Isometric Immersions, Energy Minimization and Branch Points in Non-Euclidean Elastic Sheets

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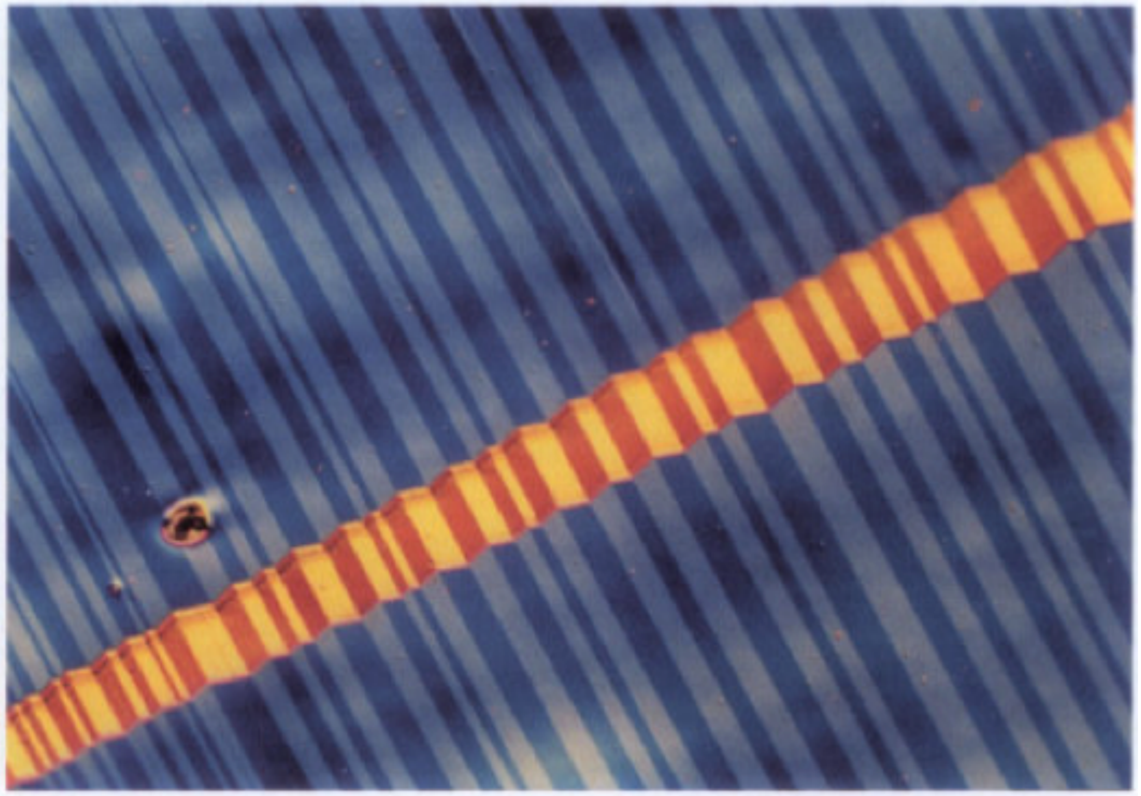
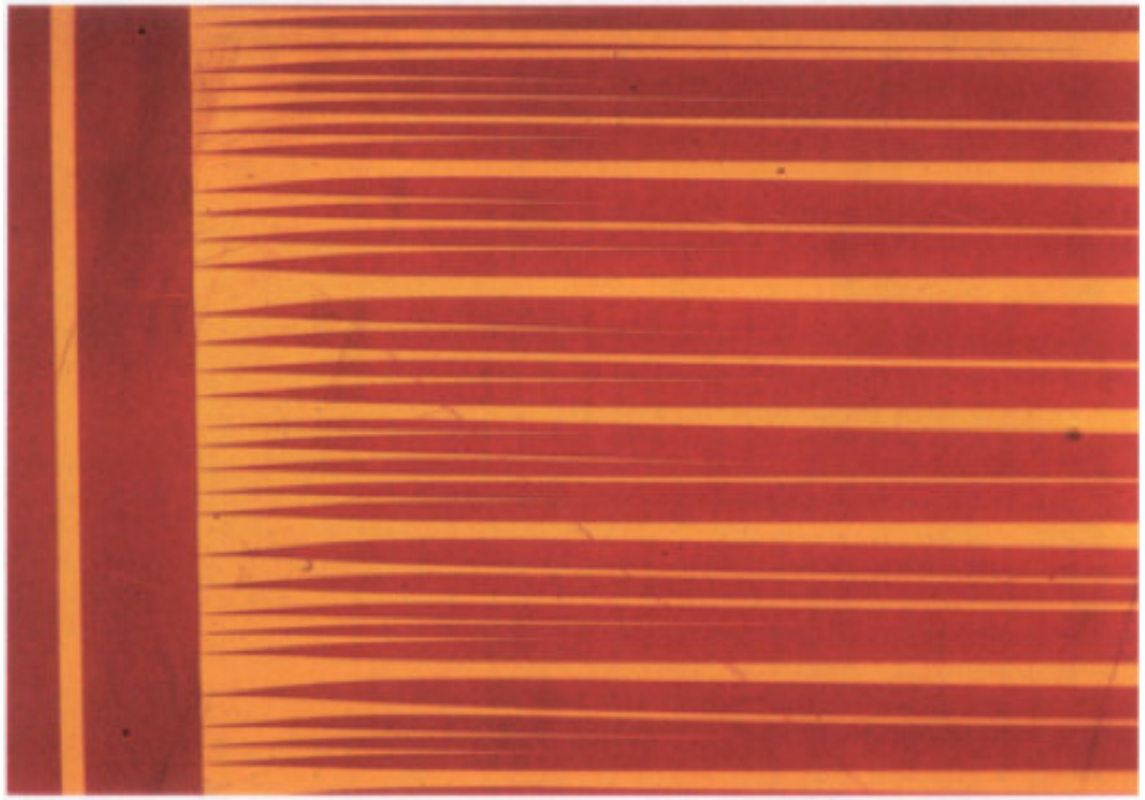
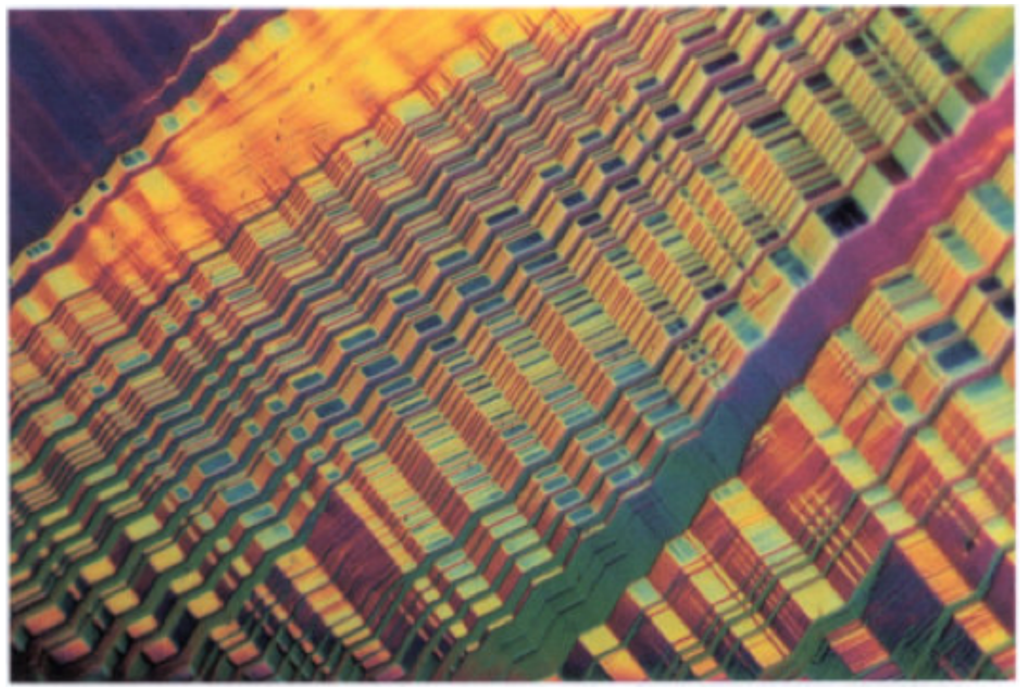
Swelling Thin Elastic Sheets

