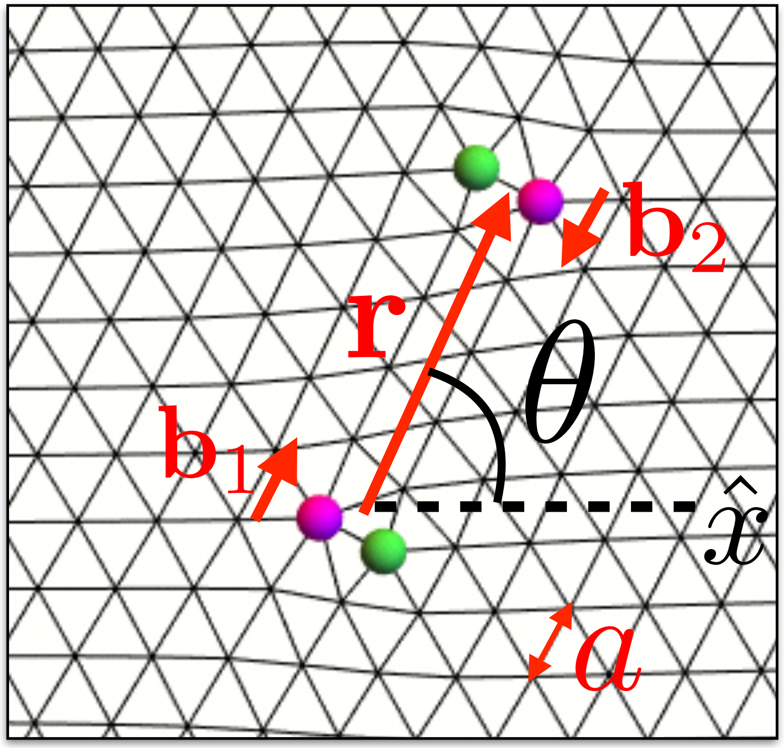

$$E_s(r) = Aa^2\ln(r/a) + s\cdot a\cdot r\cdot\left[\tfrac{1}{2}\sin(2\theta)\left(\sigma_{xx}^{\text{ext}} - \sigma_{yy}^{\text{ext}}\right) - \cos(2\theta)\sigma_{xy}^{\text{ext}}\right]$$

$$s = \text{sign}[\cos(\theta)]$$

Nelson, PRB 18:2318 (1978)

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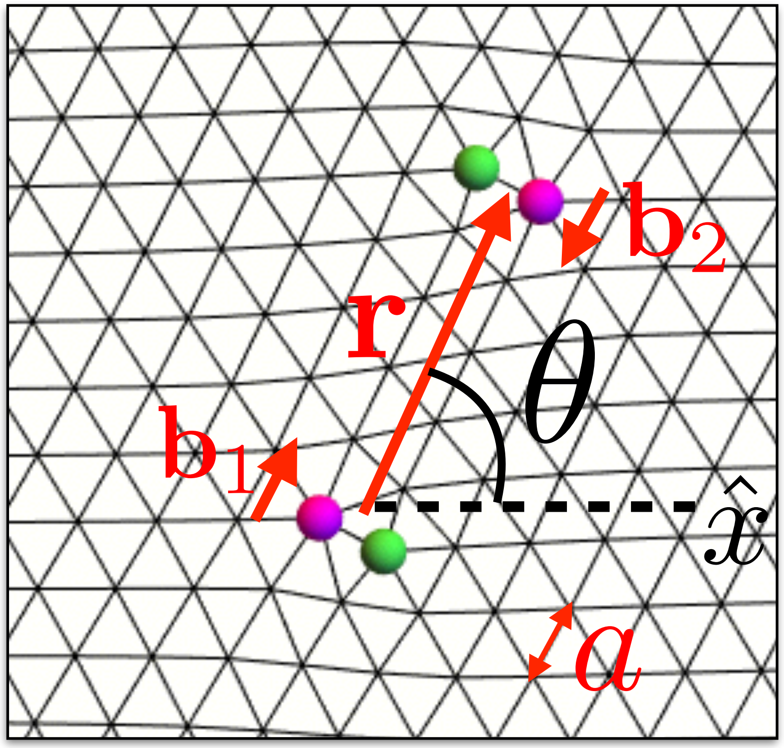

$$\begin{aligned} E_s(r) &= Aa^2\ln(r/a) \quad \leftarrow \text{dislocations' interaction} \\ &\quad + s\cdot a\cdot r\cdot\left[\tfrac{1}{2}\sin(2\theta)\left(\sigma_{xx}^{\text{ext}} - \sigma_{yy}^{\text{ext}}\right) - \cos(2\theta)\sigma_{xy}^{\text{ext}}\right] \end{aligned}$$

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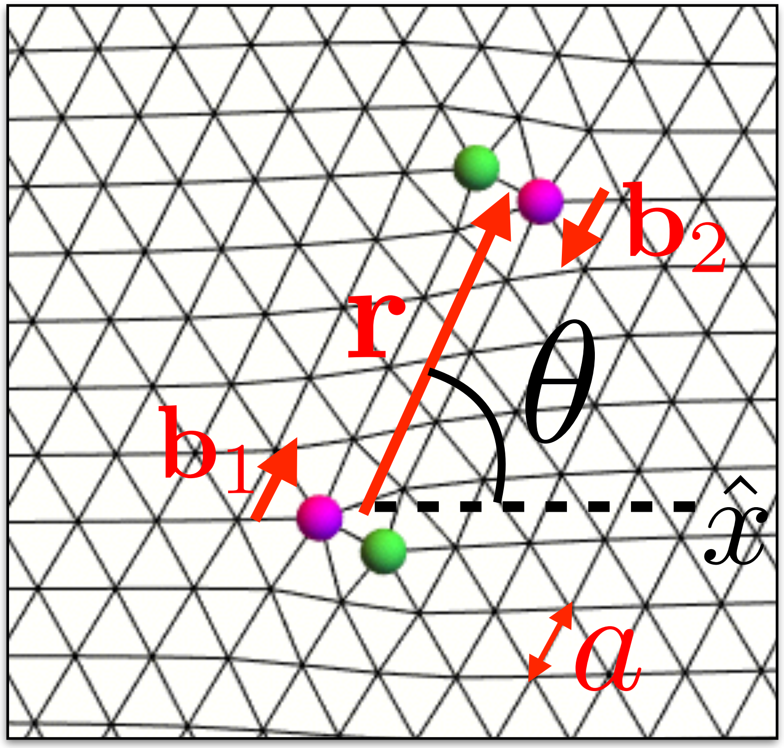


$$E_s(r) = Aa^2\ln(r/a)$$ ← dislocations' interaction

$$+\, s\cdot a\cdot r\cdot\left[\frac{1}{2}\sin(2\theta)\left(\sigma_{xx}^{\text{ext}} - \sigma_{yy}^{\text{ext}}\right) - \cos(2\theta)\sigma_{xy}^{\text{ext}}\right]$$

coupling to external stress

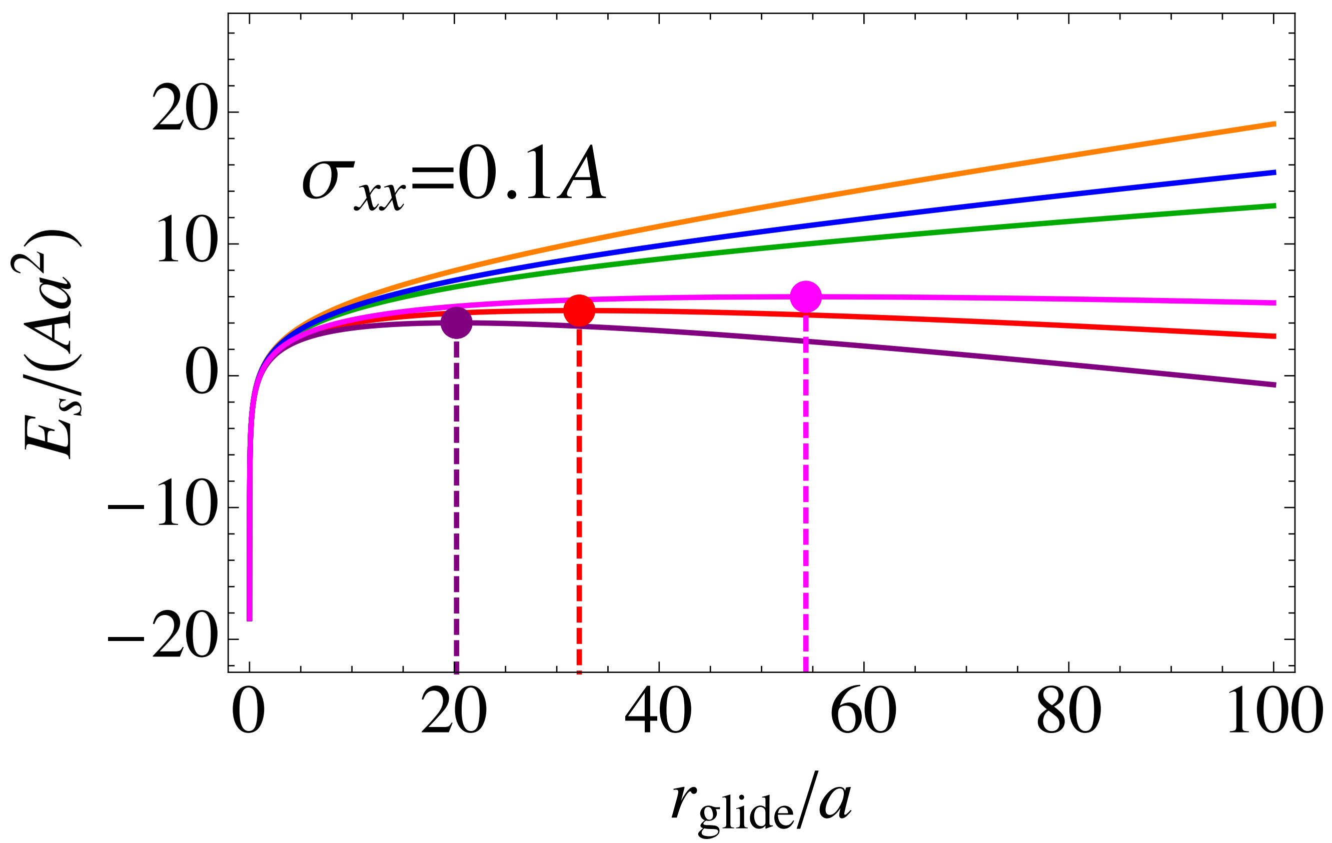
$$s = \text{sign}[\cos(\theta)]$$

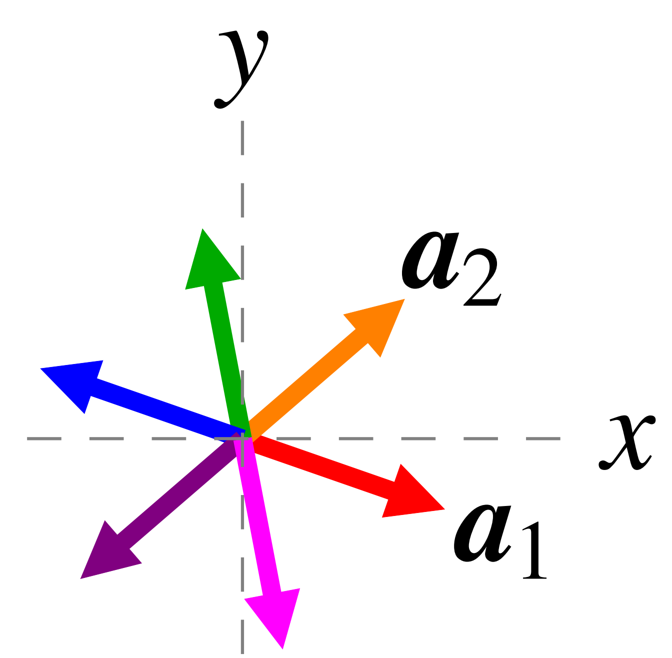
Nelson, PRB 18:2318 (1978)

Dislocation pair energy landscape

$$E_s(r) = Aa^2 \ln(r/a) + s \cdot a \cdot r \cdot \left[\tfrac{1}{2}\sin(2\theta)\left(\sigma_{xx}^{\text{ext}} - \sigma_{yy}^{\text{ext}}\right) - \cos(2\theta)\sigma_{xy}^{\text{ext}}\right] + \text{const.}$$

$$s = \text{sign}[\cos(\theta)]$$





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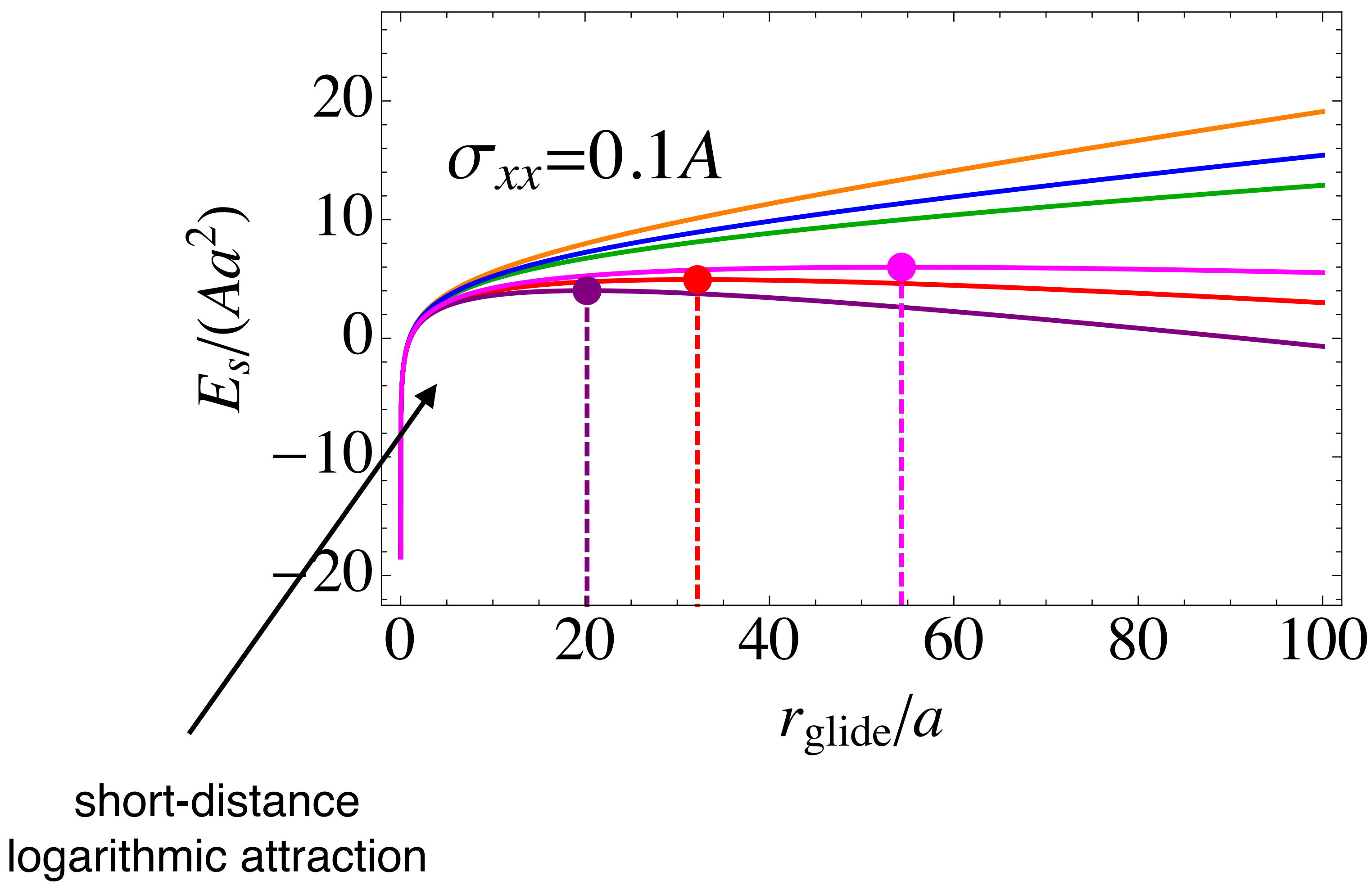

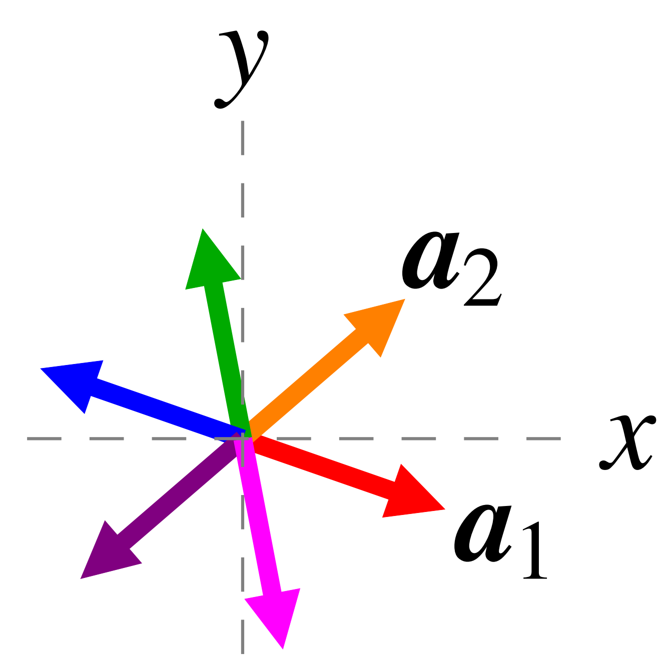

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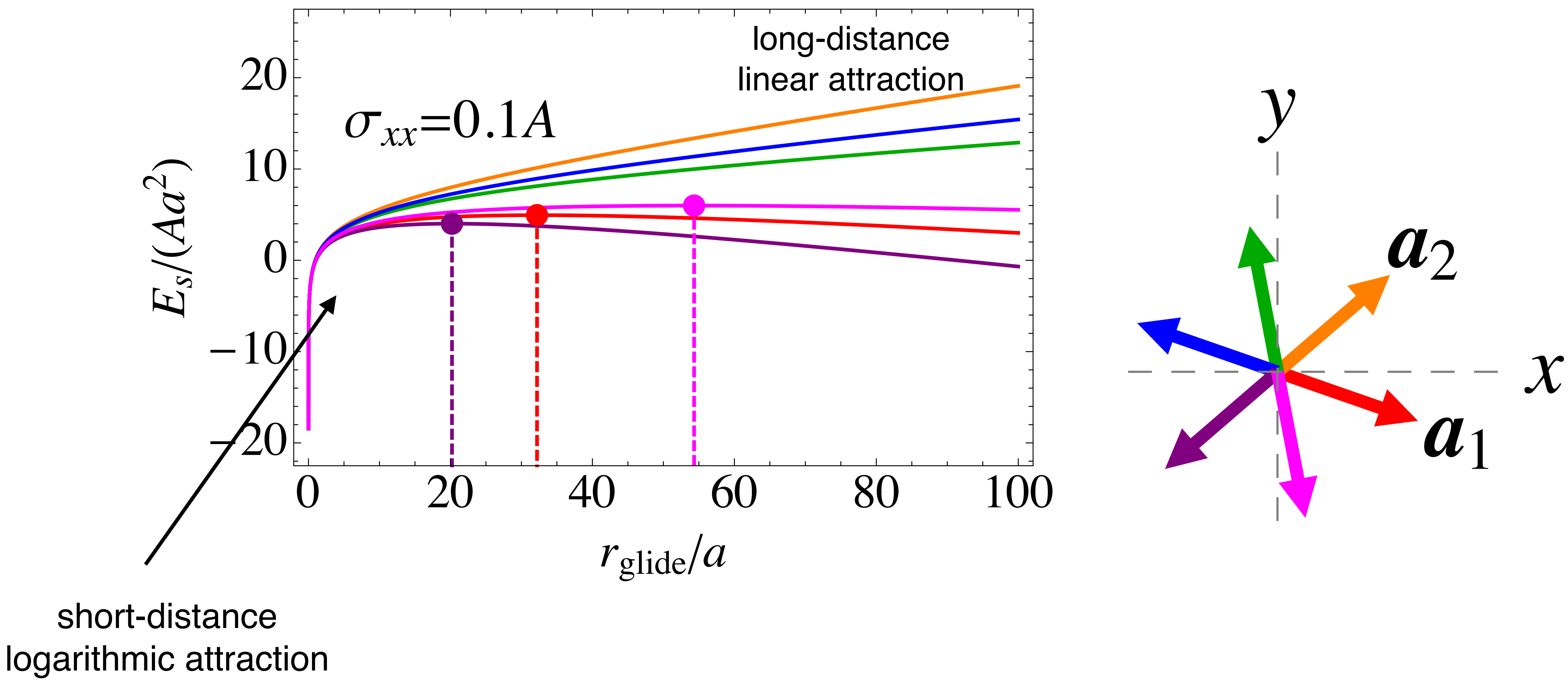

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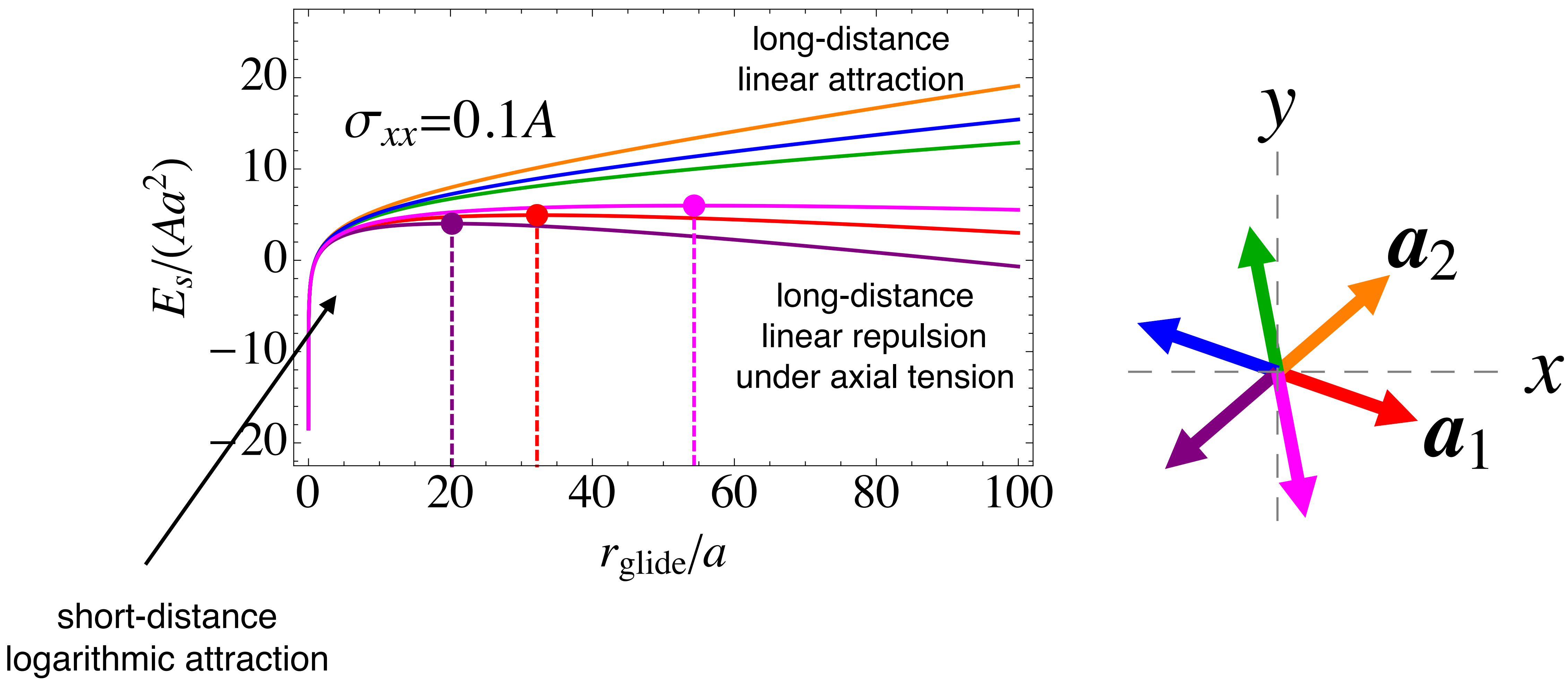

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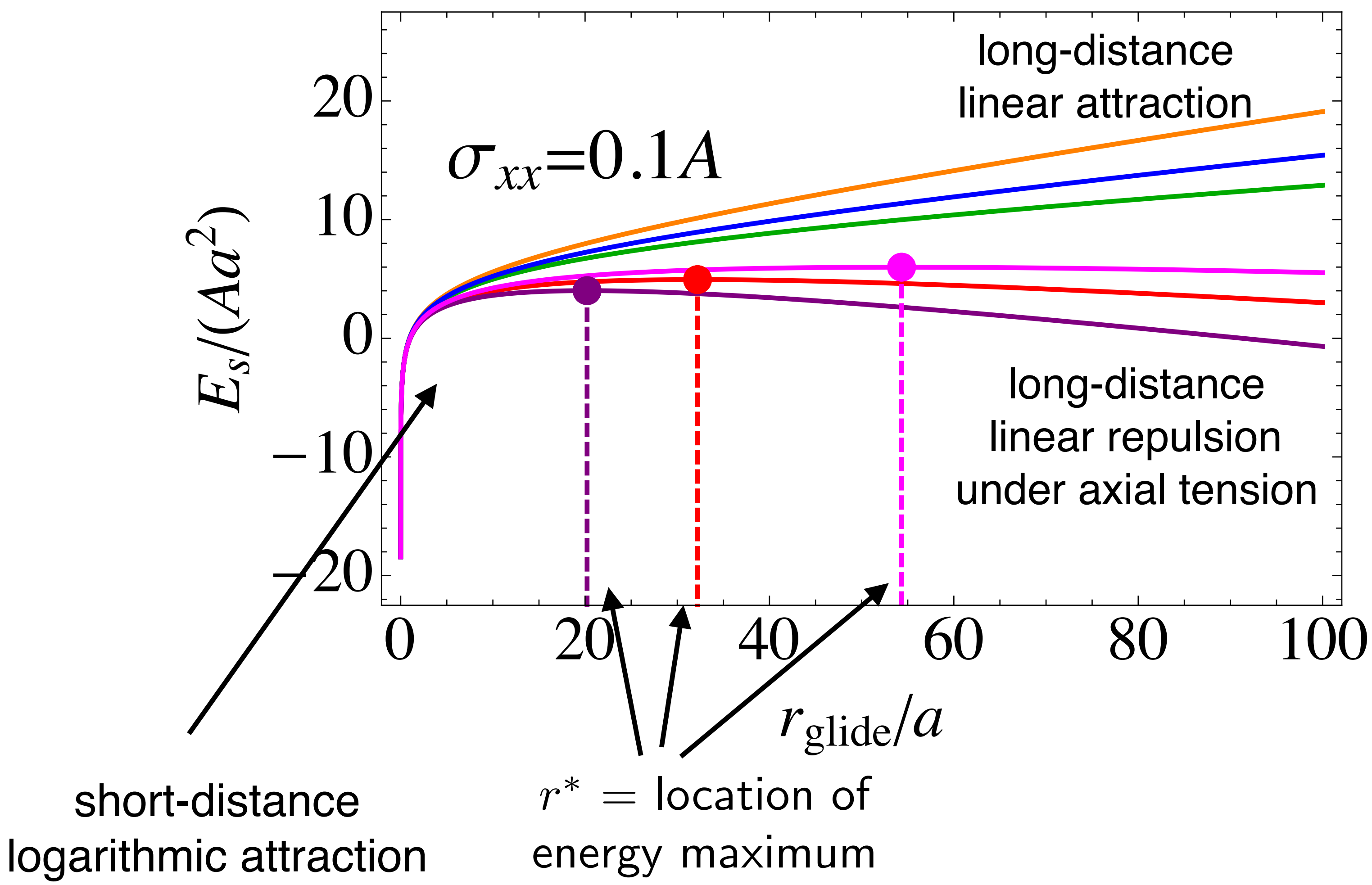

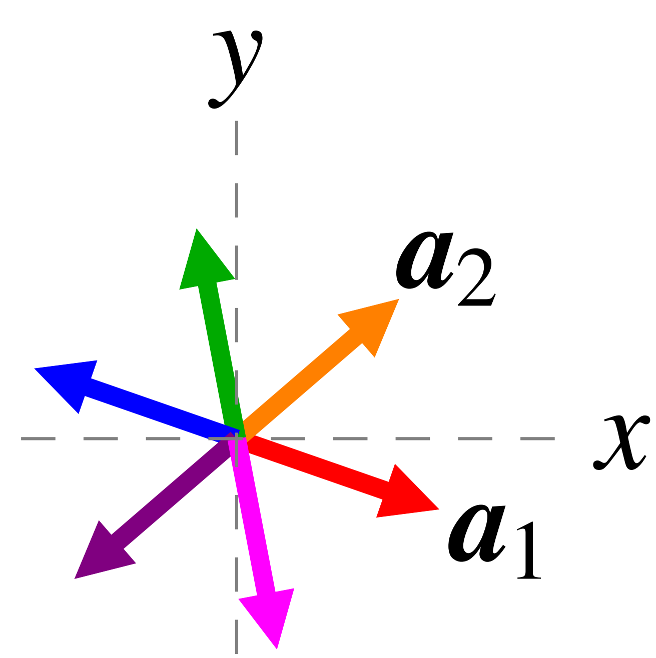

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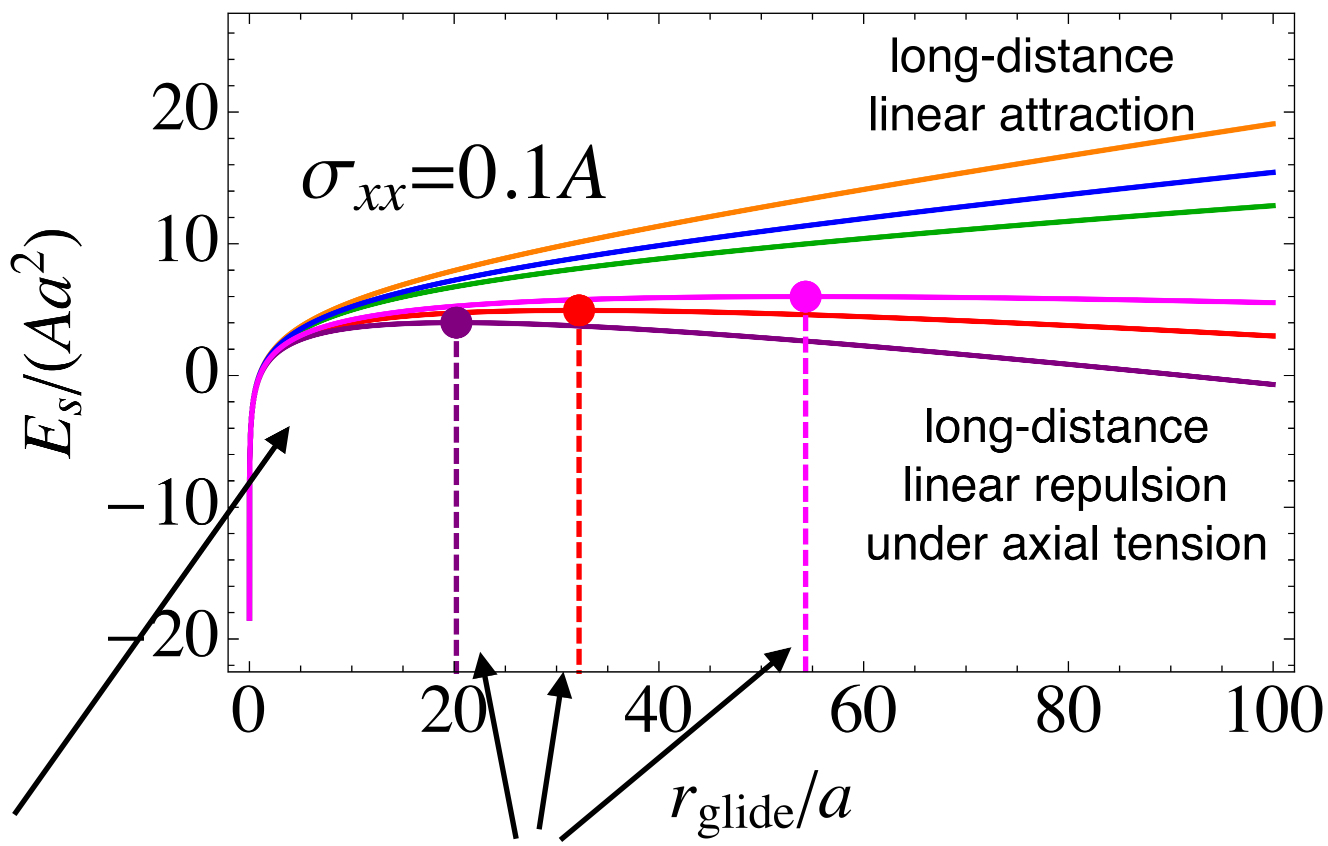

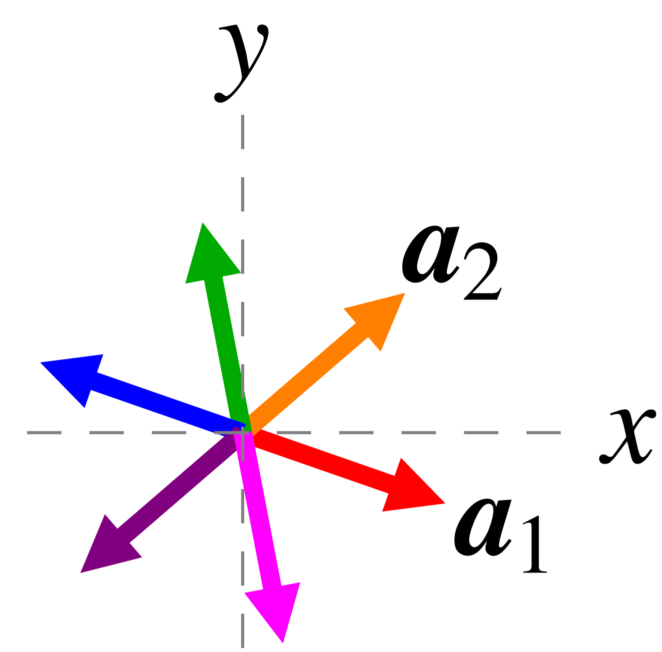

short-distance
logarithmic attraction

r^* = location of
energy maximum

$$0 = \left.\frac{dE_s}{dr}\right|_{r=r^*} \qquad r^* = \frac{sAa}{\sigma_{xy}^{\text{ext}}\cos(2\theta) - \frac{1}{2}\left(\sigma_{xx}^{\text{ext}} - \sigma_{yy}^{\text{ext}}\right)\sin(2\theta)}$$

