# Critical Mechanical Structures: Topology and Entropy

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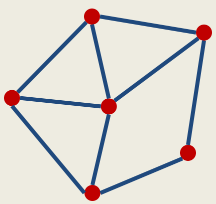
Department of Physics, University of Michigan

KITP workshop: "Geometry, elasticity, fluctuations, and order in 2D soft matter"
Feb 2, 2016

# What Are "Critical Mechanical Structures"?

Full machinery of floppy modes counting (c.f. Tom's talk)



Bond tension

Site force

$$Q \cdot t = f$$
$$Q^T \cdot u = e$$

Site displacement

Bond extension

$$N_B - N_S = N_{d.o.f.} - N_0$$

Rigid-Body

Mechanically Stable: $N_0 = d(d+1)/2$

Stress-Free: $N_S = 0$

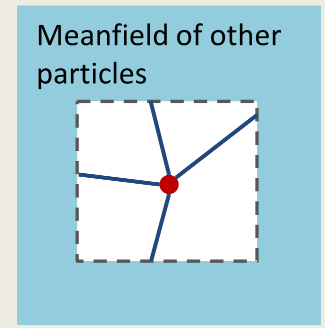
**"Isostaticity = Stable + Stress-free"**

Kindergarten version:

Onset of mechanical stability is at

$$\langle z \rangle = 2d$$

Mean coordination number




Look at site averaged quantities

Mechanical stability

(ignoring self-stress)

$$d = \langle z \rangle / 2$$

Site d.o.f

Mean # of constraints per site


- J. C. Maxwell, Phil. Mag. **27**, 598 (1864).
- C. R. Calladine, Int. J. Solids Struct. 14, 161 (1978).
- Thorpe, J. Non-Cryst. Solids, 57, 355 (1983).
- K. Sun, A. Souslov, X. Mao, and T. C. Lubensky, PNAS 109, 12369 (2012).
- C. L. Kane and T. C. Lubensky, Nat. Phys. 10, 39 (2014).
- Lubensky et al, Rep. Prog. Phys., 78, 073901 (2015)

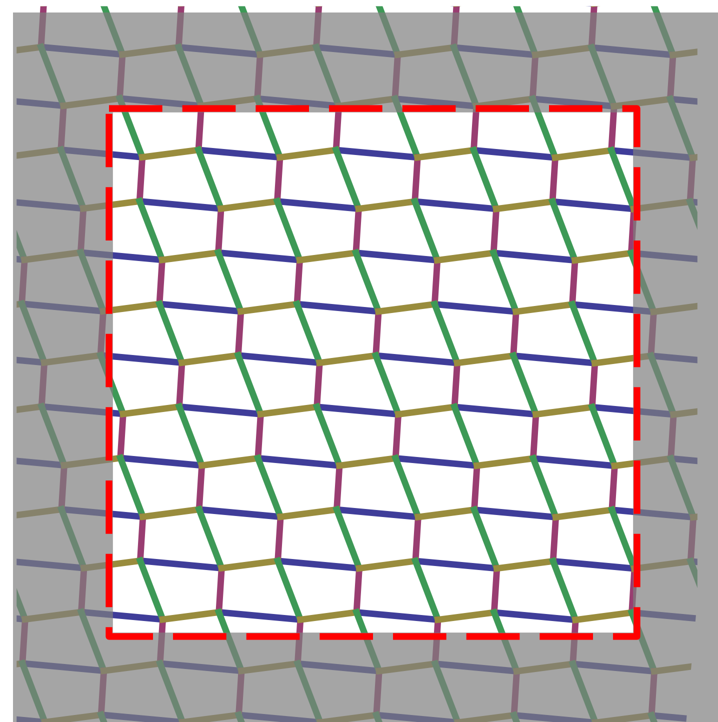
# Why Critical Mechanical Structures Are Interesting?

- Central to understanding a wide range of real systems (will discuss later)

Critical mechanical structures

Mechanically unstable (liquid) → Mechanically stable (solid)

- Interesting fundamental questions:
  Start from one simple example:
  Cut a finite piece of Maxwell lattice ($z = 2d$)
  Deficit of constraints on the boundary
  →
  **# of floppy modes $\propto$ Size of boundary**



**But where are these modes located?**

**Control of floppy modes** → **Program mechanical Response**

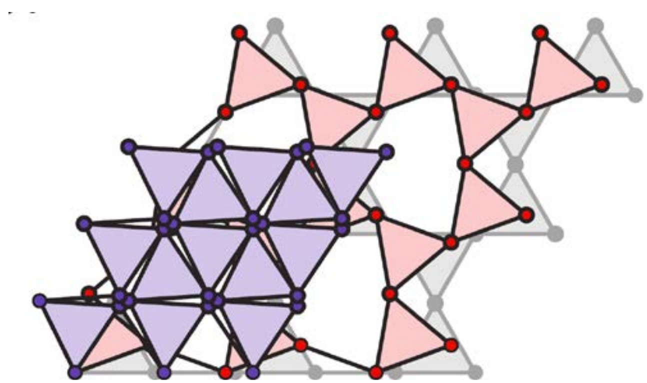
# Where Are The Floppy Modes?

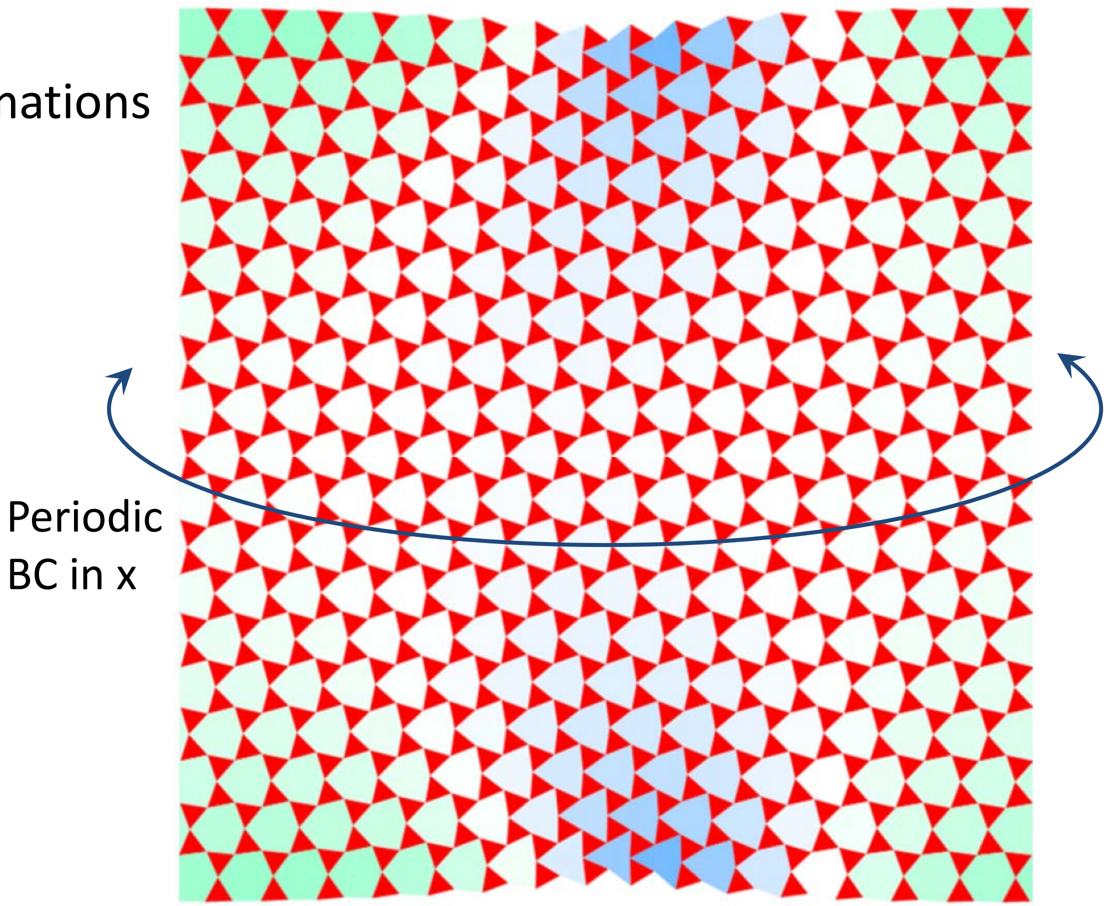
Case I. Floppy modes **localized at where you cut**