Spontaneous pattern formation from localized swelling

Twinning in metal alloys

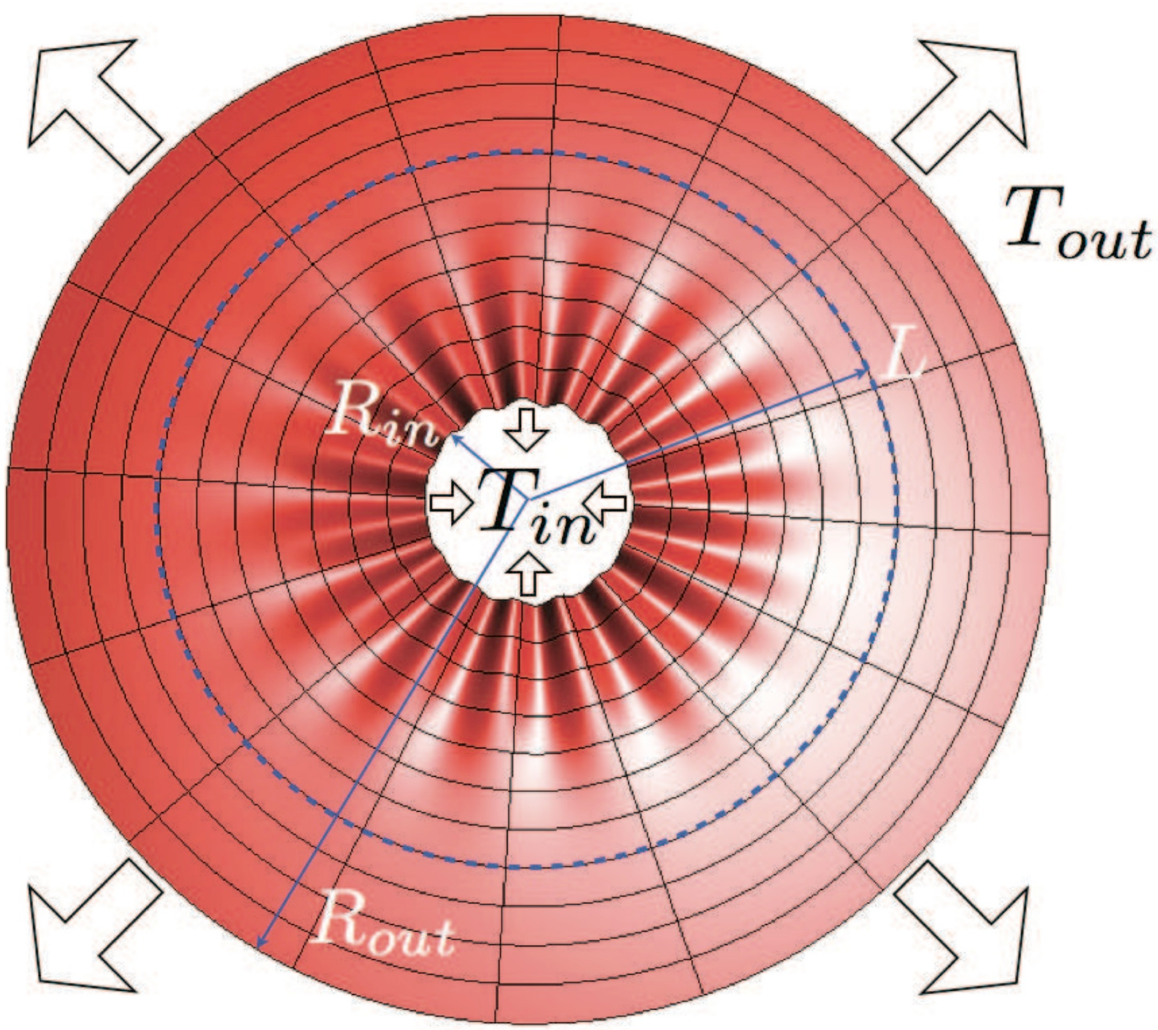


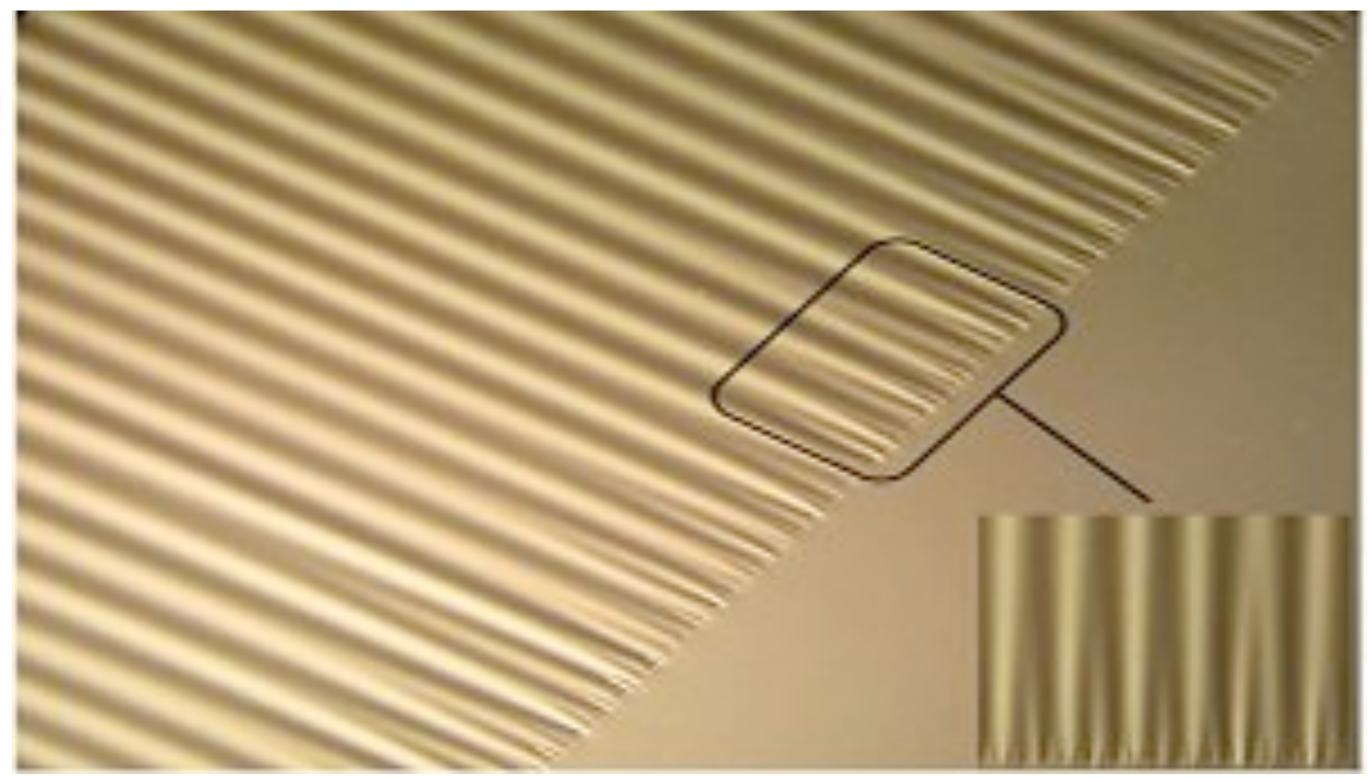




Bhattacharya, Kaushik. Microstructure of martensite: why it forms and how it gives rise to the shape-memory effect. Vol. 2. Oxford University Press, 2003.

Thin elastic sheets



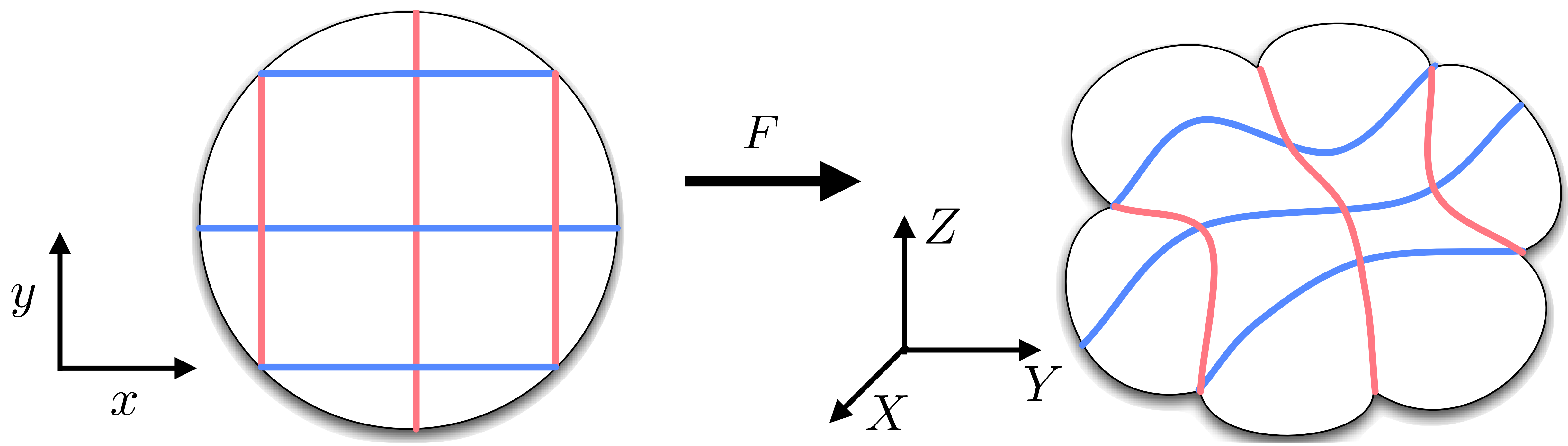


Huang, Jiangshui, et al. "Smooth cascade of wrinkles at the edge of a floating elastic film." Physical review letters 105.3 (2010): 038302.

Davidovitch, Benny, et al. "Prototypical model for tensional wrinkling in thin sheets." Proceedings of the National Academy of Sciences 108.45 (2011): 18227-18232.

Non-Euclidean Model

$\Omega \subset \mathbb{R}^2$ $\qquad\qquad$ $F(\Omega) \subset \mathbb{R}^3$



$$\mathbf{g} = \underbrace{g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2}_{\text{"metric induced by growth"}}$$

$$F^*\mathbf{g}_{\mathbb{R}^3} = \underbrace{d\mathbf{x}(\nabla F)^T(\nabla F)d\mathbf{x}^T}_{\text{"metric induced by } \mathbb{R}^3\text{"}}$$

The equilibrium configuration of a sheet of thickness t is a $W^{2,2}$ map that **minimizes:**

$$E[F] = \underbrace{\int_\Omega \|(\nabla F)^T\nabla F - \mathbf{g}\|^2 \, dA_\mathbf{g}}_{\text{"stretching energy"}} + t^2 \underbrace{\int_\Omega |D^2F|^2 \, dA_\mathbf{g}}_{\text{"bending energy"}}$$

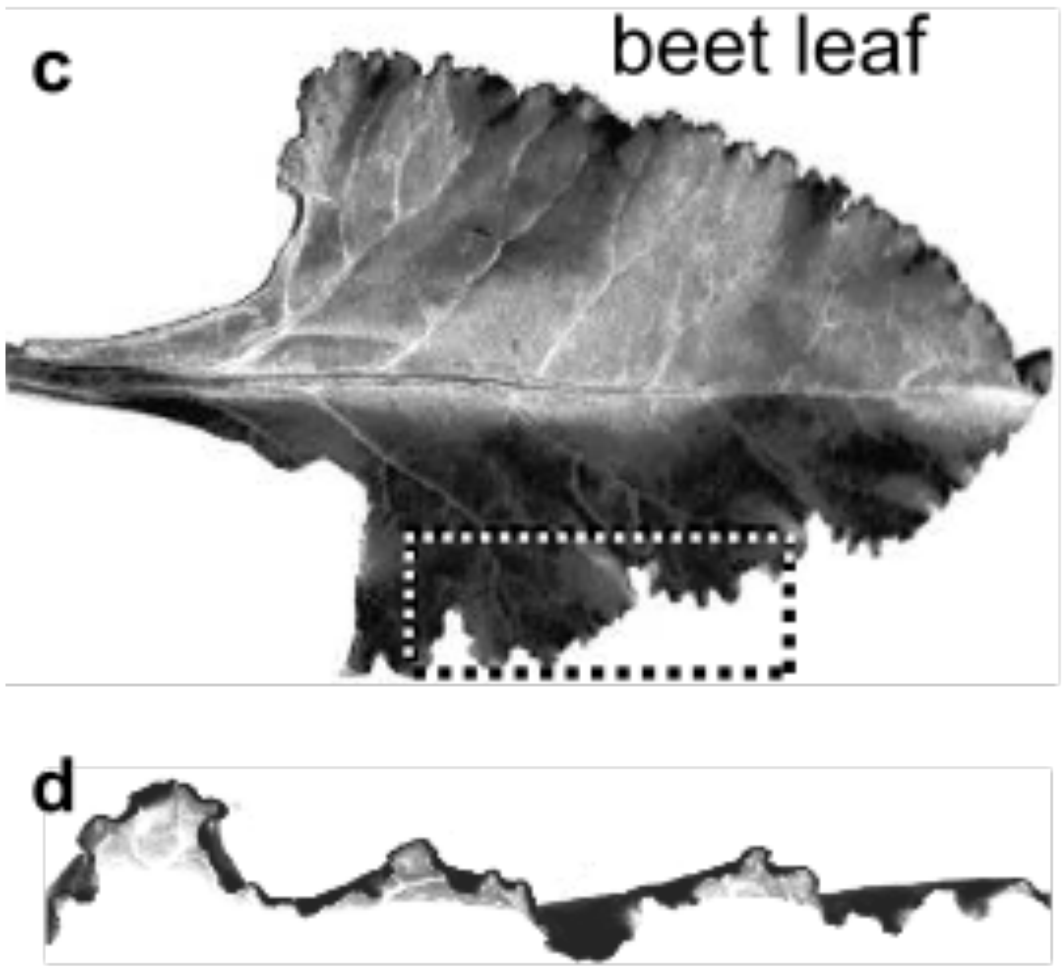
0 $\qquad$ $k_1^2 + k_2^2$

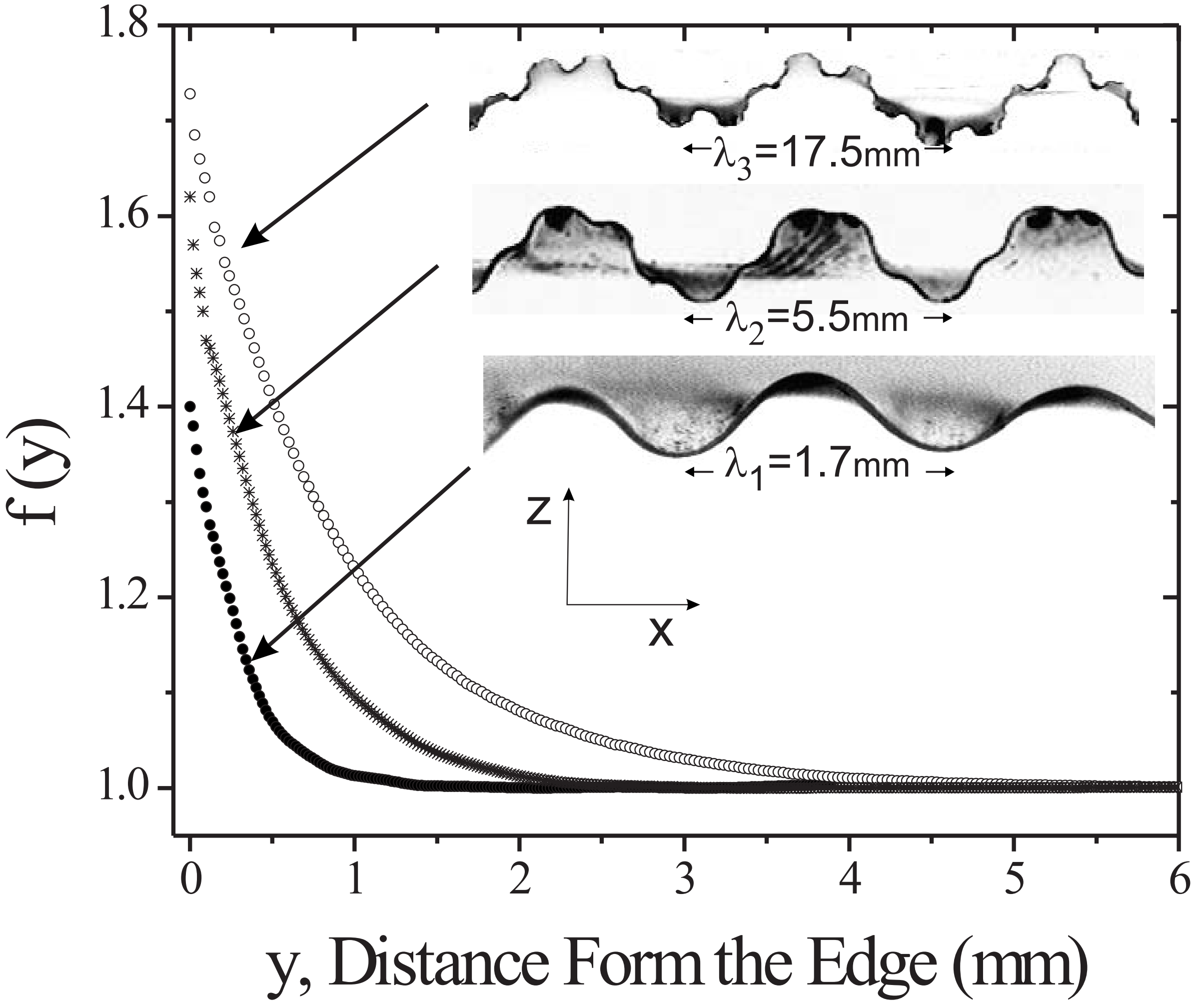
zero stretching $\iff$ $(\nabla F)^T\nabla F = \mathbf{g}$ $\iff$ isometric immersion.