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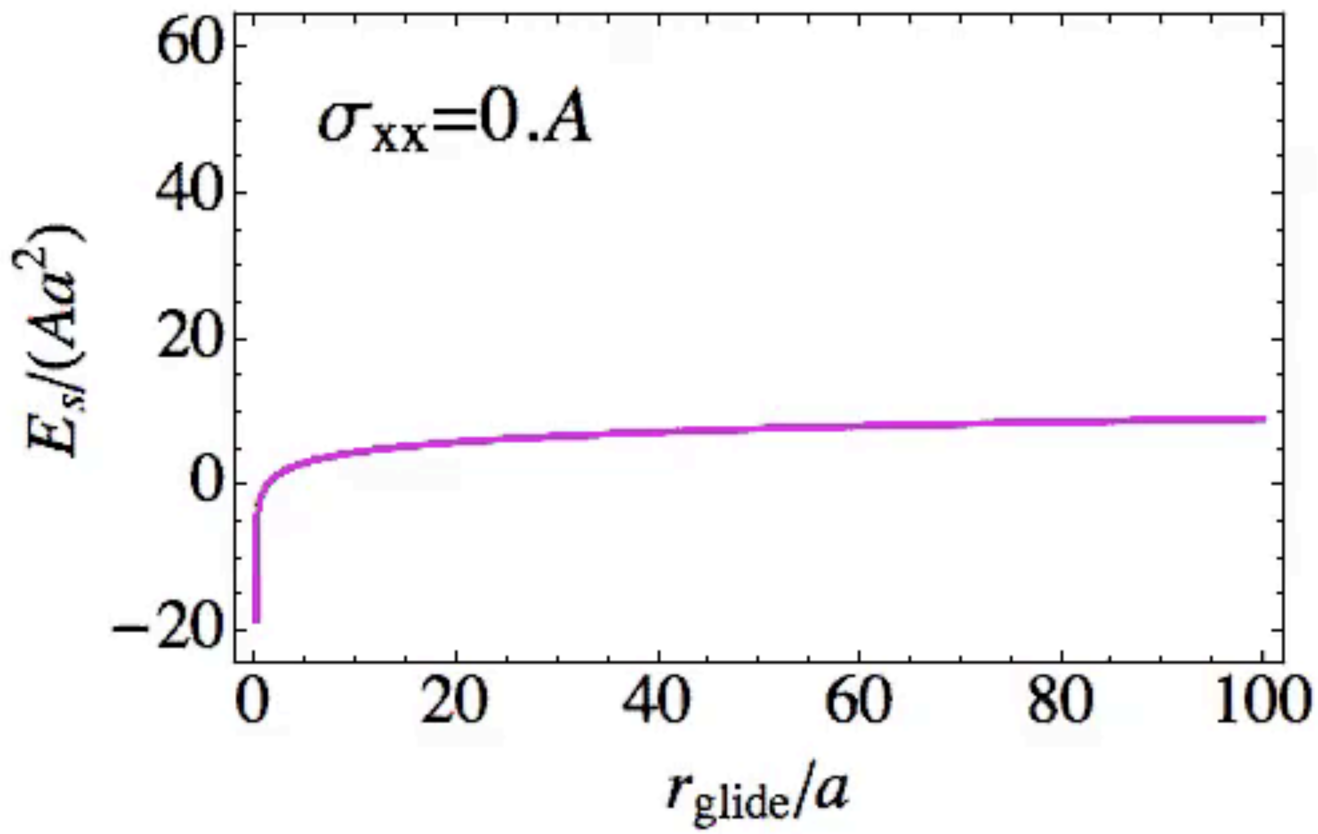

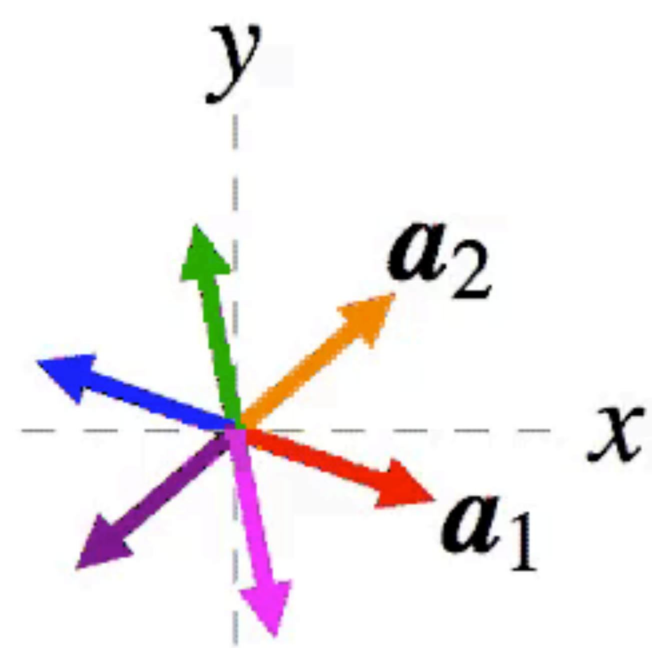

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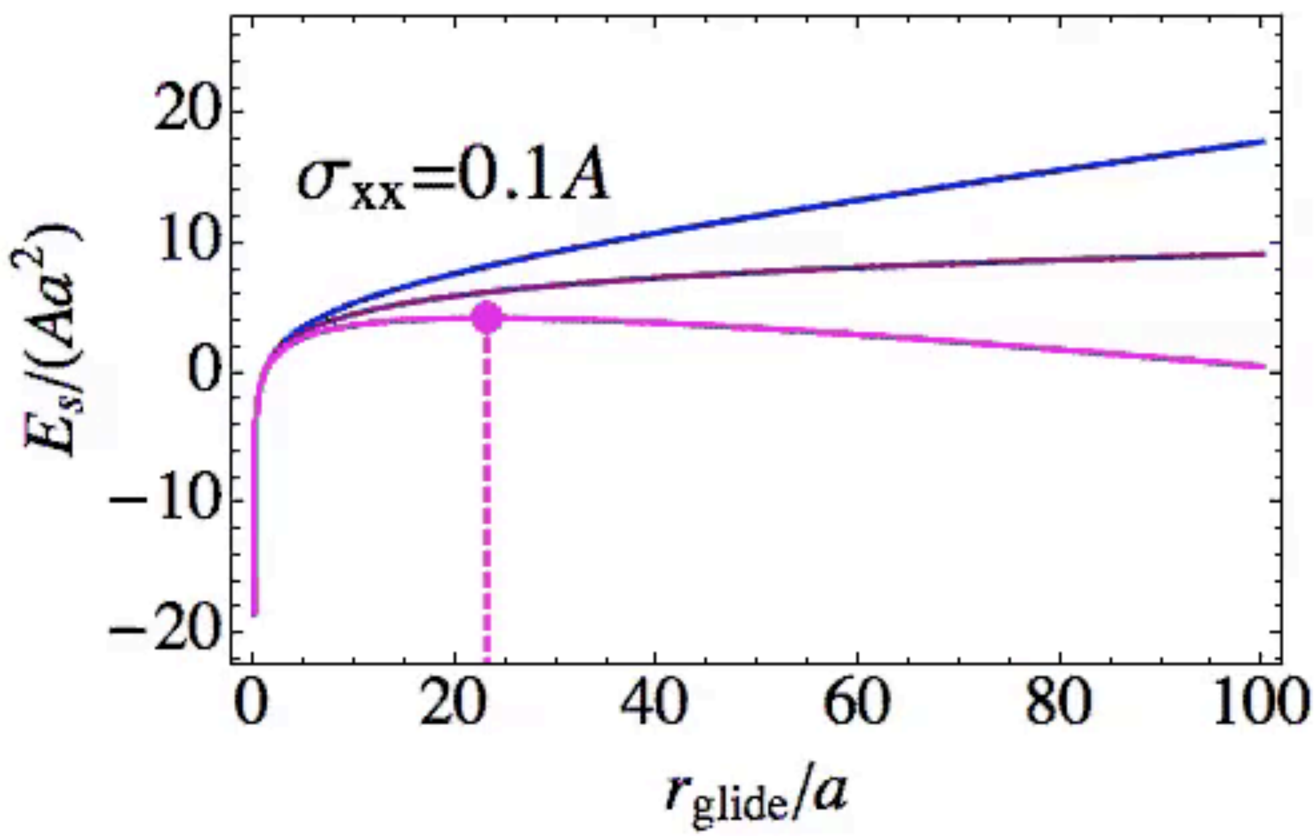

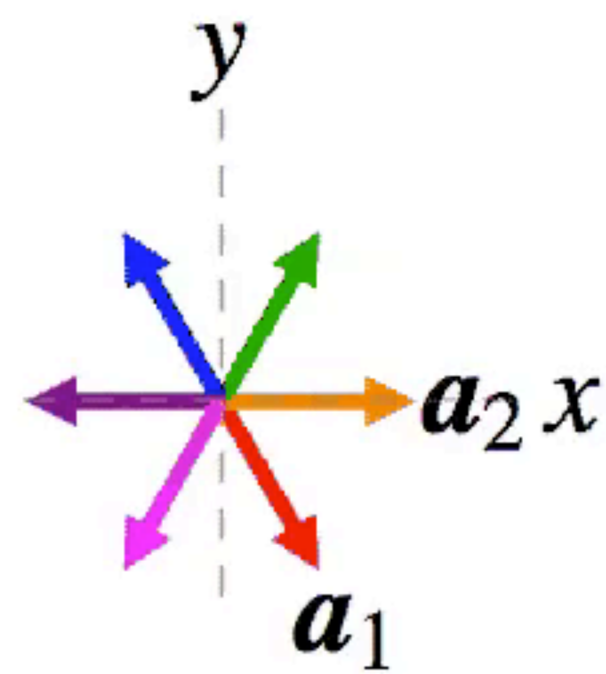

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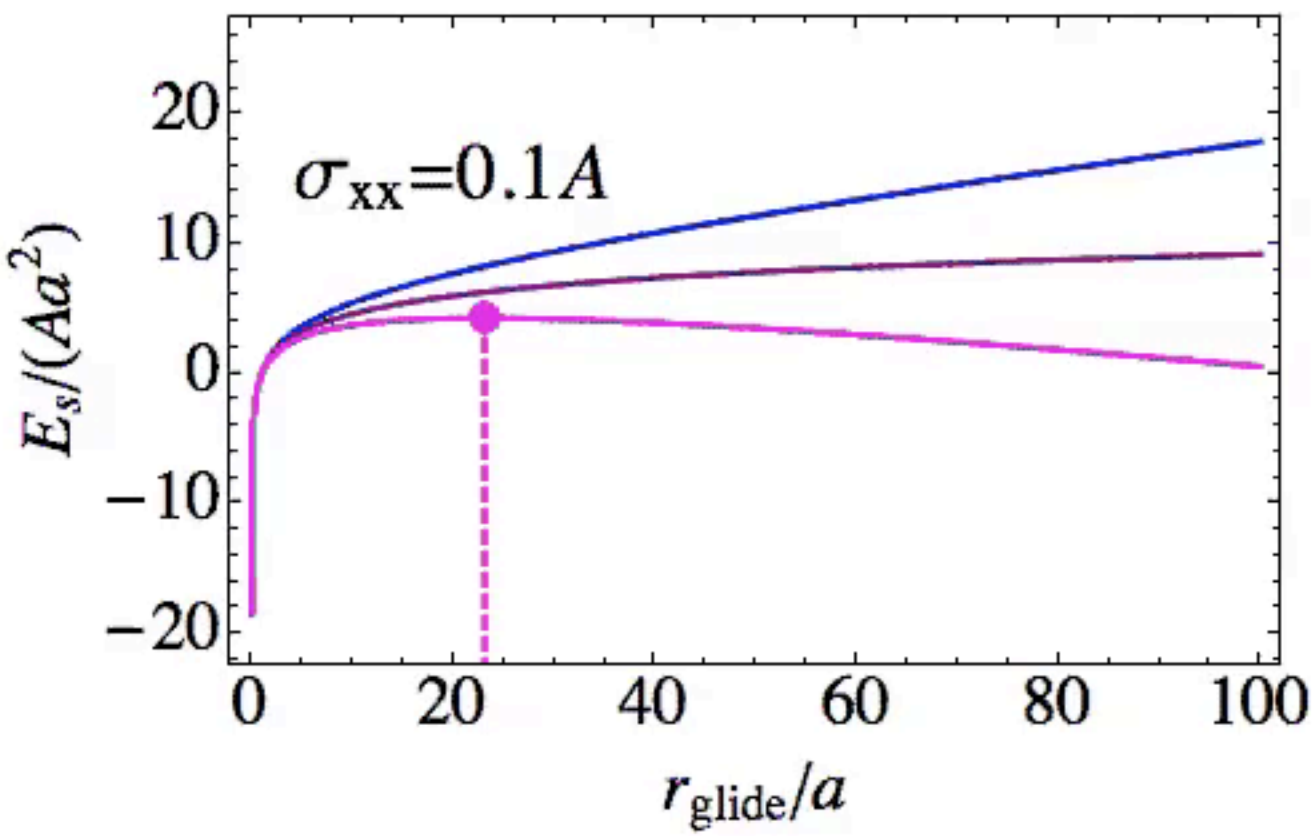

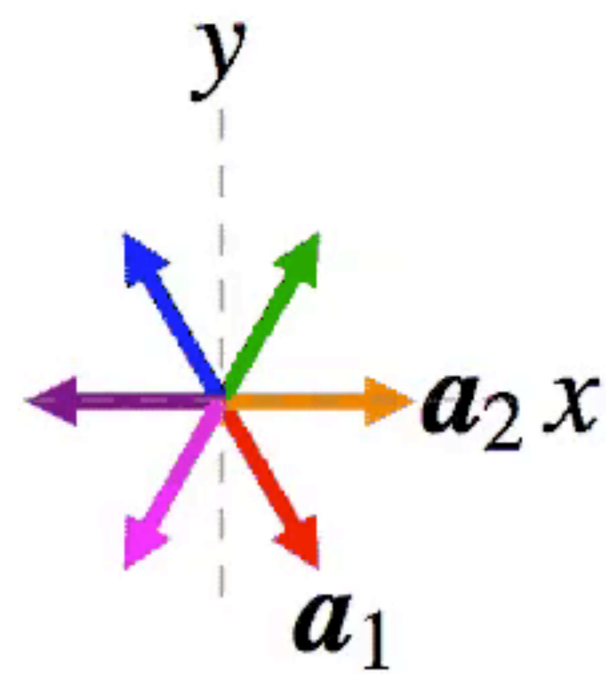

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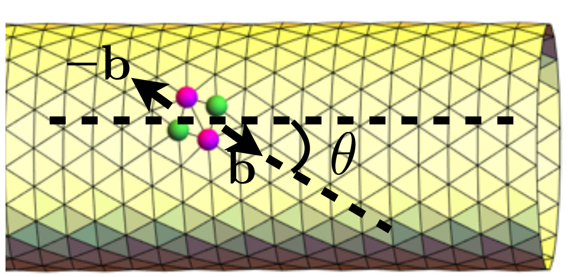
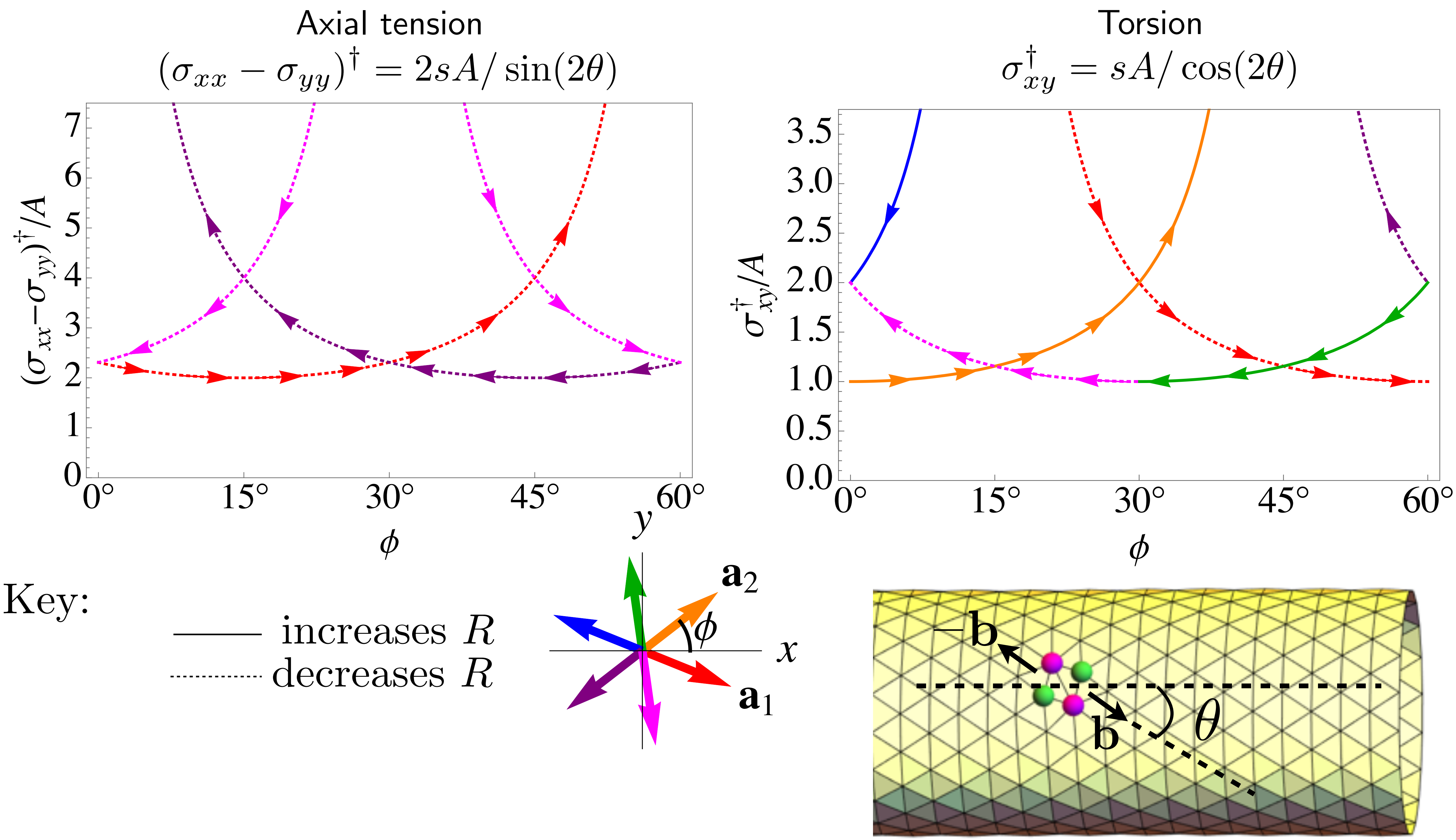
Critical stress for plastic deformation of pristine lattice:
How strong must σ^{ext} be to make $r^* = a$?



Critical stress required for instability to dislocation pair unbinding

Critical stress $\sigma^\dagger$ to unbind an *elementary* dislocation pair ($|\mathbf{b}| = r = a$) and plastically deform the tube by dislocation glide:



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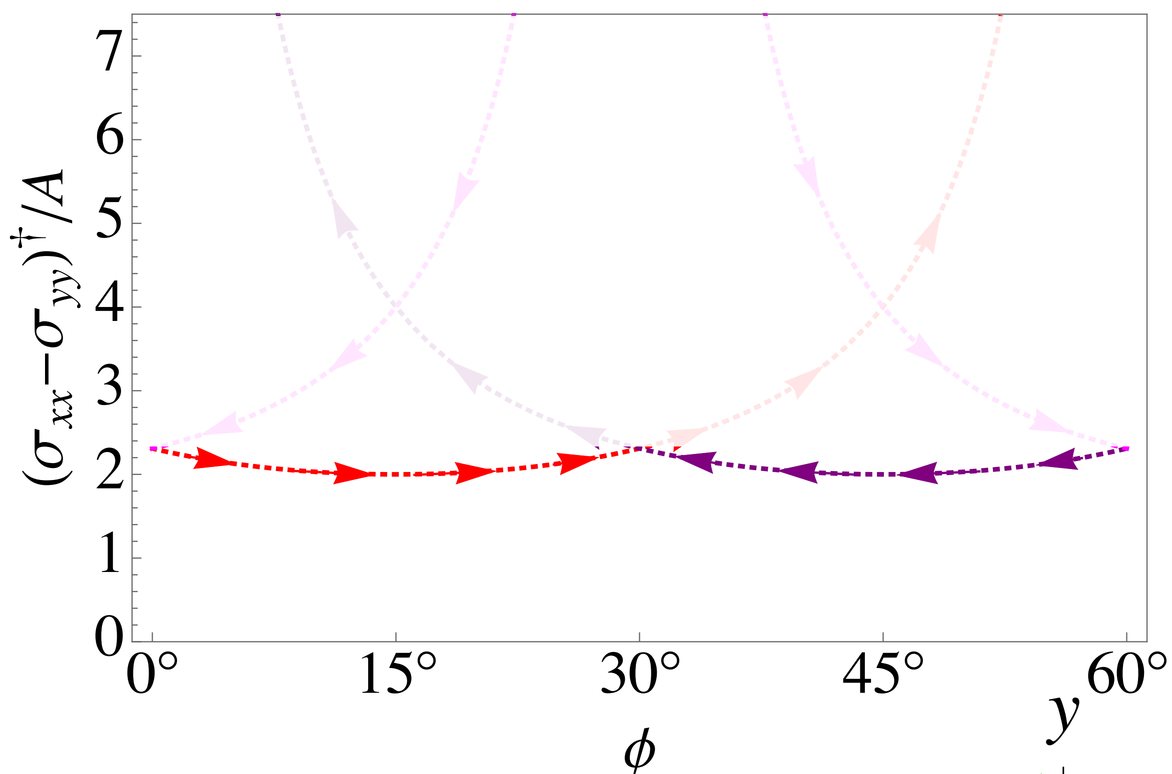
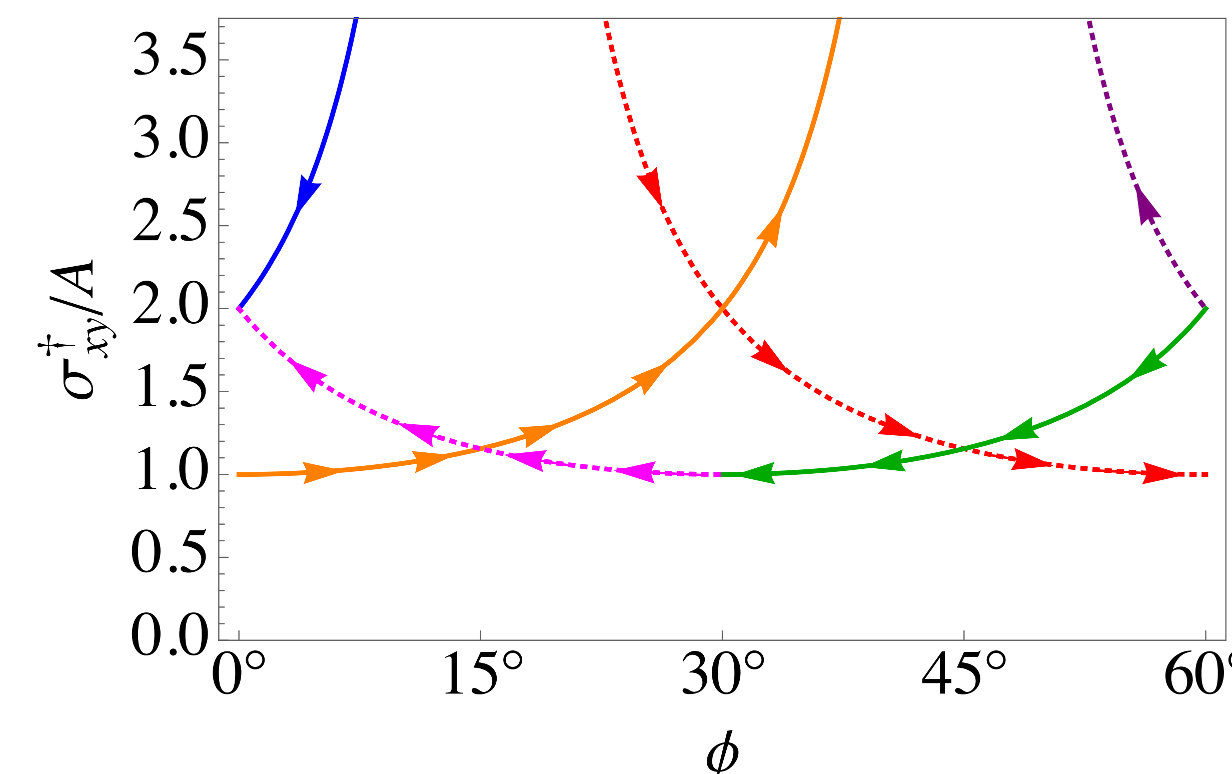
Axial tension

$(\sigma_{xx} - \sigma_{yy})^\dagger = 2sA/\sin(2\theta)$



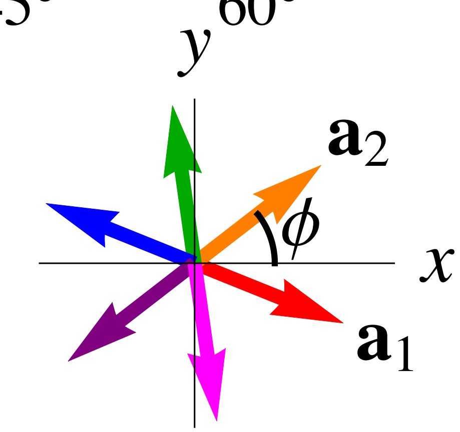
Torsion

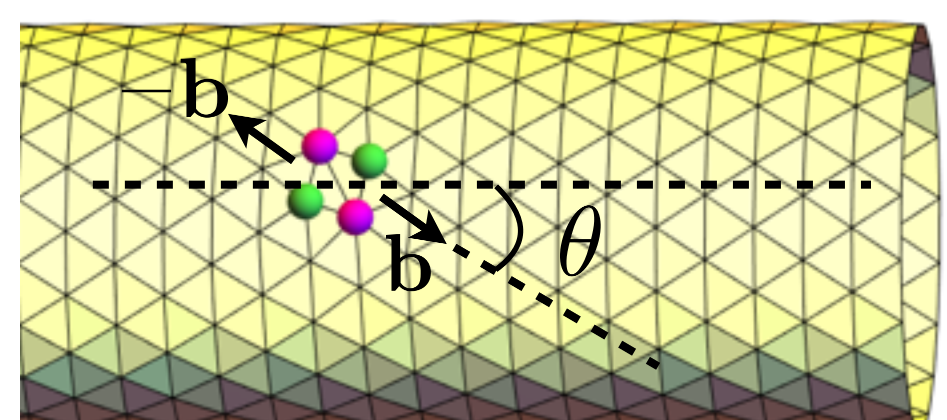
$\sigma^\dagger_{xy} = sA/\cos(2\theta)$



Key:

——	increases R
········	decreases R





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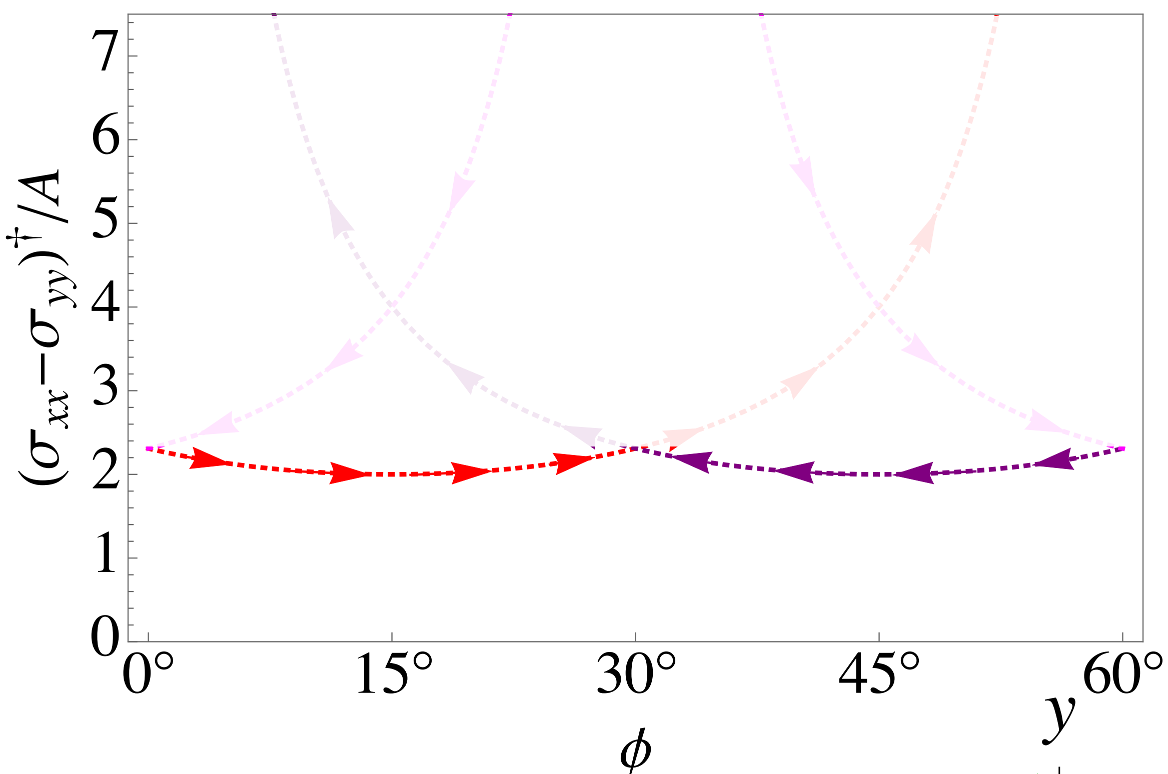

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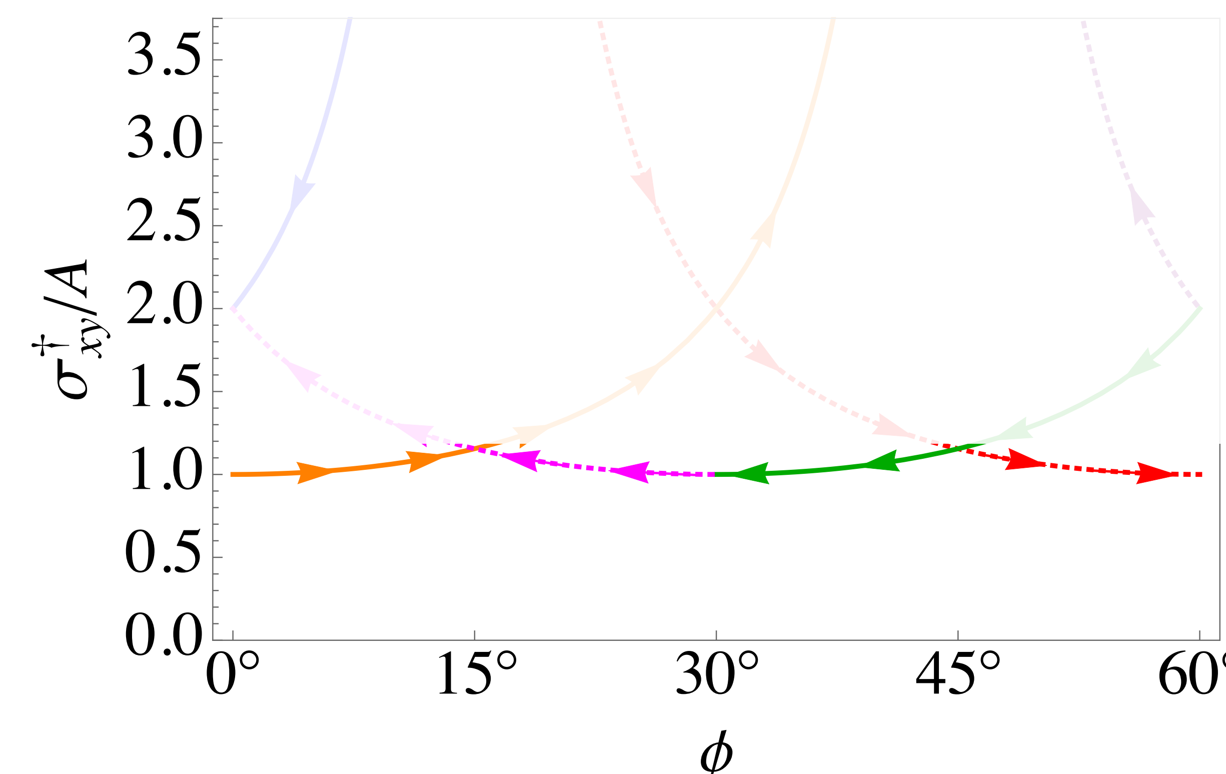

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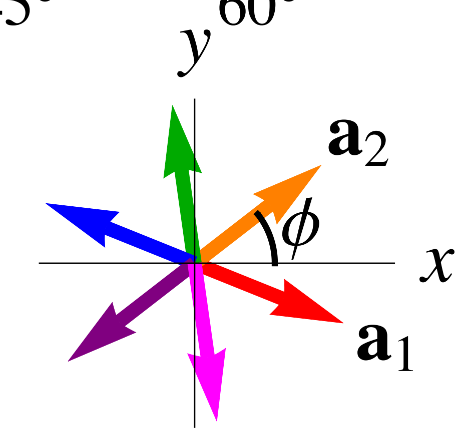