Example: twisted kagome

Floppy modes on all edges

Described by conformal transformations





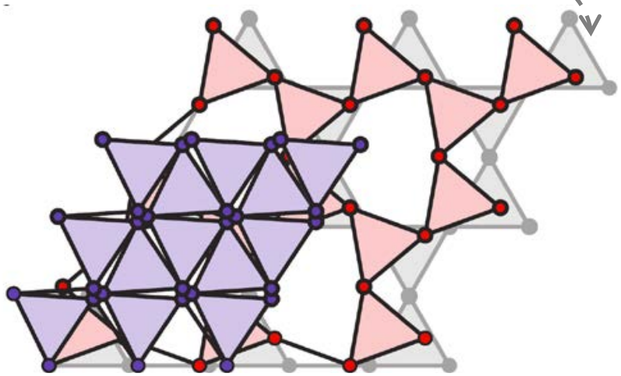

K. Sun, A. Souslov, X. Mao, and T. C. Lubensky, PNAS 109, 12369 (2012).

# Where Are The Floppy Modes?

Case II. Floppy modes are **plane waves**

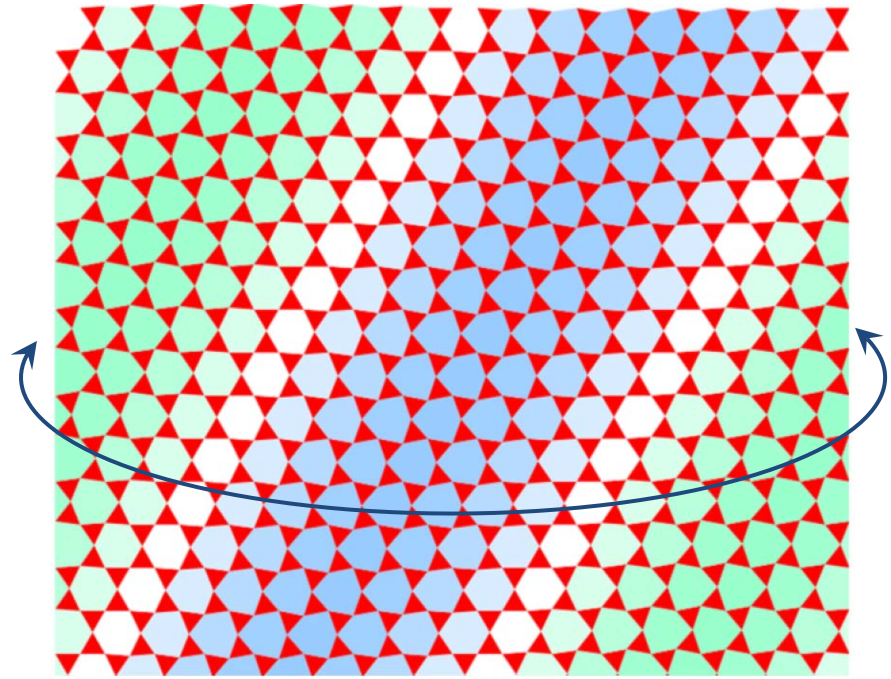
Example: regular kagome

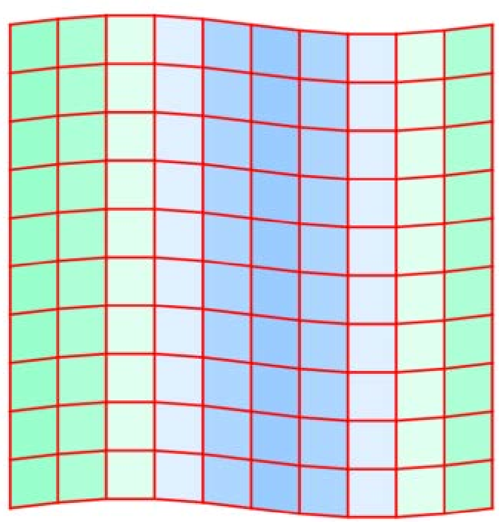
regular square



Periodic

BC in x



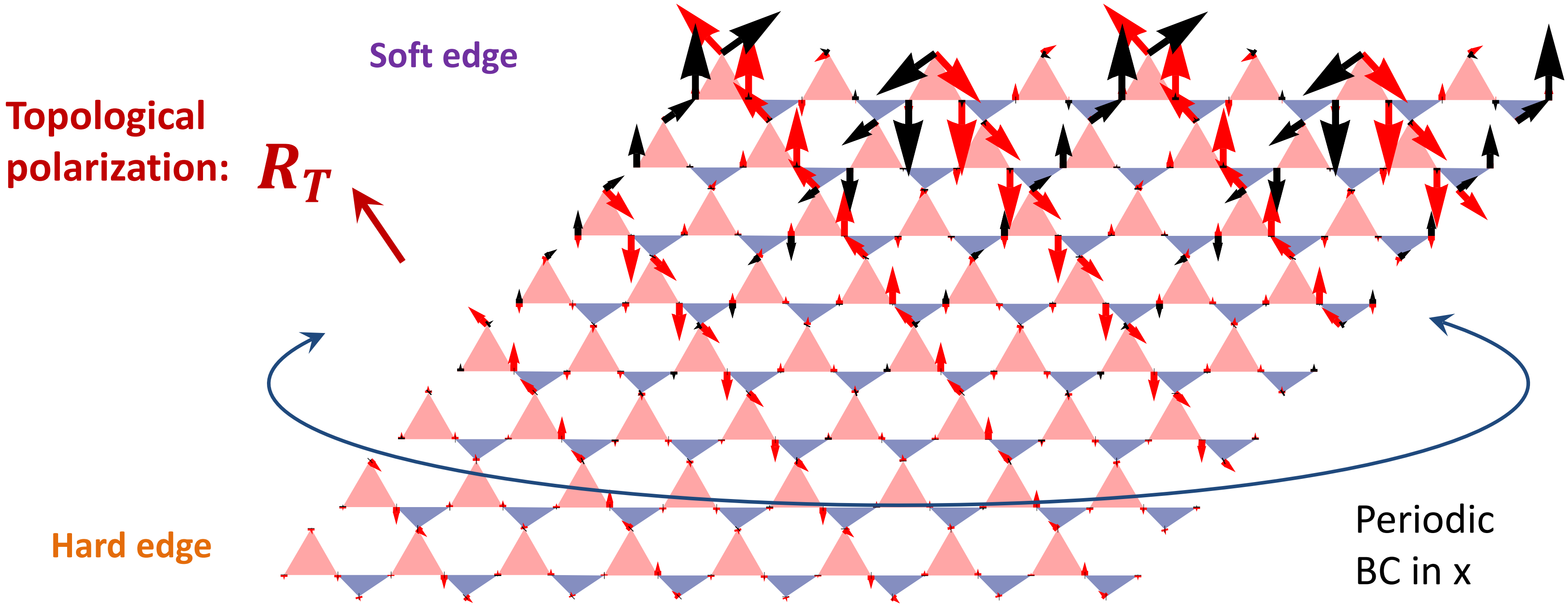


# Where Are The Floppy Modes?

Case III. Floppy modes are **topologically protected edge modes on one side**