Experimental Observations

Ω is a **strip geometry** with metric:

$$\mathbf{g} = (1 + f(y))dx^2 + dy^2$$





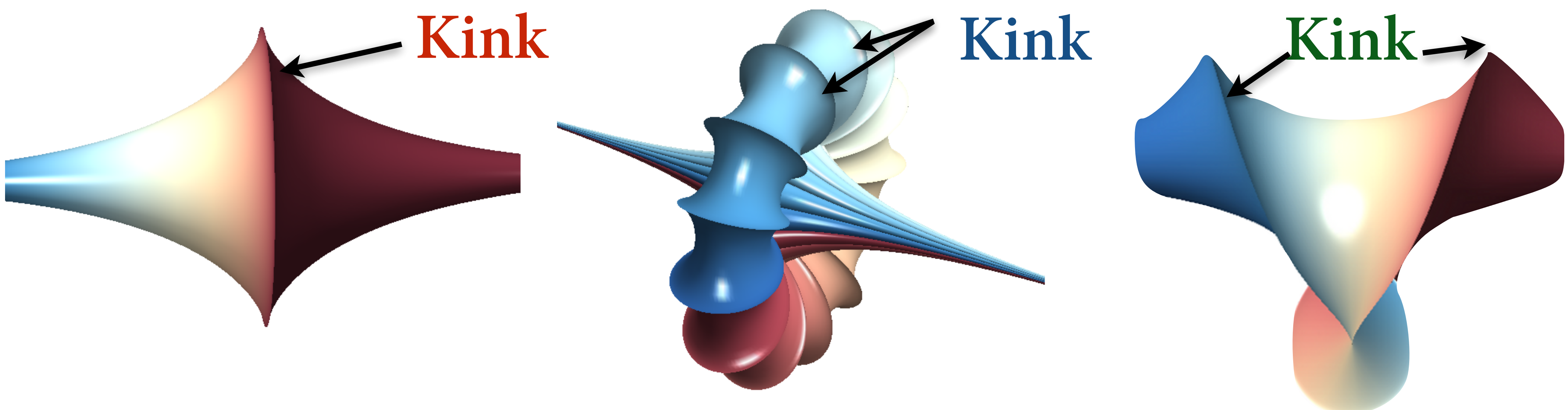
Multiple scale buckling

Sharon, E., Roman, B., & Swinney, H. L. (2007). Geometrically driven wrinkling observed in free plastic sheets and leaves. Physical Review E, 75(4), 046211.

Toy Problem

Hyperbolic Plane: Assume the metric has **constant negative Gaussian curvature.**

Summary of Known Results: Given a local smooth isometric immersion of a metric with negative Gaussian curvature, this immersion cannot be extended smoothly beyond a finite distance *d*. Moreover, the singularities form a "singular edge", i.e. a one-dimensional submanifold on which the surface fails to be C^2.



Pseudosphere | Breather Surface | Kuen's Surface

A natural question is what is the relationship between the existence of these singularities and the observed morphologies in thin elastic sheets.

Negative Curvature: Disk Geometry

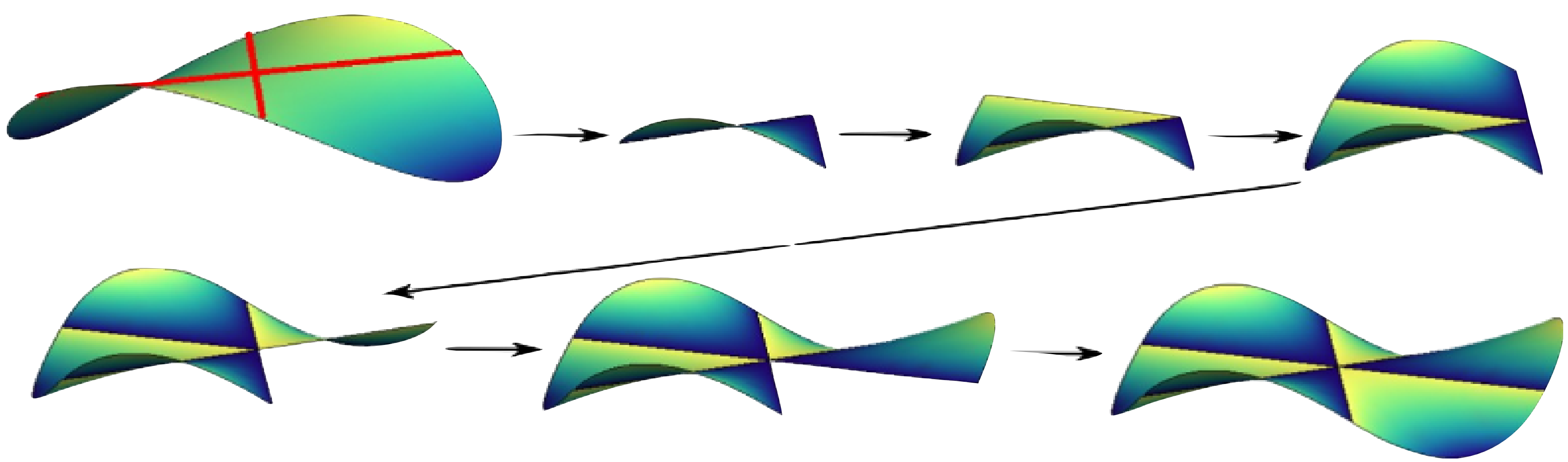
Small slopes approximation: Introduce dimensionless curvature $\epsilon = \sqrt{-K}R$.

Ansatz: $X = x + \epsilon^2 u_1(x,y), \quad Y = y + \epsilon^2 u_2(x,y), \quad Z = \epsilon\eta(x,y)$

Solvability Condition: $\det(D^2\eta) = -1.$

One parameter family of solutions: $\eta_a = \frac{1}{2}\left(ax^2 - \frac{1}{a}y^2\right)$

Pick: $a = \cot(\pi/n).$

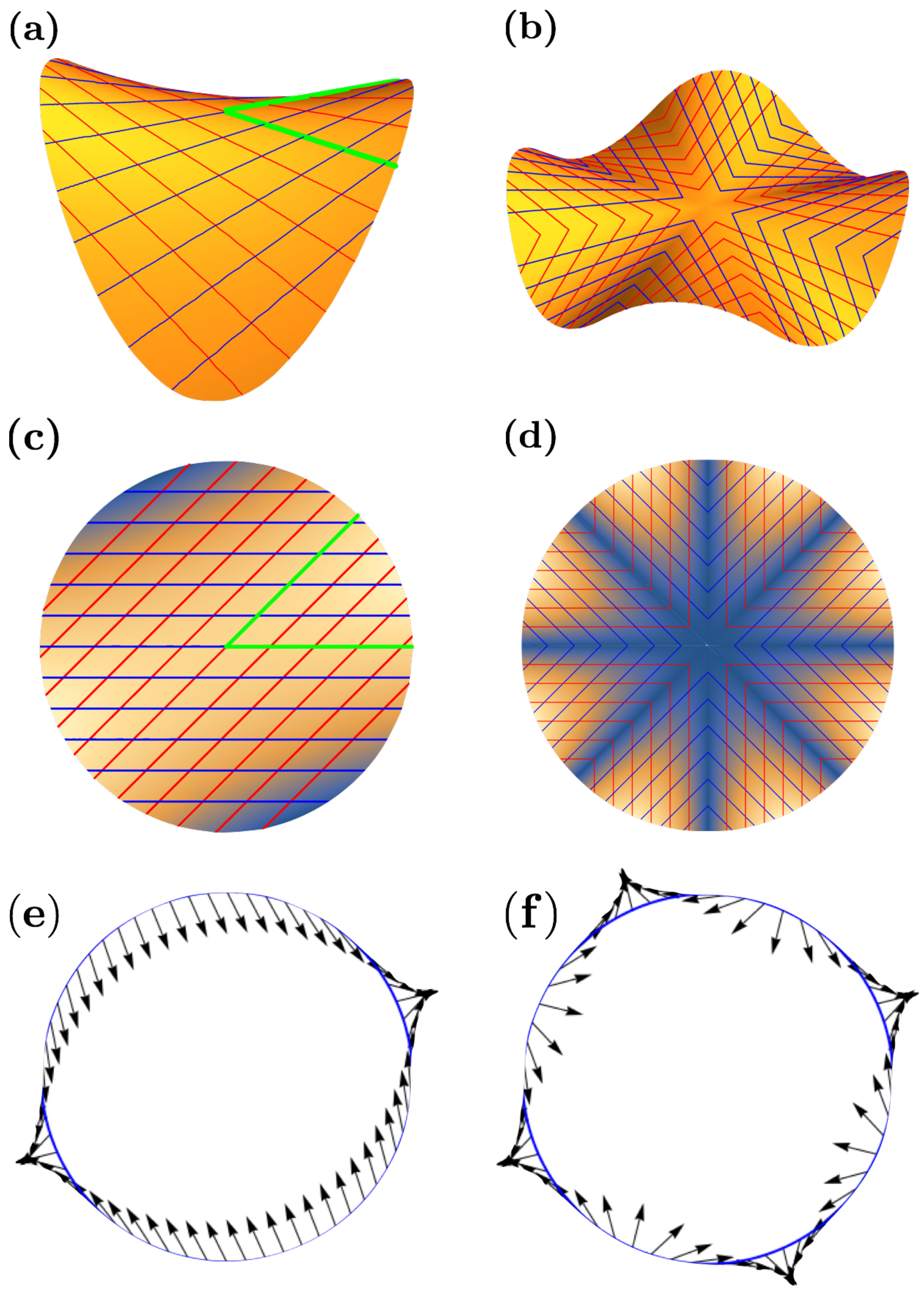


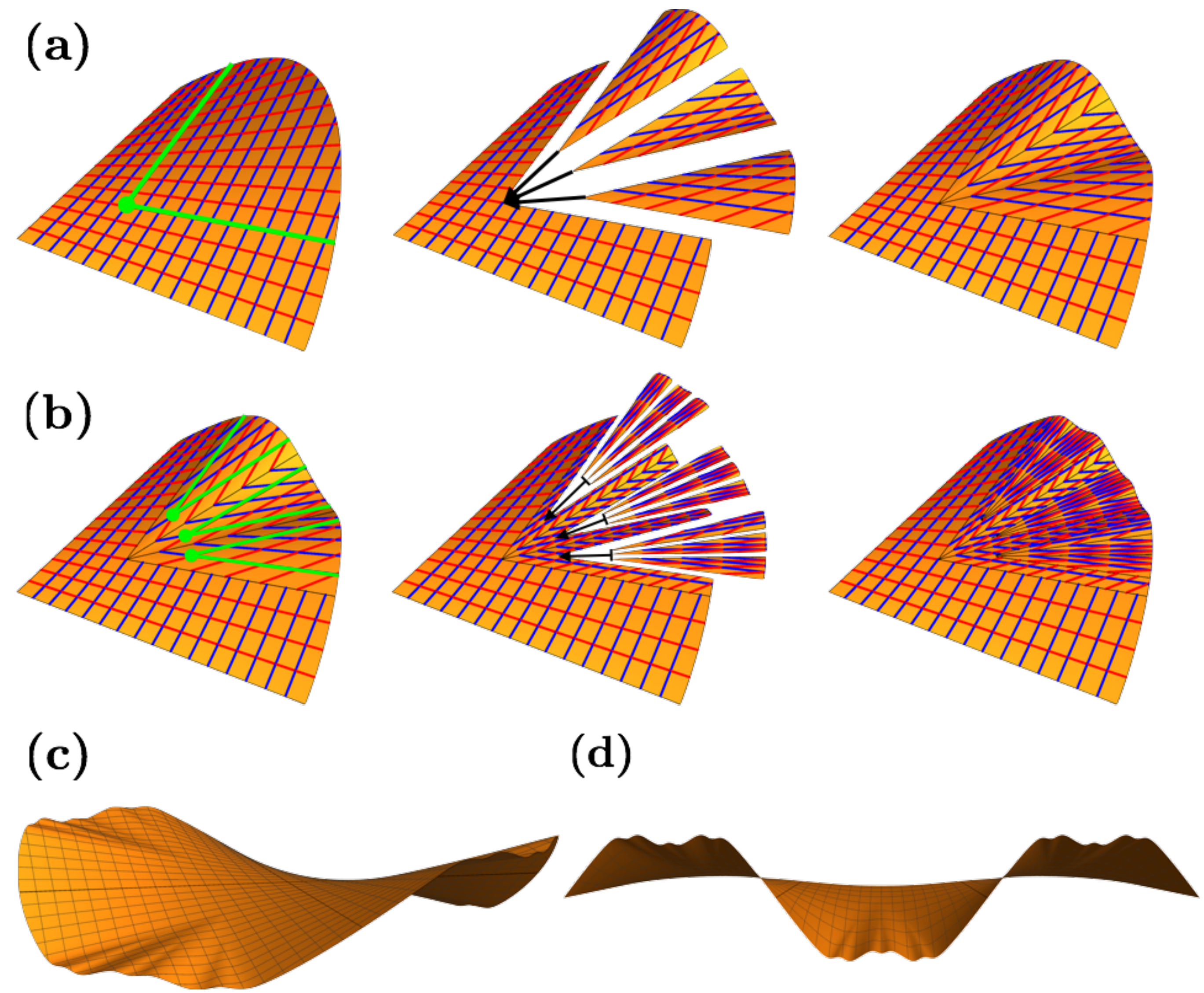
Theorem:(J. Gemmer, SV). D is the unit disk with a metric whose FvK curvature is -1. For all $n \in \mathbb{N}$, we have a n-periodic local minimizer for the elastic energy, whose energy satisfies the bounds