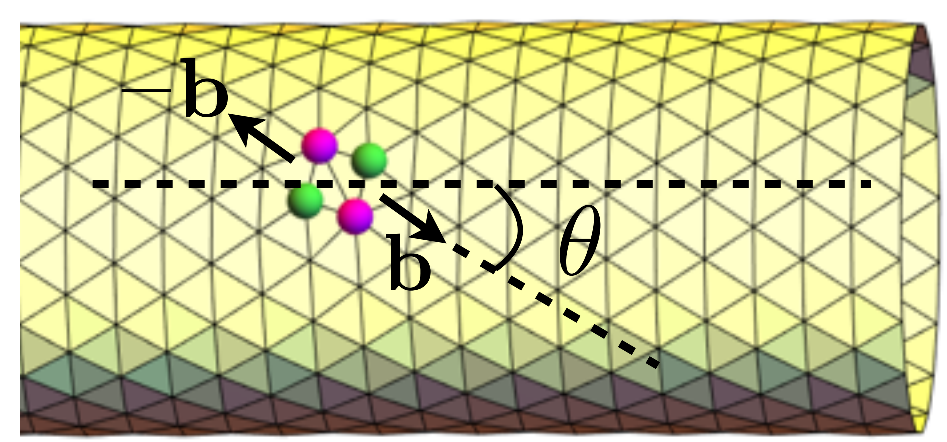

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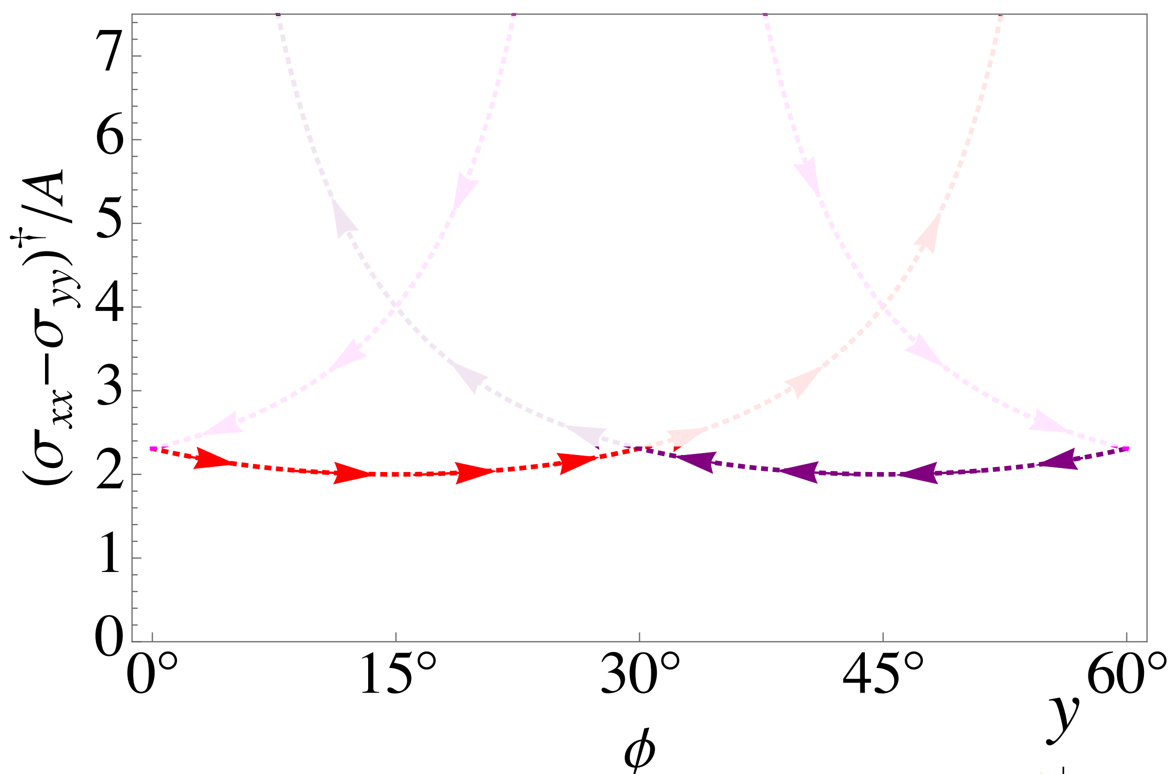

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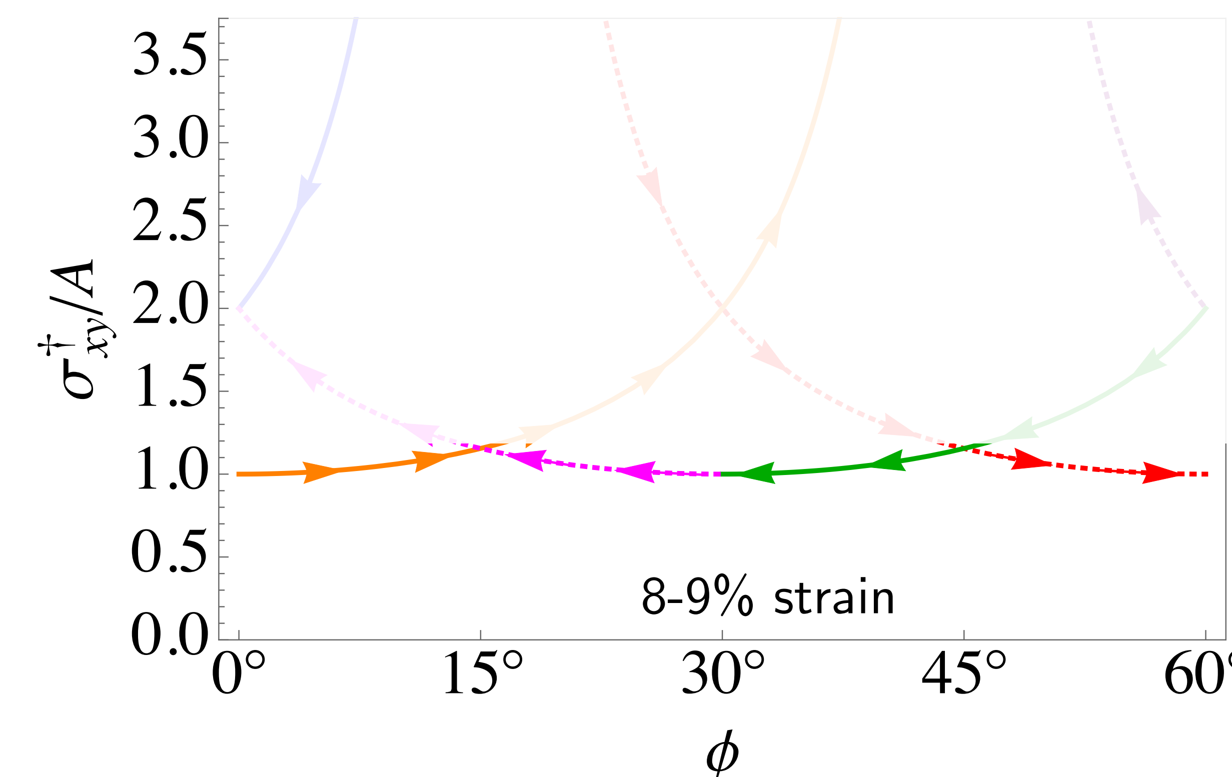

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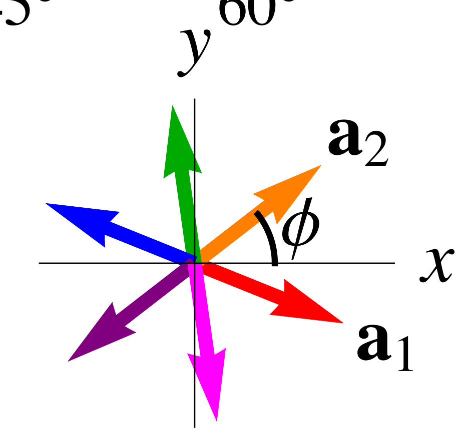

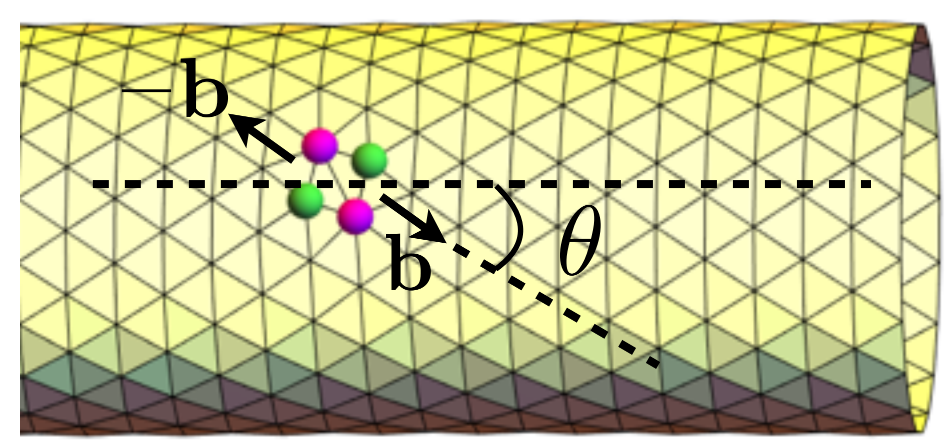

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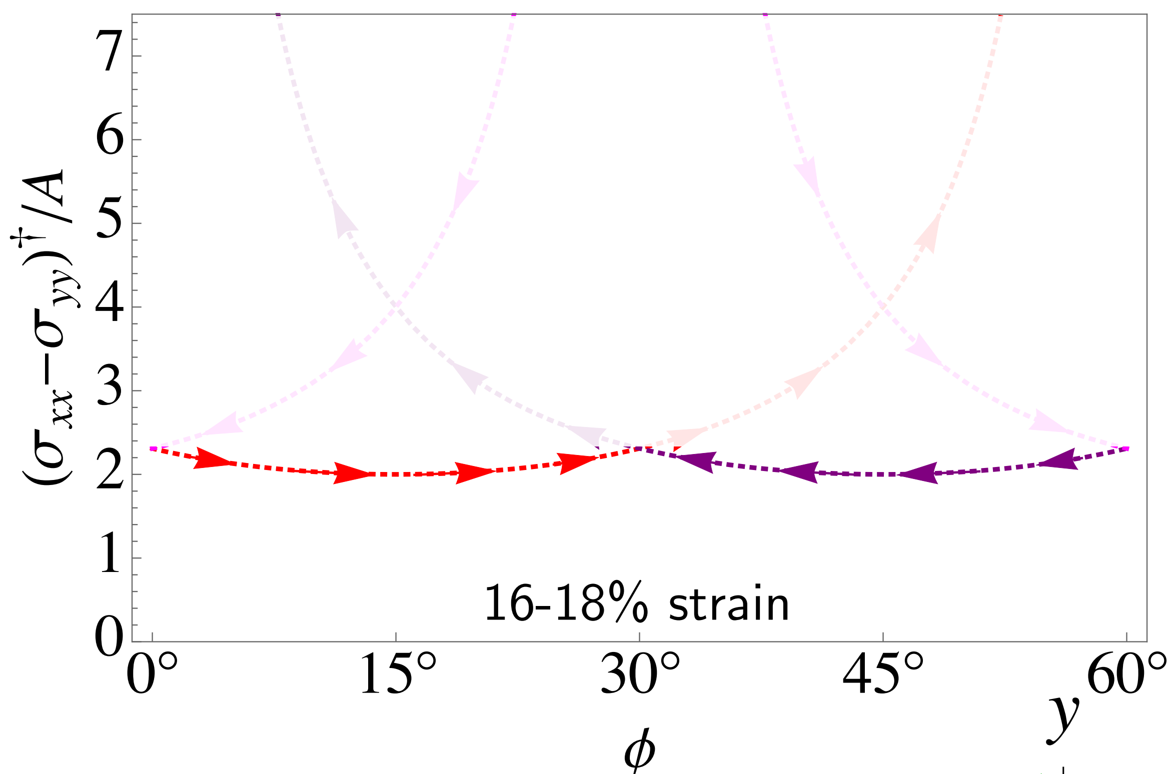

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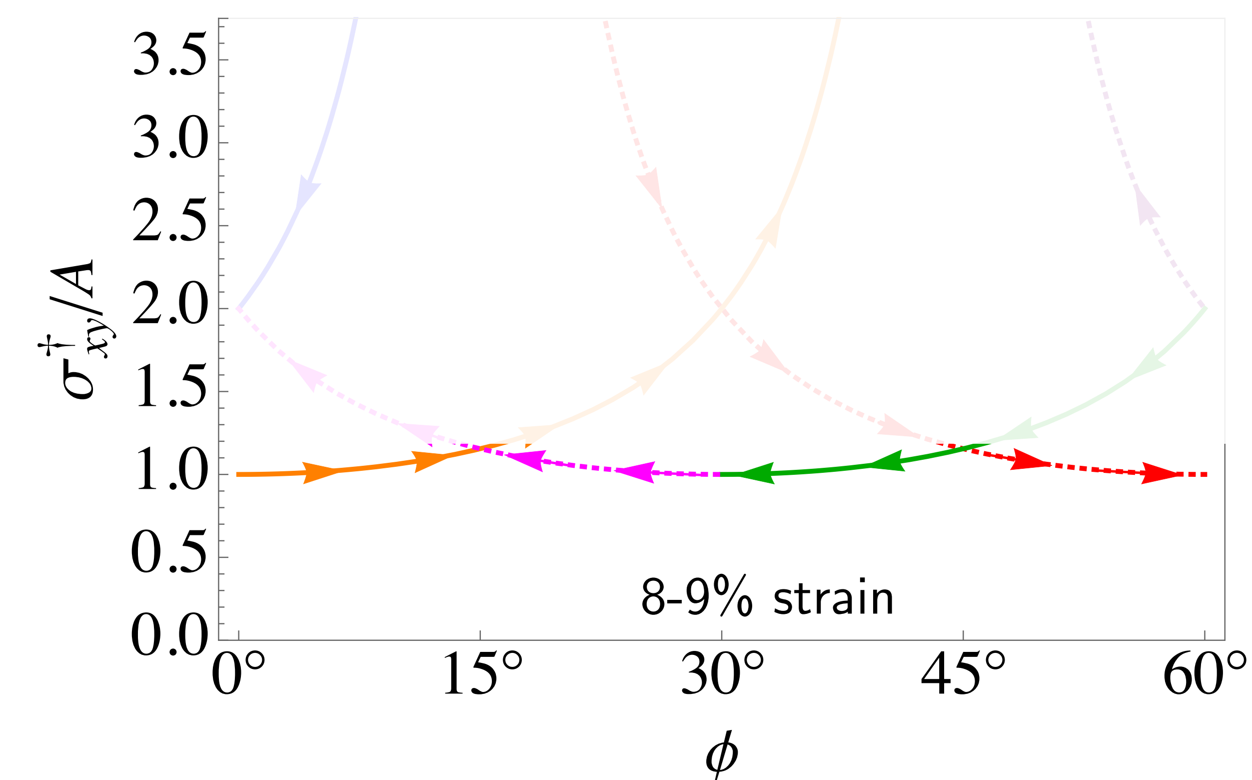

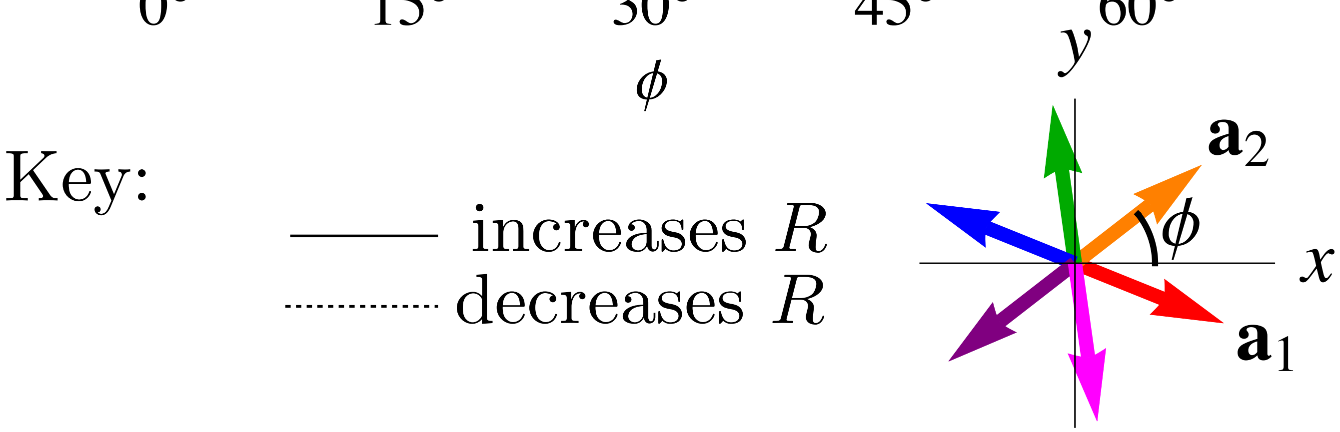

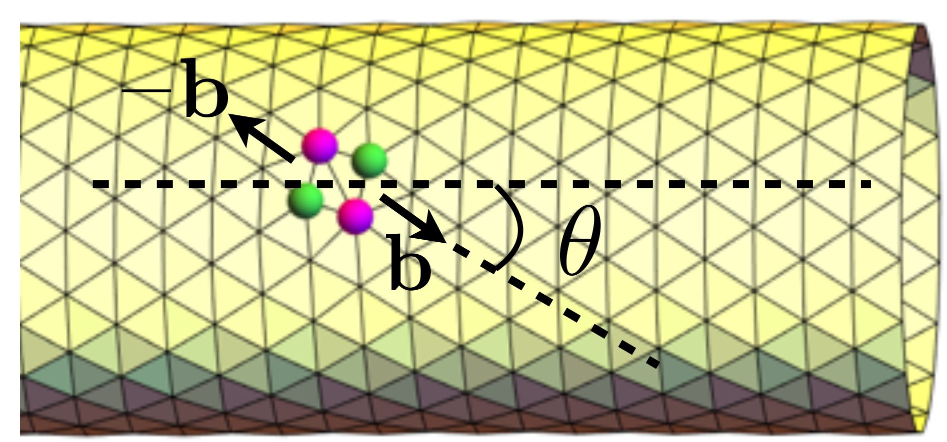

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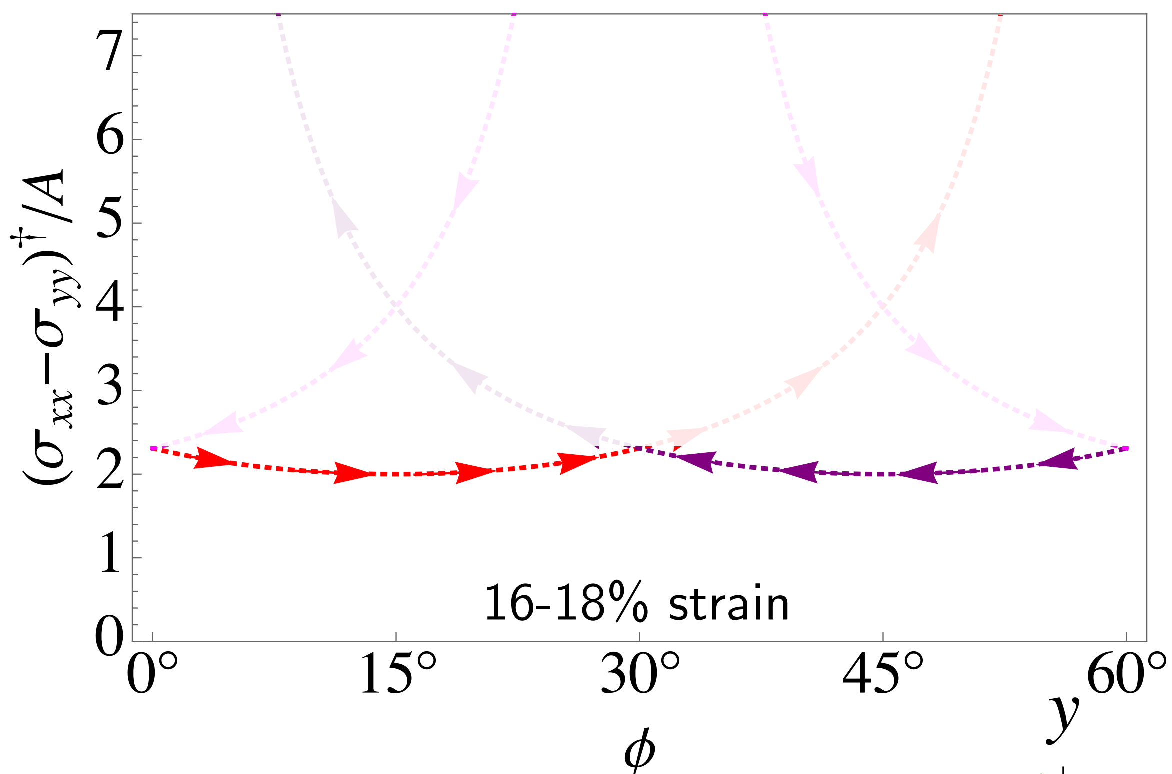

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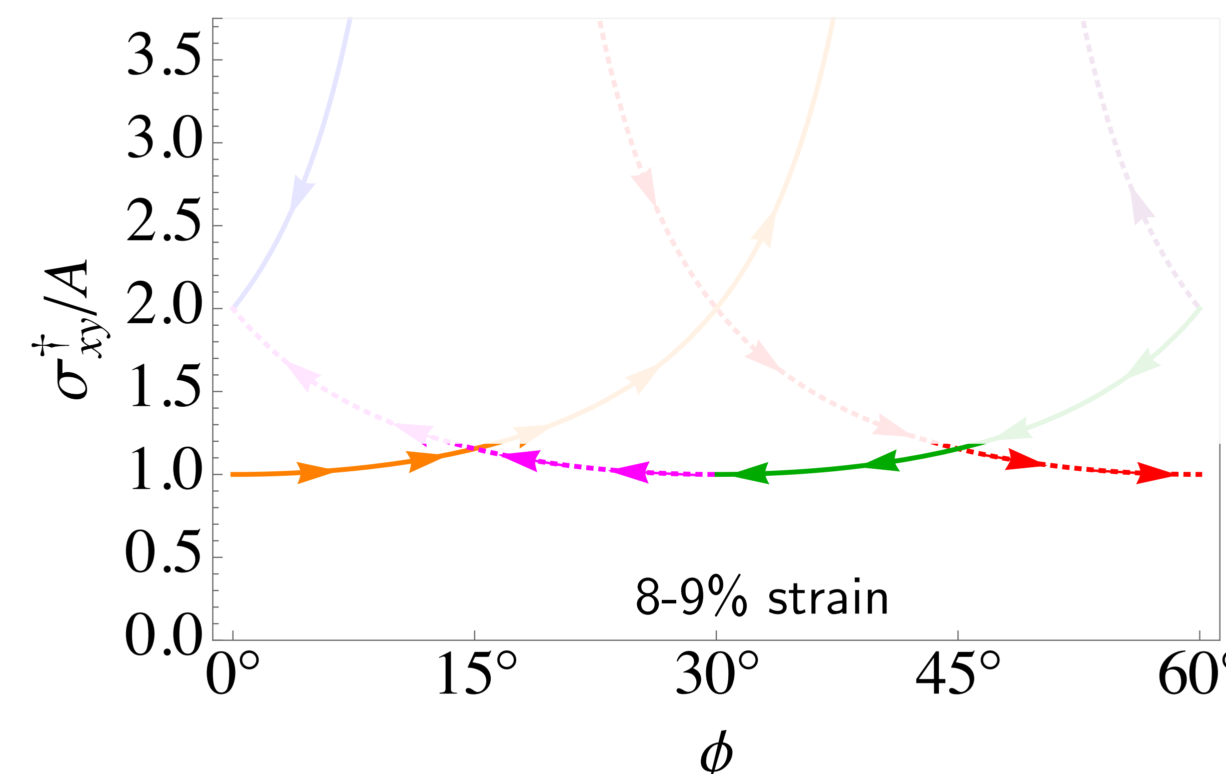

Key:

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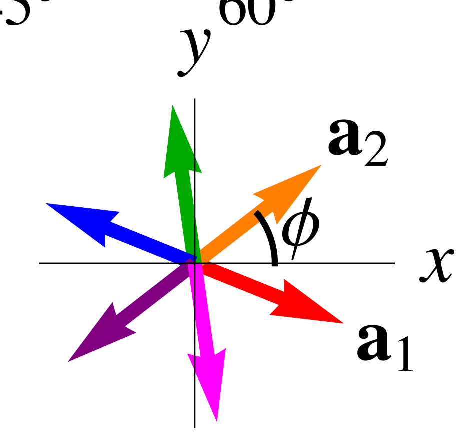

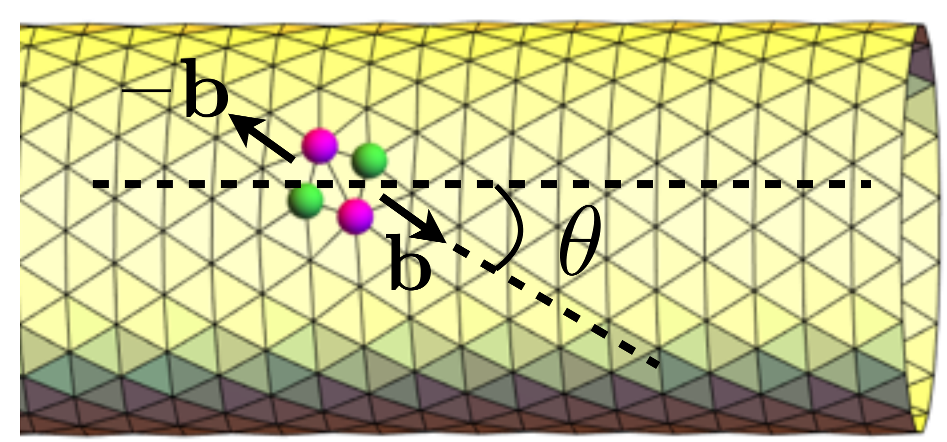

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So far we have only been considering planar dislocation interactions. How does being on a cylinder change defect interactions?

The bending energy

- Helfrich free energy for a bent membrane

$$E_b = \int d\mathbf{x} \left[\tfrac{1}{2}\kappa (H(\mathbf{x}))^2 + \bar{\kappa} K(\mathbf{x}) \right]$$

- Infinite cylinders/periodic B.C.'s $\Rightarrow \int d\mathbf{x} K(\mathbf{x}) = 0$.

- For a perfect cylinder, $H = 1/R$ so bending energy per unit length is $\boxed{E_b/L = \pi\kappa/R}$.

Mean curvature $H = \dfrac{1}{R_1} + \dfrac{1}{R_2}$

Gaussian curvature $K = \dfrac{1}{R_1 R_2}$

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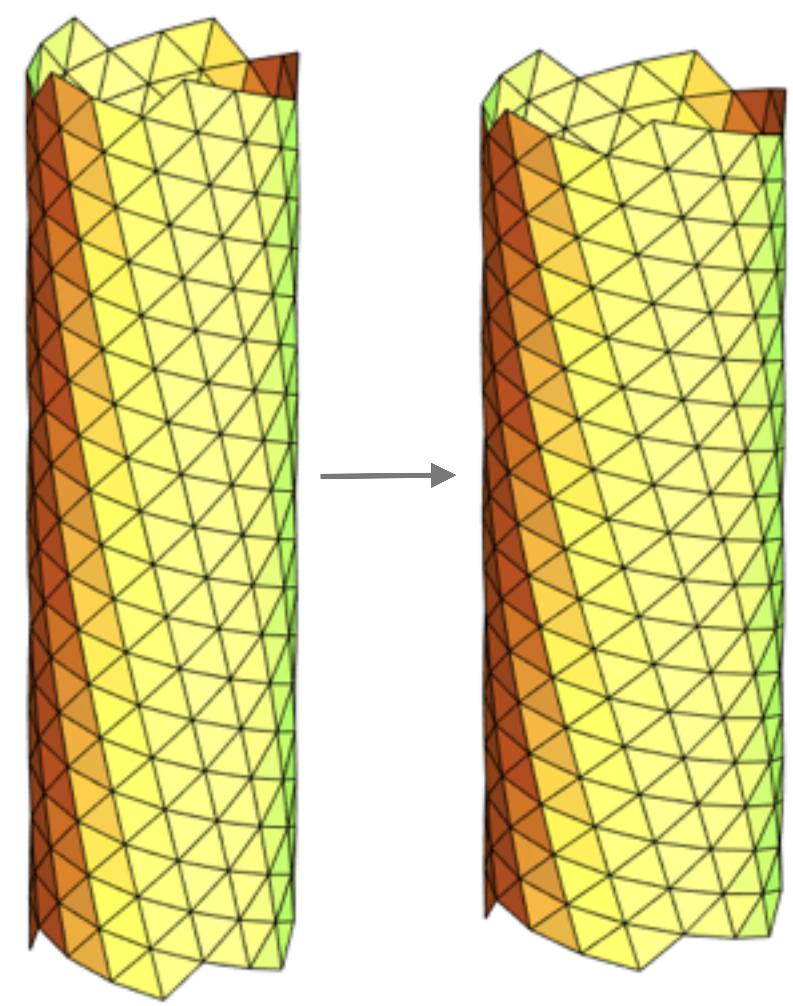

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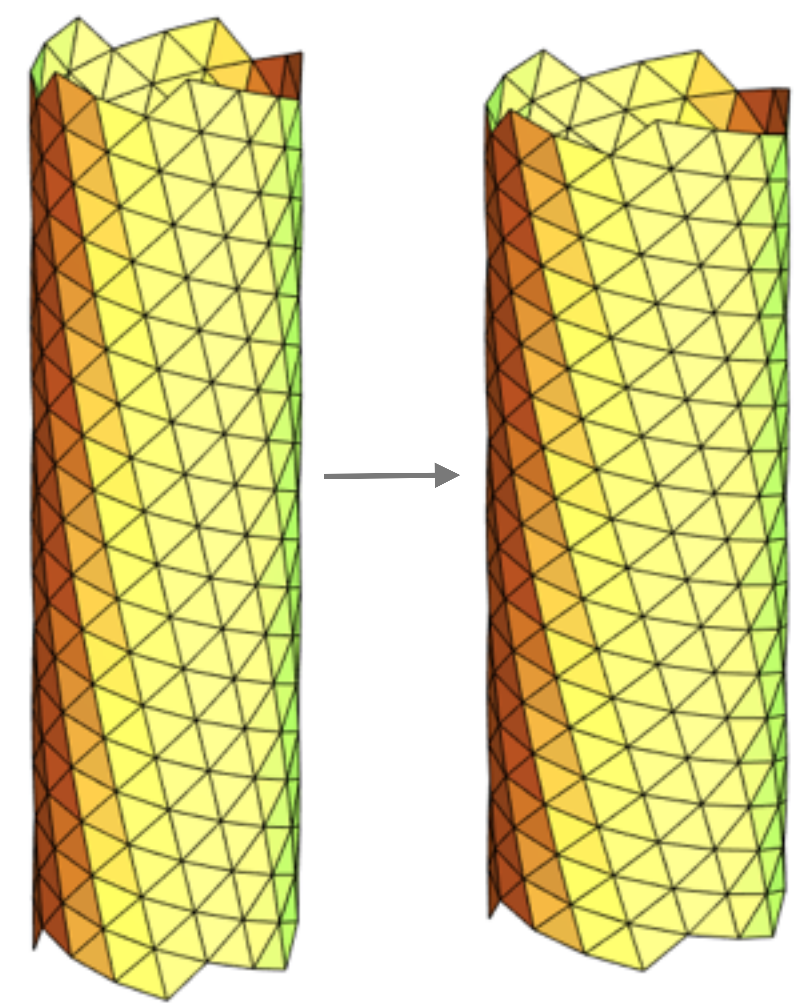
- How important is bending energy E_b compared to stretching energy E_s?
- Dimesionless ratio: the **Föppl-van Kármán number**

$$\boxed{\gamma \equiv \frac{YR^2}{\kappa}}$$

- For large γ, bending is easier than stretching.
- *E.g.* For single-walled carbon nanotubes, $\gamma \sim 10^2 - 10^3$.