Example: deformed kagome

(arrows show a pair of modes)



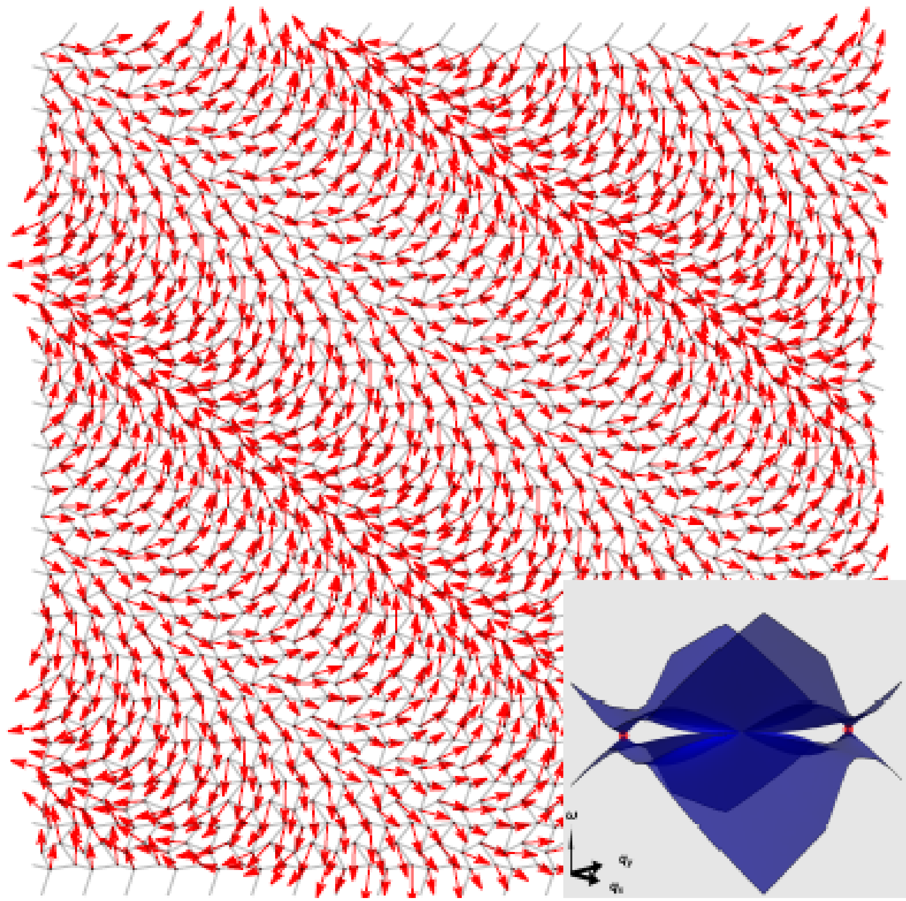

- C. L. Kane and T. C. Lubensky, Nat. Phys. 10, 39 (2014).
- Lubensky, Kane, Mao, Souslov, Sun, Rep. Prog. Phys., 78, 073901 (2015)

# Where Are The Floppy Modes?

Case IV. Floppy modes are **topologically protected Weyl modes in the bulk**

Example: deformed square (4 site per cell)

(arrows show a Weyl mode)

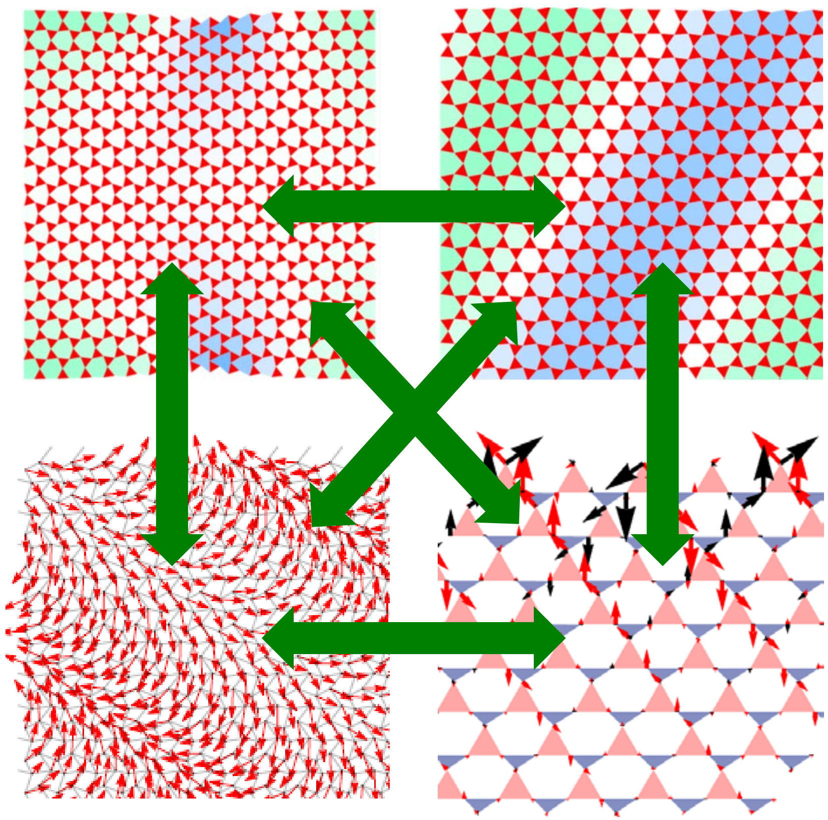


Rocklin, Chen, Falk, Vitelli & Lubensky, arXiv:1510.04970 (2015)