$$\min(C_1, C_2 n t^2) \leq E_{FvK} \leq \min(c_1, c_2 n^2 t^2).$$

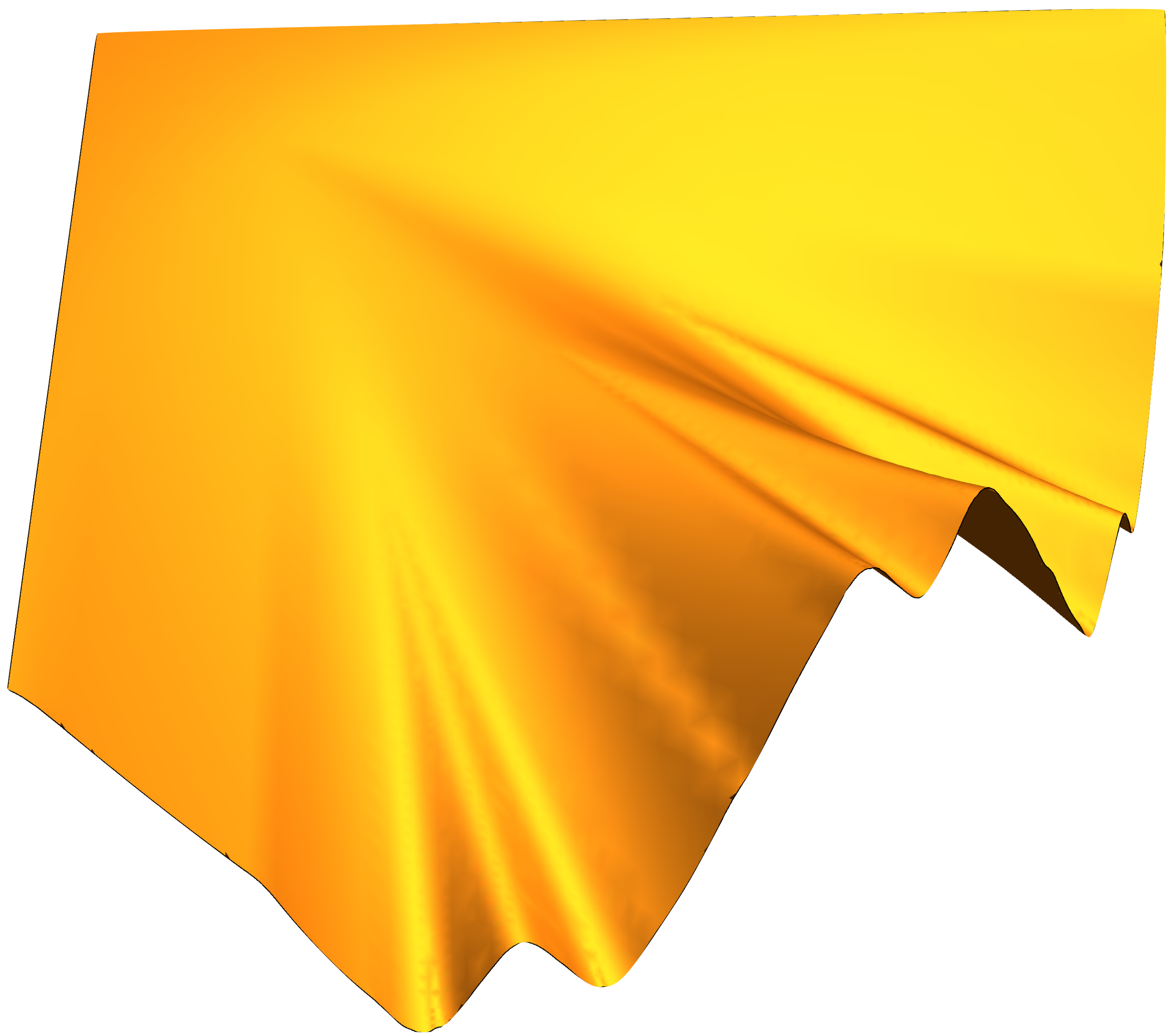
Multiple Branch Points

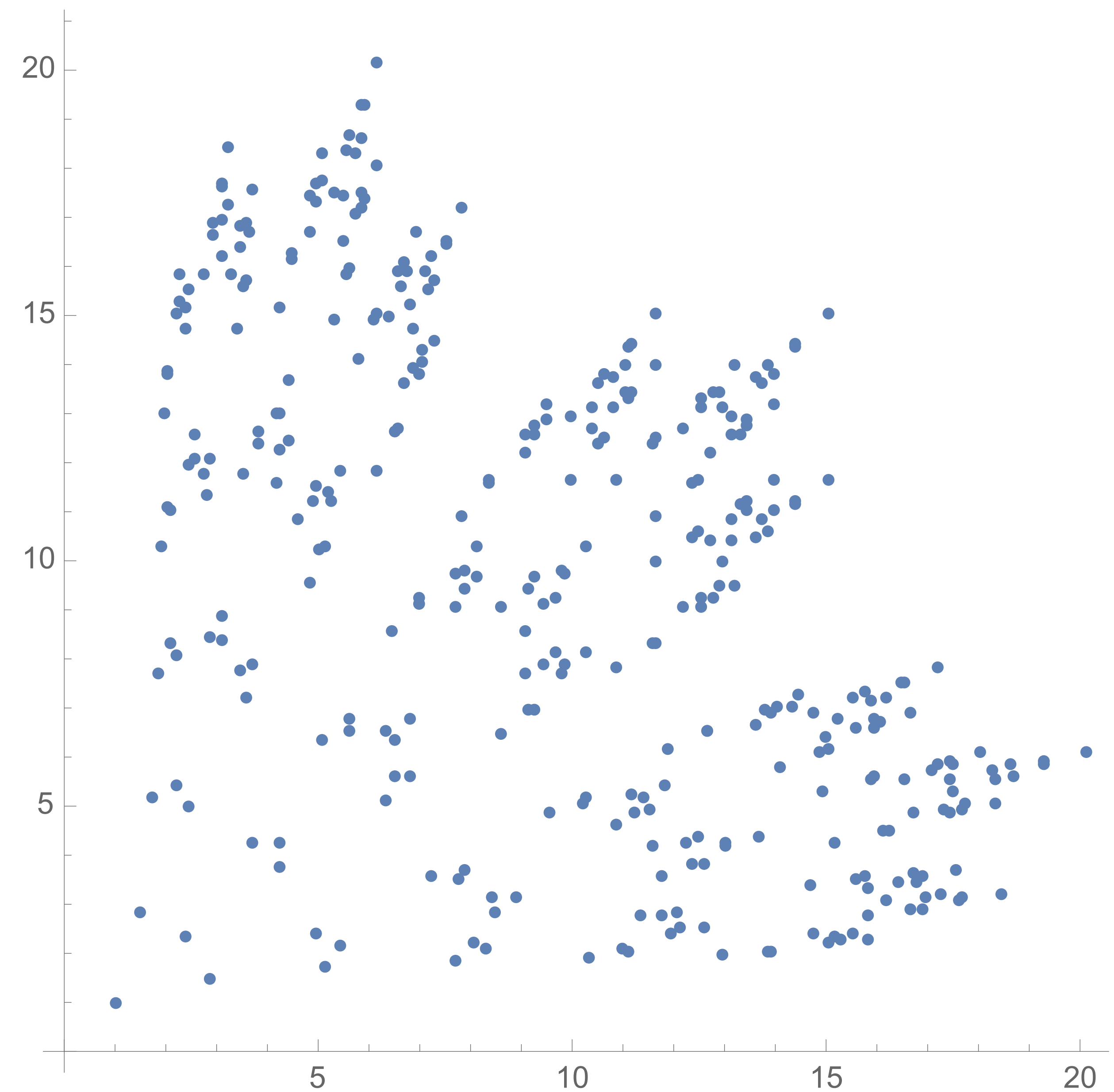
The origin is not special and multiple branch points can be introduced