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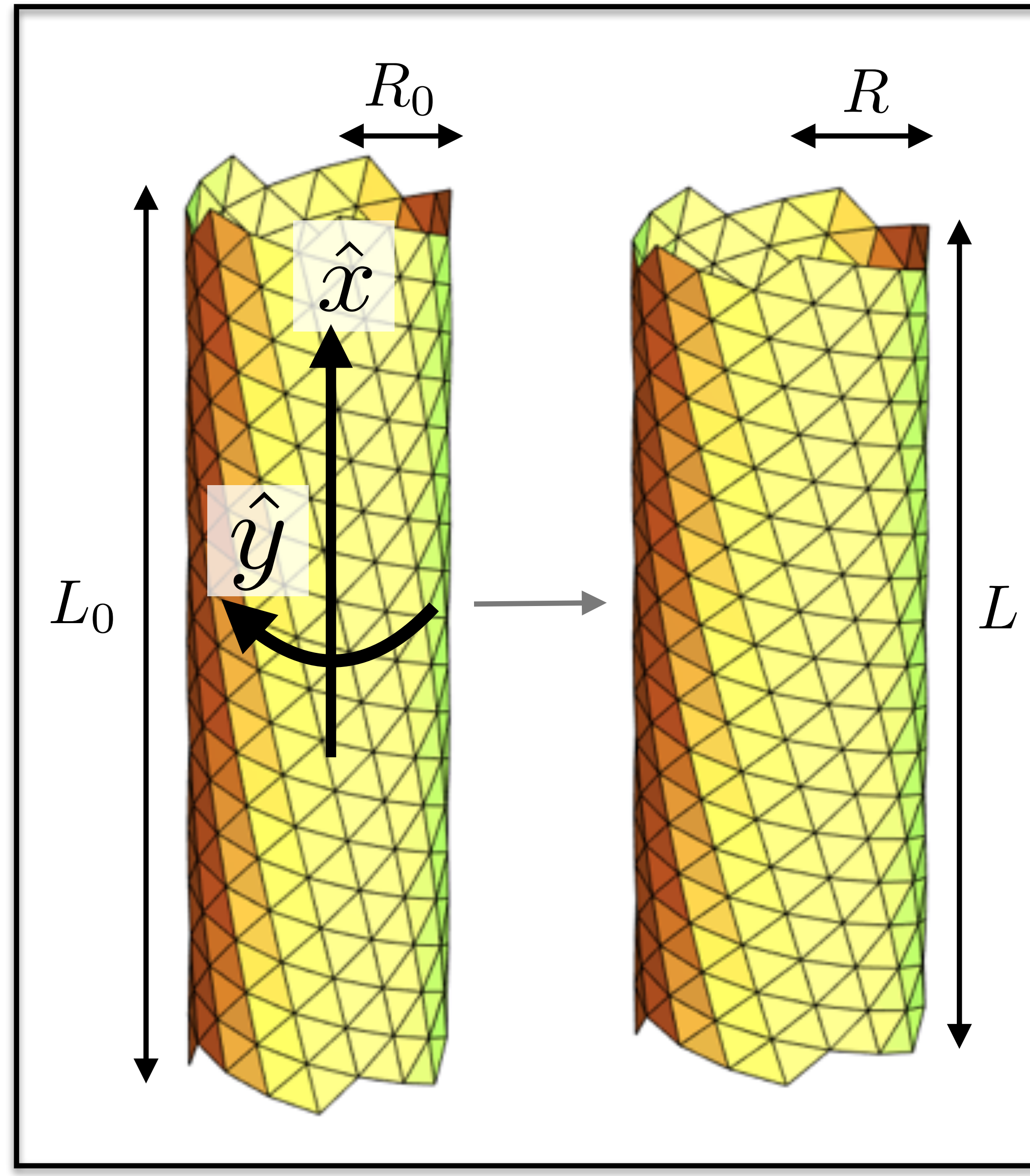
The bending energy as a perturbation

Föppl-van Kármán number $\gamma \equiv \dfrac{Y R^2}{\kappa} \gg 1$



The bending energy as a perturbation

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- Radius preferred by stretching energy:

$$\boxed{R_0 \approx \frac{a}{2\pi}\sqrt{m^2+n^2-mn}}$$

- With stretching *and* bending energies,

$$L_0 \to L = L_0(1+u_{xx})$$
$$R_0 \to R = R_0(1+u_{yy})$$

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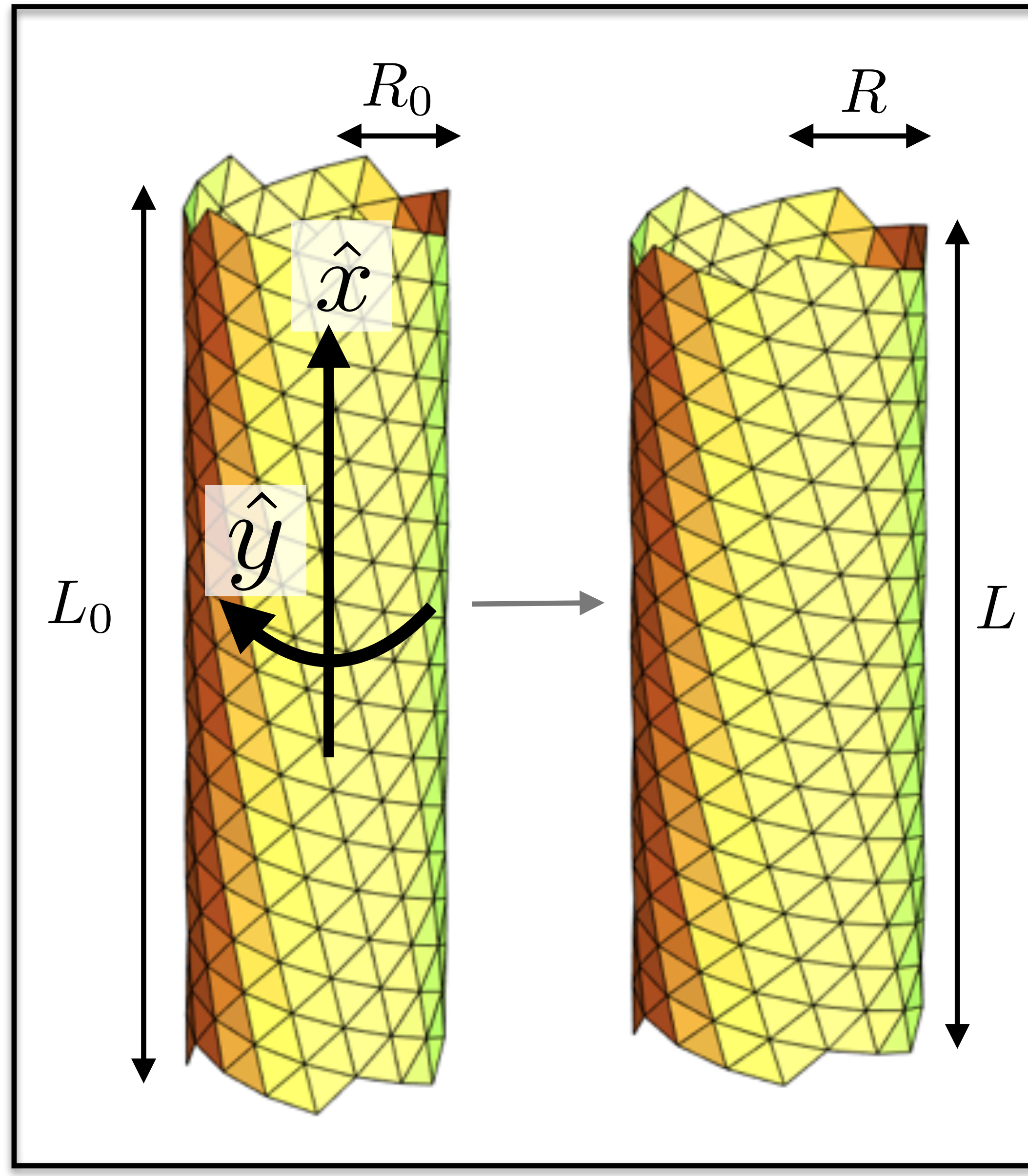

The bending energy as a perturbation

$$\boxed{\text{Föppl-van Kármán number } \gamma \equiv \frac{YR^2}{\kappa} \gg 1}$$

- Radius preferred by stretching energy:

$$\boxed{R_0 \approx \frac{a}{2\pi}\sqrt{m^2+n^2-mn}}$$

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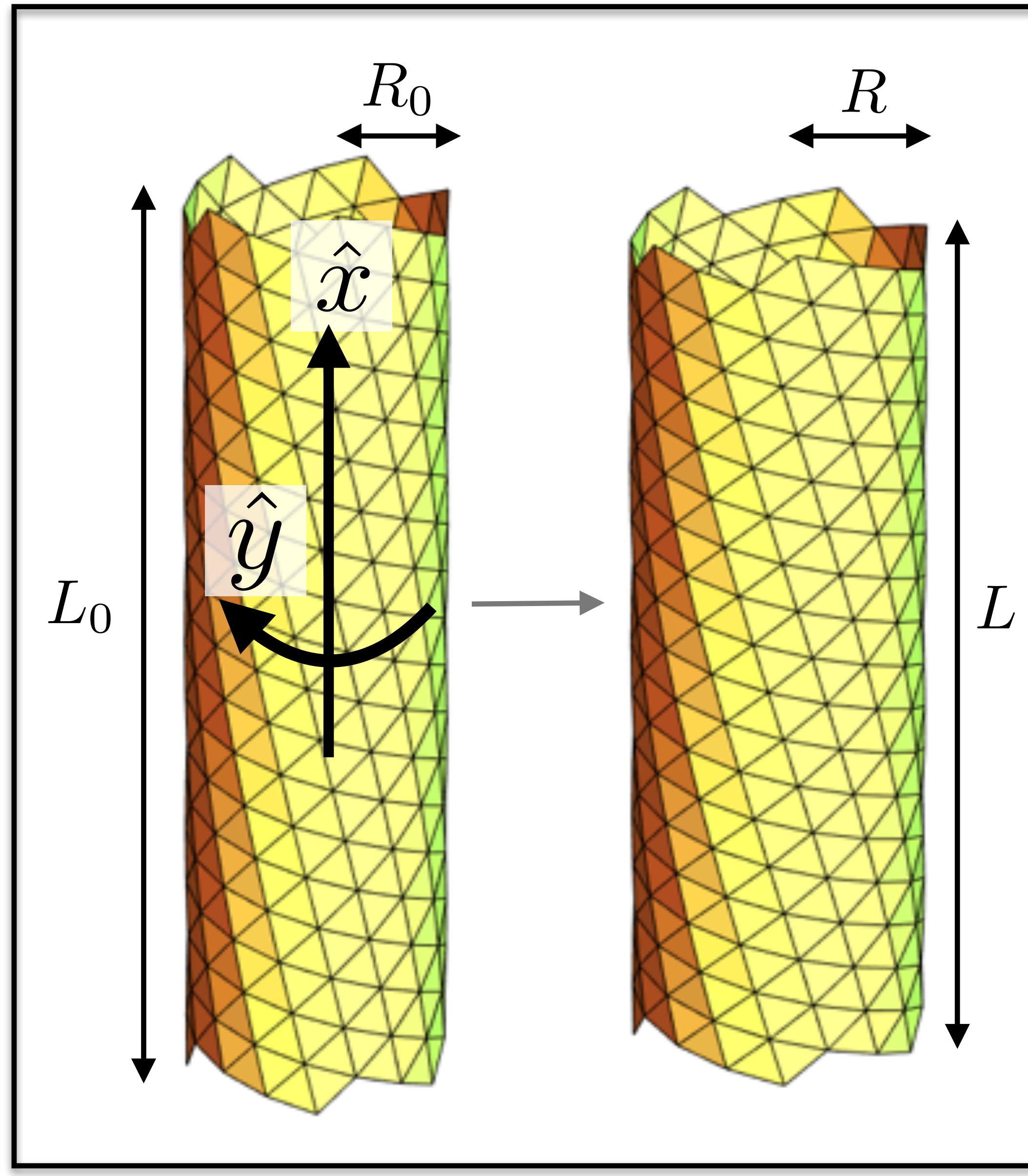

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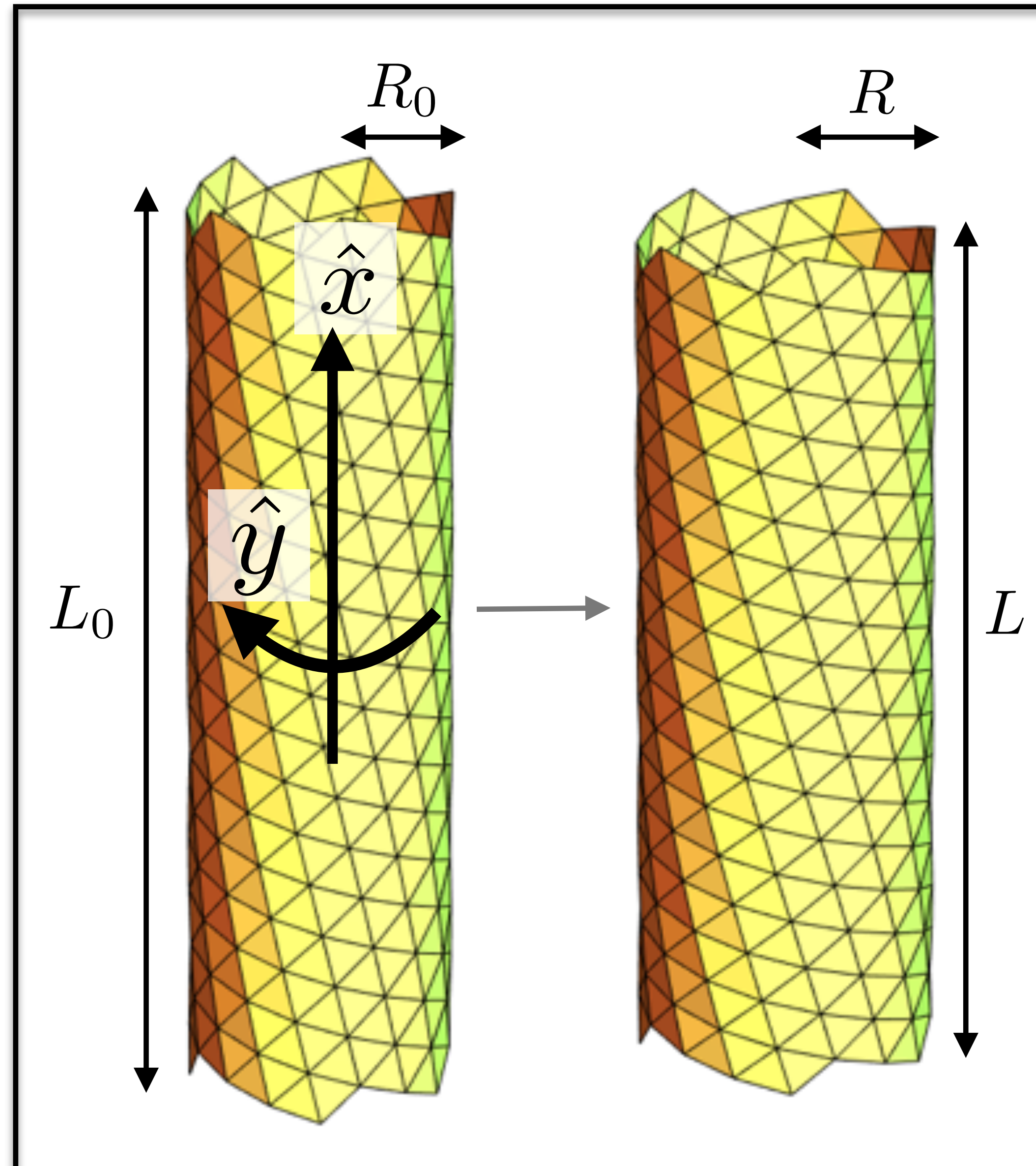

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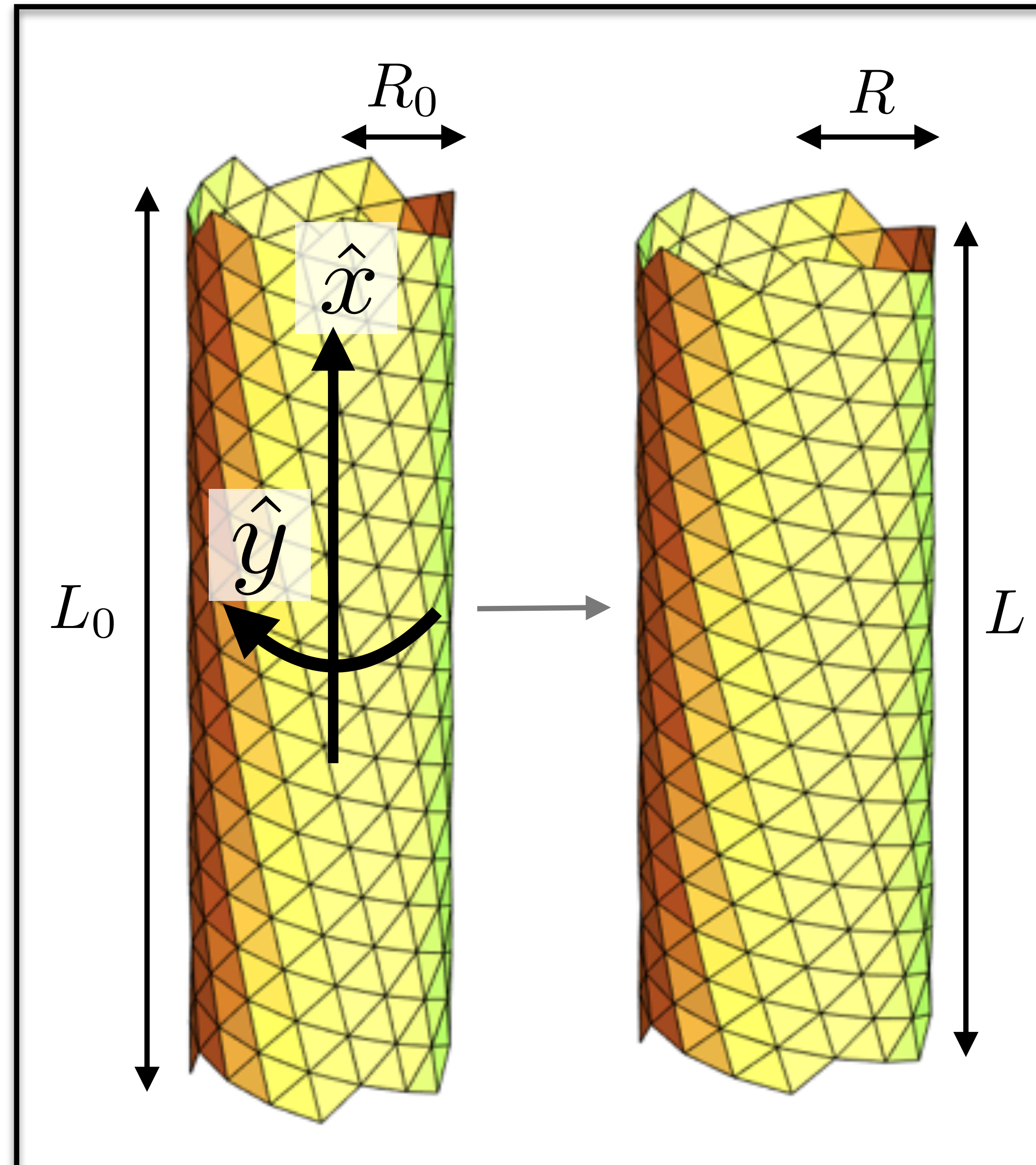

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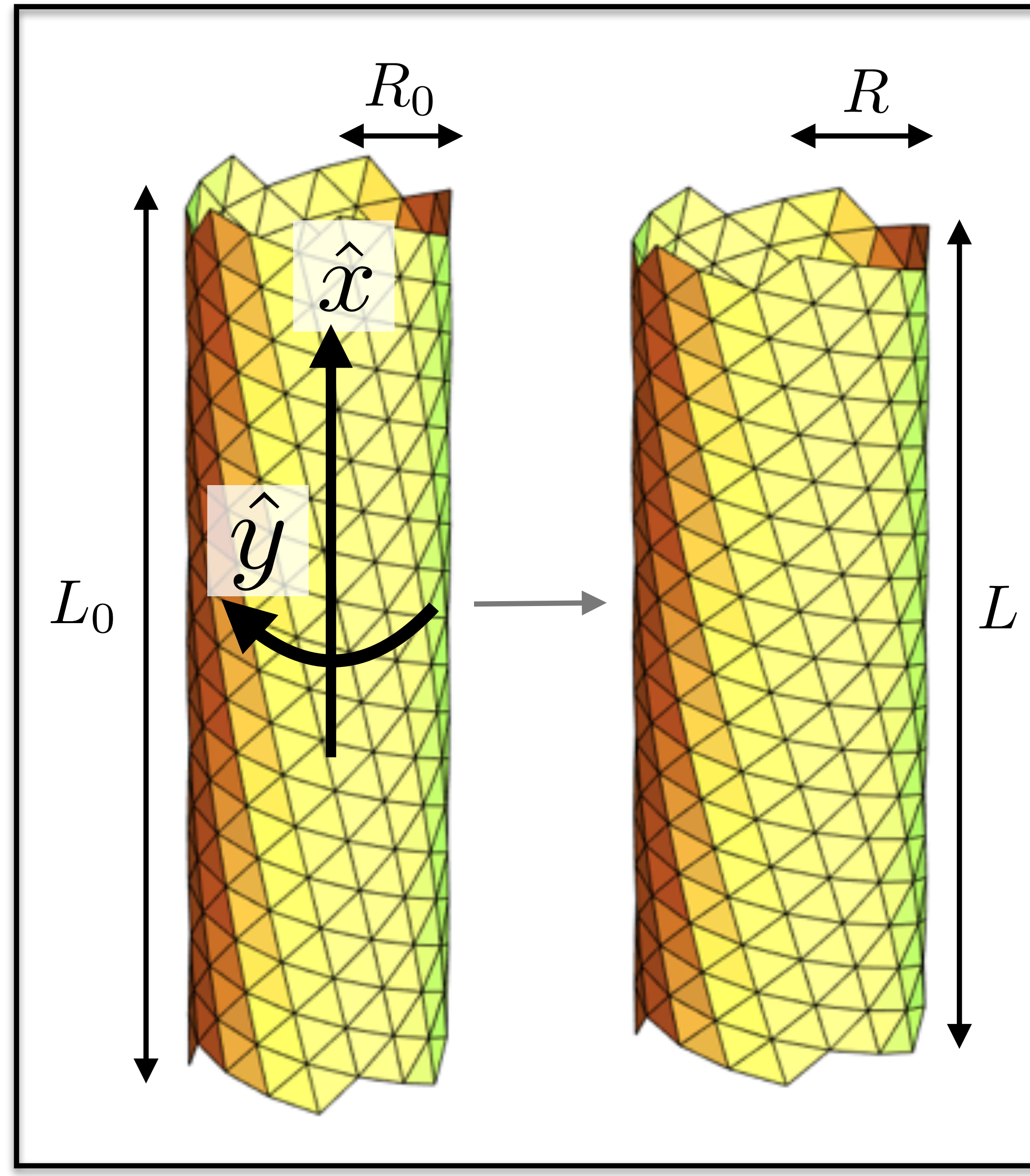

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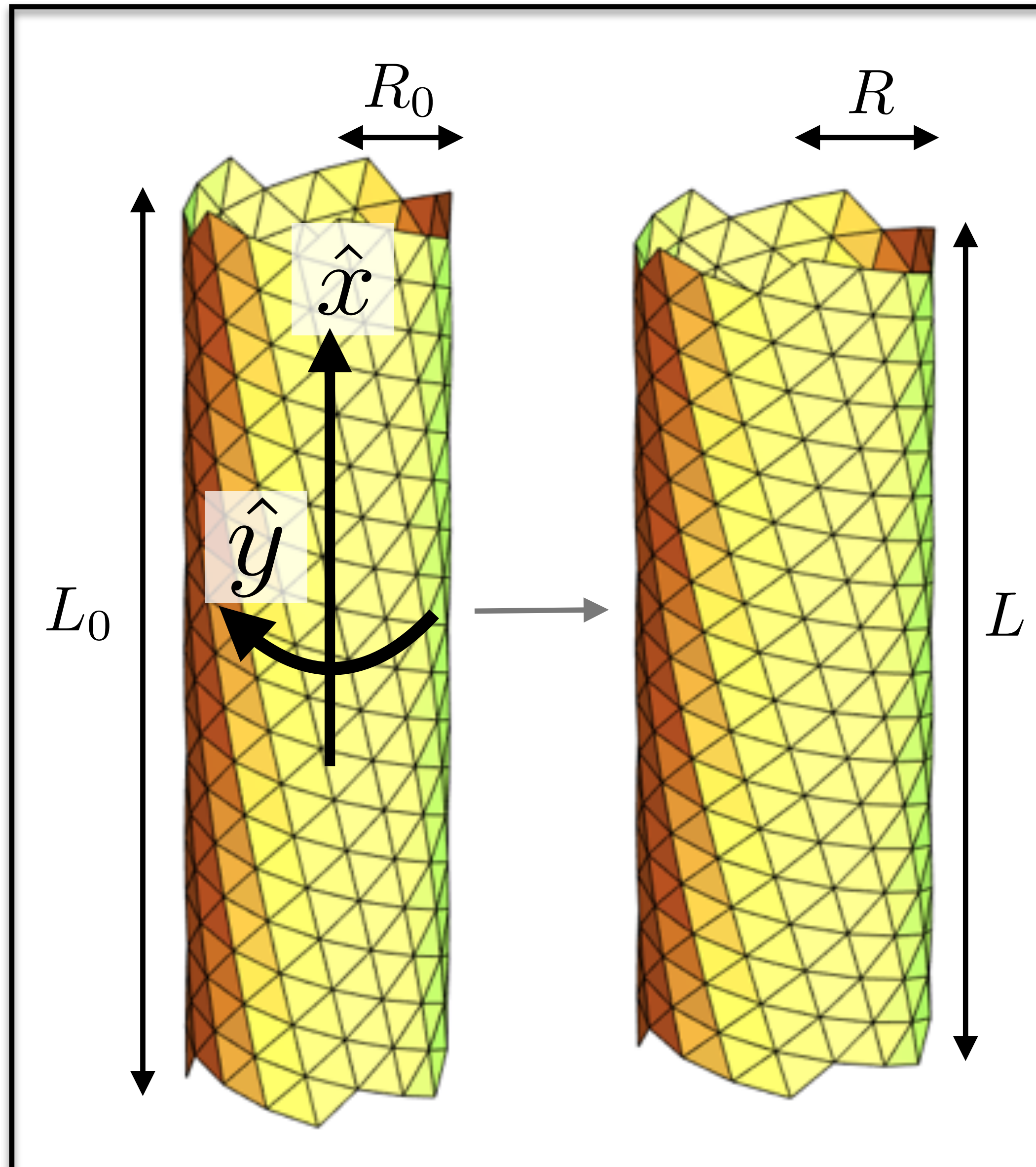

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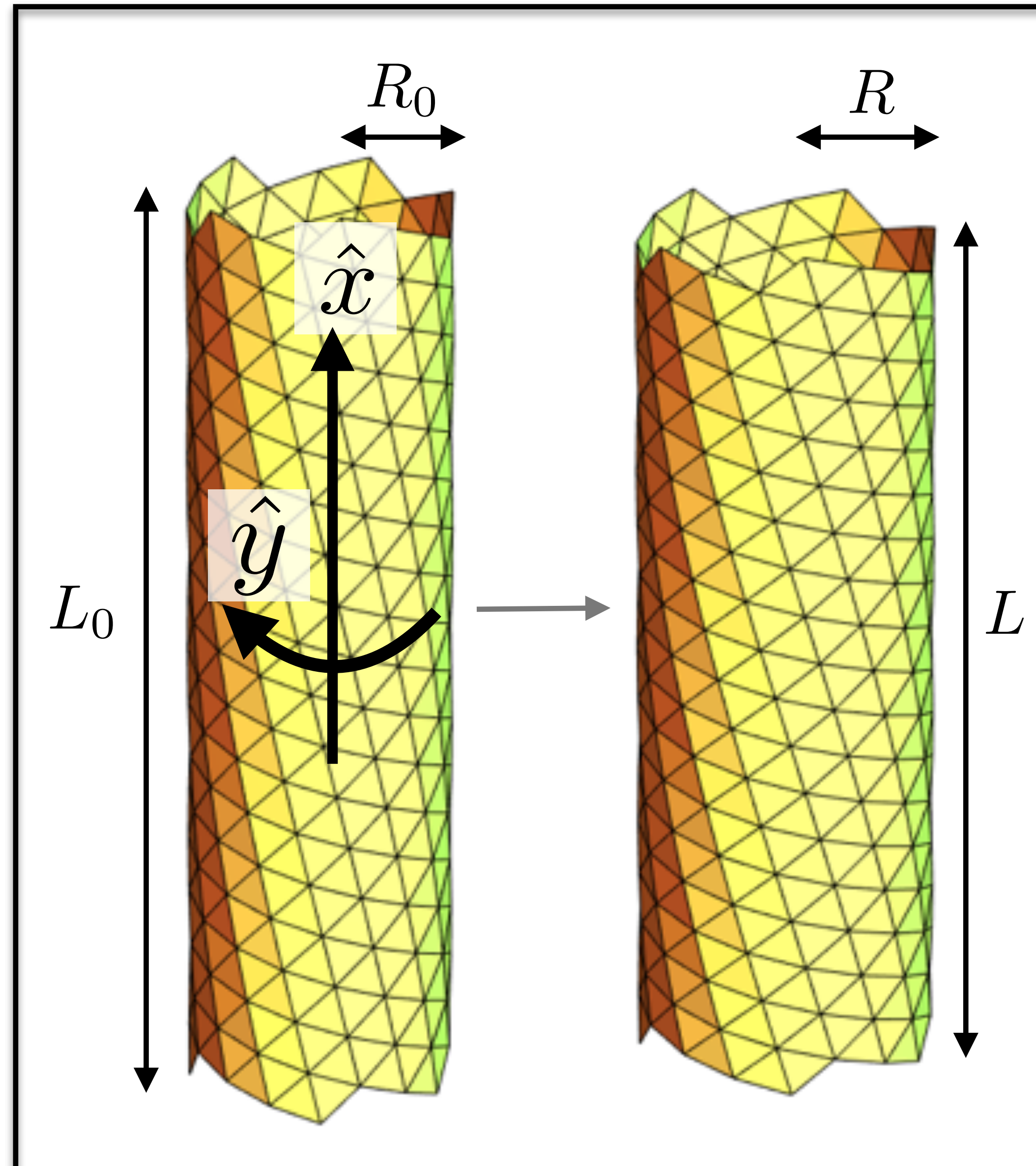

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H^2

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γ^{-1}

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The bending energy as an effective stress

Föppl-van Kármán number $\gamma \equiv \frac{YR^2}{\kappa} \gg 1$

- $u_{yy} = -u_{xx} \approx \frac{2}{3}\gamma^{-1}$
- Bending energy has same effect as an external stress tensor

$$\sigma^b = \begin{pmatrix} \sigma^b_{xx} & \sigma^b_{xy} \\ \sigma^b_{xy} & \sigma^b_{yy} \end{pmatrix} = \frac{1}{2} Y \gamma^{-1} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$E_b = \frac{1}{2}\kappa \int d\mathbf{x} H^2 \to -\int d\mathbf{x} \sigma^b_{ij} u_{ij}$$

- Therefore, the *effective* critical tensile stress contains a simple "curvature offset",

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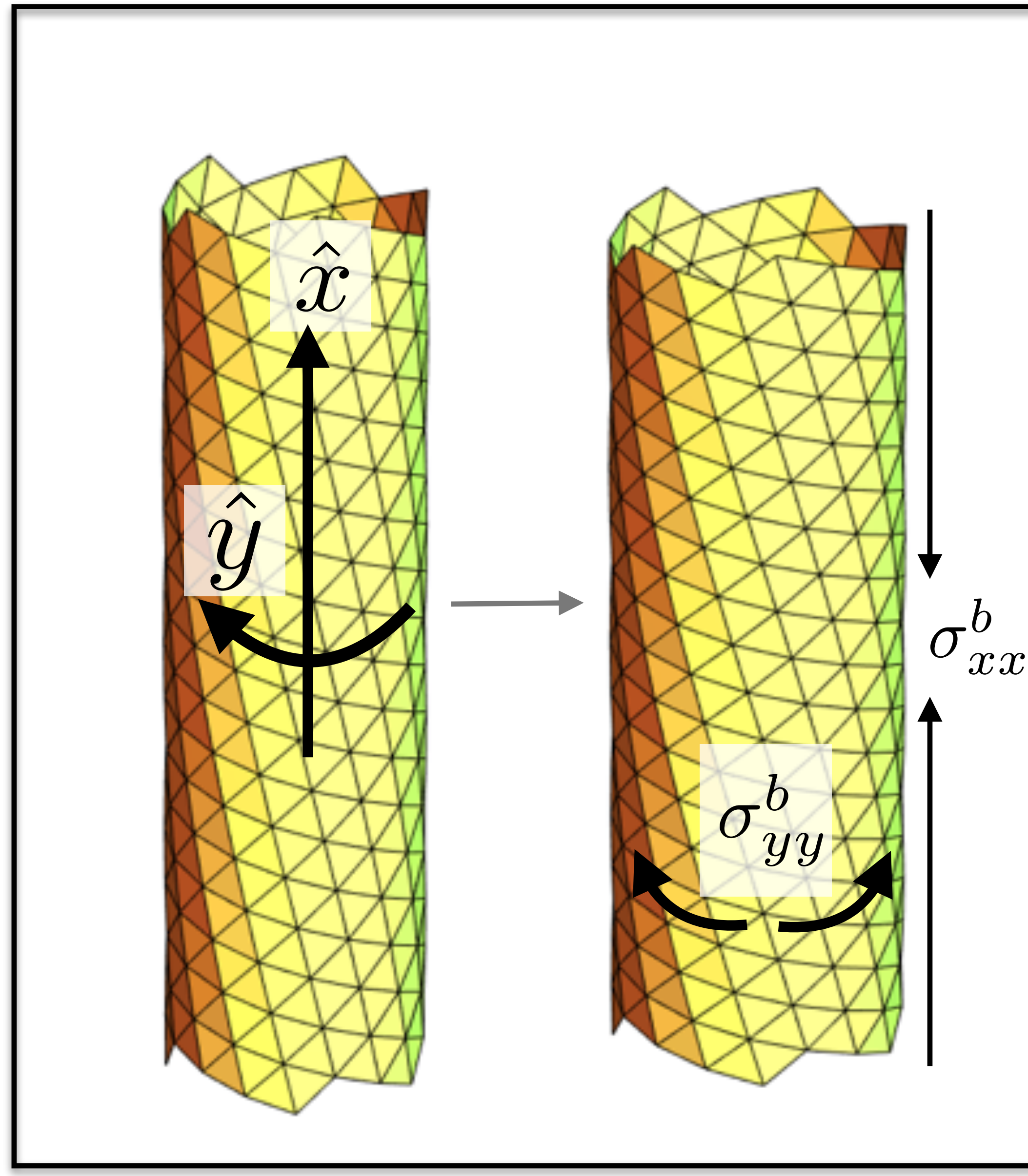

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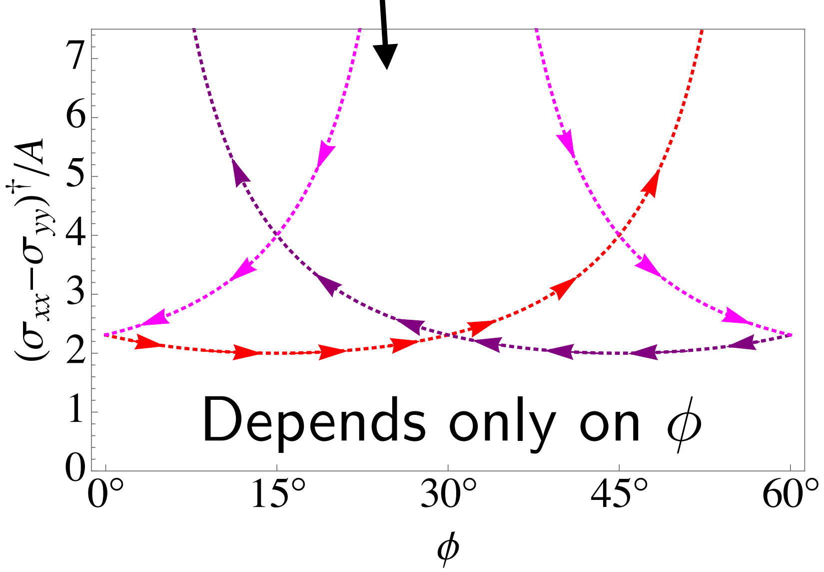