# Talk Outline

In this talk we will try to answer 2 questions:

1. **In a real system can you switch between different classes of floppy mode behaviors?**
   - **Analogous to metal-insulator transitions**
   - **Transformable mechanical metamaterials**
   - **General classification of critical mechanical structures**
2. **What are the thermodynamic signatures when these transitions occur?**
   - **Large number of competing instabilities**
   - **"Is a mechanism (floppy mode) still a mechanism at finite T?"**

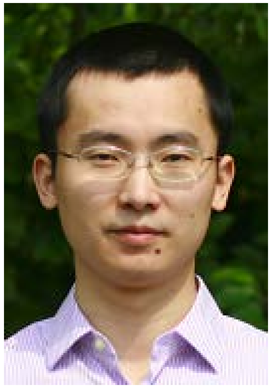


# Collaborators

- Topological transitions, mechanical metamaterials, general classifications

Zeb Rocklin
(Michigan)

Shangnan Zhou
(Michigan)



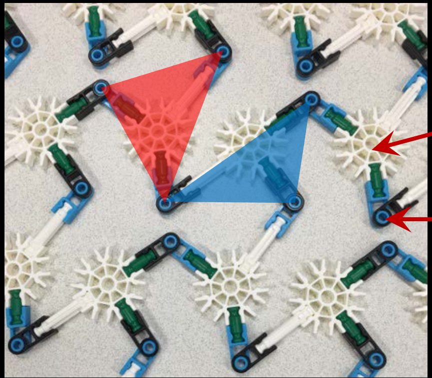
Kai Sun
(Michigan)

Rocklin, Zhou, Sun, and Mao, [arXiv:1510.06389](arXiv:1510.06389) [cond-mat.soft]

# Transformable topological mechanical metamaterials

D. Zeb Rocklin, Shangnan Zhou,
Kai Sun, Xiaoming Mao

University of Michigan



Rigid connector

Hinge

Webpage of video:
http://www-personal.umich.edu/~maox/research/TTMM/TTMM.html

# What Is The Operation?

## On the determinacy of repetitive structures

S.D. Guest[a,*], J.W. Hutchinson[b]

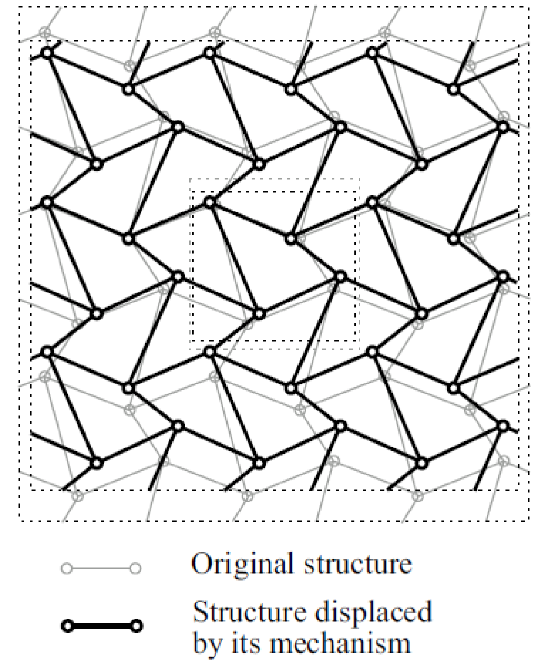
[a] *Department of Engineering, University of Cambridge, Trumpington Street, Cambridge, CB2 1PZ, UK*
[b] *Department of Engineering and Applied Sciences, Harvard University, Cambridge, MA 02138, USA*

Received 5 June 2002; accepted 24 September 2002

"Any $z = 2d$ lattice must have at least $d(d-1)/2$ homogeneous elastic deformations that are soft (stretch no bonds)"

**"Guest Modes"**

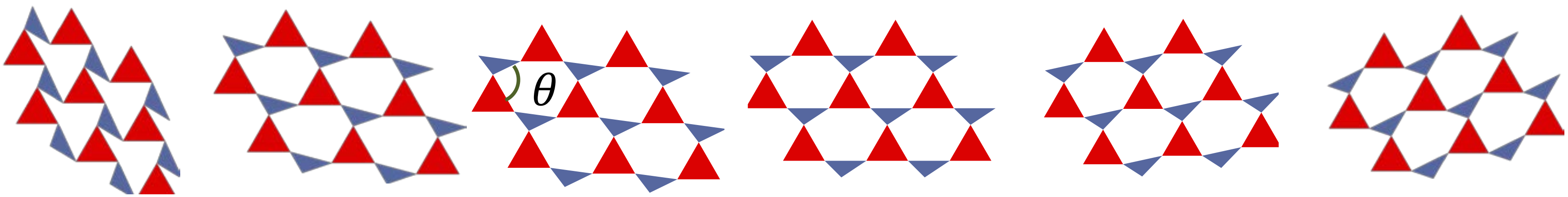
**A convenient way to TUNE Maxwell lattices!**



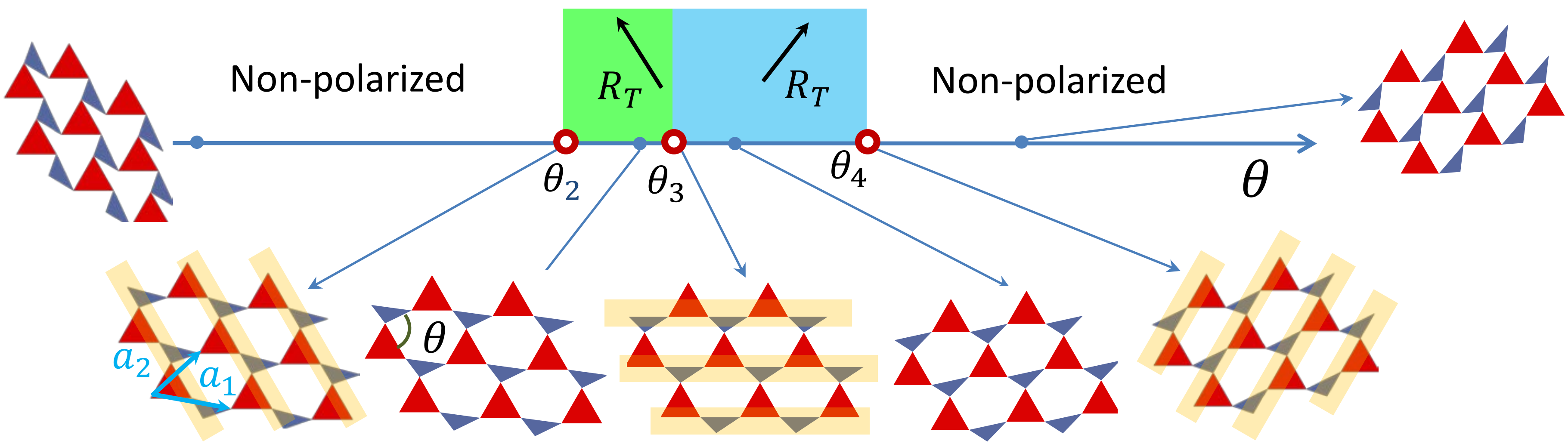

- Lubensky, Kane, Mao, Souslov, Sun, Rep. Prog. Phys., 78, 073901 (2015)

# Uniform Soft Twistings of The Deformed Kagome

- Uniform soft twisting is an easy way to transform a lattice




- Does it change the topological structures of the phonons?