“Generic” isometric immersions

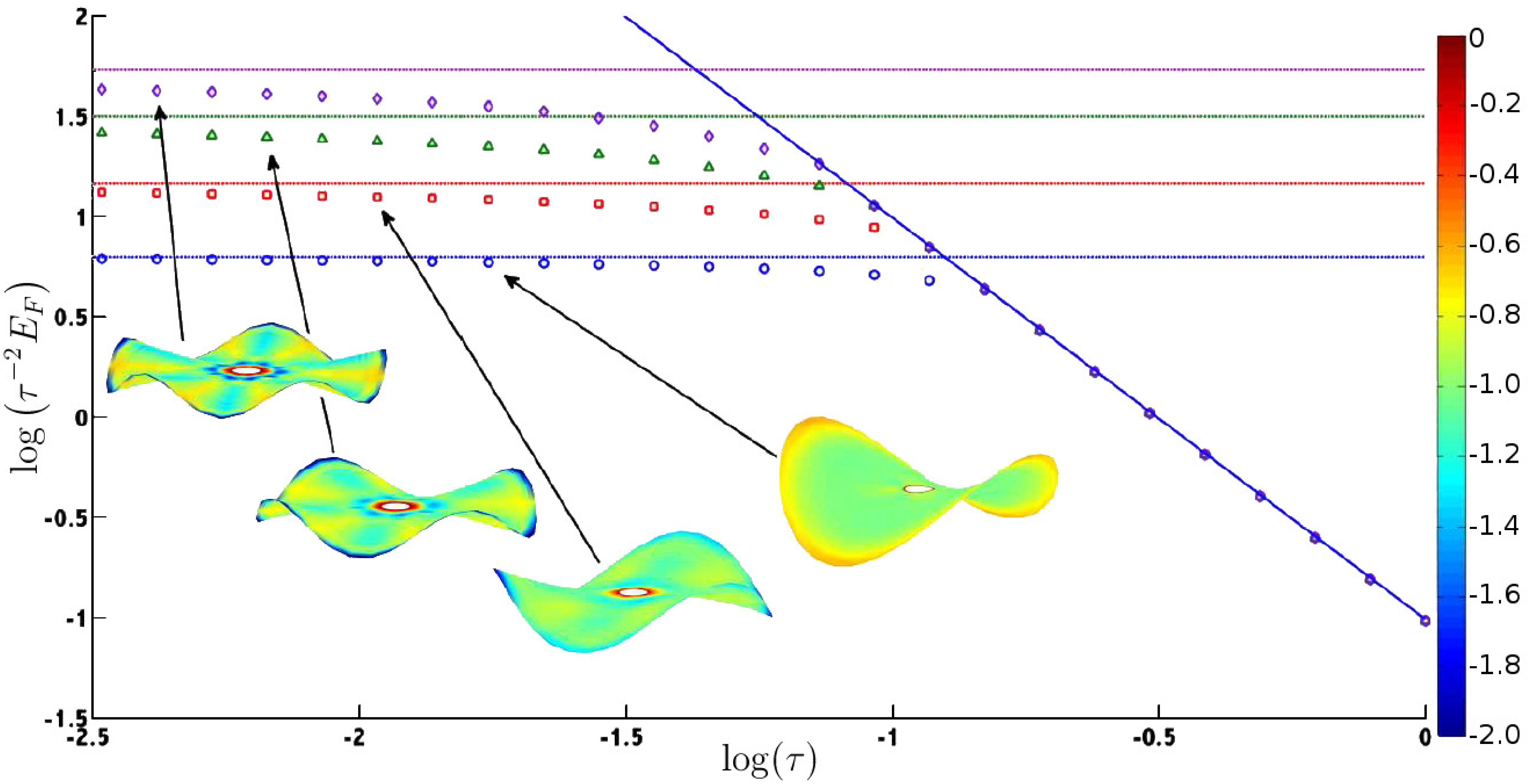




Bifurcation points

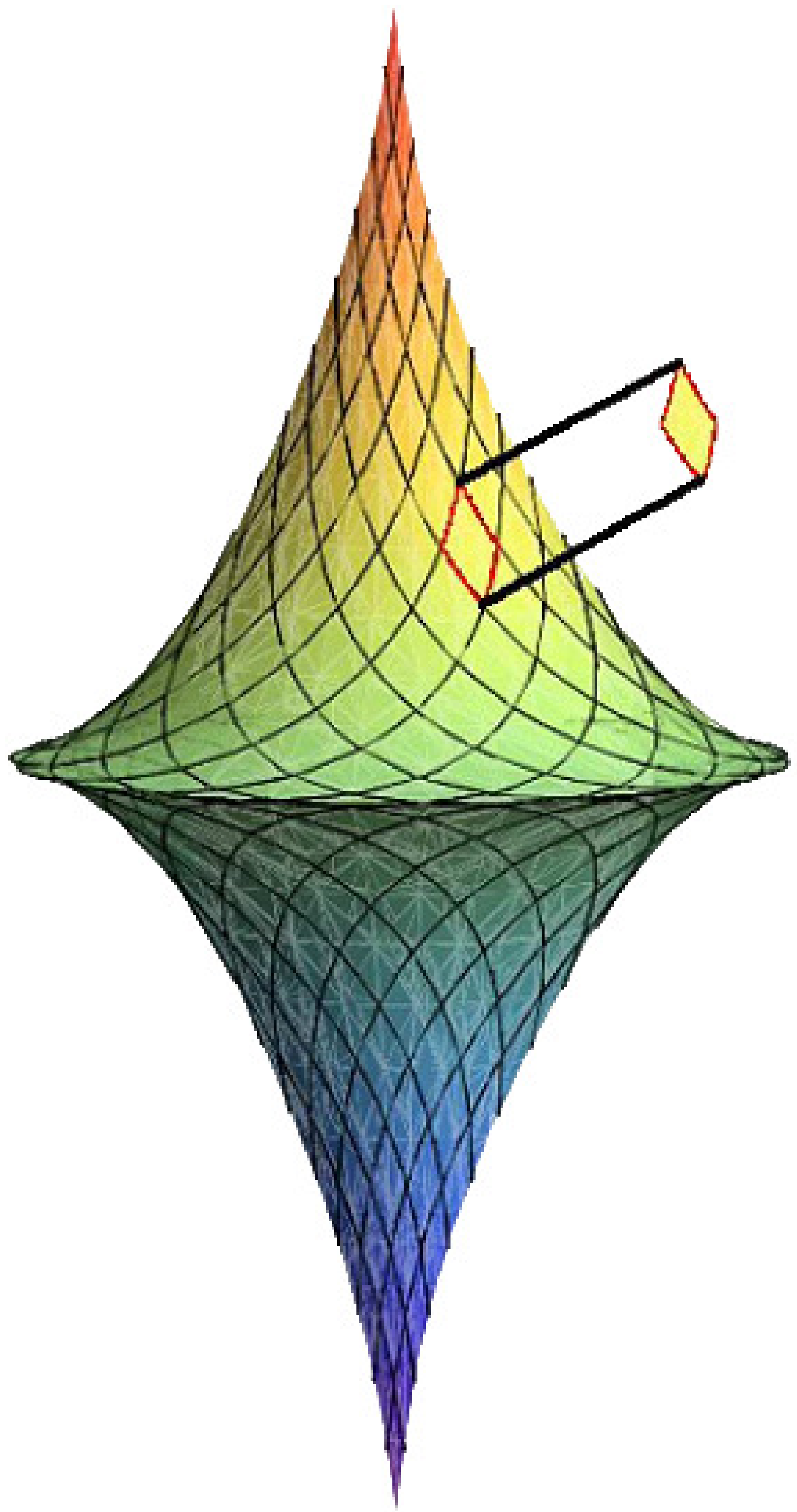
Small Slopes Decreasing Thickness

In this asymptotic regime, the saddle shape is energetically preferred.



Small slopes theory always predicts a saddle shape.

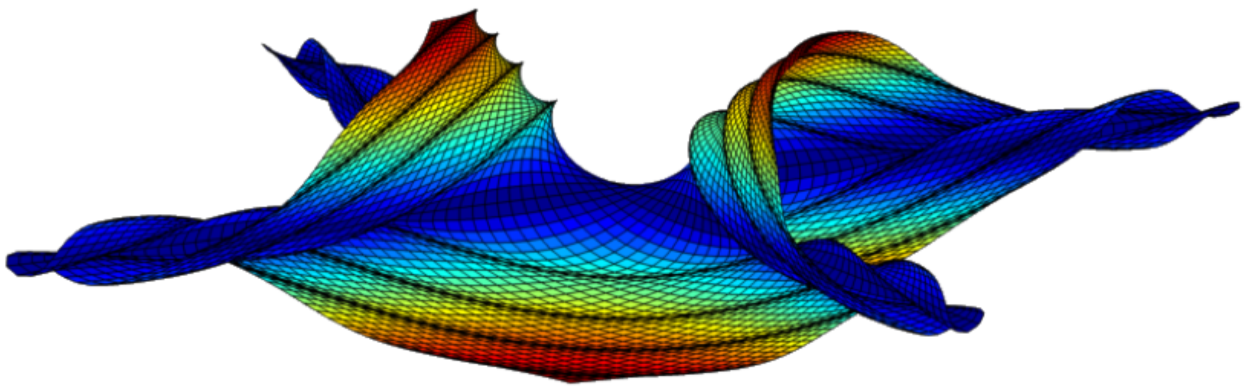
Chebychev Nets and the Hyperbolic Plane



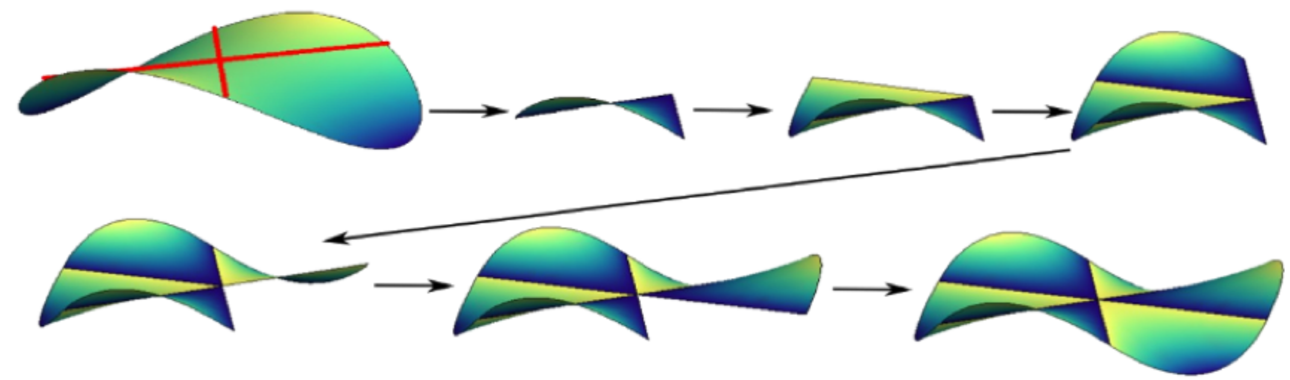
- A Chebychev Net is a configuration $\mathbf{x}(u, v)$ with metric