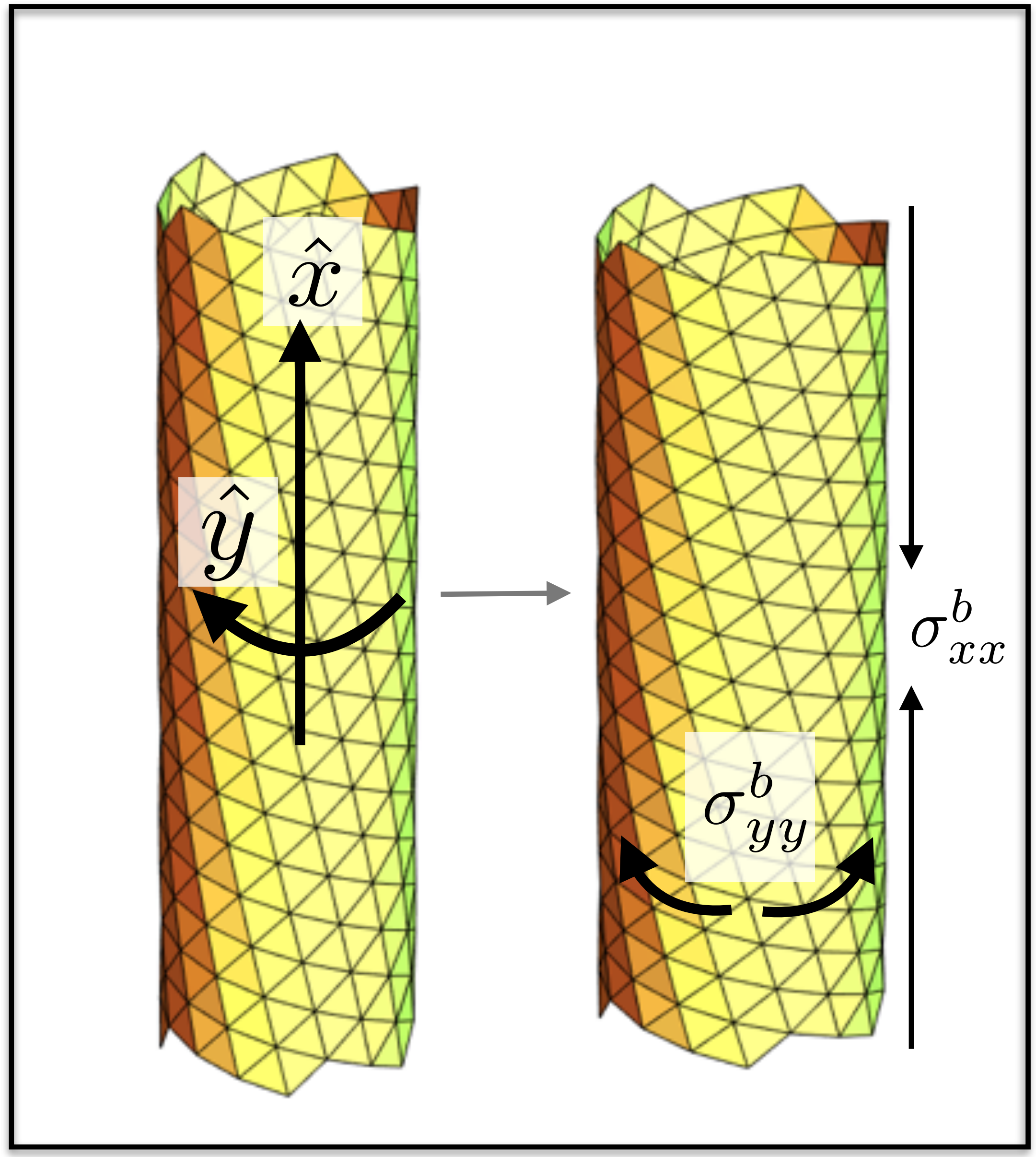

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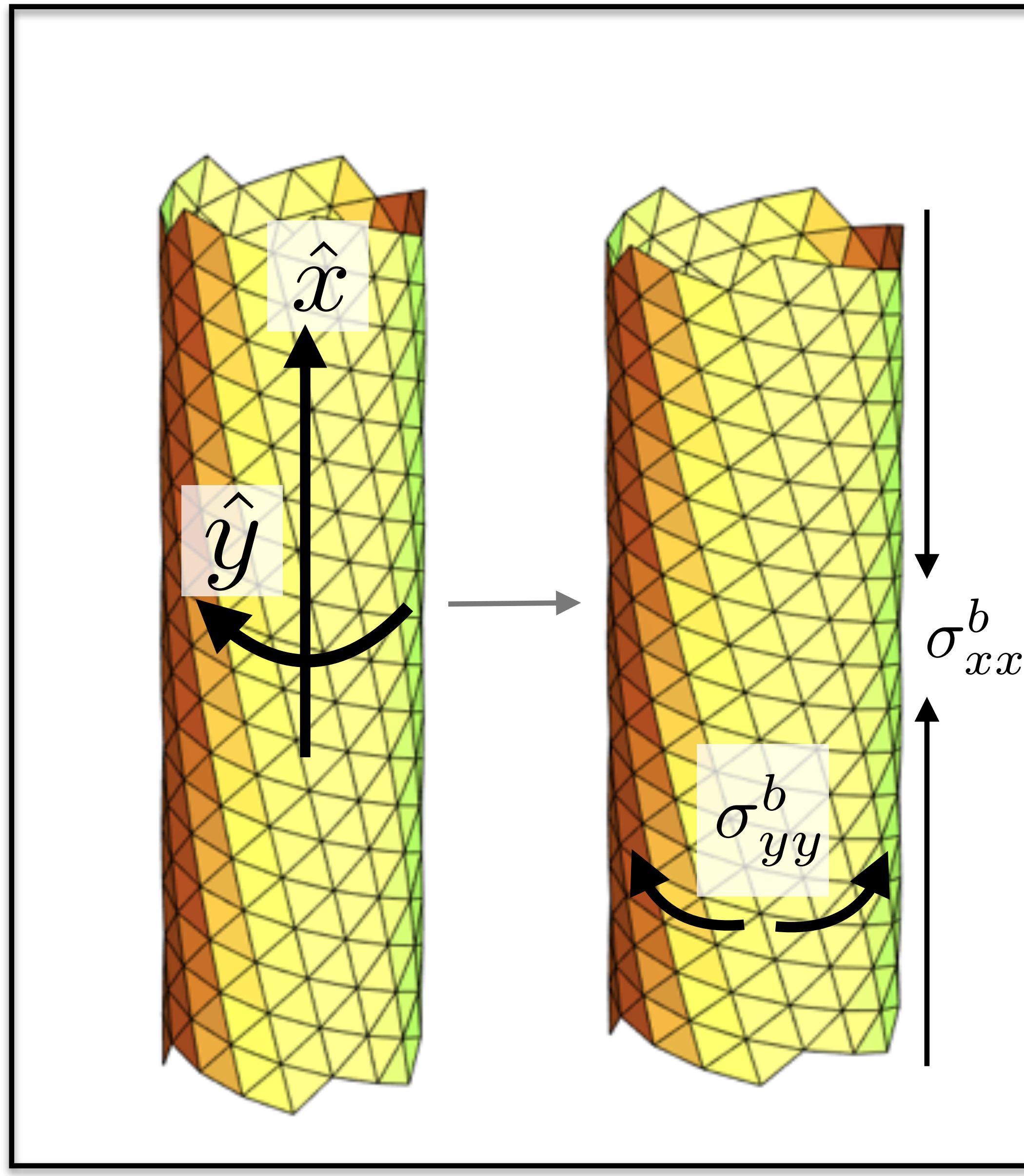

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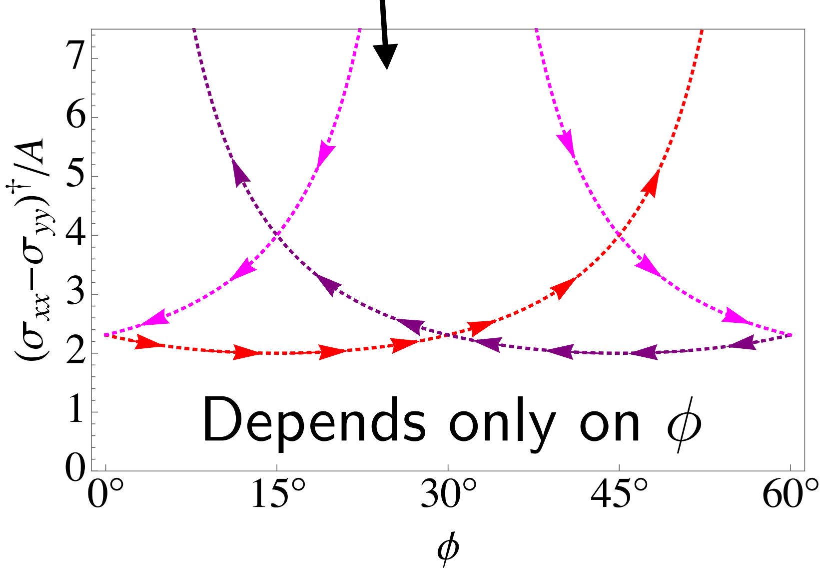
Depends only on R



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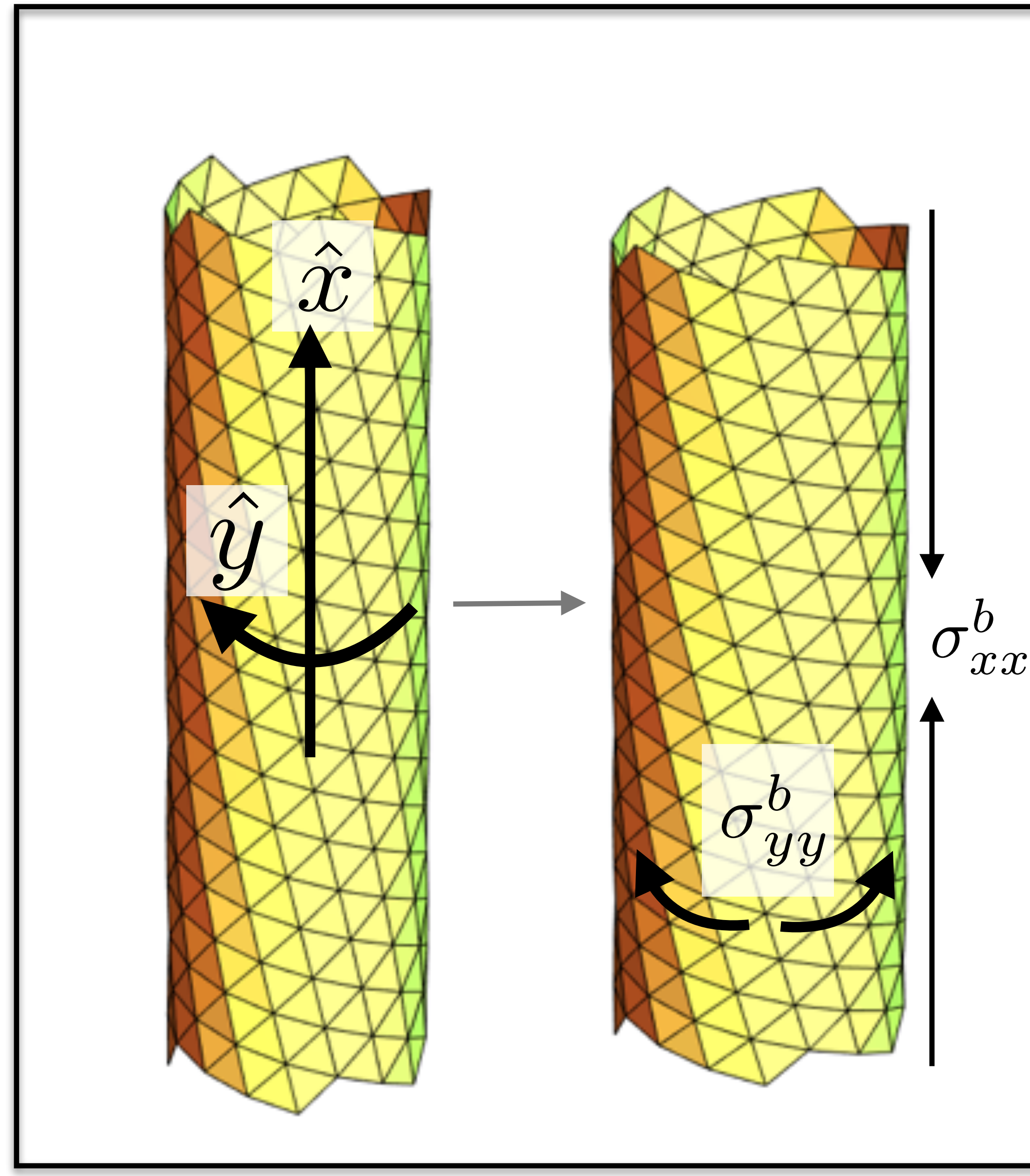

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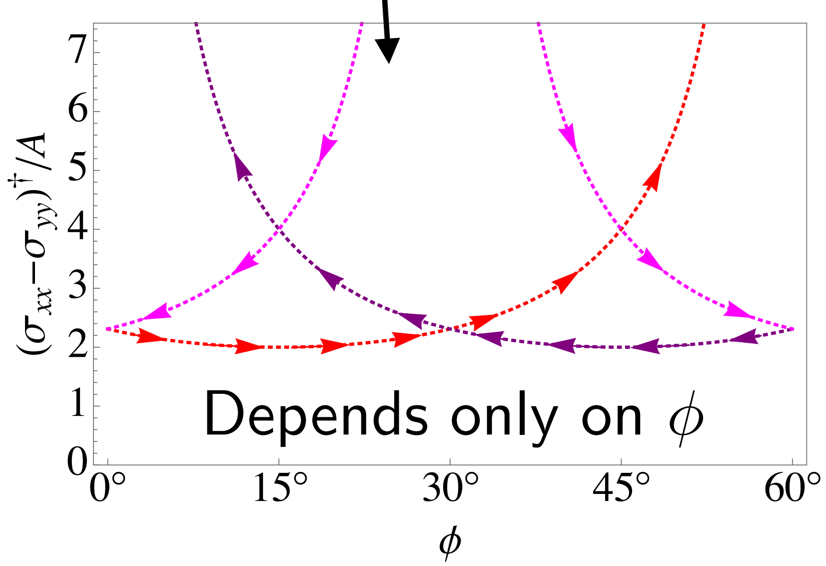


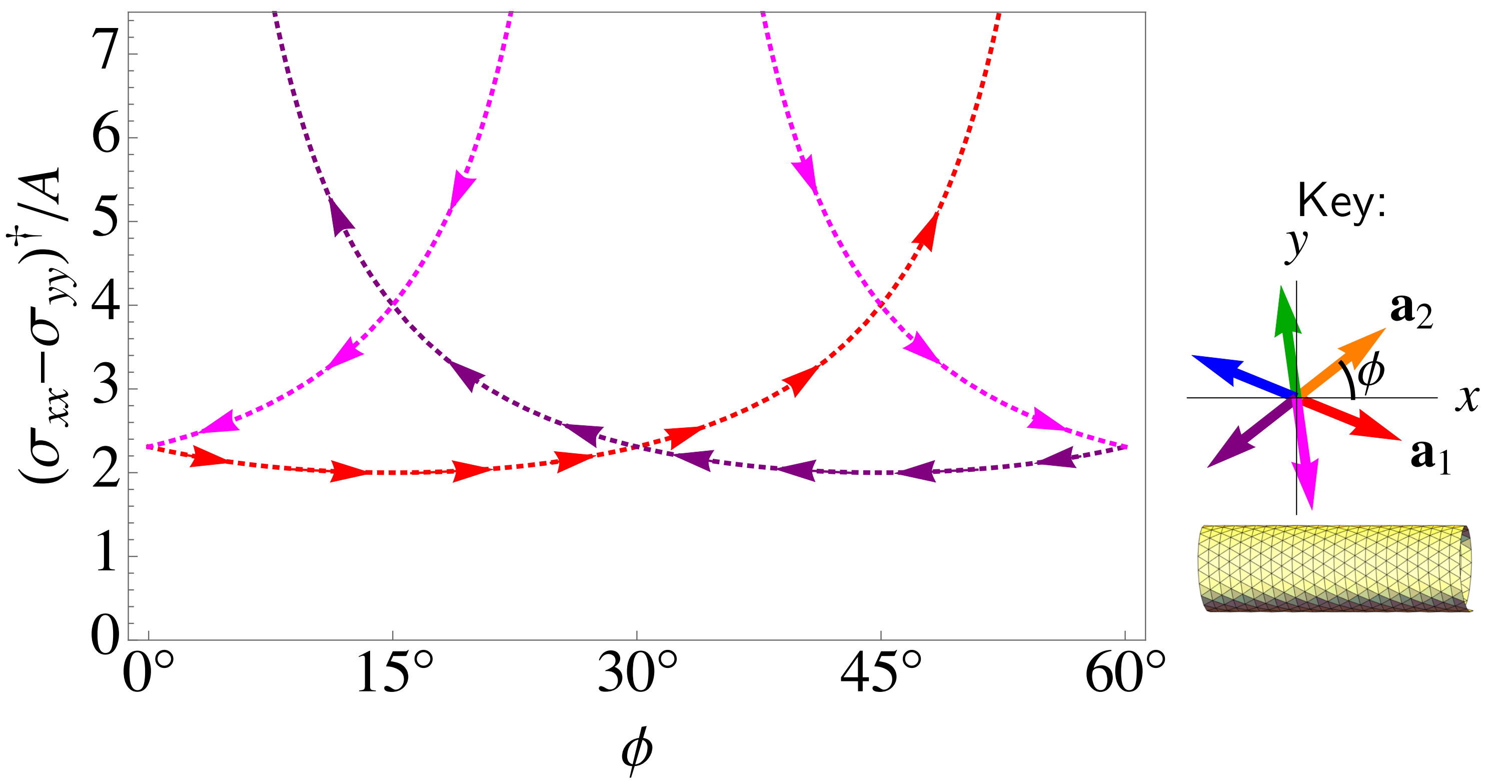
- Bending energy opposes plastic deformations that decrease R.
- Larger $\kappa \Rightarrow$ larger $\gamma^{-1} \Rightarrow$ greater stress required to unbind dislocations.

Bending energy may make very narrow tubes unstable

- $(\sigma_{xx} - \sigma_{yy})^{\dagger\text{eff}} = (\sigma_{xx} - \sigma_{yy})^{\dagger} - Y\gamma^{-1}$
- What happens when $Y\gamma^{-1} > \sigma_c \approx 2A$?
- Then, with zero external stress, it is energetically favorable to unbind dislocation pairs that *widen* the tube.
- Tubes are unstable if

$$R < \boxed{R_c = \sqrt{\kappa/\sigma_c(\phi)}}$$

- (Need $\tilde{\kappa} \equiv \kappa Y/a^2 \gtrsim 0.2$ in order for $R < R_c$ to be geometrically possible.)



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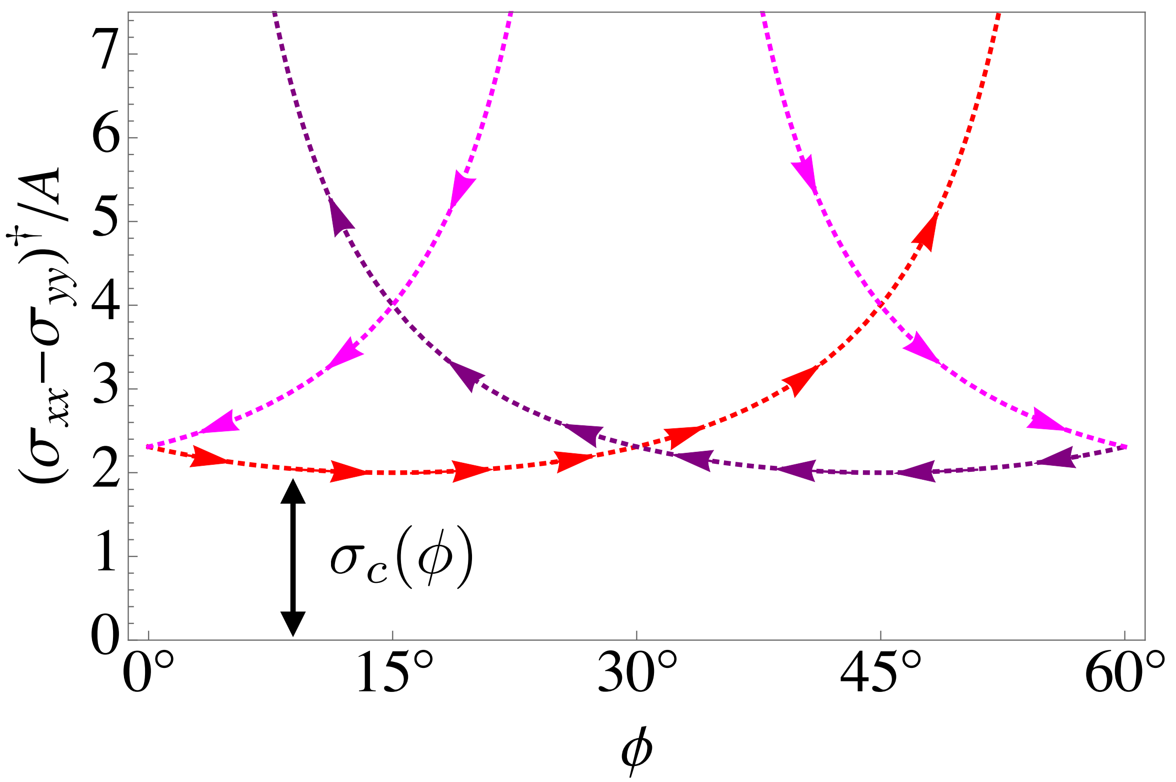

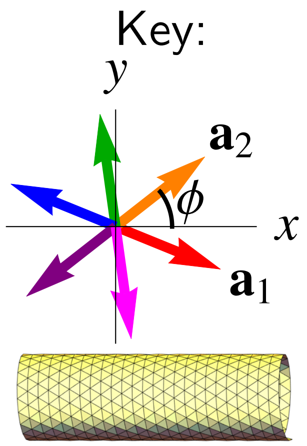

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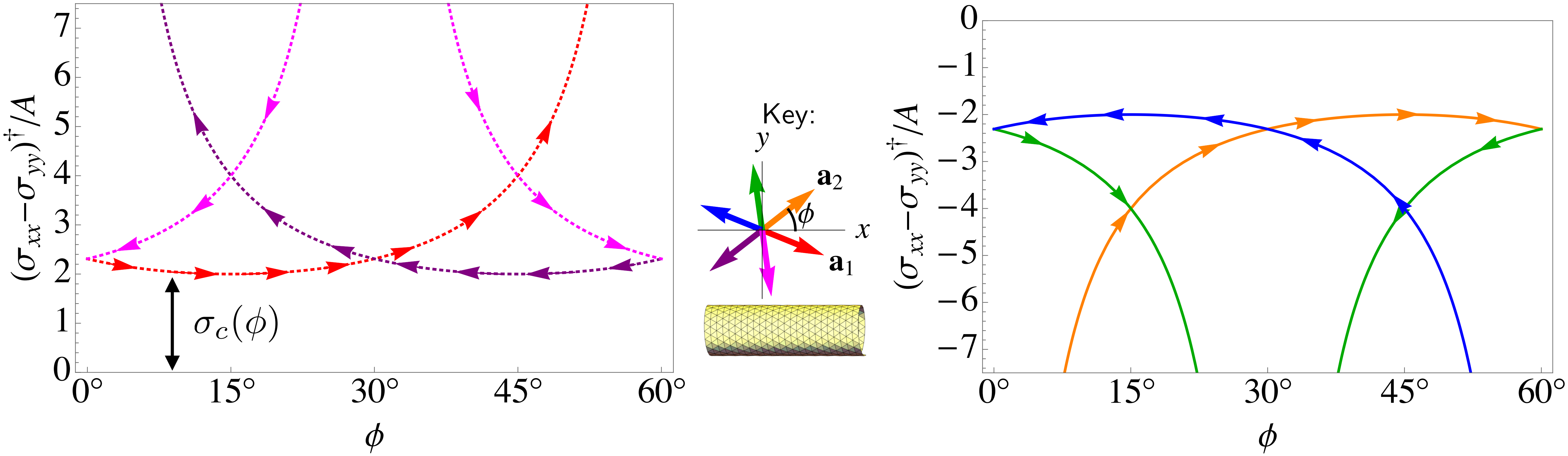

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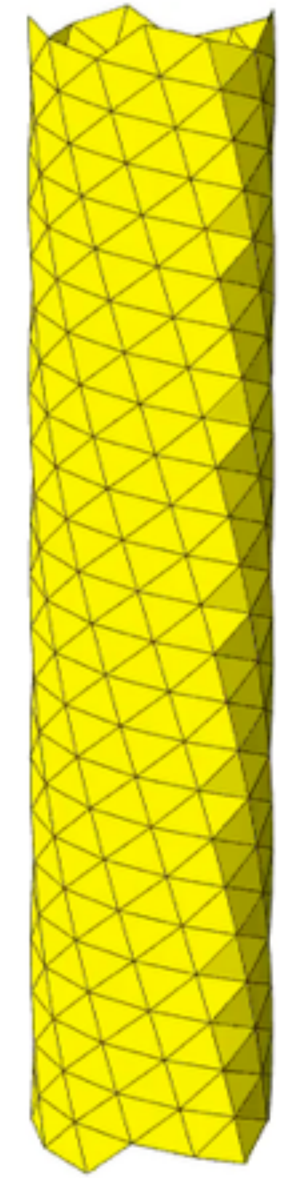

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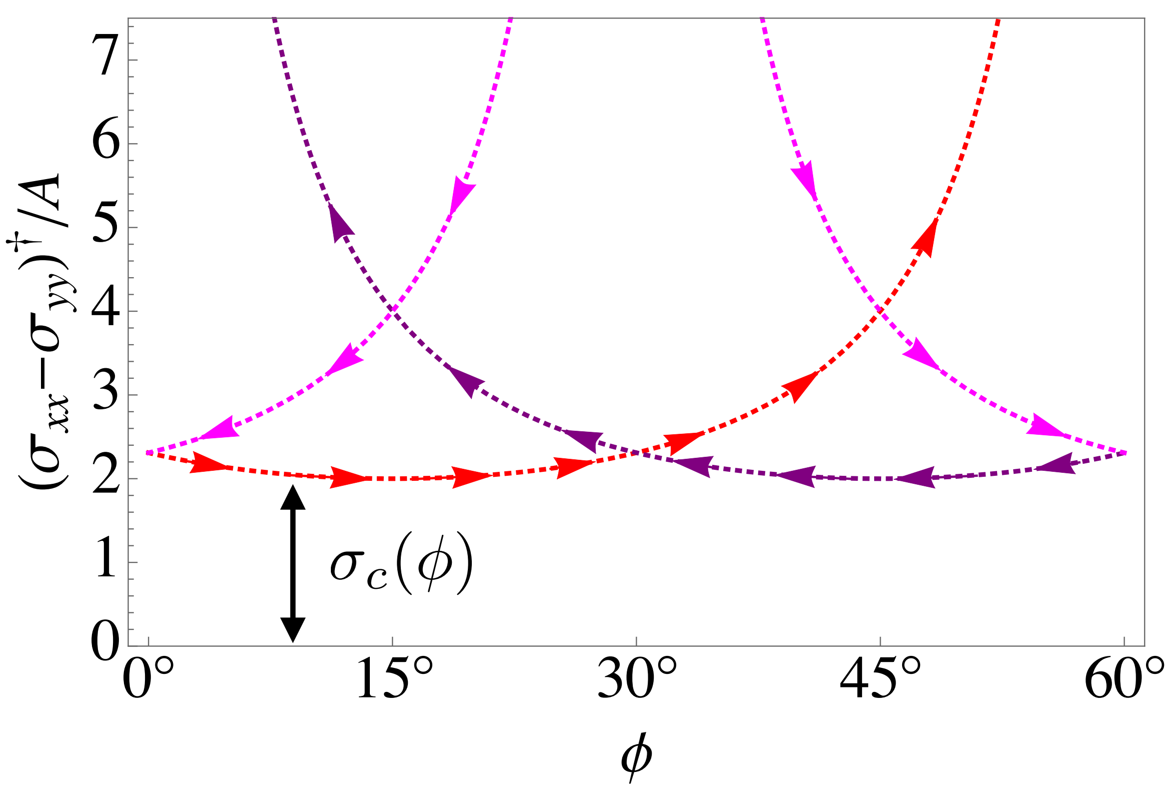
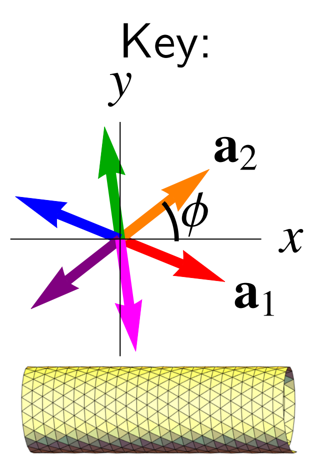
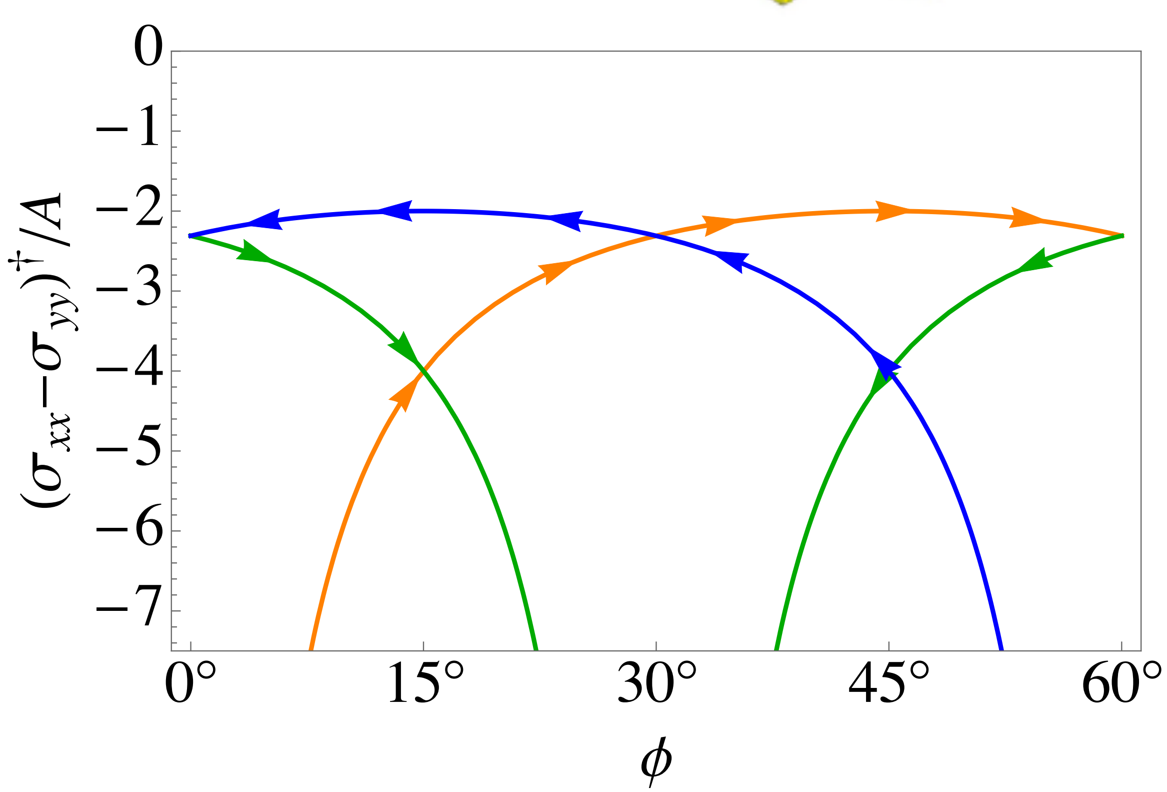


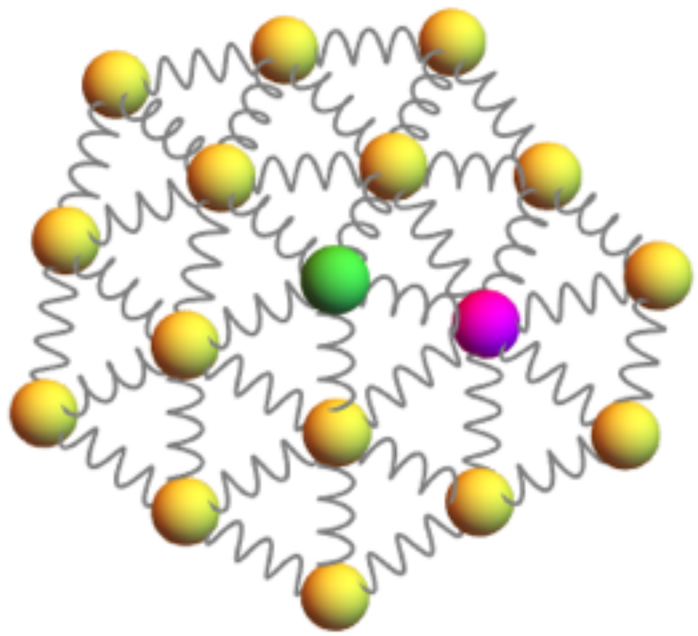




Plastic deformation of tubular crystals

- Background: Phyllotactic geometry of tubular crystals
- Mechanics of plastic deformation: Analytic predictions
- <u>Numerical modeling</u>
- Necks in tubes: Radius profiles near dislocations

Numerical modeling of tubular crystals

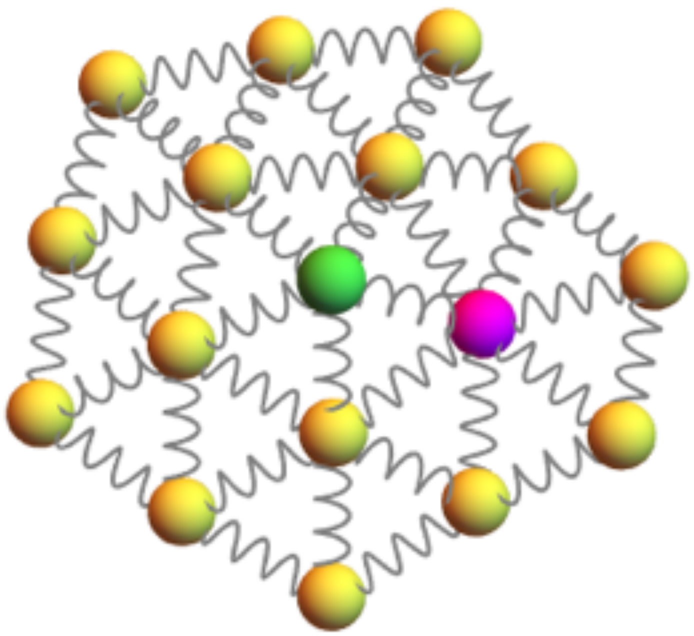


- "Ball and spring model": Nodes connected by harmonic springs*
  - Rest length $a = 1$
  - Spring constant $k = (\sqrt{3}/2)Y$
- Bending energy penalizes mean curvature when neighboring nodes are not coplanar**.
  - Bending rigidity $\tilde{\kappa} = \kappa Y/a^2$

* Seung and Nelson, Phys. Rev. A 38:1005 (1988)
** Gompper and Kroll, J. de Physique I, 6:1305 (1996)

- Periodic boundary conditions along the cylinder axis:
  - No end effects for dislocations
  - Zero total Gaussian curvature
- Reconnect right end to left end by a translation $-L_X \hat{X}$ and a rotation β about $\hat{X}$, found by energy minimization.
- To apply tensile strain, change L_X.
- To apply torsional strain, change β.