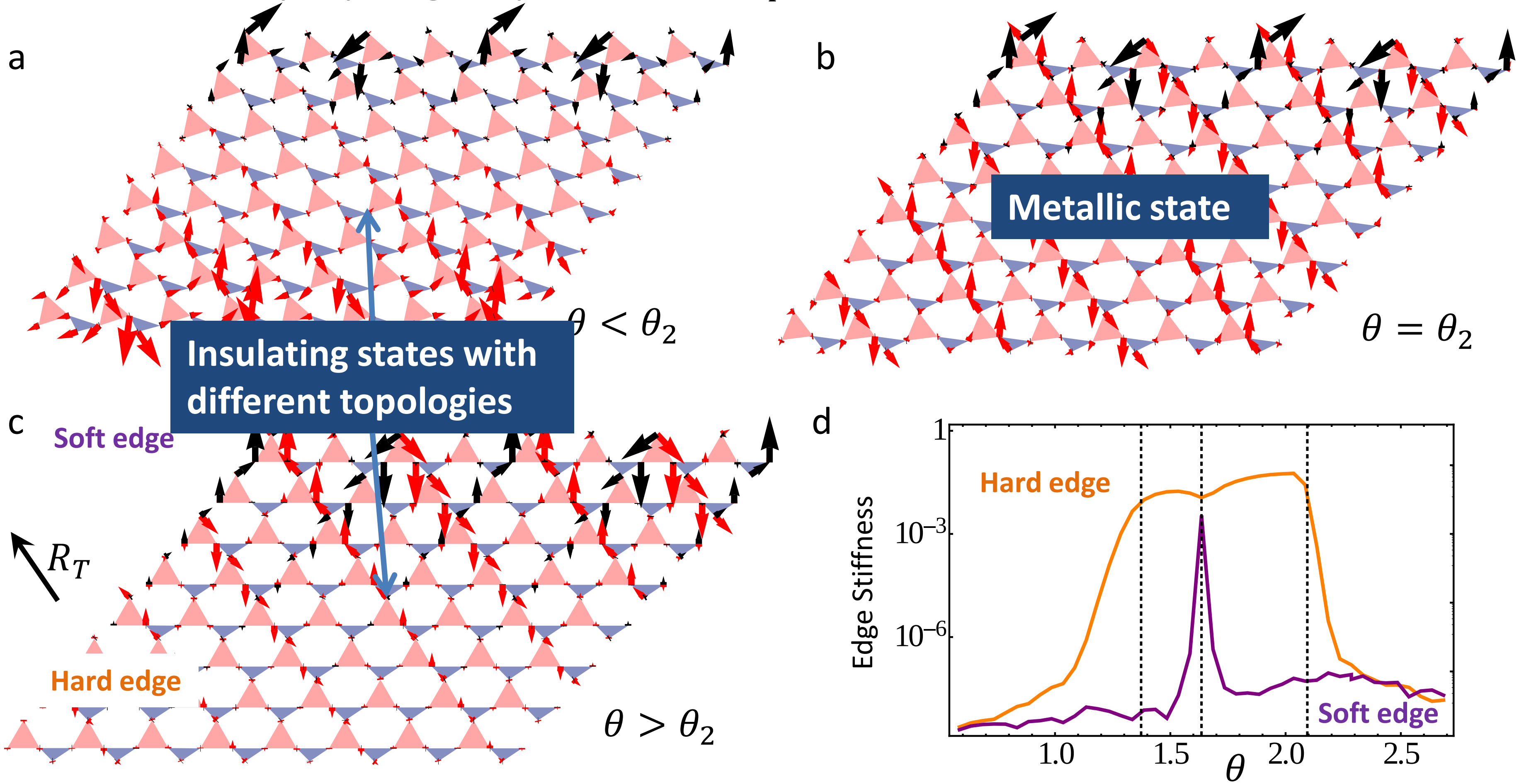



What happens at the topological transitions $\theta_2$, $\theta_3$, $\theta_4$ ?

Rocklin, Zhou, Sun, and Mao, arXiv:1510.06389 [cond-mat.soft]

# What happens at the transition?

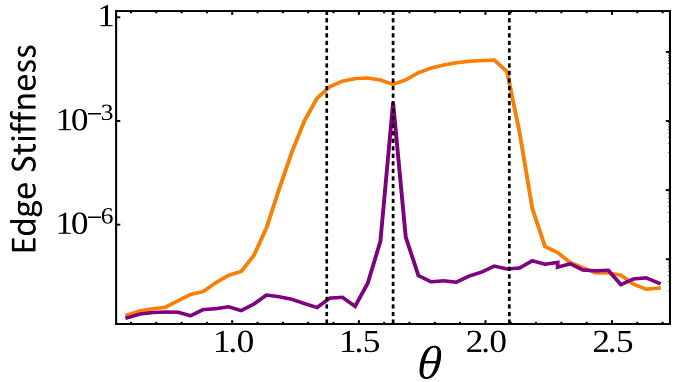
**Evolution of a pair of floppy edge modes**
(determined by topological polarization $R_T$)




**→ Transformable mechanical metamaterial**

Rocklin, Zhou, Sun, and Mao, arXiv:1510.06389 [cond-mat.soft]

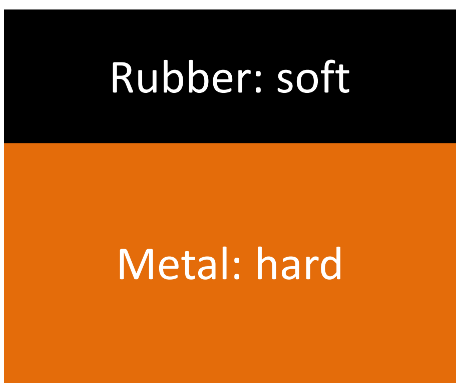
# Switch of Stiffness Is Protected by Topology




**→ Transformable mechanical metamaterial**

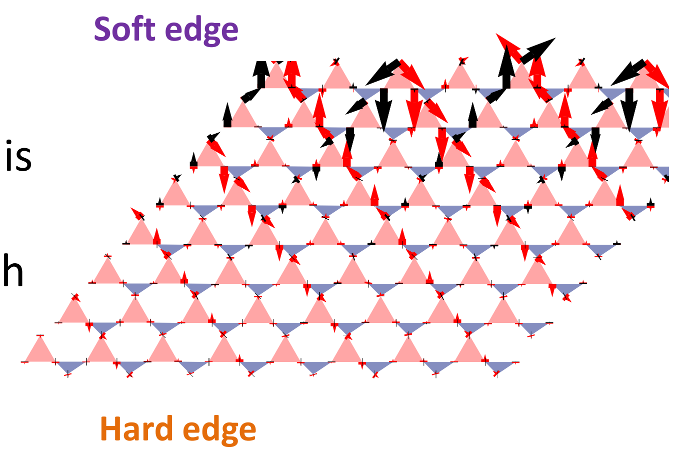
Compare 2 approaches to get contrasting edge stiffness