$$\mathbf{g} = du^2 + \cos(\phi(u, v))dudv + dv^2.$$

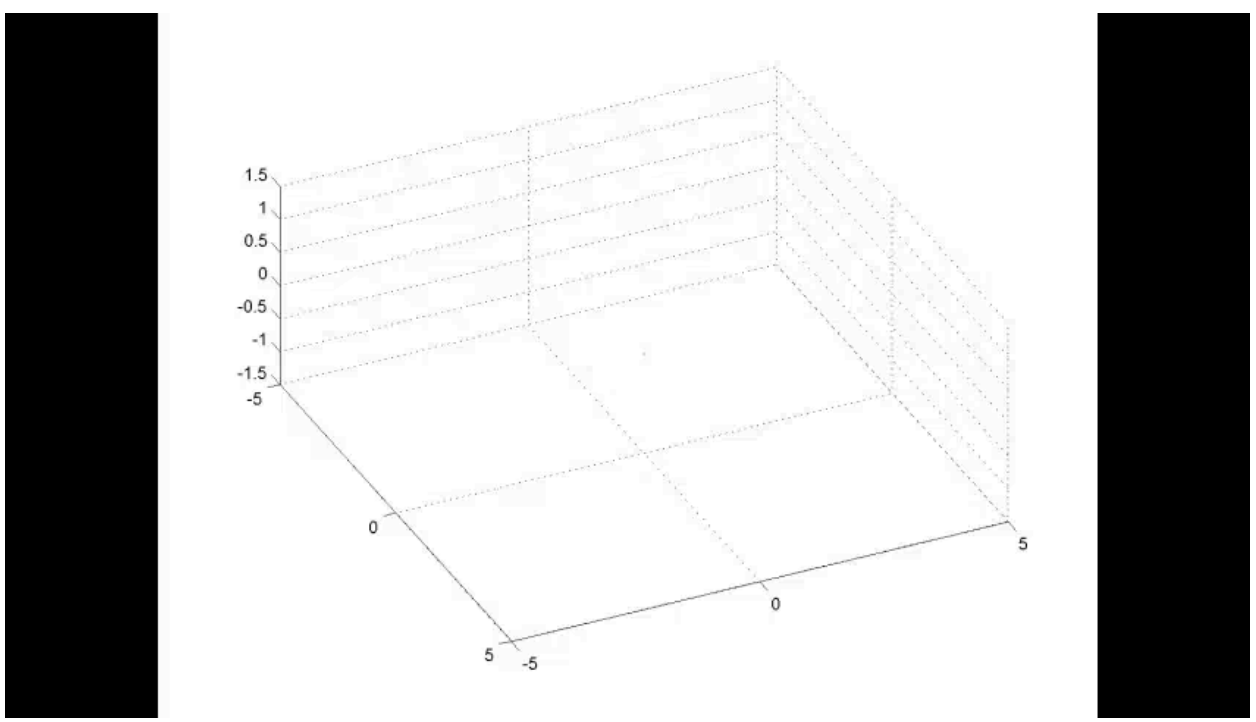
Small Slopes Lifted to Exact Isometry



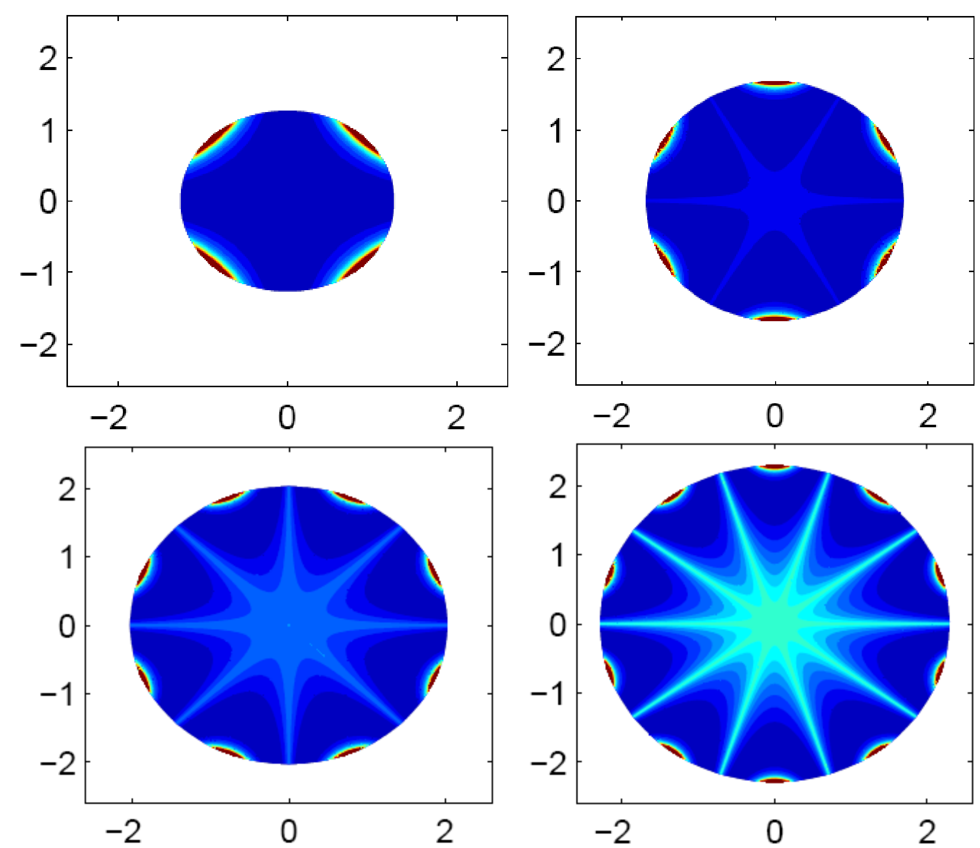
Isometric Immersion



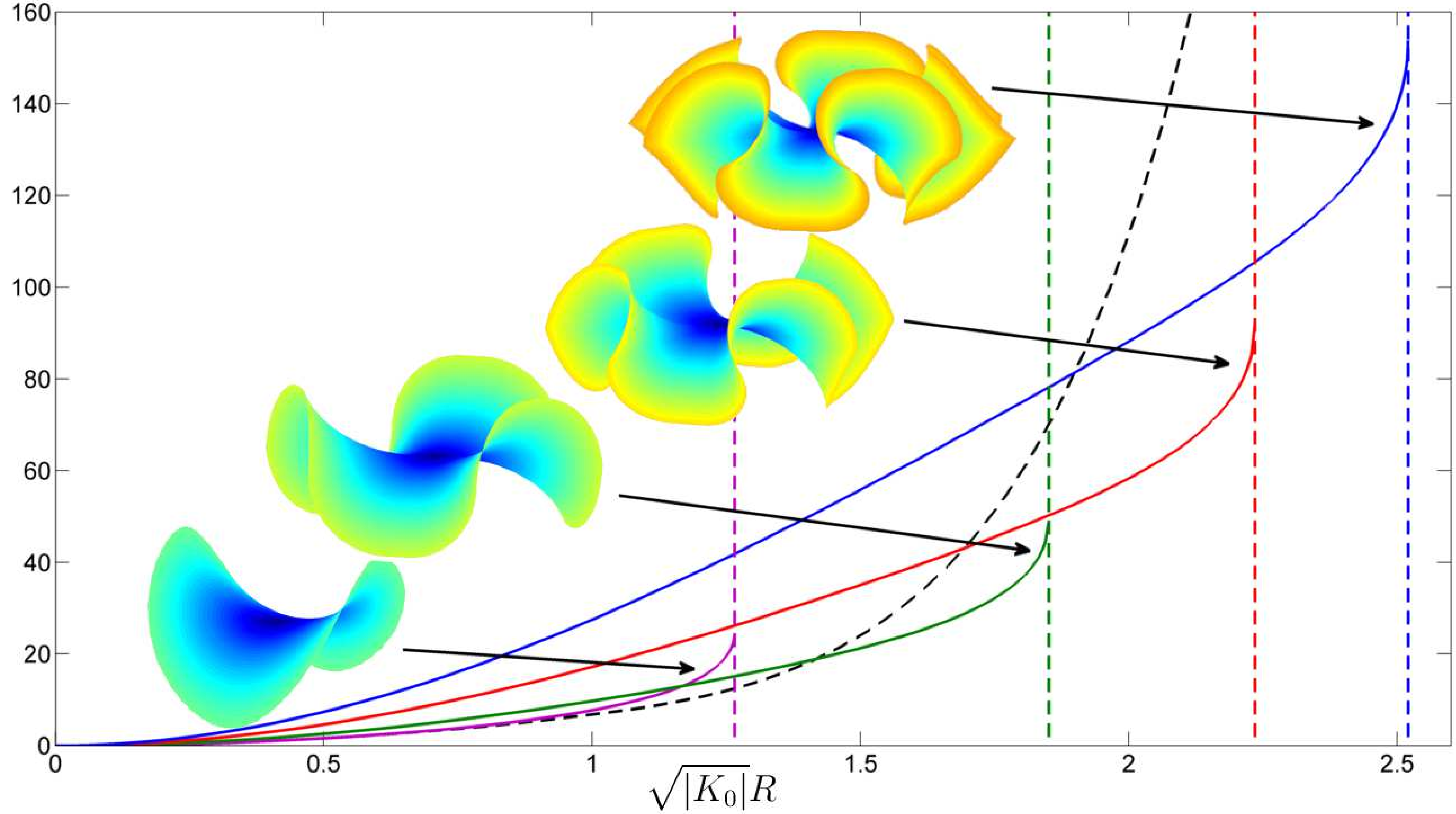
Periodic Reflections



$$n \sim \exp\left(\sqrt{K_0}R\right)$$



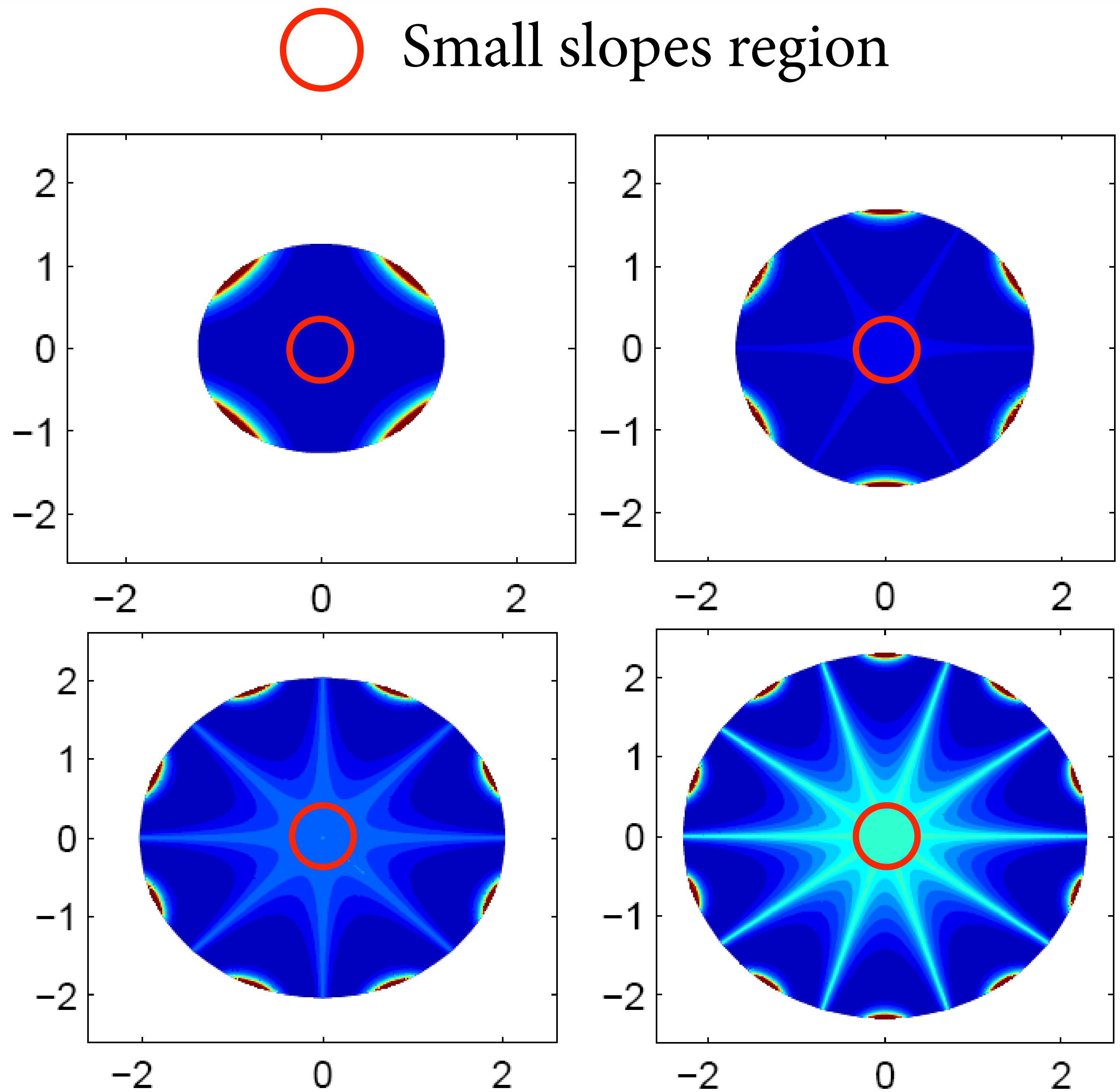
Piecewise Smooth Exact Isometries



Non-smooth isometries have lower energy than their smooth counterparts. This cannot be captured by the small slopes approximations.

$$\max\{|k_1|, |k_2|\} \geq \frac{1}{64} \exp\left(|K_0|^{\frac{1}{2}} R\right),$$

Concentration of Energy



Conjecture: Branch points can be introduced near the singularities to lower bending energy. The introduction of branch points is energetically favorable to global refinement of the wavelength.

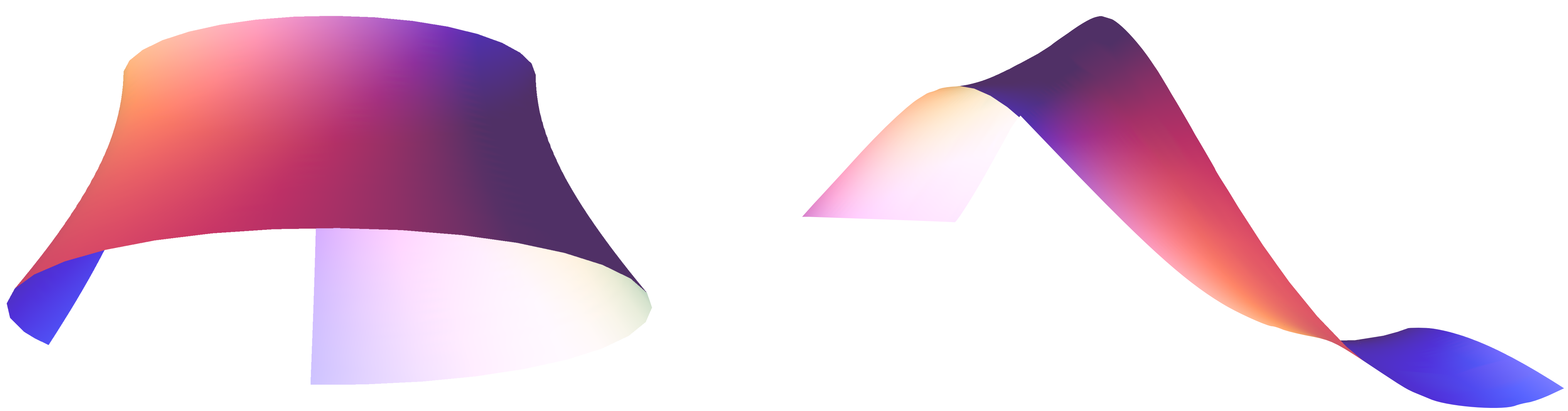
Strip Geometry

Ω is a **strip geometry** $\Omega = \mathbb{R} \times [0, W]$ with metric:

$$\mathbf{g} = (1 + 2\epsilon^2 f(y))\, dx^2 + dy^2$$

where $\epsilon > 0$ and for $\alpha \in (0, \infty)$:

$$f(y) \sim (1 + y/l)^{-\alpha}.$$



Isometric Immersions Exist

Strip Geometry

To match the metric to lowest order in ϵ we assume an ***ansatz*** of the form:

$$F(x,y) = \left(x + \epsilon^2 u(x,y), y + \epsilon^2 v(x,y), \epsilon w(x,y)\right)$$

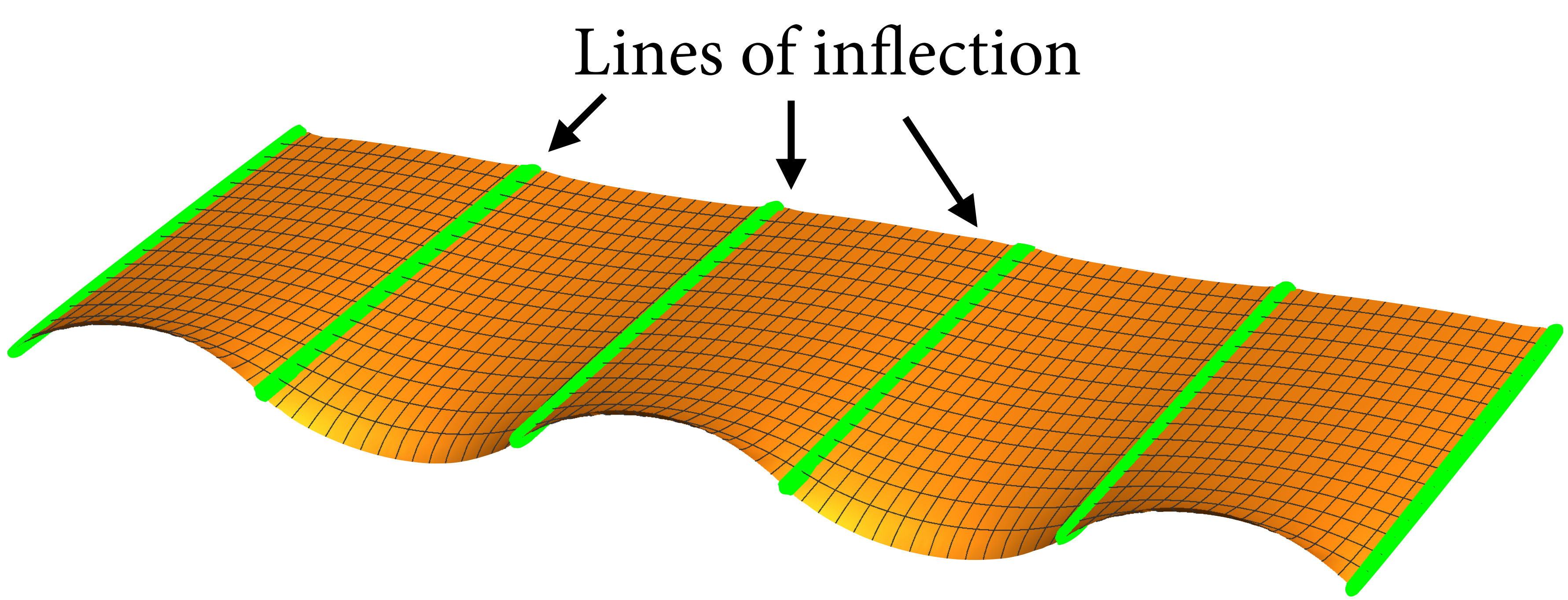
The lowest order condition for an isometry is the following small-slope version of **Gauss's Theorema Egregium:**

$$\det(D^2(w(x,y)) = w_{xx}w_{yy} - (w_{xy})^2 = -f''(y).$$

We can solve the **Monge-Ampere equation** by assuming $\omega(x,y) = \phi(y)\psi(x)$,

$$\phi(y) = (1 + y/l)^{-\alpha/2}$$

$$\psi'^2 \pm k^{\frac{2\alpha}{1+\alpha}} |\psi|^{\frac{2\alpha}{1+\alpha}} = 1.$$



Energy of Single Wavelength Isometries

For a single wavelength isometry with wavenumber k the bending content per unit length $\bar{B}$ satsifies:

$$\bar{B} \sim C_1 k^2 \int_0^W \frac{dy}{(1+y/l)^{\alpha}} + \frac{C_2}{k^2 l^4} \int_0^W \frac{dy}{(1+y/l)^{\alpha+4}}$$

Optimizing over k the **"global" wavelength** satisfies:

$$\lambda_{glob} \sim l \left| \frac{(1+W/l)^{1-\alpha} - 1}{(1+W/l)^{-3-\alpha} - 1} \right|^{\frac{1}{4}}.$$

However the optimal **"local" wavelength** satisfies:

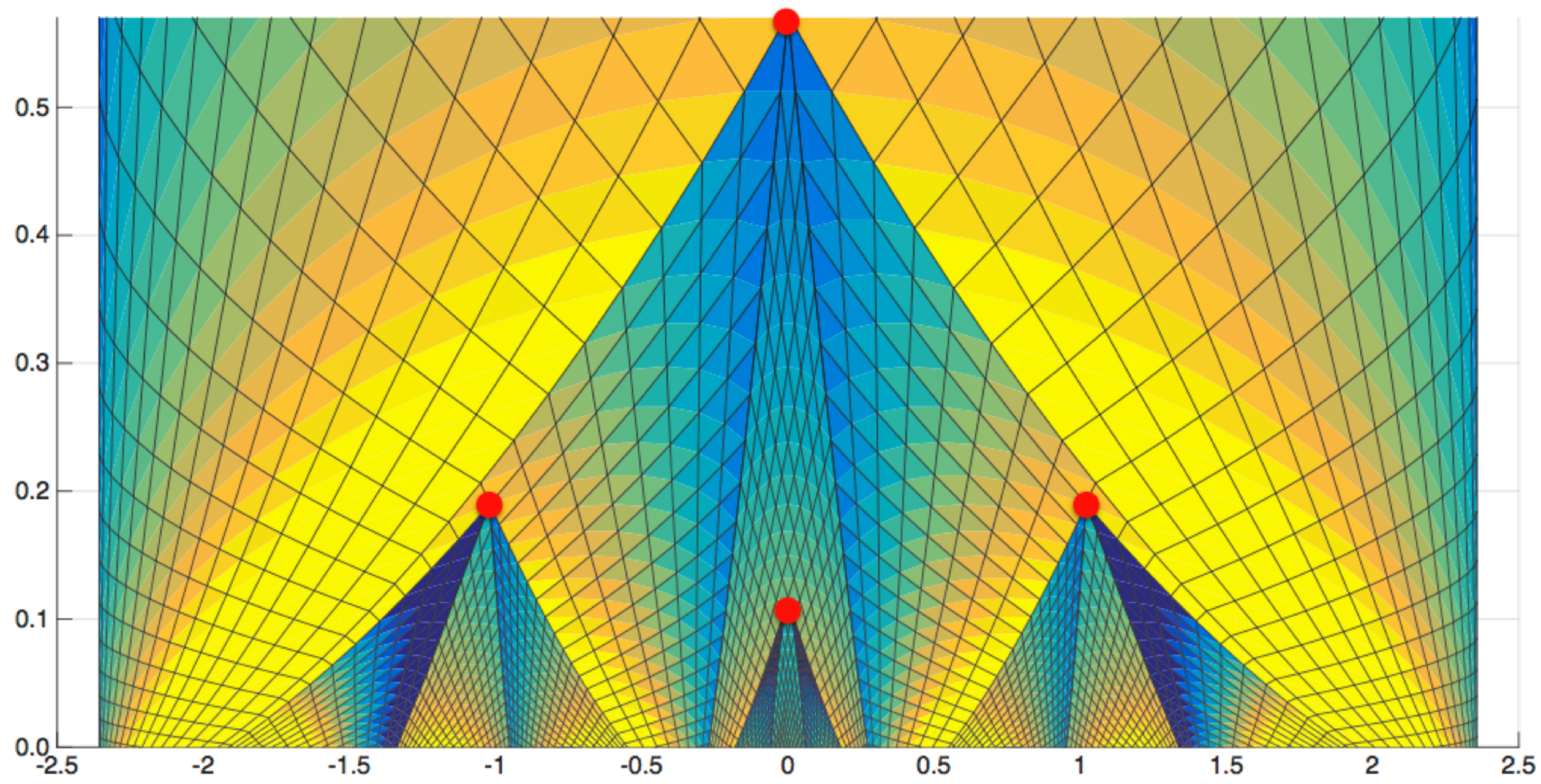
$$\lambda_{loc}(\zeta) \sim l(1+y/l) = (y+l).$$

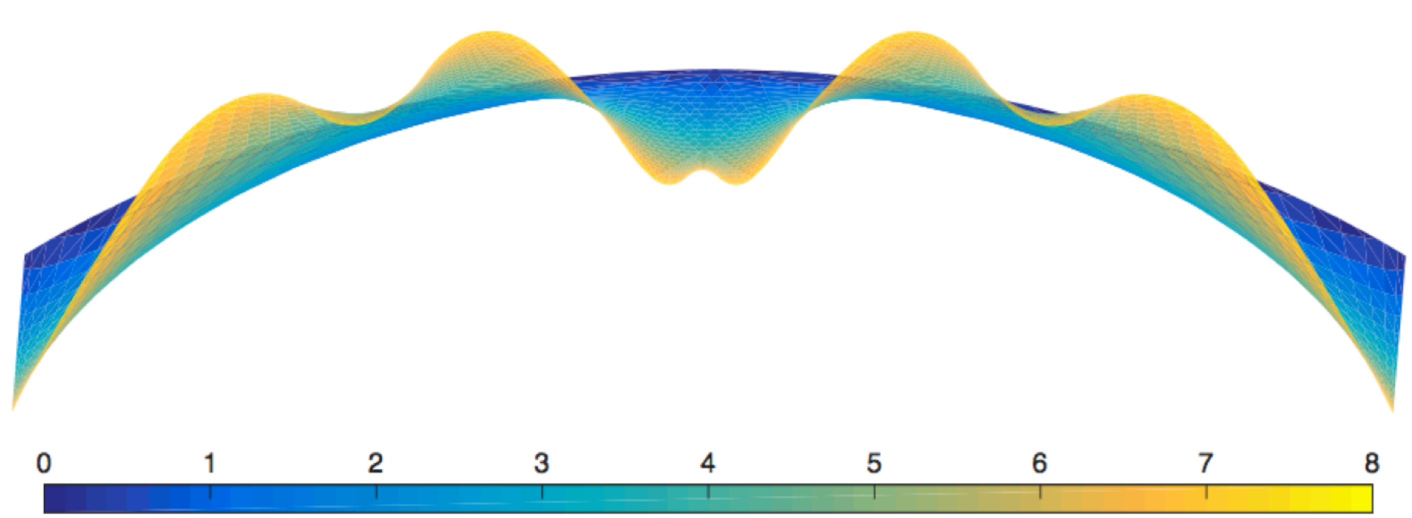
There is a competition between the two principal curvatures in the sheet.

Branch Points

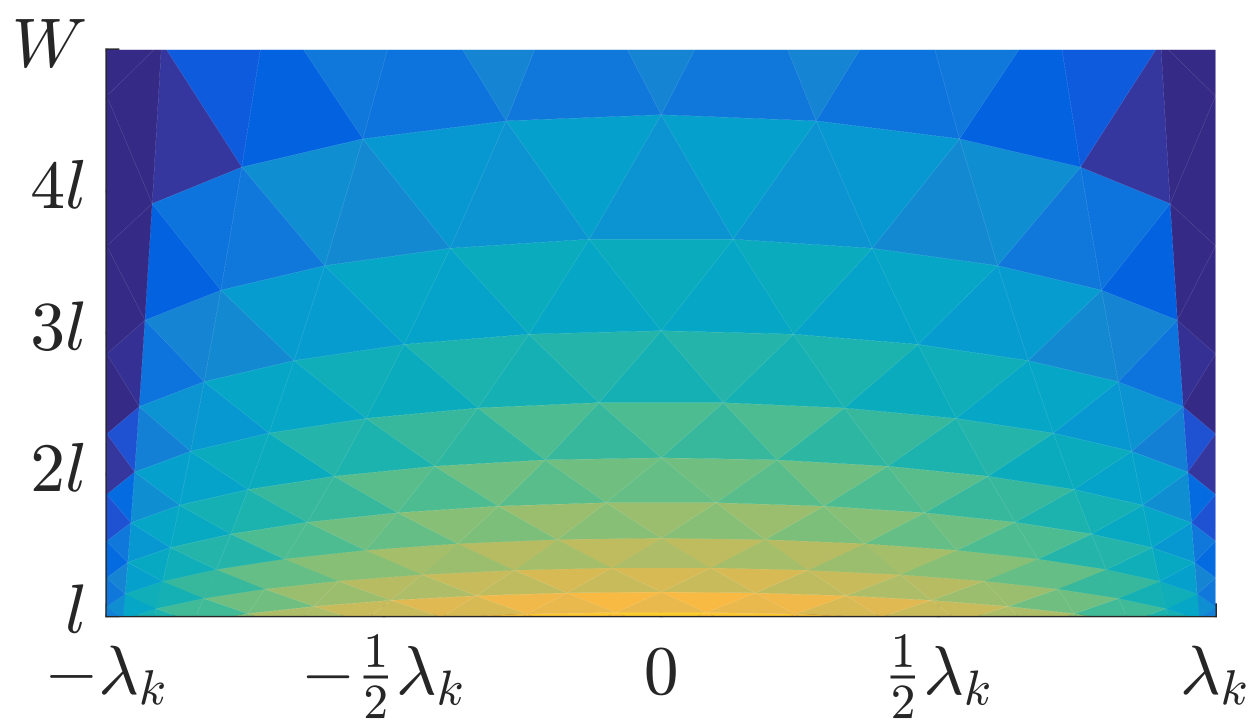
Beltrami-Enneper Theorem: The rate of rotation of the tangent plane along an asymptotic line is proportional to the square root of the Gaussian curvature.

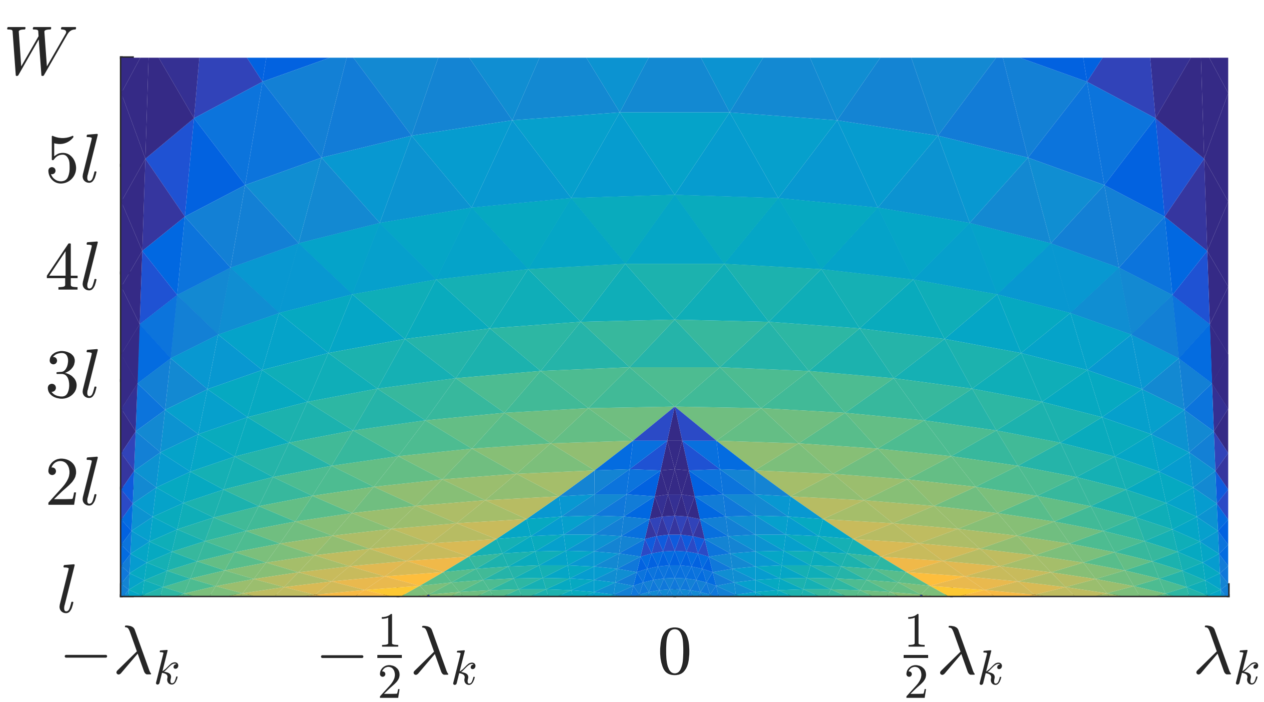
Disparity: $\eta = \frac{H}{\sqrt{|K|}} = \sqrt{\frac{k_1}{k_2}} + \sqrt{\frac{k_2}{k_1}}$