Numerical modeling of tubular crystals



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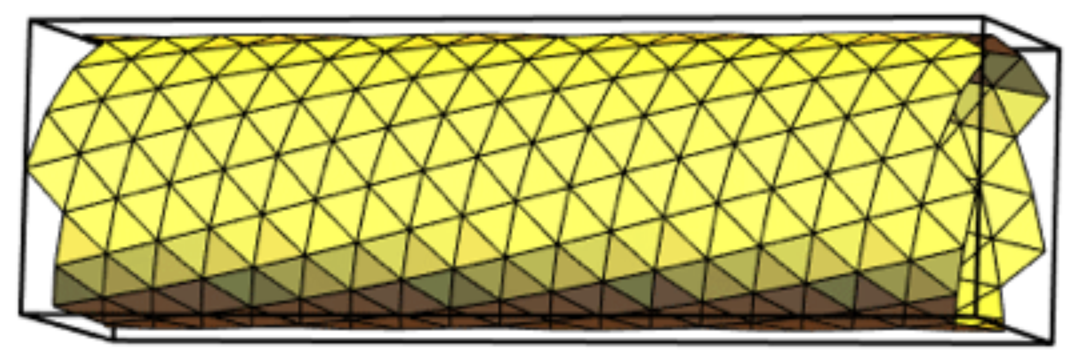

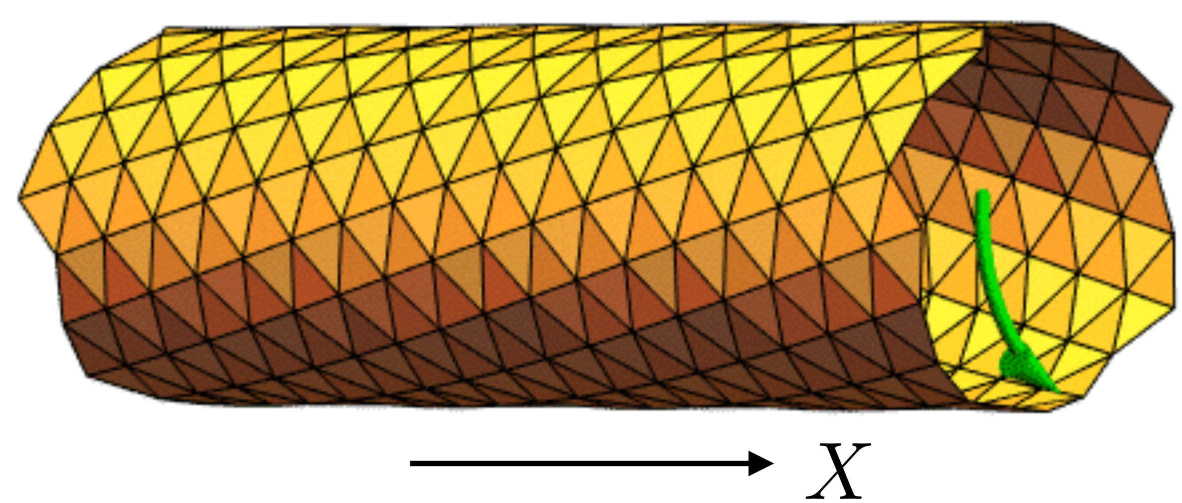

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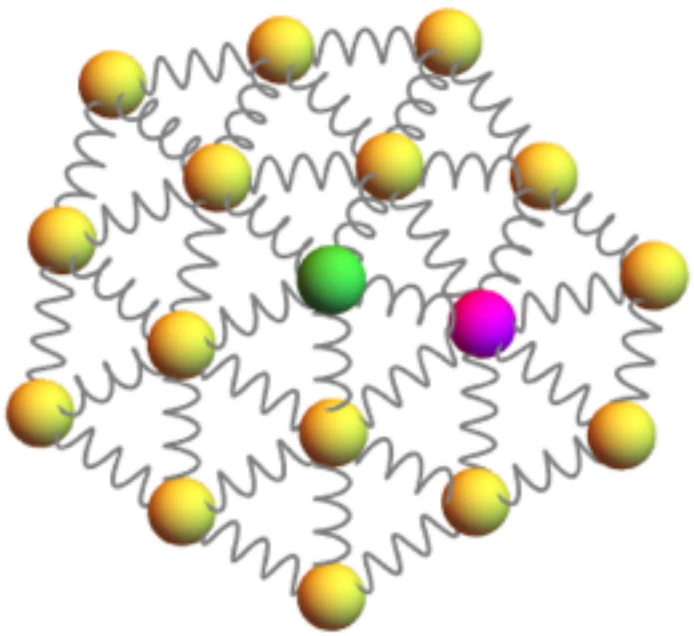

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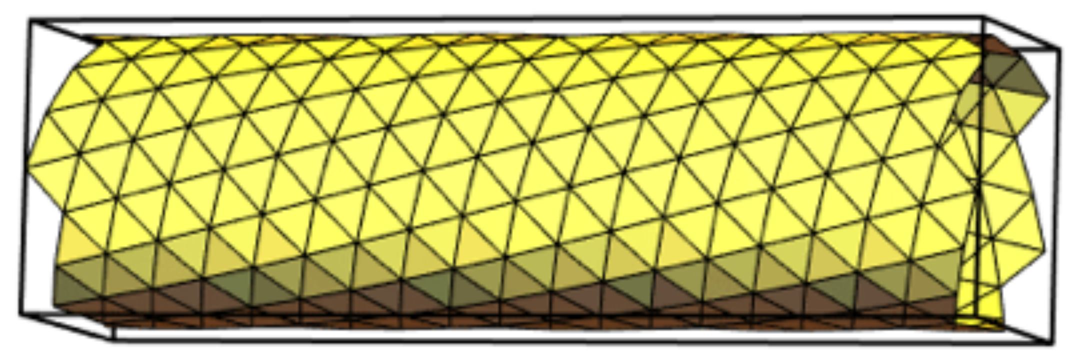
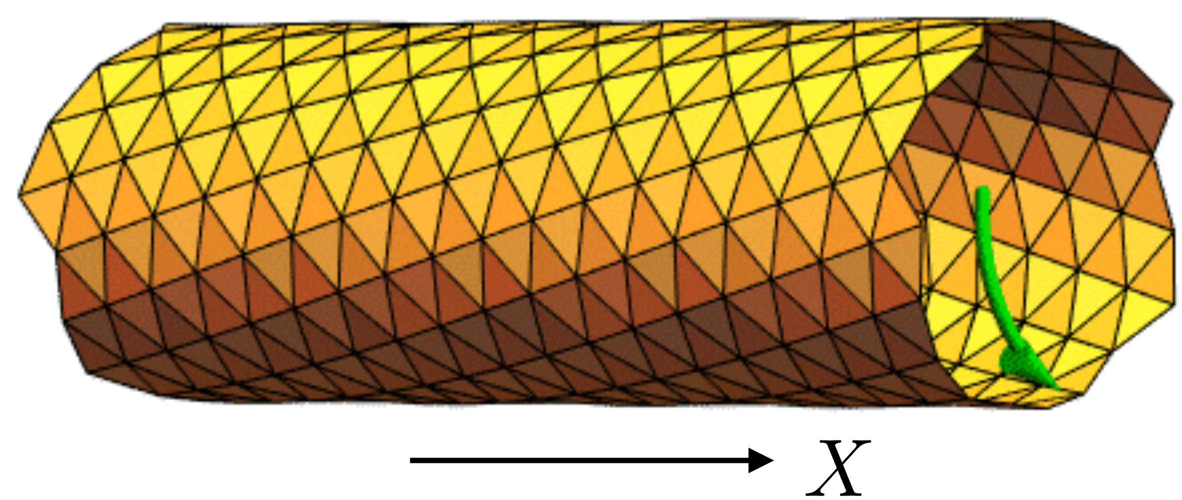
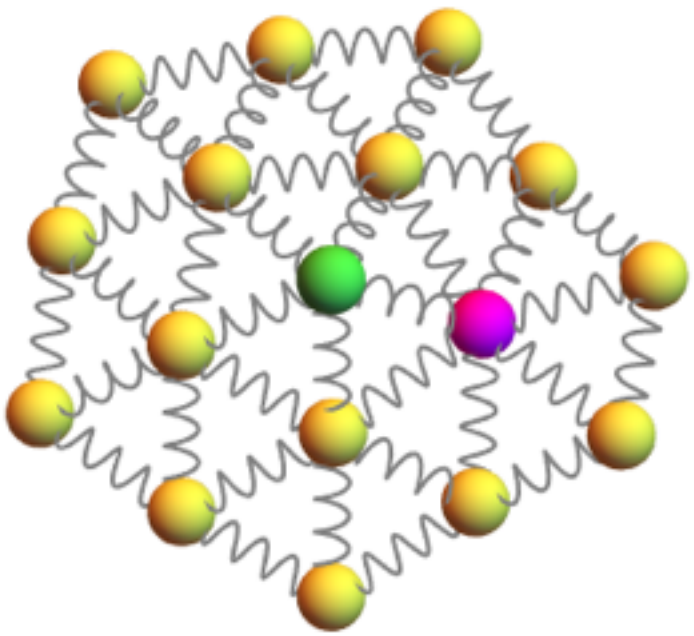




Numerical modeling of tubular crystals



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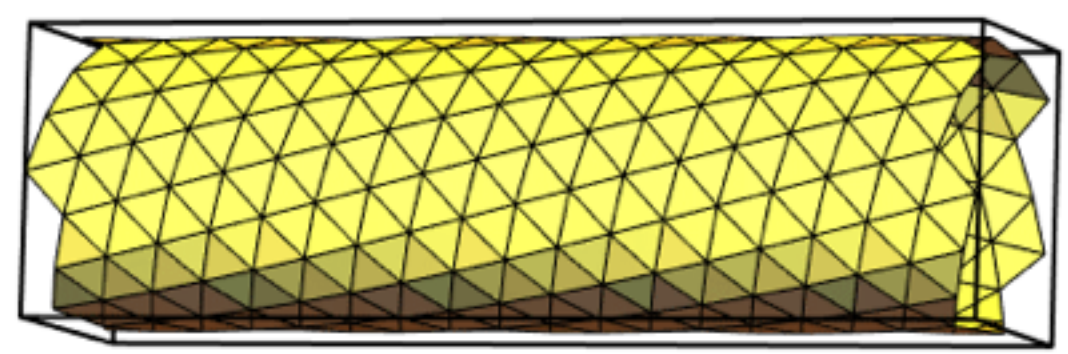

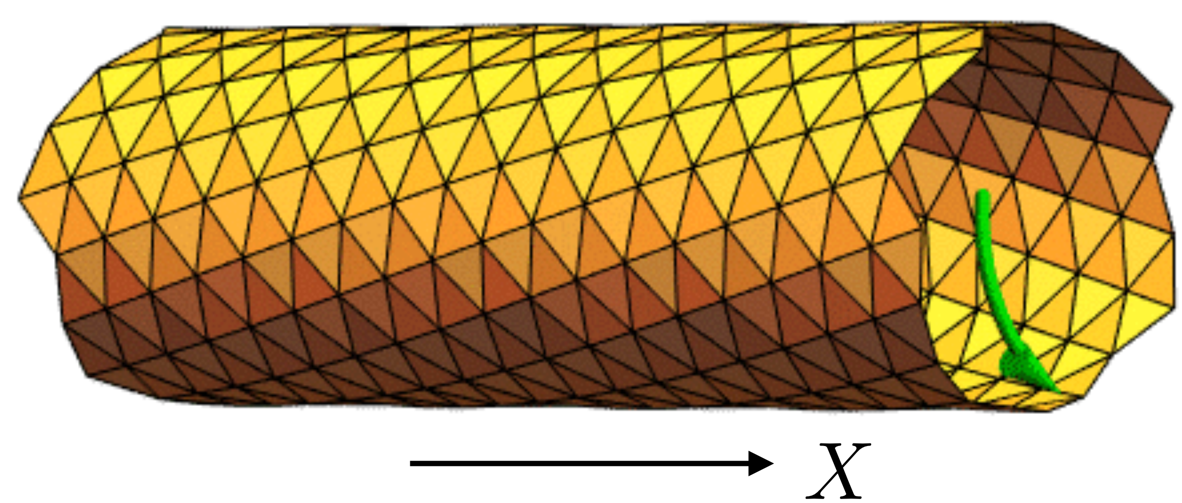

Numerical modeling of tubular crystals

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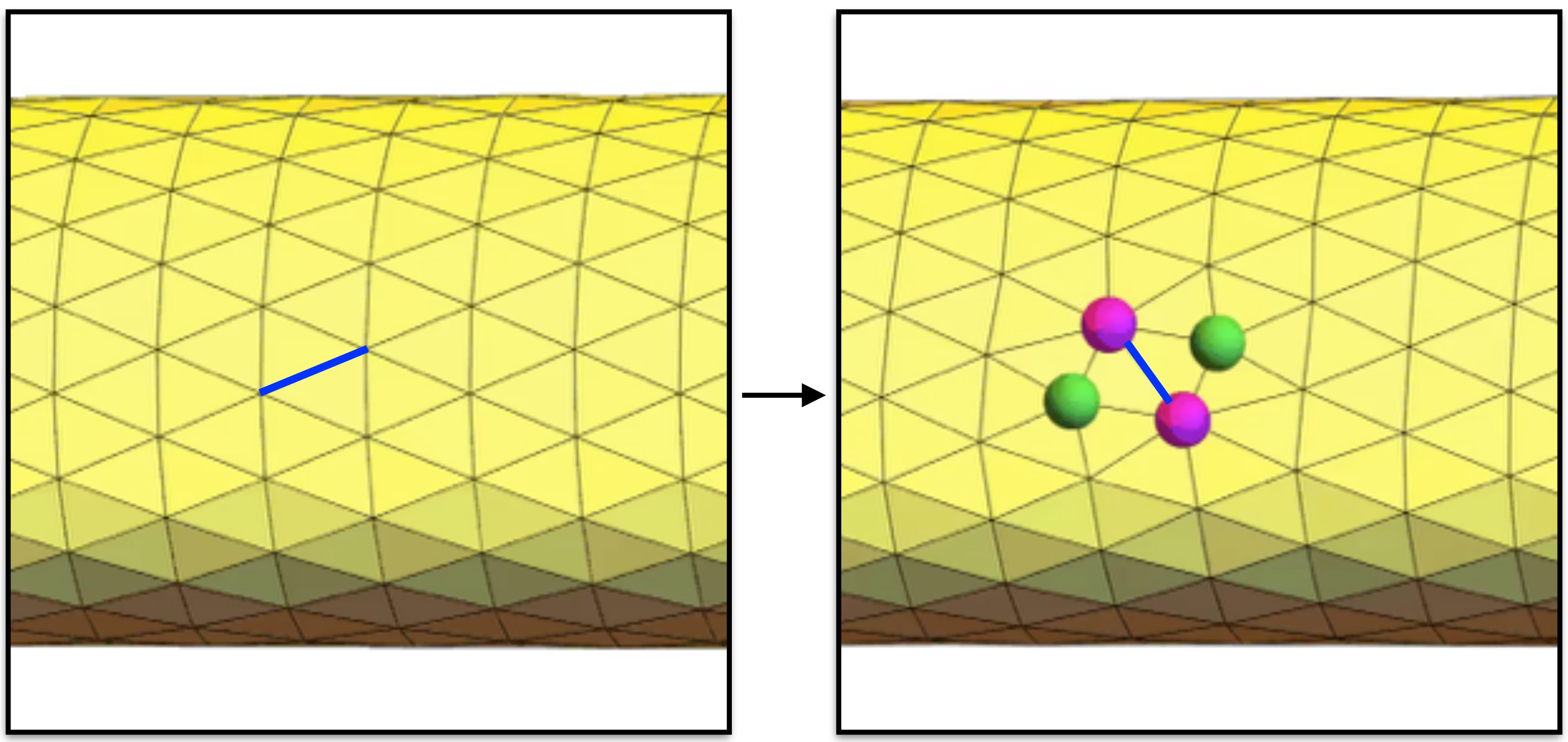
- Glide move accepted only if it lowers the energy



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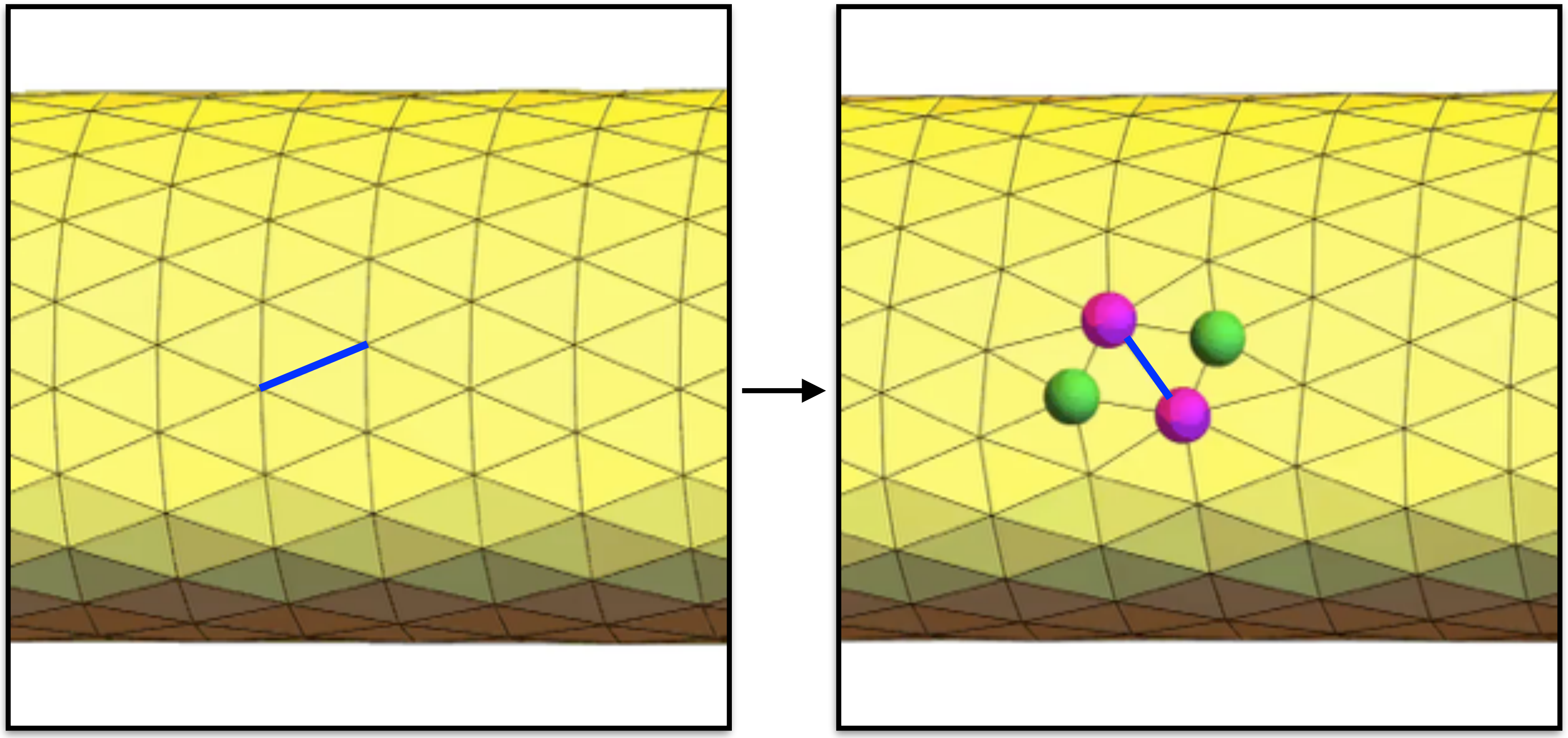

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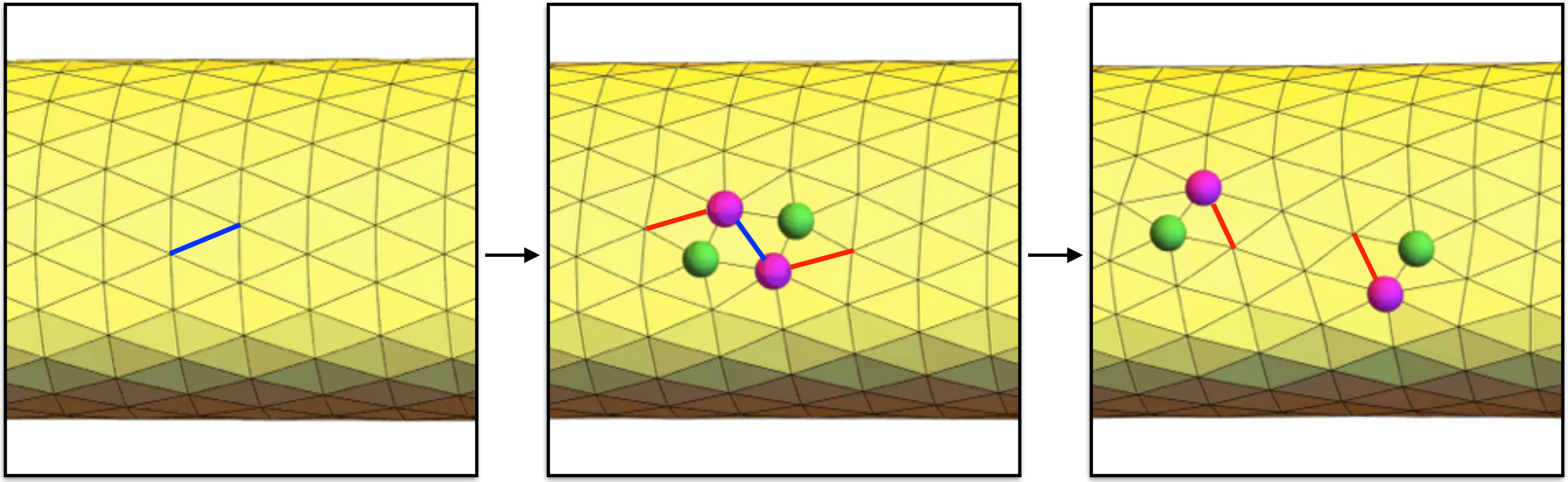
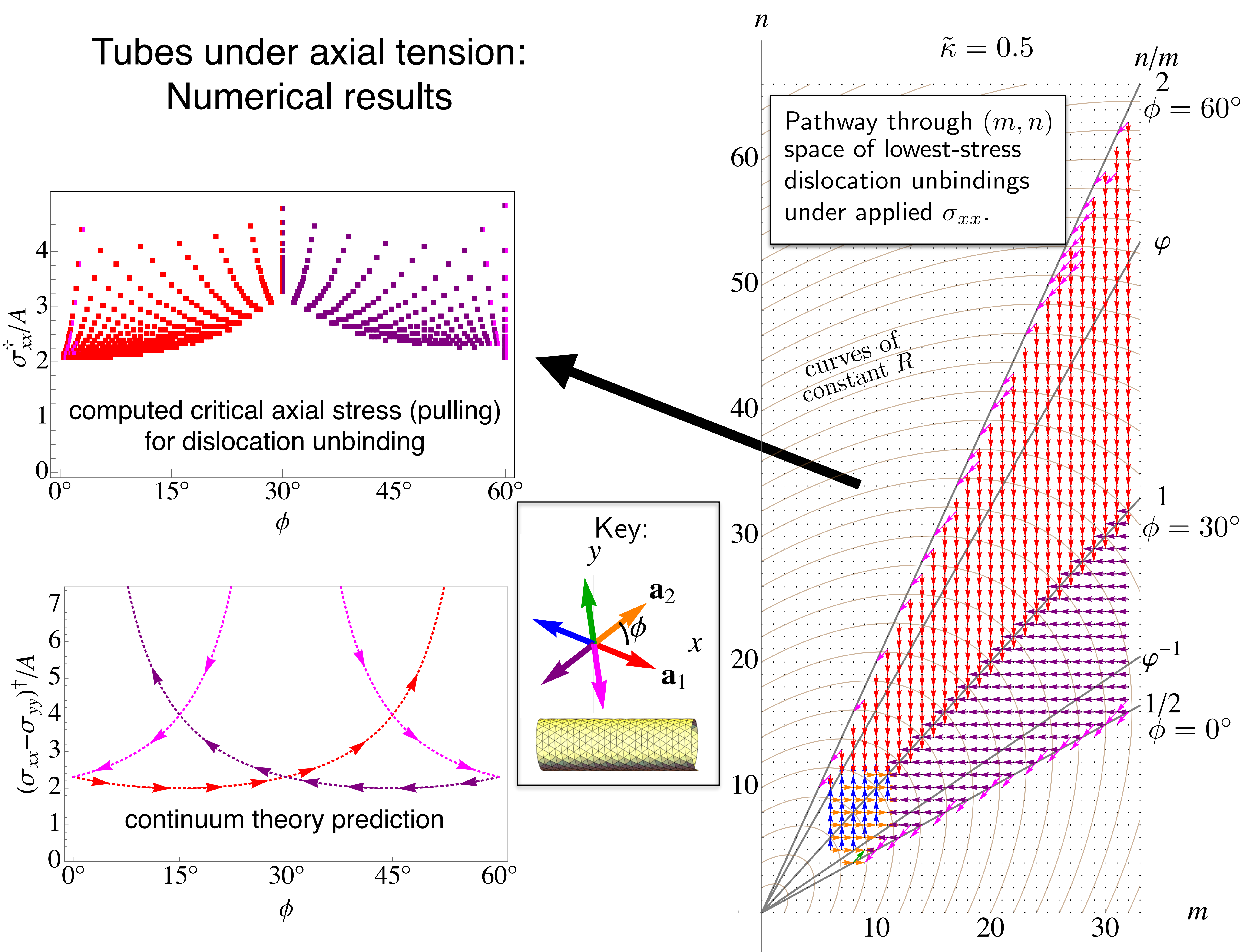


Tubes under axial tension: Numerical results



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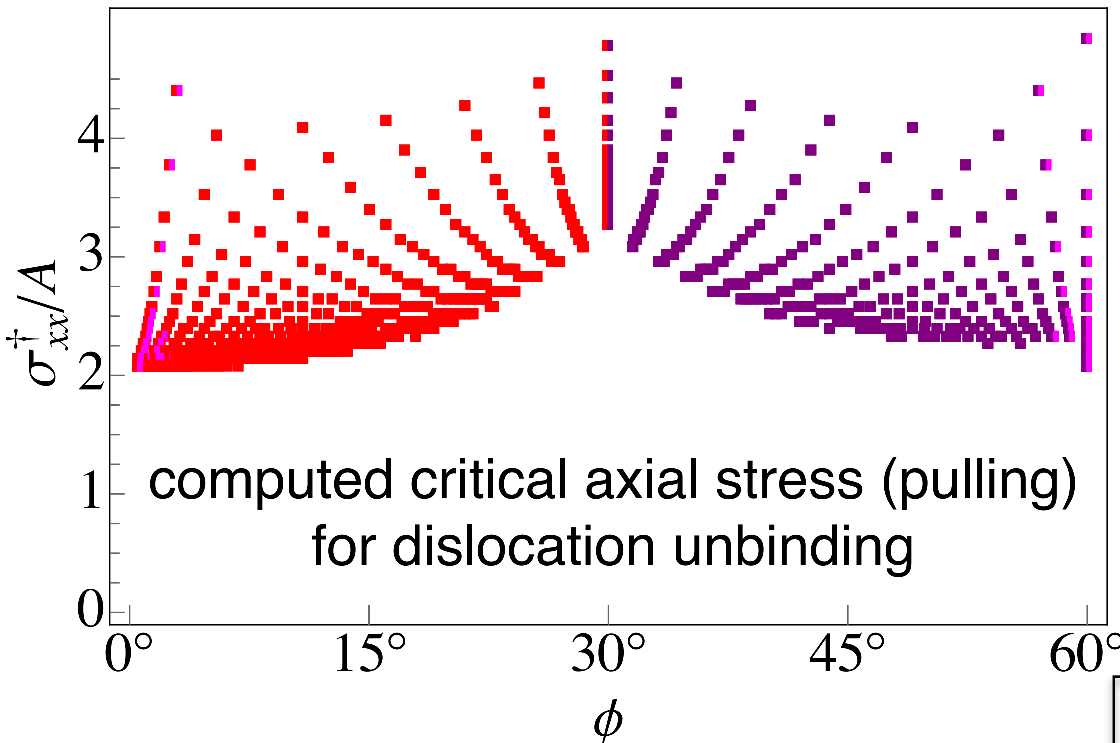
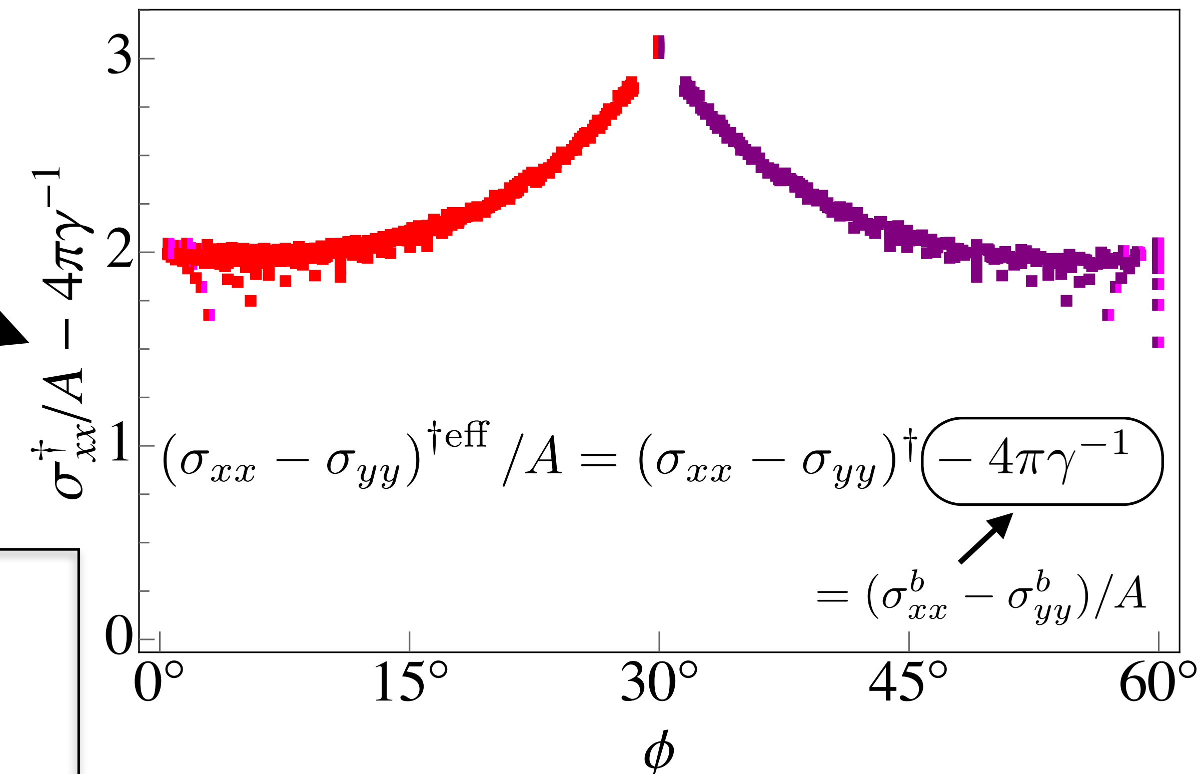
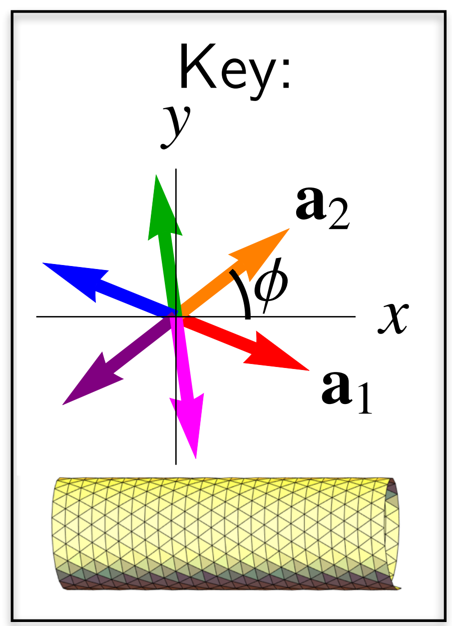
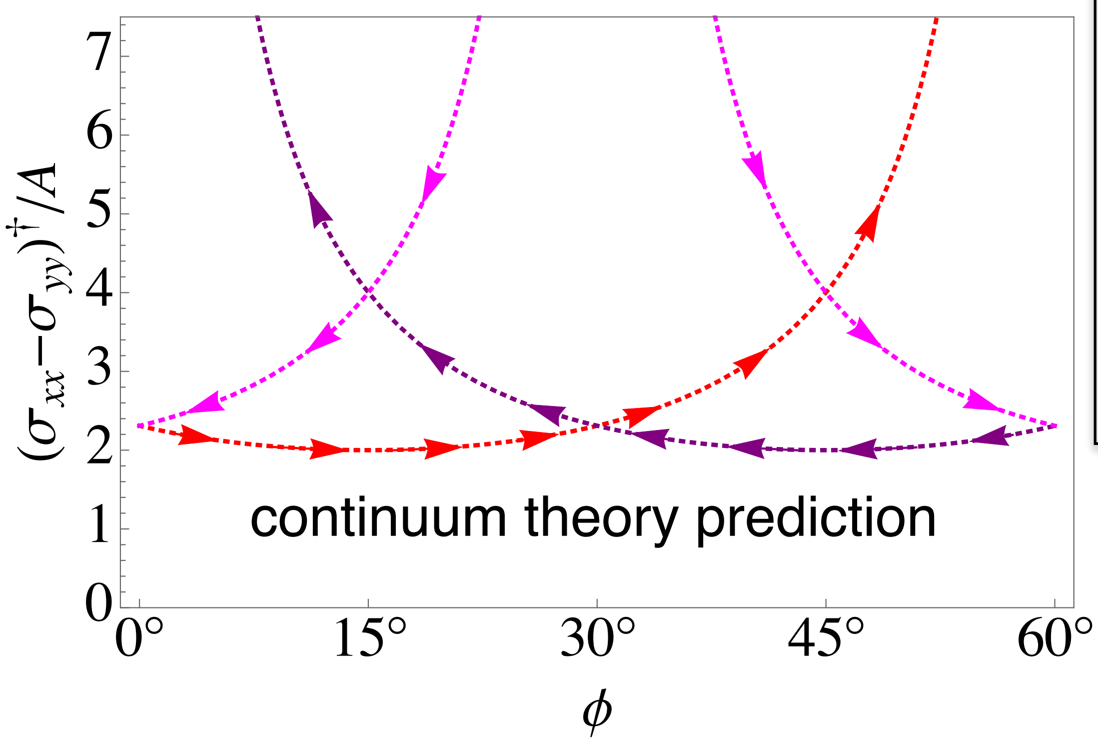