(1) composite




- If rubber wears out, protection is lost
- Can't switch

(2) topological




- If top layers wear off, newly exposed surface becomes soft
- Can switch

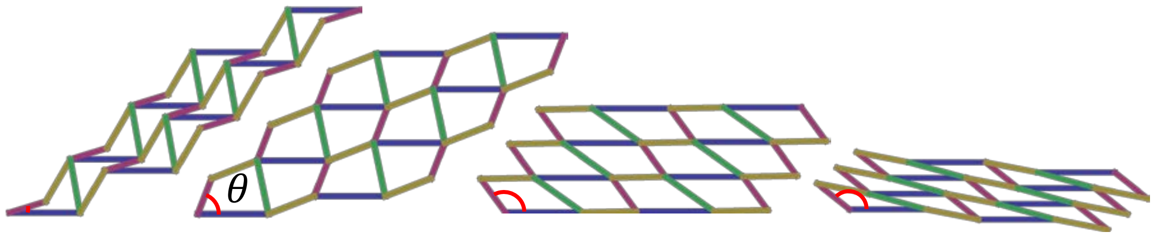
Rocklin, Zhou, Sun, and Mao, [arXiv:1510.06389](https://arxiv.org/abs/1510.06389) [cond-mat.soft]

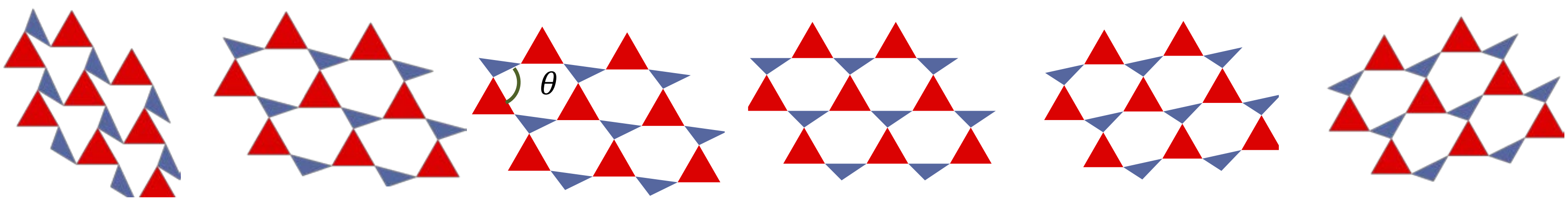
# Which Lattices Exhibit Topological Transitions?

Many structures exhibit soft elastic deformations:

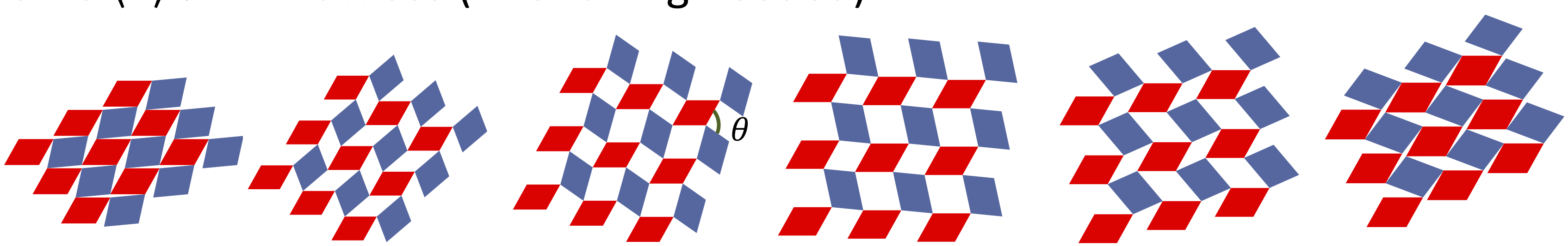
**“Uniform soft twistings”**

- ALL $\langle z \rangle = 2d$ lattices (guaranteed by the Guest modes)




- Some $\langle z \rangle > 2d$ lattices (fine tuning needed)




How many “universality classes” are there for mechanical instability?

Rocklin, Zhou, Sun, and Mao, arXiv:1510.06389 [cond-mat.soft]

# Our Approach: General Classification

*A soft homogeneous deformation*

*("uniform soft twisting")*

→ Spatially varying waves that are floppy

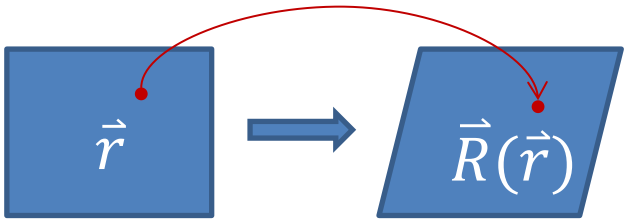
→ Topological Transitions

Bulk modes (plane waves)

Edge modes (surface waves)

Rocklin, Zhou, Sun, and Mao, arXiv:1510.06389 [cond-mat.soft]

# Floppy Modes




Deformation gradient: $\Lambda_{ij} \equiv \frac{\partial R_i}{\partial r_j}$

Metric Tensor: $g = \Lambda^T \Lambda$

(Right Cauchy-Green) strain tensor: $\epsilon \equiv (g - I)/2$

- General form of elastic energy (assume no stress)

Elastic constants

$$F = \int \left[ \underline{c_{ijkl}} \epsilon_{ij} \epsilon_{kl} + O(\epsilon^3) \right] d^2 r$$

- The uniform soft twisting can be written as

$$\tilde{\epsilon} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{xy} & \epsilon_{yy} \end{pmatrix}$$ and they satisfy $c_{ijkl} \tilde{\epsilon}_{ij} \tilde{\epsilon}_{kl} = 0$

- So all strain tensors of the form $\boldsymbol{\epsilon}(\boldsymbol{r}) = \underline{\mathbf{f}(\boldsymbol{r})} \tilde{\boldsymbol{\epsilon}}$ must also be soft

Arbitrary Scalar

$$F = \int \left[ f(r)^2 c_{ijkl} \tilde{\epsilon}_{ij} \tilde{\epsilon}_{kl} + O(\tilde{\epsilon}^3) \right] d^2 r = O(\tilde{\epsilon}^3)$$

Rocklin, Zhou, Sun, and Mao, arXiv:1510.06389 [cond-mat.soft]

# Floppy Modes

- Not all arbitrary scalar function $f(r)$ correspond to physical deformations

- Extra condition on $g(r)$ for deformations in flat space

$$\Gamma_{kij} = \frac{1}{2}(\partial_j g_{ki} + \partial_i g_{kj} - \partial_k g_{ij})$$