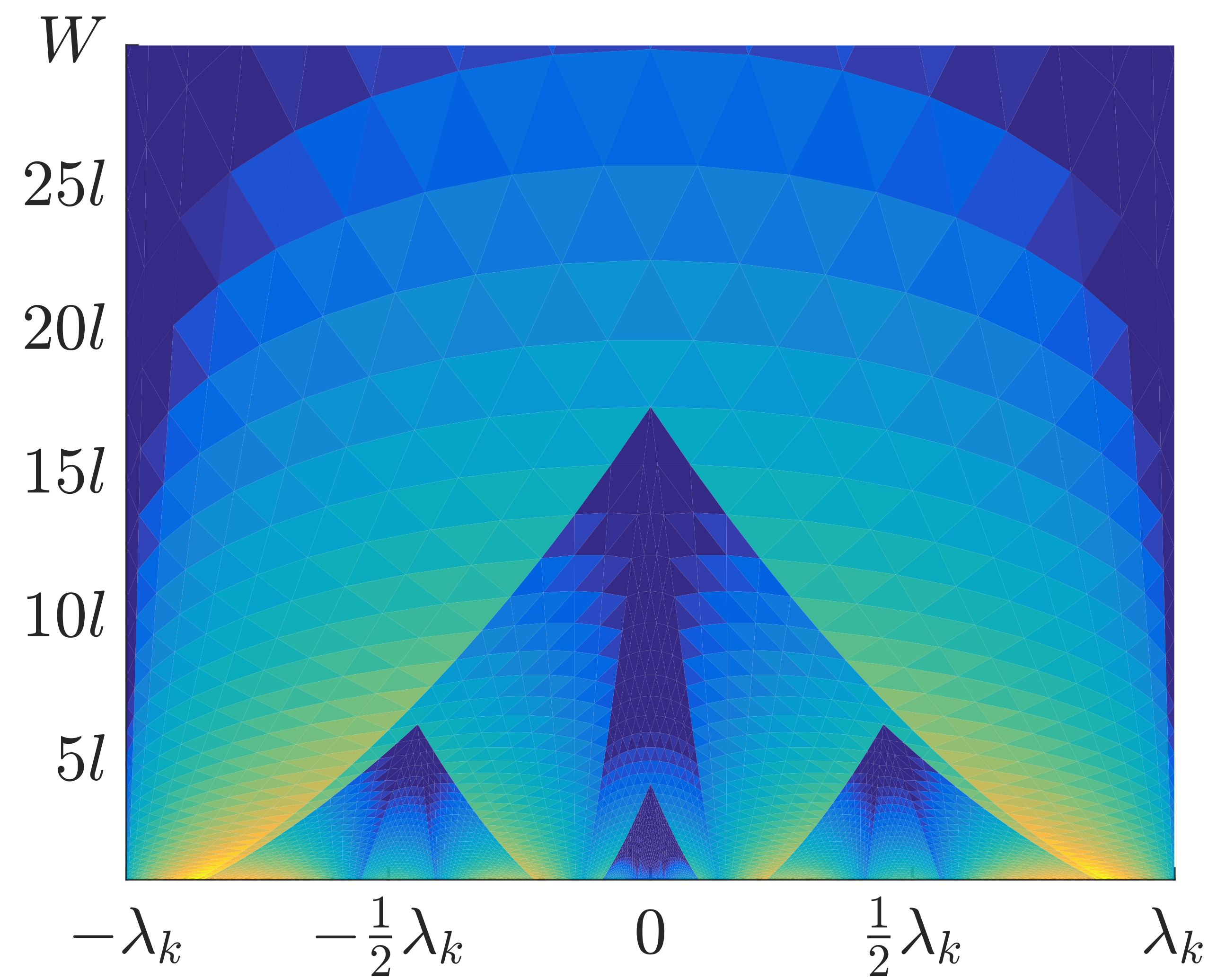




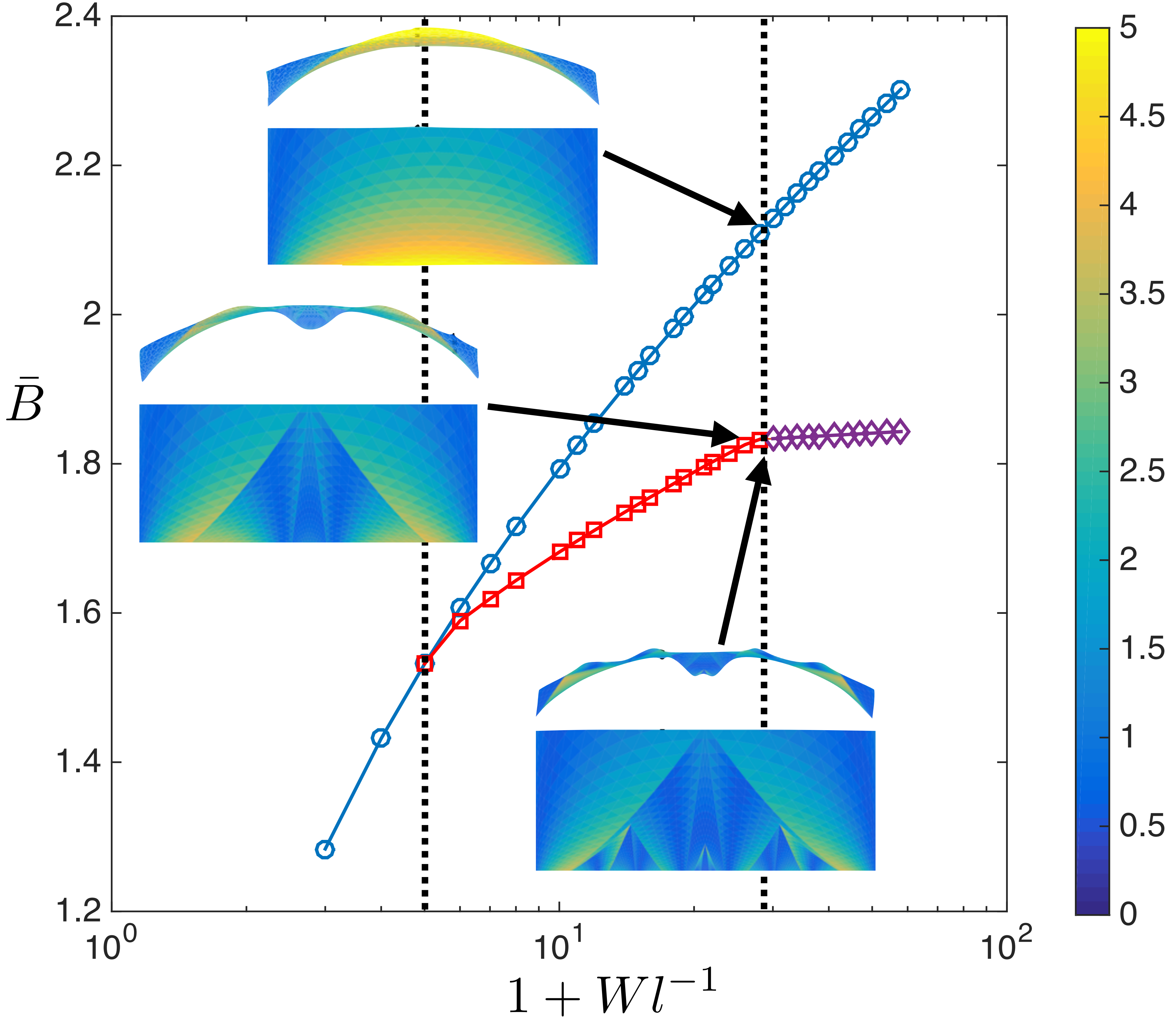
Bifurcation with Disparity







Energy of Branch Points



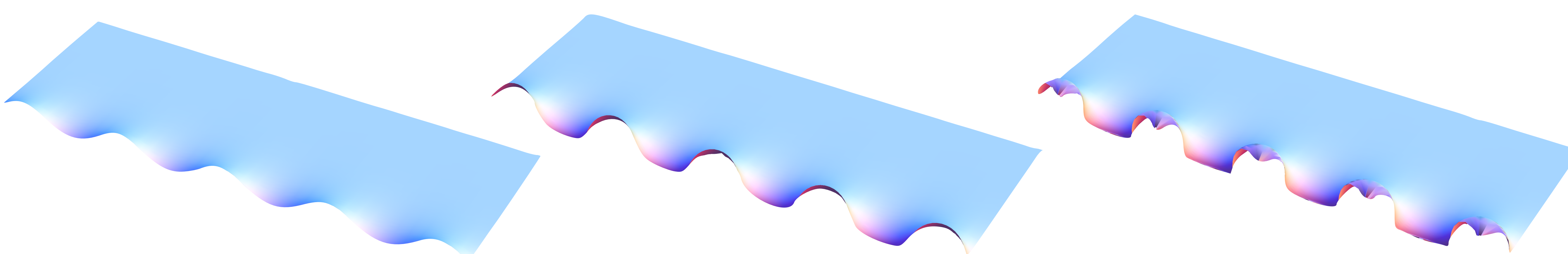
Series Solution

$$\alpha = \infty \text{ (Exponential Case)}$$

$$\omega_0(x,y) = \frac{\exp(-\beta y)\cos(kx)}{\sqrt{2}k}$$

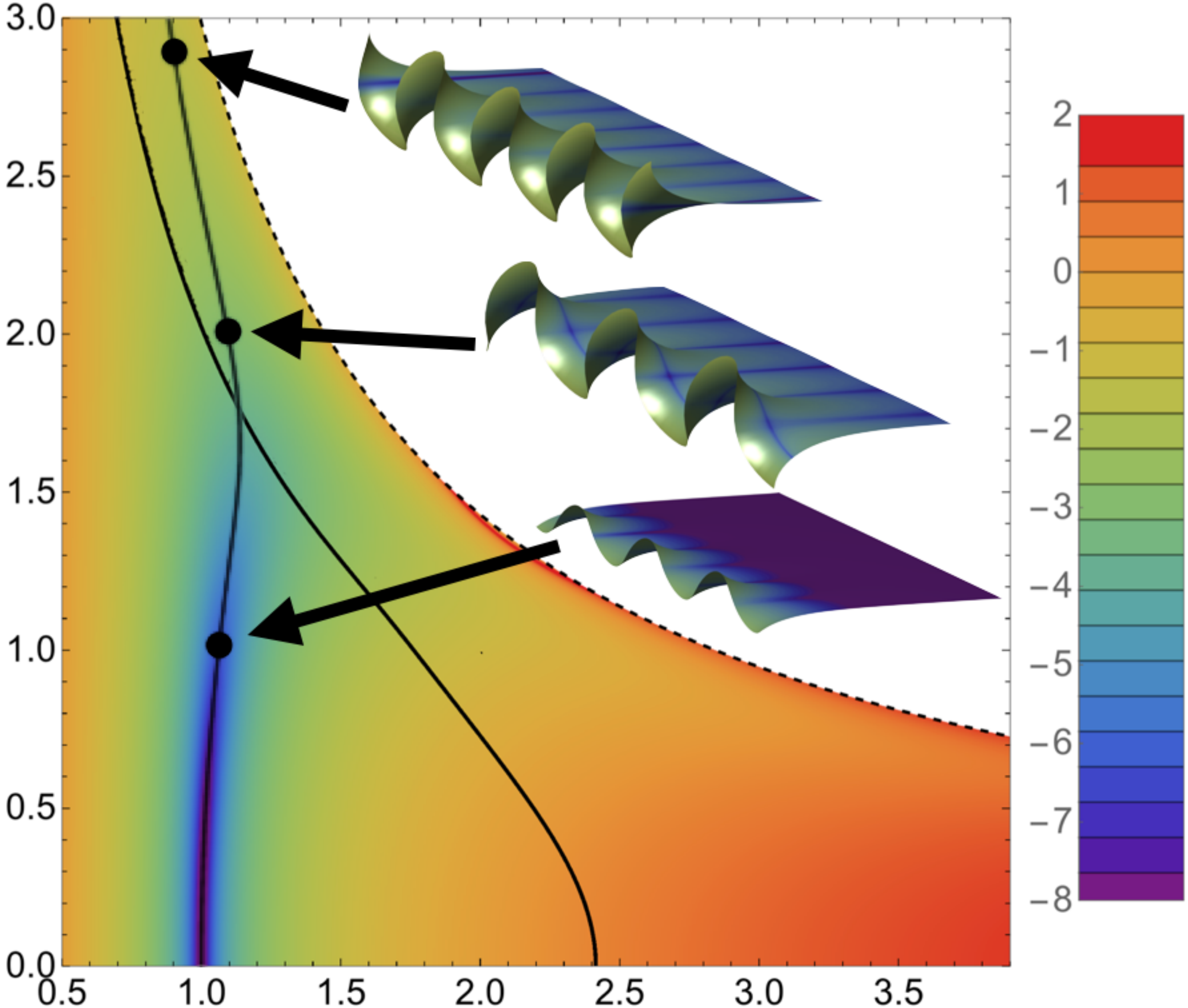
$$\omega_1(x,y) = e^{-3\beta y}\left(\frac{\left(k^2-3\beta^2\right)\cos(3kx)}{576\sqrt{2}k^3} - \frac{\left(\beta^2+k^2\right)\cos(kx)}{64\sqrt{2}k^3}\right)$$

$$\omega_2(x,y) = \left(\frac{\left(-9\beta^4+43k^4+42\beta^2k^2\right)\cos(kx)}{36864\sqrt{2}k^5} + \frac{\left(-9\beta^4+7k^4+42\beta^2k^2\right)\cos(3kx)}{73728\sqrt{2}k^5} - \frac{\left(9\beta^4+17k^4-42\beta^2k^2\right)\cos(5kx)}{368640\sqrt{2}k^5}\right)e^{-5\beta y}$$



$\epsilon = 1$ $\epsilon = 2$ $\epsilon = 3$

Convergence of Series



Summary

1. Differential growth can lead to non-Euclidean geometries. **A fundamental question is can we deduce the three dimensional shape from exact knowledge of the swelling pattern.**

2. This is a problem with multiple scales. **Can we classify all asymptotic regimes.**

3. Growth is a highly dynamic process. **Perhaps local minimizers are selected along particular dynamic pathways.**

4. **What is the role of the piecewise smooth solutions to the physically observed patterns?**