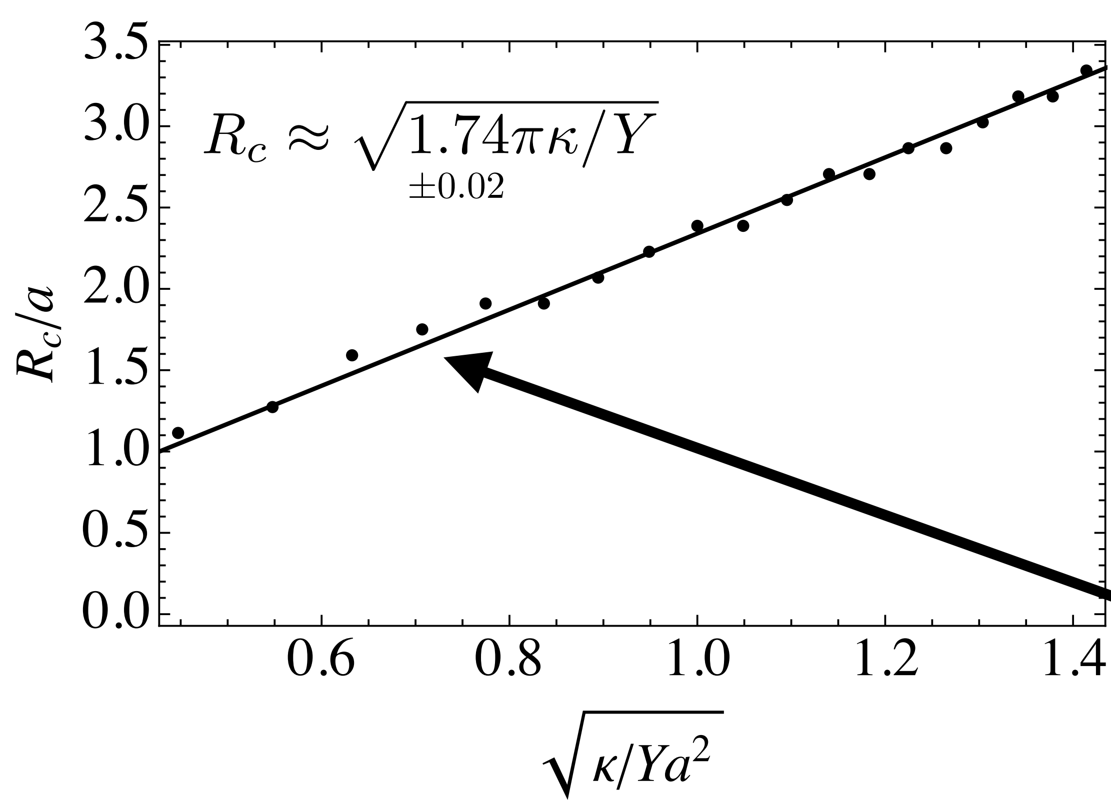


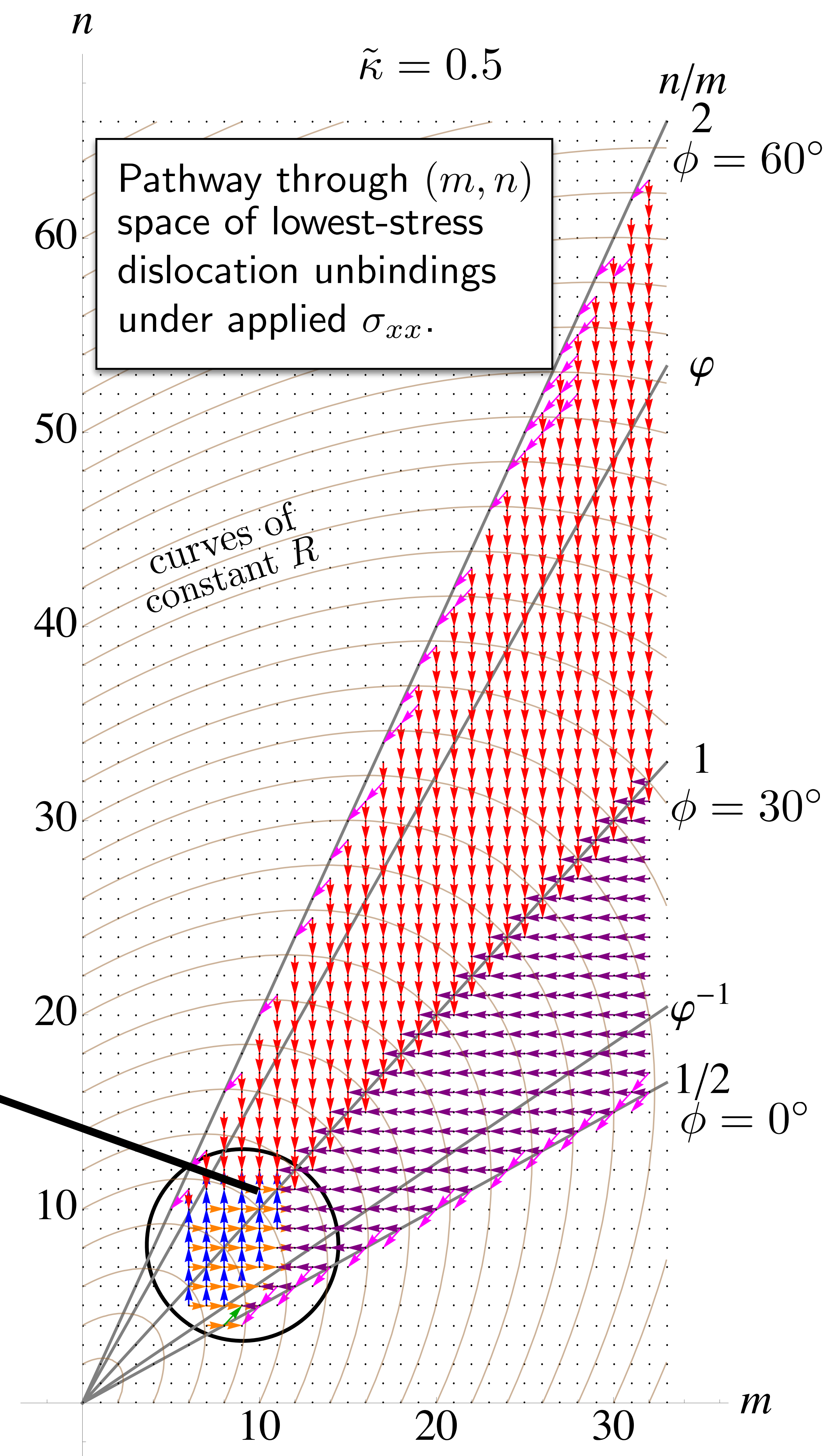


Computed $\sigma^{\dagger}_{xx}$ collapses to R-independent curve $\sigma^{\dagger\mathrm{eff}}_{xx}$ when we offset the applied stress by the bending energy's effective stress.

Tubes under axial tension: Numerical results

The bending energy makes narrow tubes with $R < R_c$ unstable to spontaneous dislocation unbindings that widen the tube.





Continuum theory predicts

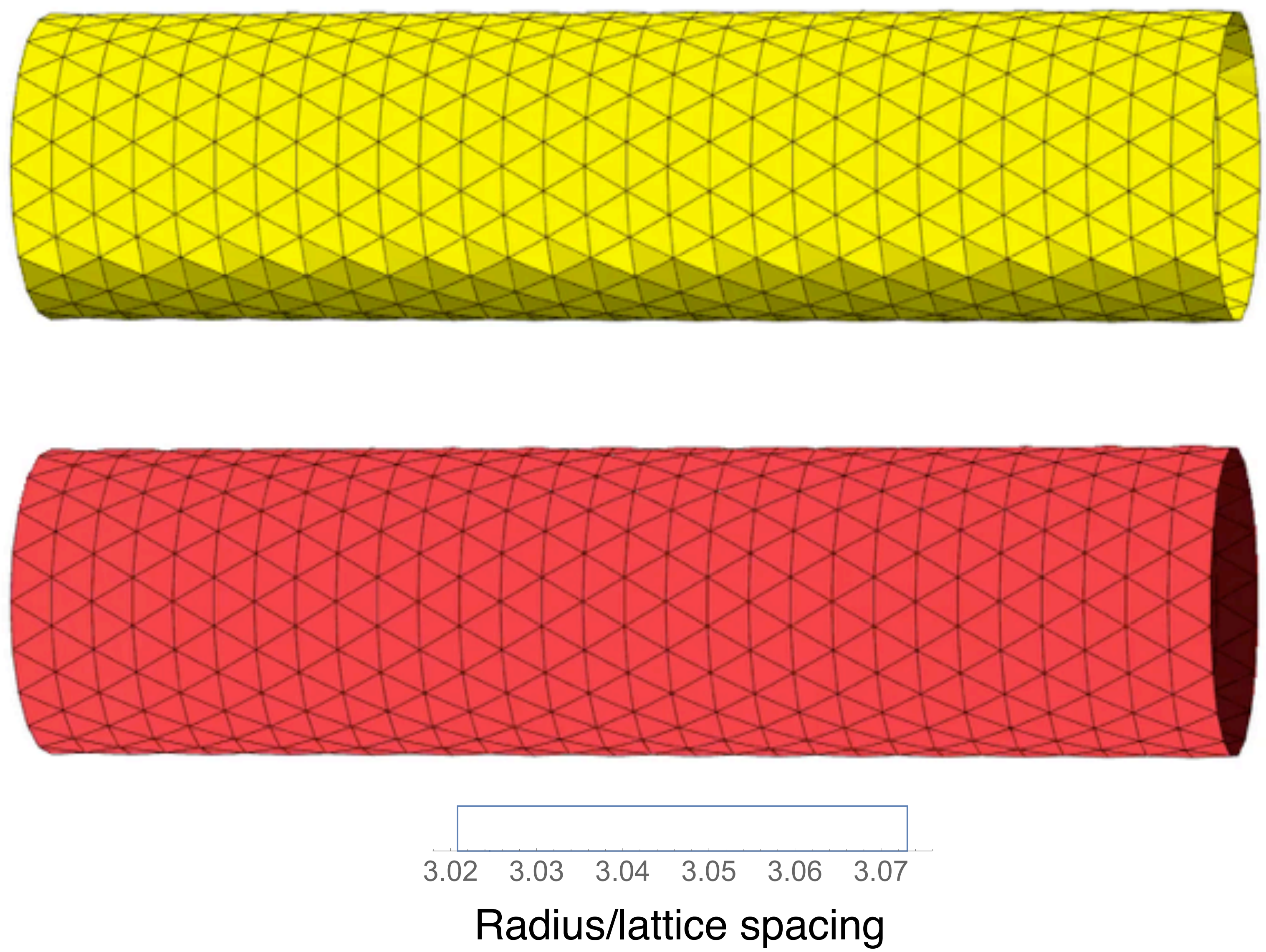
$$R_c = \sqrt{\kappa/\sigma_c} = \sqrt{\sqrt{3}\pi\kappa/Y} \approx \sqrt{1.73\pi\kappa/Y}$$

Plastic deformation of tubular crystals

- Background: Phyllotactic geometry of tubular crystals
- Mechanics of plastic deformation: Analytic predictions
- Numerical modeling
- <u>Necks in tubes: Radius profiles near dislocations</u>

The shape of a tube containing dislocations

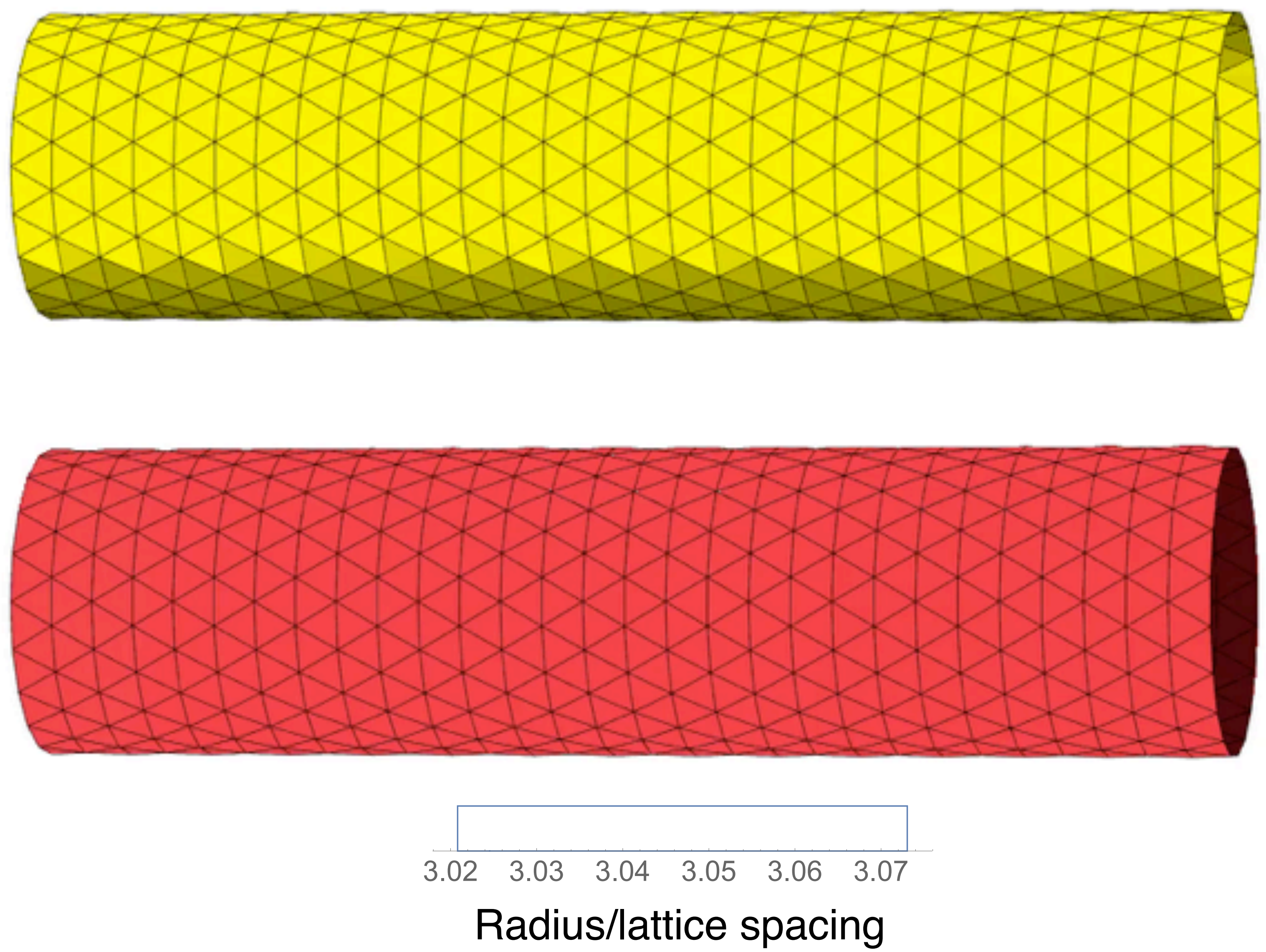
$$(20,20) \rightarrow (20,19)$$



Local radius $R(\mathbf{x})$ tracks dislocation motion

The shape of a tube containing dislocations

$$(20,20) \to (20,19)$$



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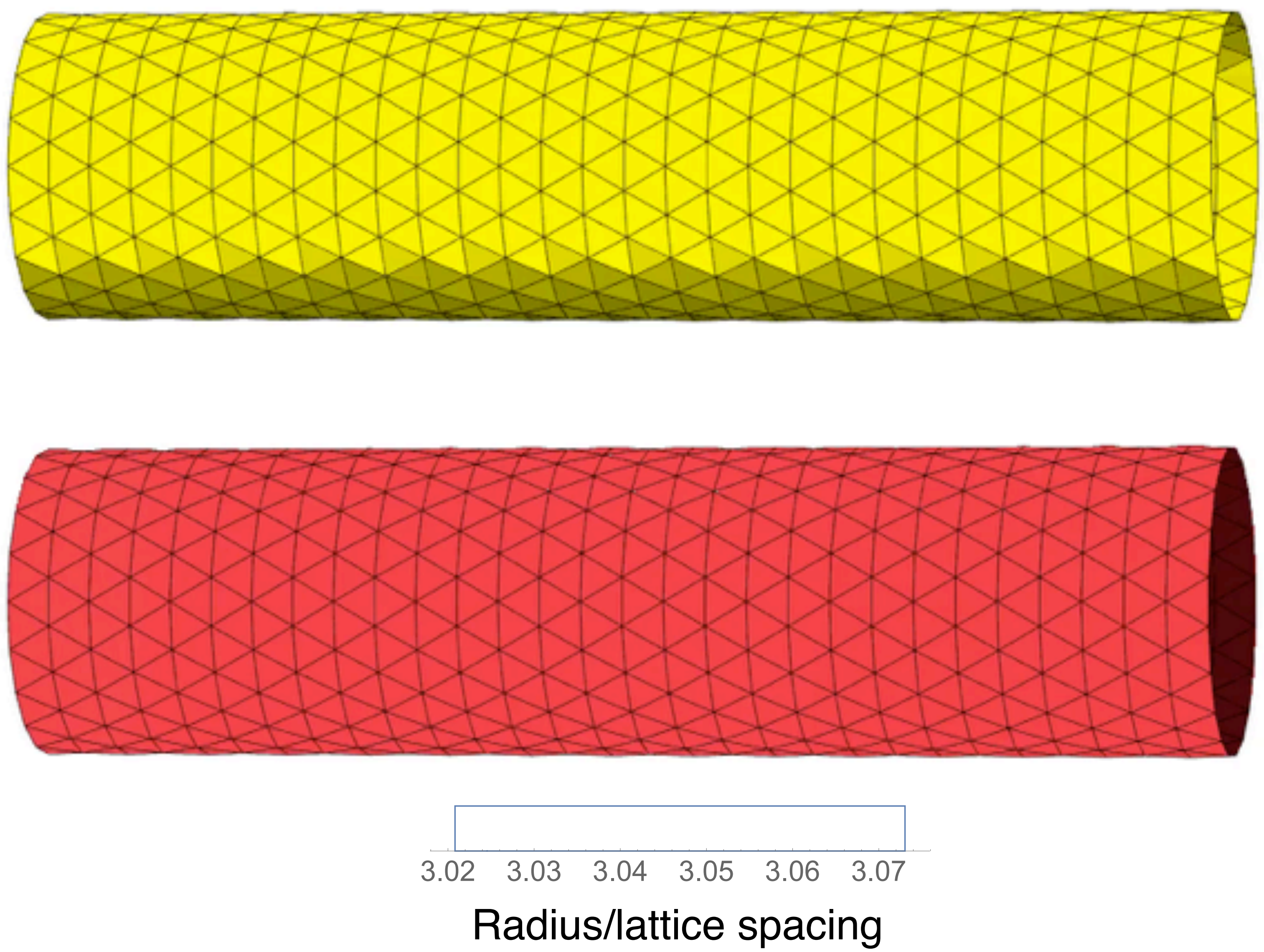

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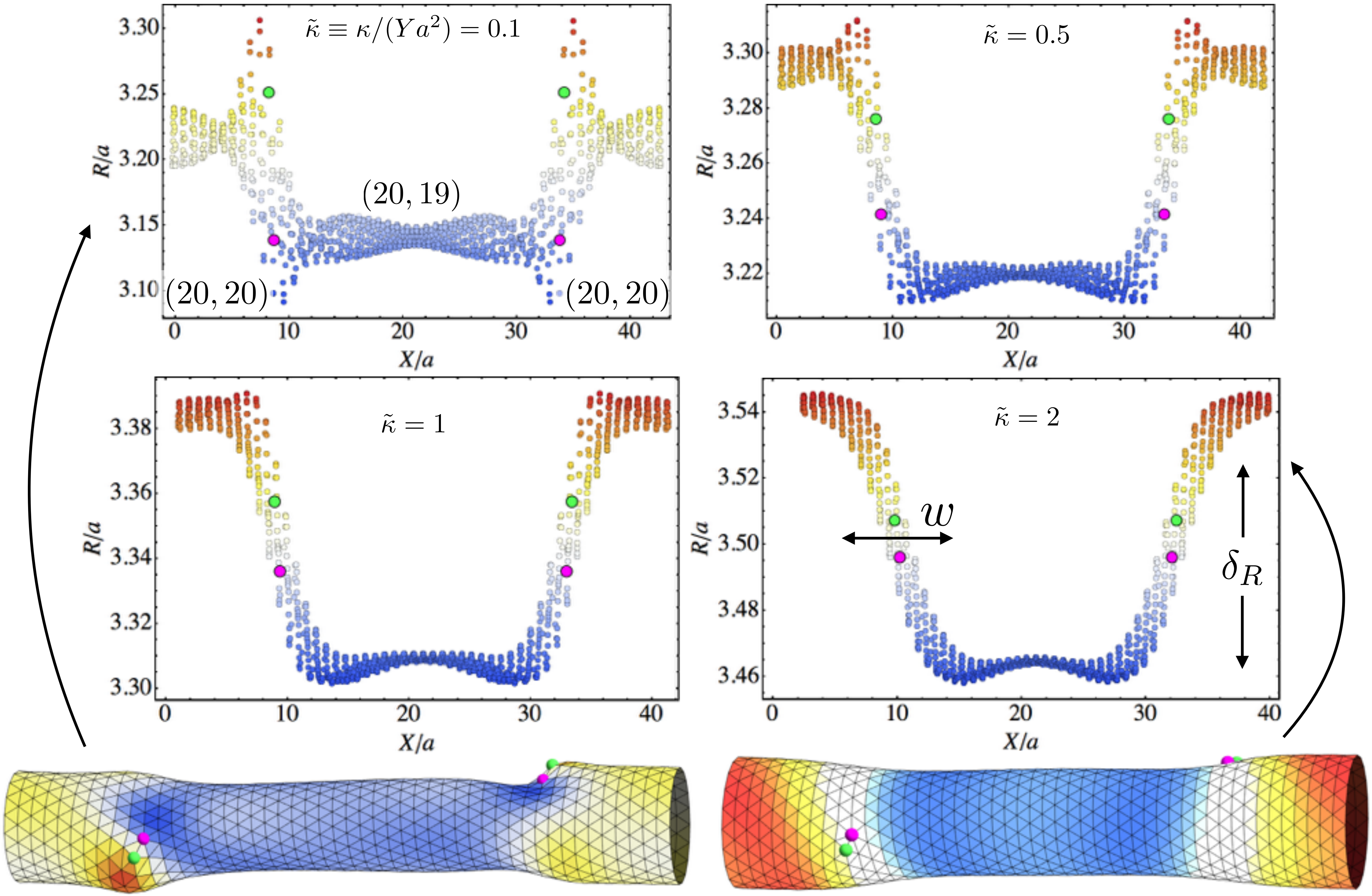
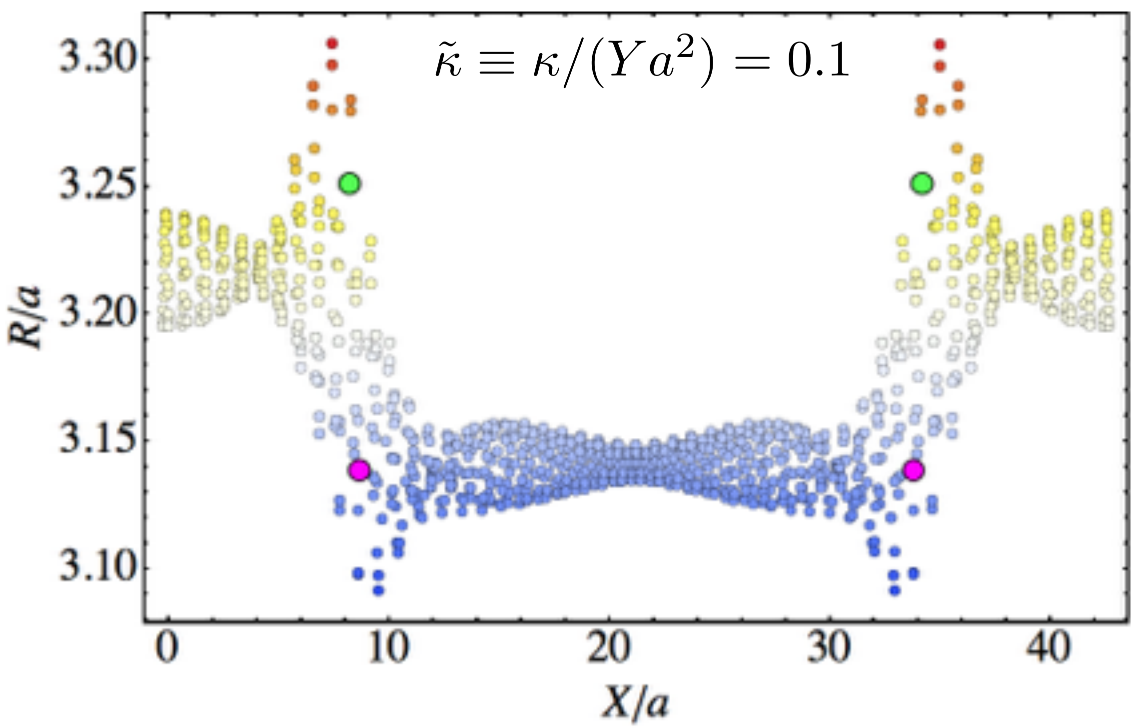
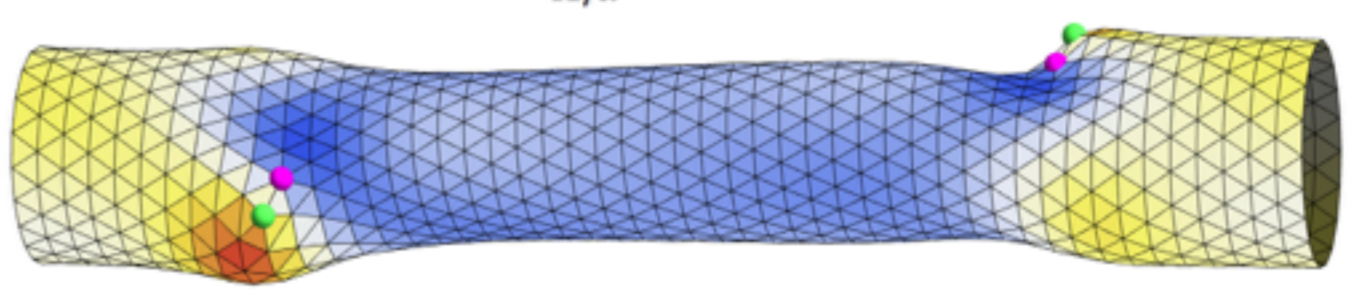
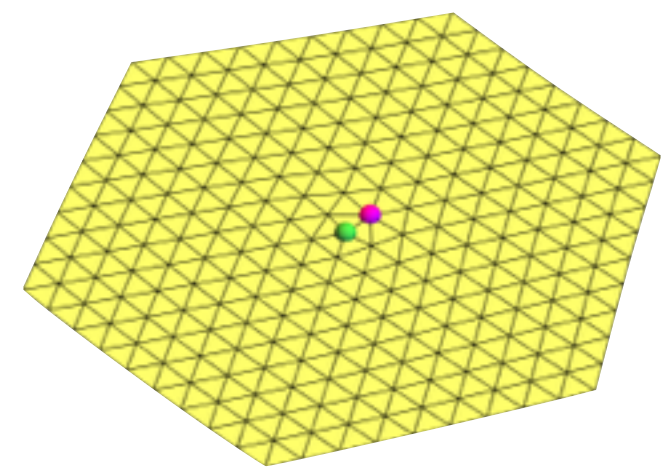


radius variations exaggerated by factor of 10 for clarity

Buckling at small $\tilde{\kappa}$

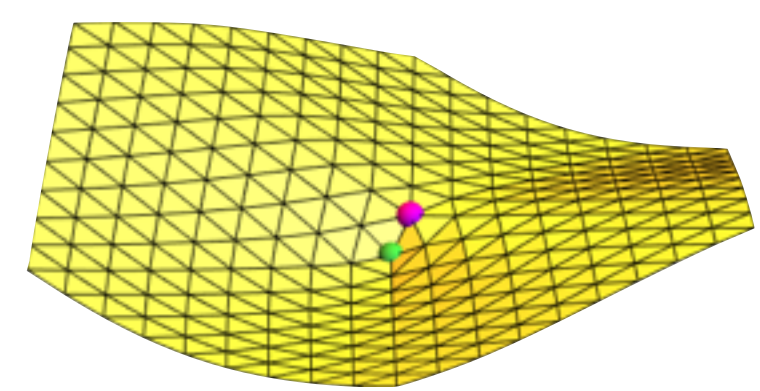






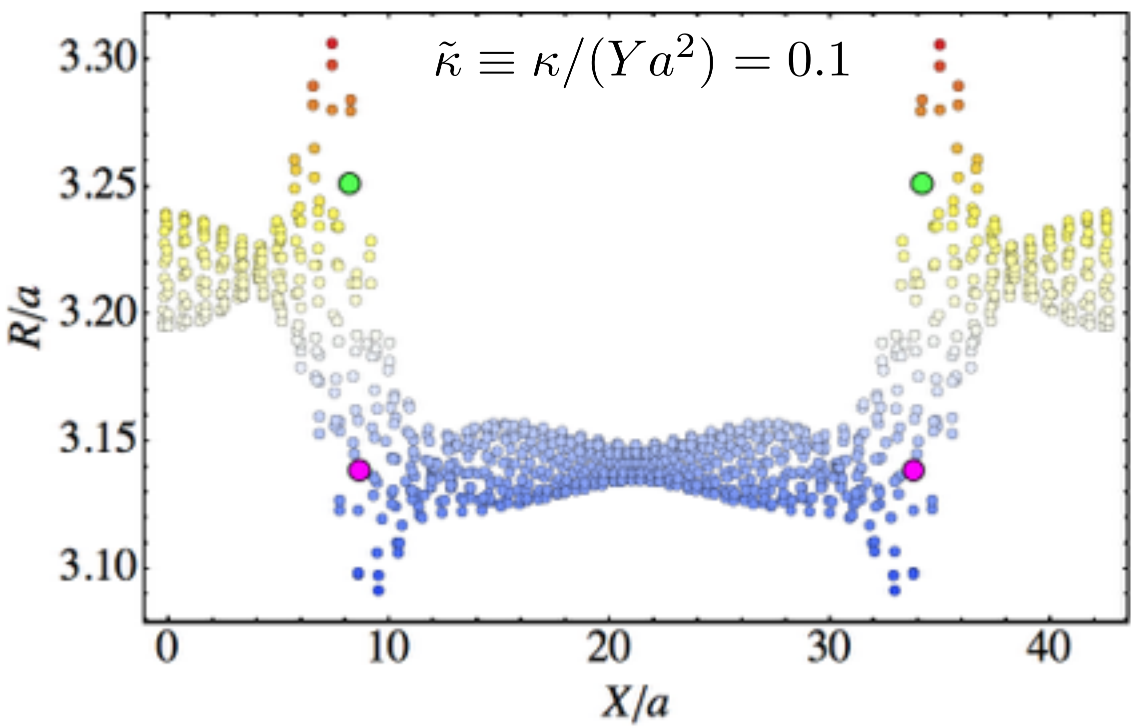
$\frac{R}{a\tilde{\kappa}} \approx 90$

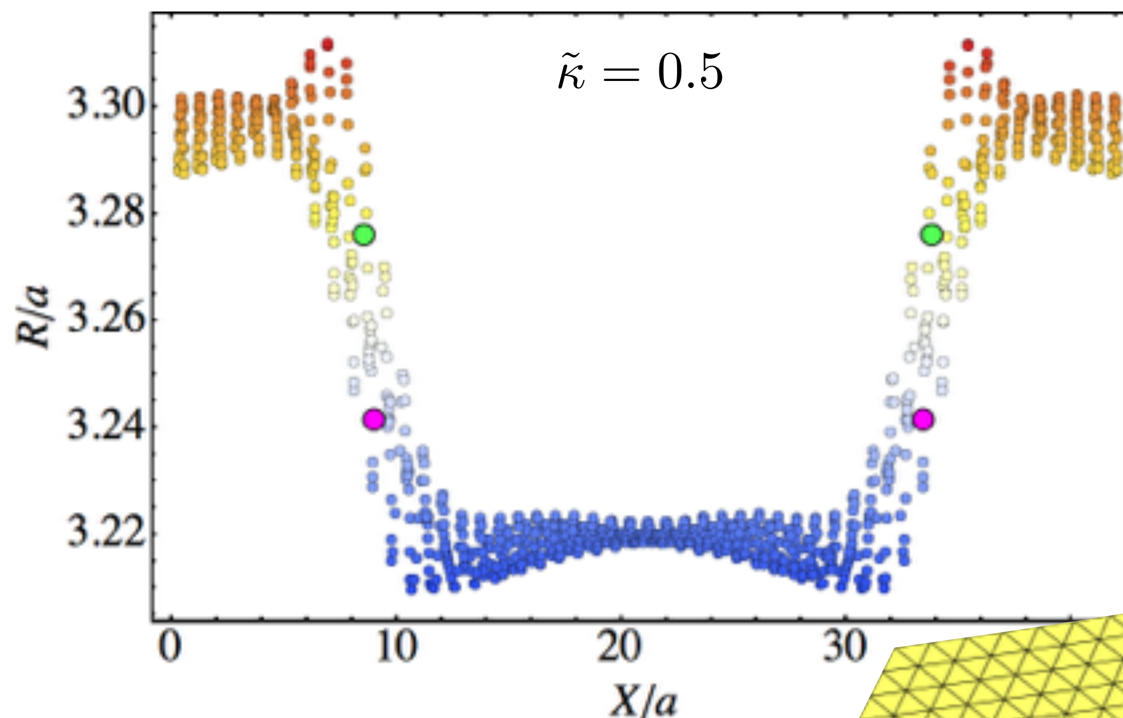
- Large local variations in R can be understood as membrane buckling.
- A membrane with an elementary dislocation at its center buckles when the system size exceeds $\approx (127\tilde{\kappa})a$.
  [Seung and Nelson, Phys. Rev. A 38:1005 (1988)]
- For tubes, this predicts buckling when
  $\tilde{\kappa} < \tilde{\kappa}_{\text{buckle}} \equiv 2\pi R_0/127 \quad \rightarrow \approx 0.16 \text{ for } (m,n) = (20,20)$