No curvature

$$R_{ijkl} = \partial_k \Gamma_{ilj} - \partial_l \Gamma_{ikj} + g^{mn}\Gamma_{ikm}\Gamma_{nlj} - g^{mn}\Gamma_{ilm}\Gamma_{nkj} = 0$$

- Two families of floppy modes must also exist

$$\boldsymbol{\epsilon}_+(\mathbf{r}) = \tilde{\boldsymbol{\epsilon}}\ f_+(x + \lambda_+ y)$$
$$\boldsymbol{\epsilon}_-(\mathbf{r}) = \tilde{\boldsymbol{\epsilon}}\ f_-(x + \lambda_- y)$$

with

$$\lambda_+ = (\tilde{\epsilon}_{xy} + \sqrt{-\det \tilde{\boldsymbol{\epsilon}}})/\tilde{\epsilon}_{xx}$$
$$\lambda_- = (\tilde{\epsilon}_{xy} - \sqrt{-\det \tilde{\boldsymbol{\epsilon}}})/\tilde{\epsilon}_{xx}$$

- Two classes of behaviors

$\det \tilde{\boldsymbol{\epsilon}} > 0$ Dilation dominant, $\lambda_\pm$ imaginary, floppy modes are **edge waves**

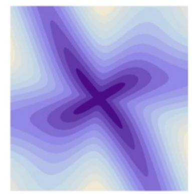
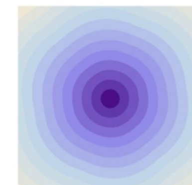
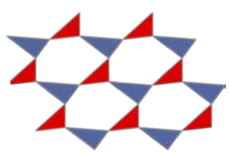
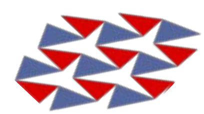
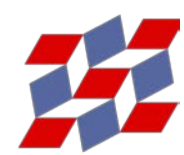
$\det \tilde{\boldsymbol{\epsilon}} < 0$ Shear dominant, $\lambda_\pm$ real, floppy modes are **bulk waves**

**Auxetic (negative Poisson's ratio)**

Independent of microscopic details

Rocklin, Zhou, Sun, and Mao, arXiv:1510.06389 [cond-mat.soft]

# Classification of Structures with "Uniform Soft Twisting"

<table>
<tr><td colspan="2">Soft twisting characteristic</td><td colspan="2">Shear dominant<br>$\det \tilde{\boldsymbol{\epsilon}} < 0$</td><td colspan="2">Dilation dominant<br>$\det \tilde{\boldsymbol{\epsilon}} > 0$</td></tr>
<tr><td colspan="2">Spatially varying floppy modes</td><td colspan="2">$k_y = \lambda_\pm\, k + O(k^2)$<br>with $\lambda_\pm \in \mathbb{R}$</td><td colspan="2">$k_y = \mathrm{Re}(\lambda_\pm)\, k + i\, \mathrm{Im}(\lambda_\pm)\, k + O(k^2)$</td></tr>
<tr><td colspan="2">Bulk phonon spectra</td><td colspan="2">Vanishing speed of sound in two directions </td><td colspan="2">Positive speed of sound in all directions </td></tr>
<tr><td colspan="2">Edge modes</td><td colspan="2">Additional floppy edge modes may arise on some edges and can be topological</td><td colspan="2">Floppy edge modes on ALL edges and can be described by conformal transformations</td></tr>
<tr><td rowspan="4">Floppy edge modes (when present)</td><td rowspan="2">Example lattices</td><td>$z = 2d$</td><td>$z > 2d$</td><td>$z = 2d$</td><td>$z > 2d$</td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td>Frequency</td><td>$\omega = 0$</td><td>$\omega = O(k^2)$</td><td>$\omega = 0$</td><td>$\omega = O(k^2)$</td></tr>
<tr><td>Decay rate</td><td>$O(k^2)$</td><td>$O(k^2)$</td><td>$O(k)$</td><td>$O(k)$</td></tr>
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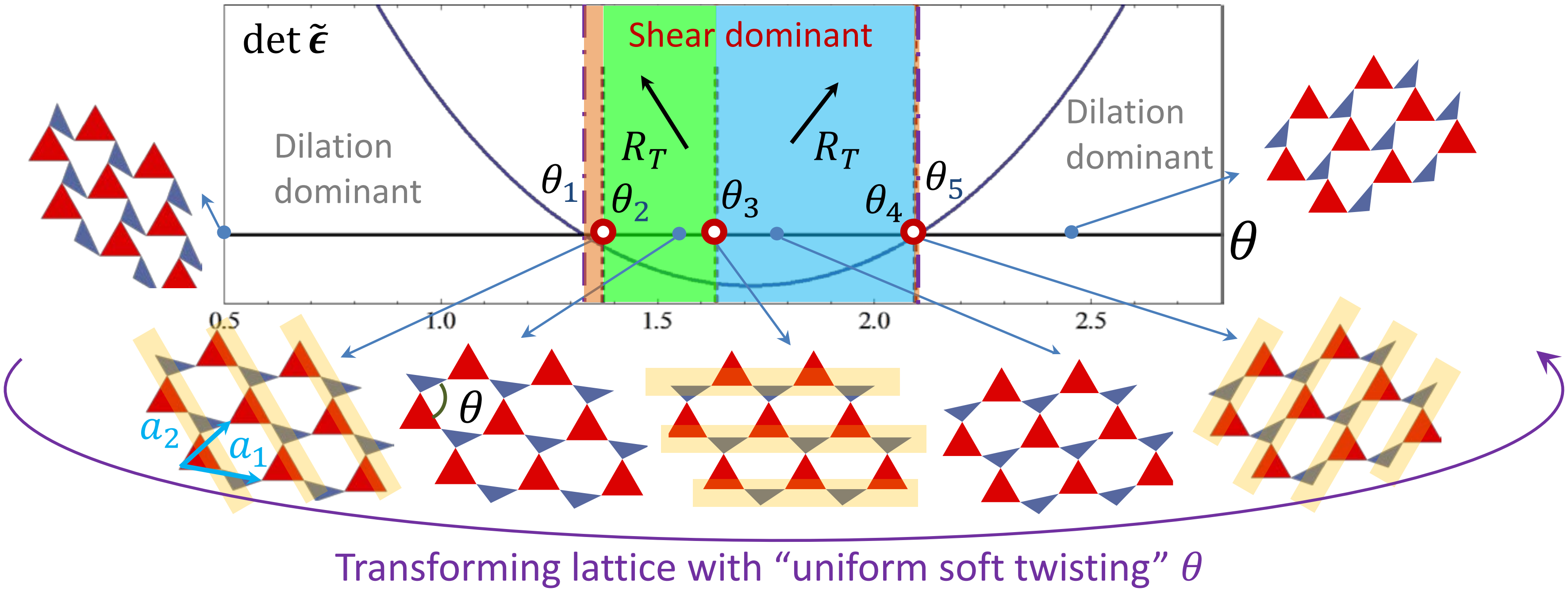
# Transitions In The Deformed Kagome