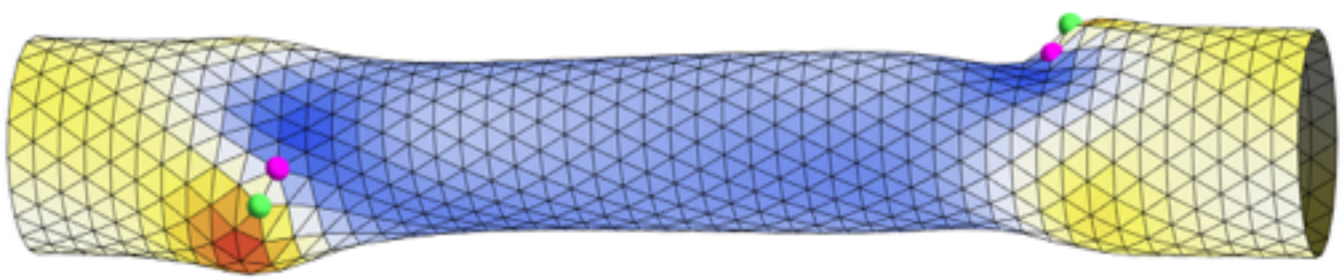


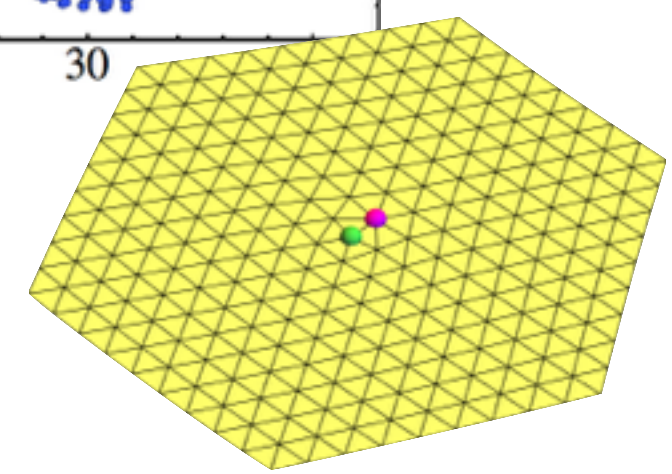
$\frac{R}{a\tilde{\kappa}} \approx 360$

Buckling at small $\tilde{\kappa}$









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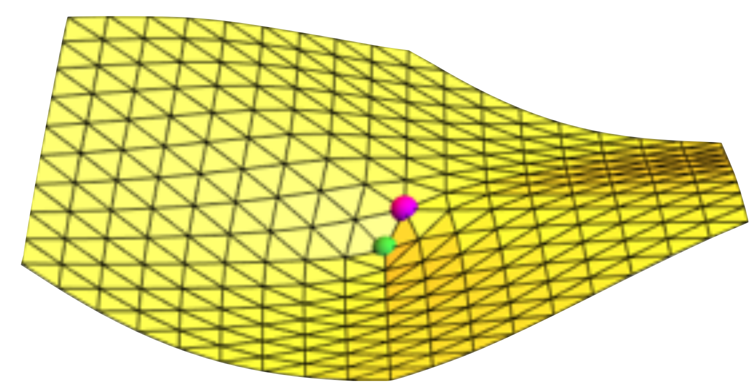

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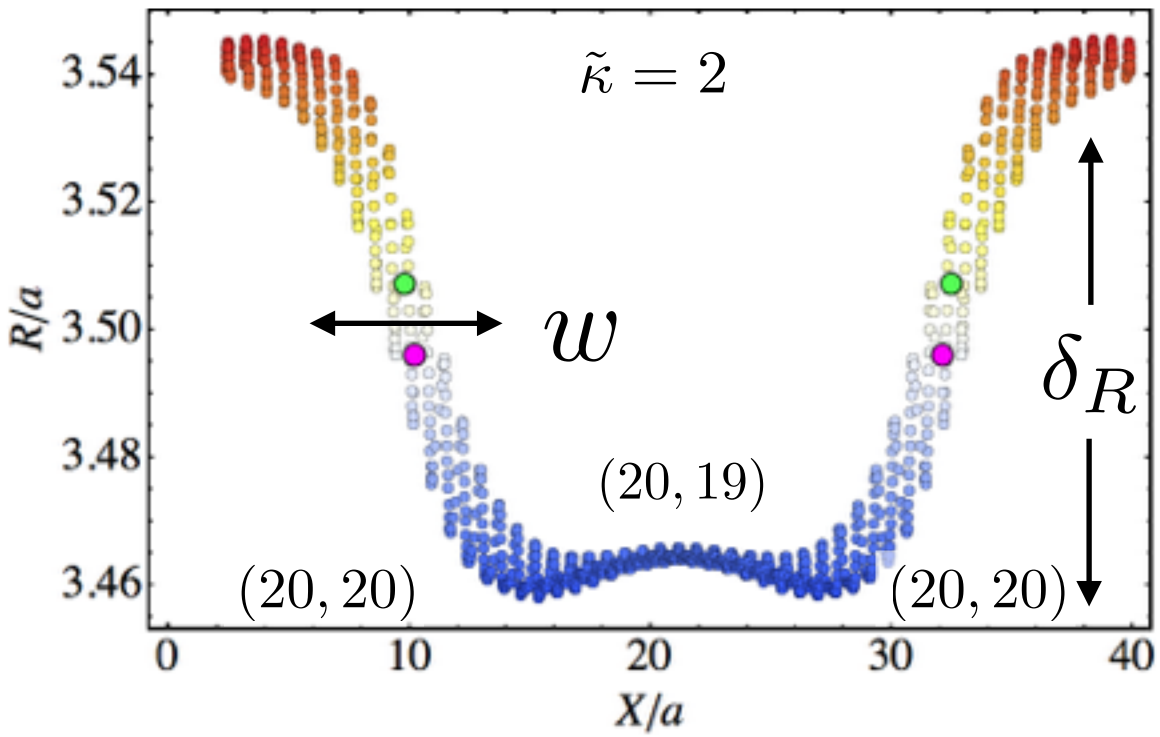
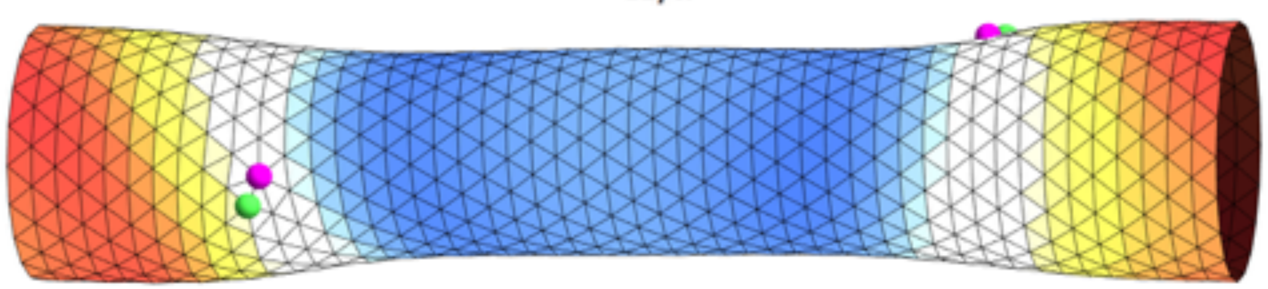
When there is a well-defined neck profile ($\tilde{\kappa} \gg \tilde{\kappa}_{\text{buckle}}$)...

What is the width of the neck?





Scaling argument

- $\delta_R \sim a$
- $\Rightarrow$ Stretching energy density $\sim Y(a/R_0)^2$
- Curvature due to neck: a/w^2
- $\Rightarrow$ Bending energy density $\sim \kappa(a/w^2)^2$
- $E_s \sim E_b \Rightarrow w \sim \left(\kappa/YR_0^2\right)^{1/4} R_0 = \boxed{\gamma^{-1/4} R_0}$

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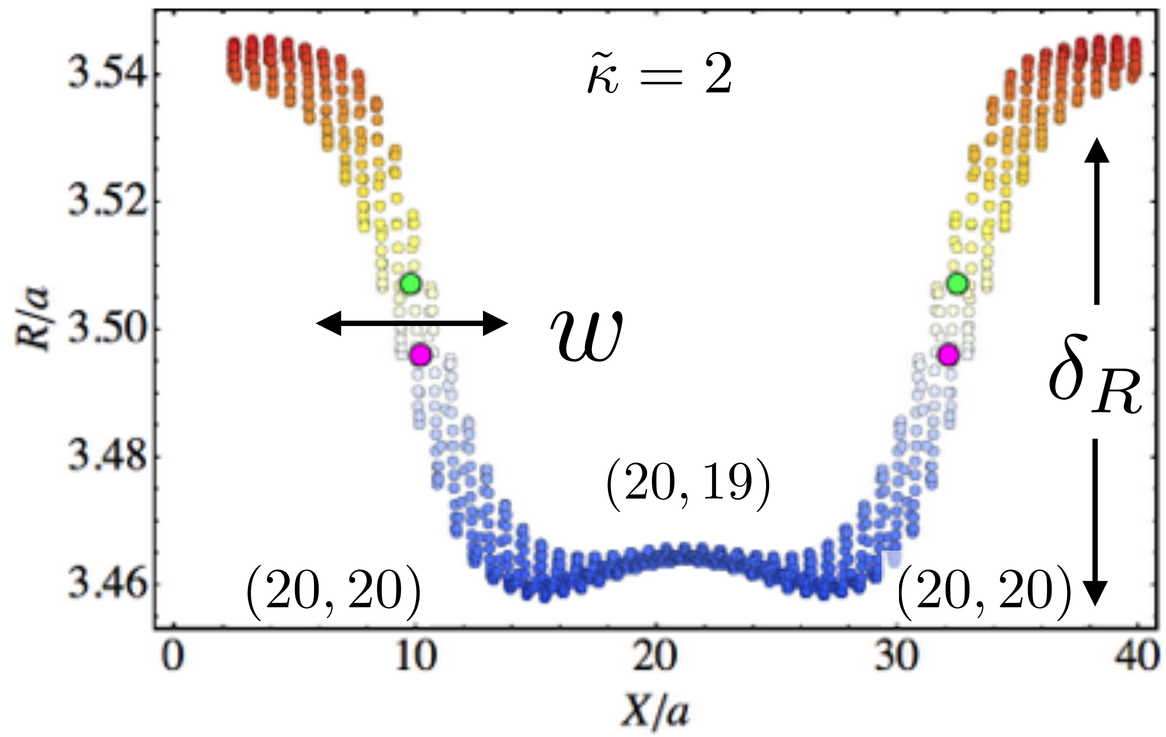

Calculation for a weakly deflected cylinder

- Suppose $R(x) = R_0 + \zeta(x)$, $\zeta \ll R_0$.
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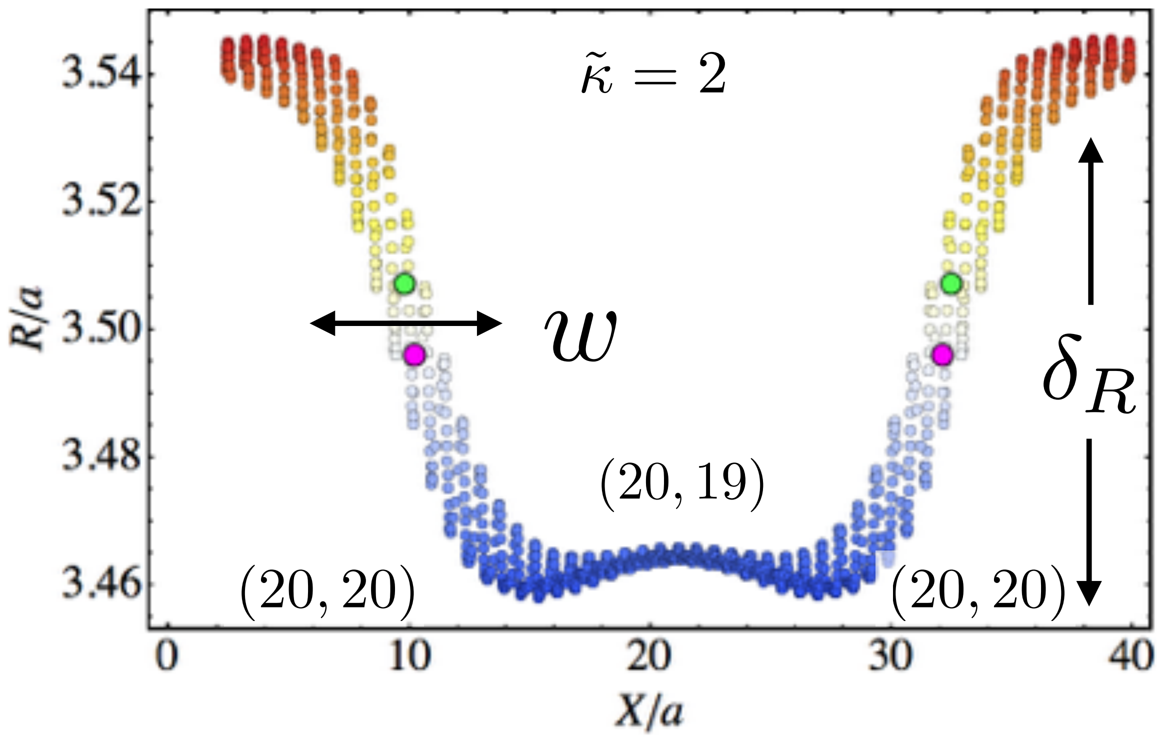

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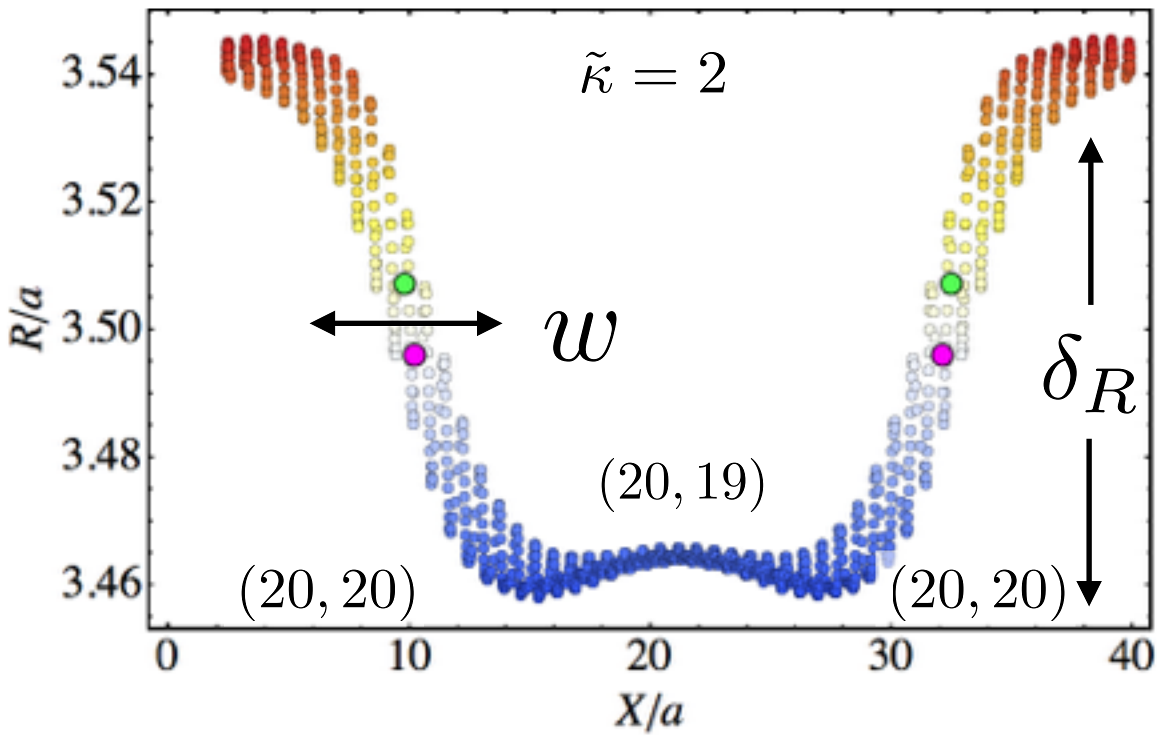

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$$\frac{E}{2\pi R_0} \approx \int dx \left\{ \tfrac{3}{4} Y \left(\zeta(x)/R_0\right)^2 + \tfrac{1}{2}\kappa H[\zeta(x)]^2 \right\}$$

- Solution: $R(x) = R_{\text{pristine}} + c\,\mathrm{Re}\left[e^{\pm x/w}\right]$, with

$$\boxed{w = R_0 \left[-1 + i\sqrt{\tfrac{3}{2}\gamma}\right]^{-1/2}} \sim \gamma^{-1/4} R_0$$

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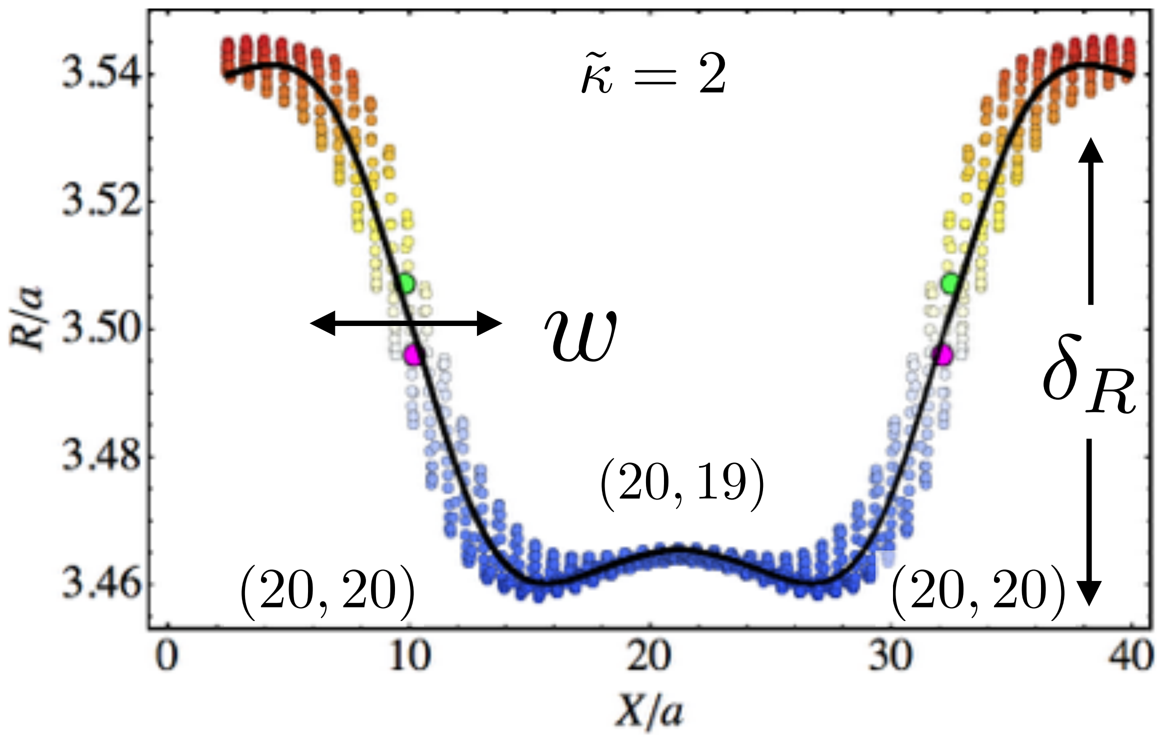

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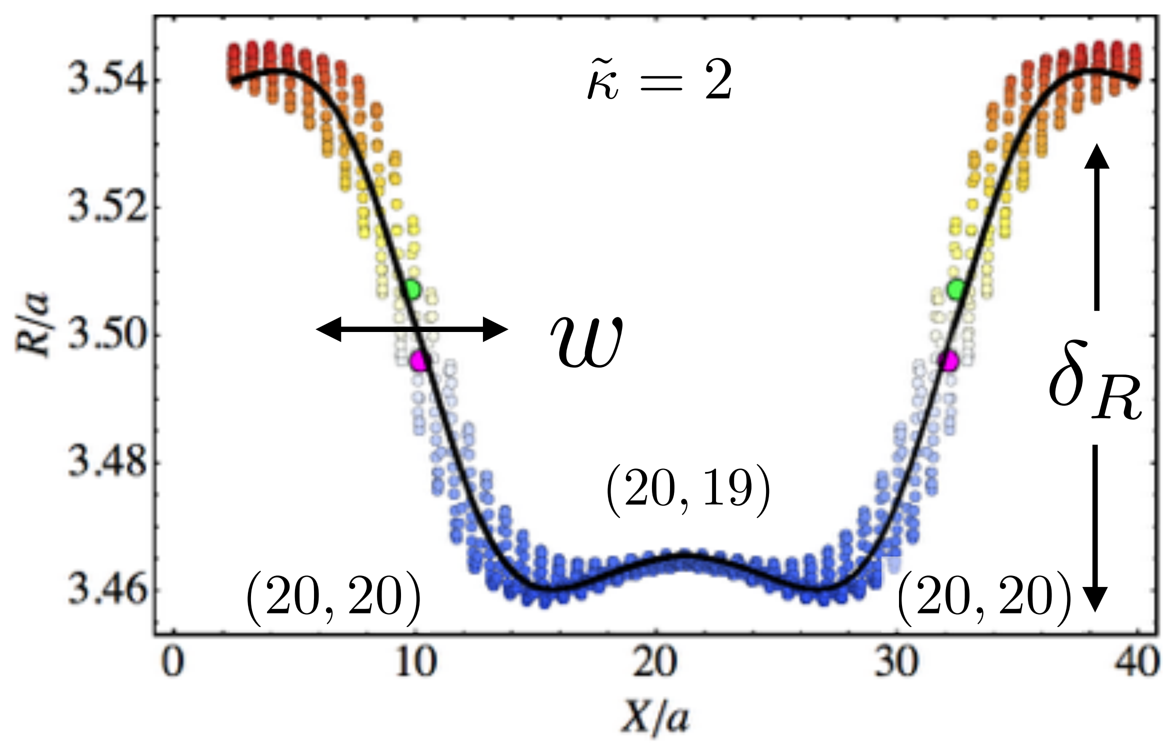

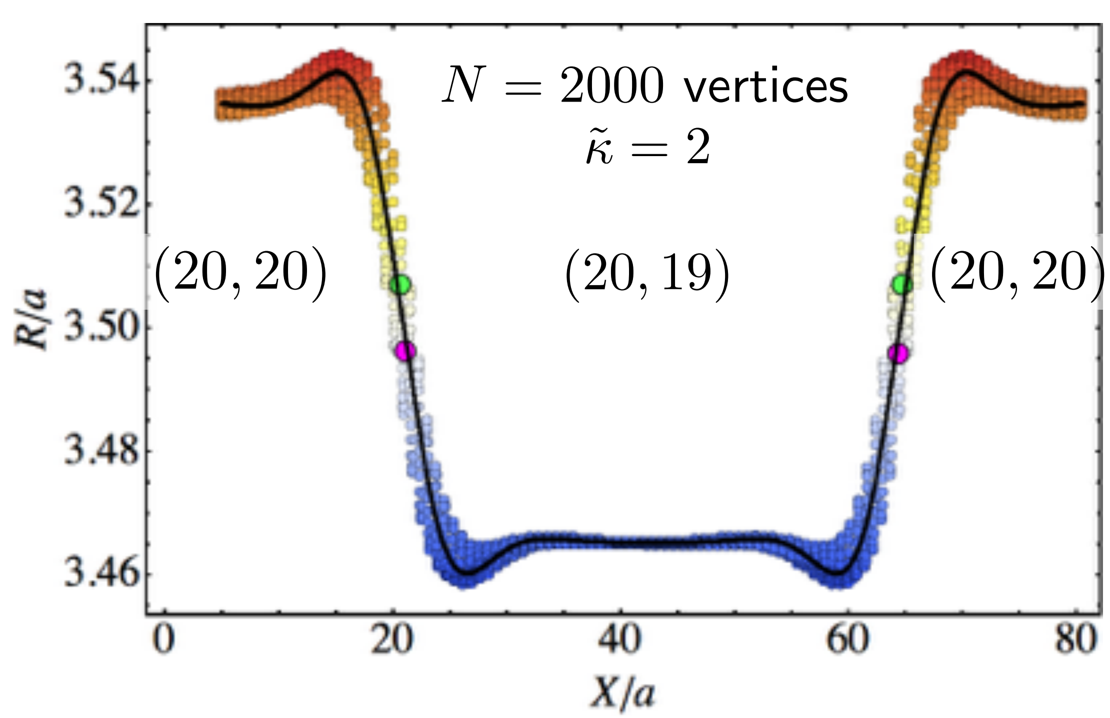

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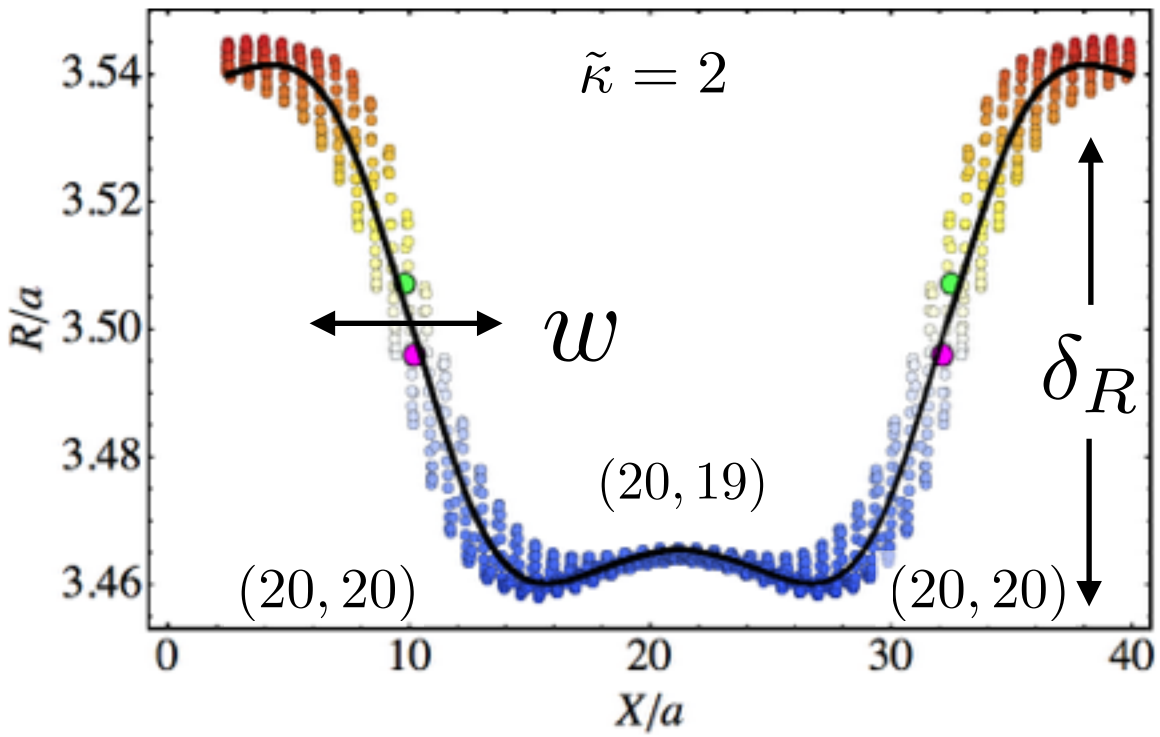

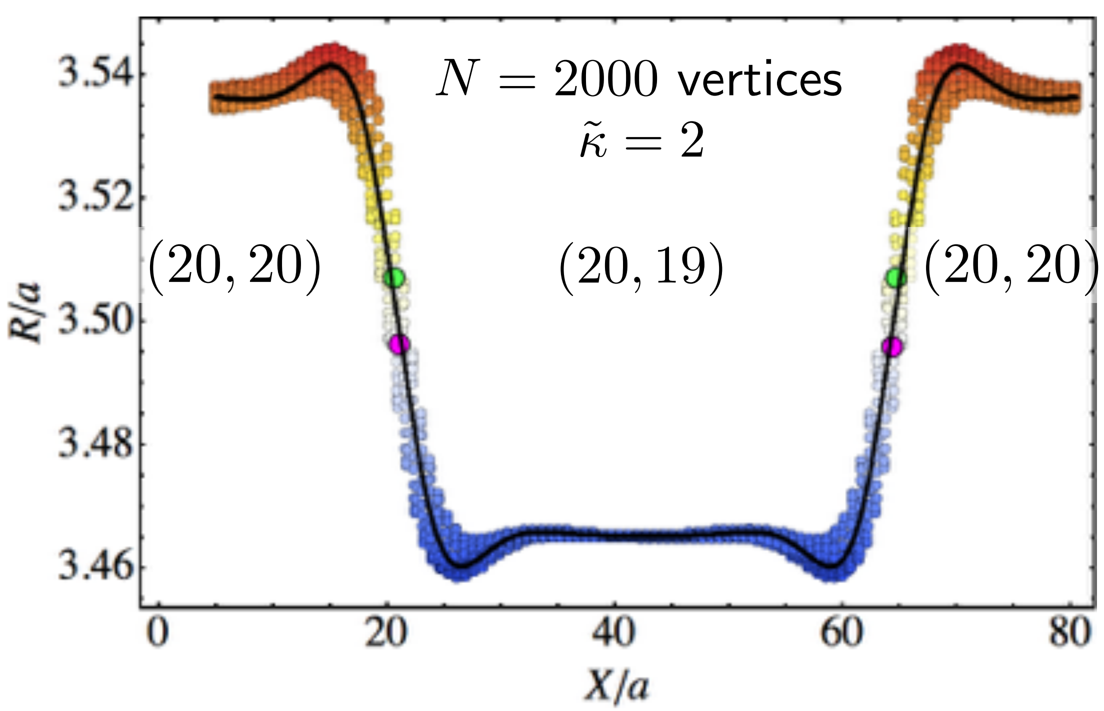

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$\Rightarrow$ predicts rise/decay *and* oscillations

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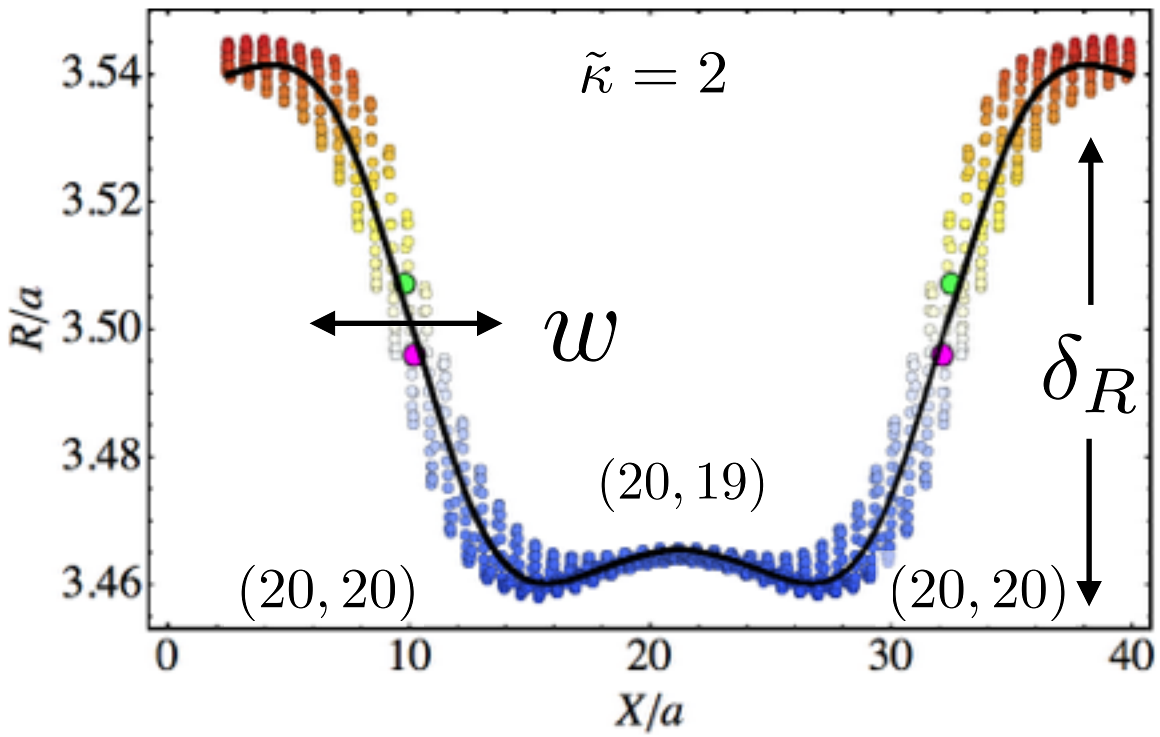

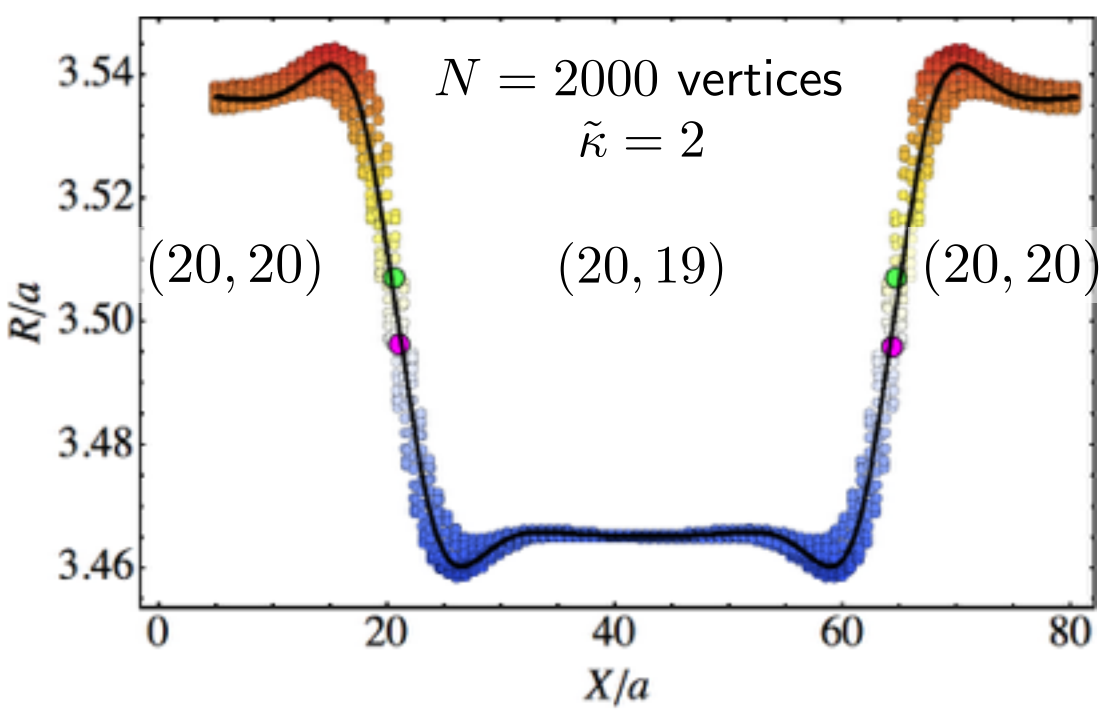

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Conclusions

- Glide separation of dislocation pairs provides a mode of plastic deformation by parastichy transition $(\Delta m, \Delta n)$.
- Tubes under axial tension σ_{xx} converge toward the stable $m = n$ achiral states while their radius shrinks.
- The bending modulus κ shifts up the critical stress $\sigma_{xx}^{\dagger}$ required to drive apart dislocations, stabilizing narrow tubes.
  - This shift contains all the R-dependence in $\sigma_{xx}^{\dagger}$.
- If κ is large enough, very small tubes may even be unstable to emission of dislocation pairs that widen the tube.
- The "neck" around a dislocation has width $w \sim \gamma^{-1/4} R$ and also oscillations in local radius.

Acknowledgements

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HARVARD
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School of Engineering
and Applied Sciences