**The "uniform soft twisting" can transform a structure between different class**

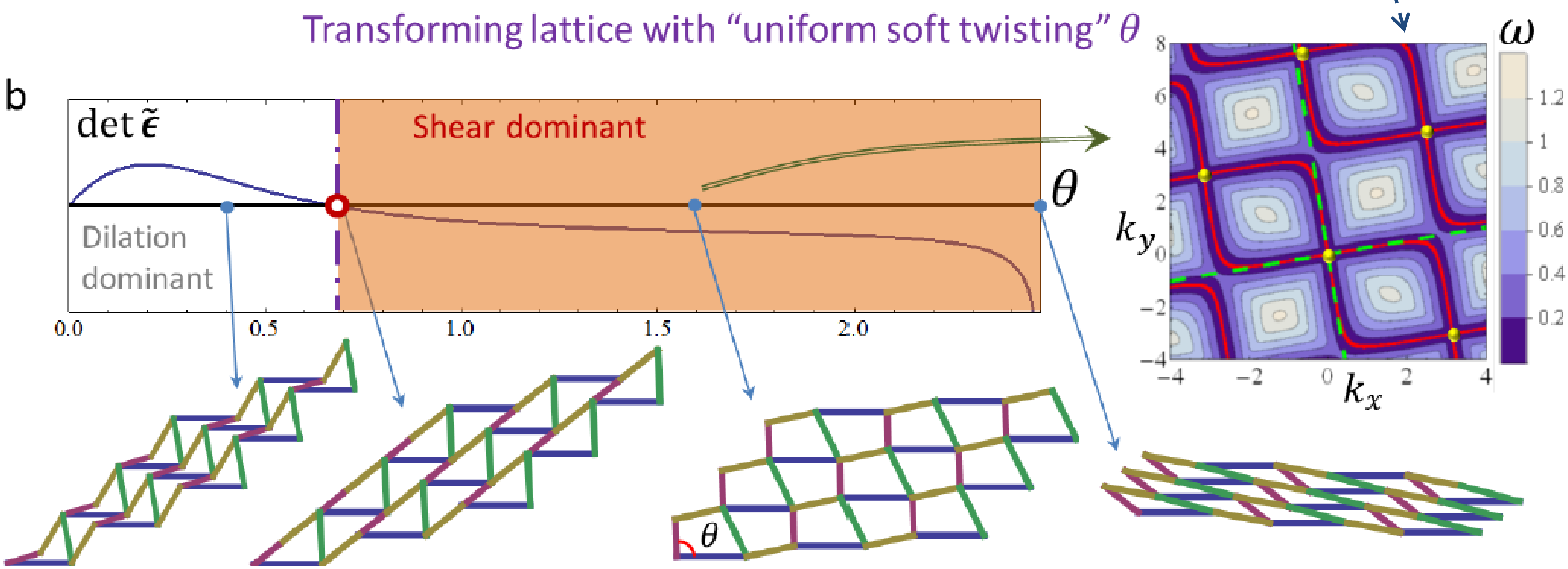



**Shear dominant is a necessary but not sufficient condition for topological states**

Rocklin, Zhou, Sun, and Mao, arXiv:1510.06389 [cond-mat.soft]

# Shear Dominant But Not Topologically Polarized

Plane wave floppy modes in the bulk

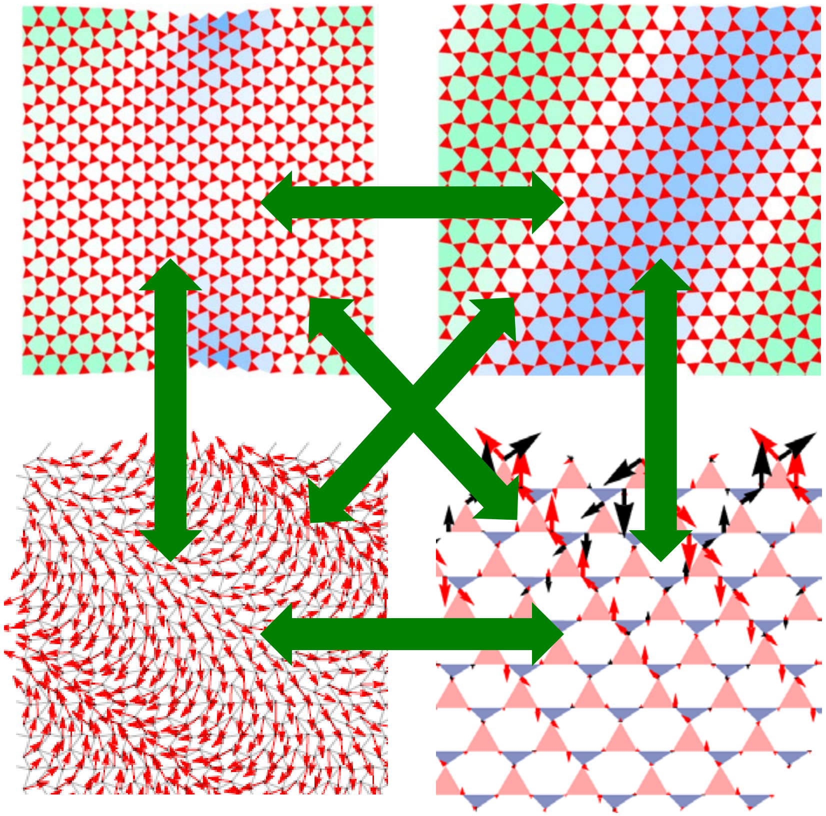



Rocklin, Zhou, Sun, and Mao, arXiv:1510.06389 [cond-mat.soft]

# Talk Outline

In this talk we will try to answer 2 questions:

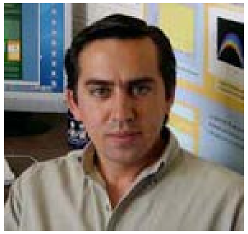
1. **In a real system can you switch between different classes of floppy mode behaviors?**
   - **Transformable mechanical metamaterials**
   - **New types of phase transitions**
   - **General classification of critical mechanical structures**
2. **What are the thermodynamic signatures when these transitions occur?**
   - **Large number of competing instabilities**
   - **"Is a mechanism (floppy mode) still a mechanism at finite T?"**



# Collaborators

- Finite-T mechanical instability in an ordered lattice

Anton Souslov
(Georgia Tech)

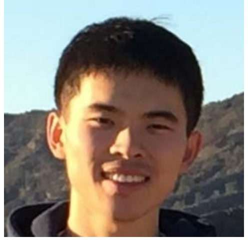


Carlos Mendoza
(Universidad Nacional
Autonoma de Mexico)

Tom Lubensky
(Penn)

  - Mao, Souslov, Mendoza, and Lubensky, *Nature Communications 6, 5968* (2015).

- Finite-T mechanical instability in disordered lattices



Leyou Zhang
(University of Michigan)

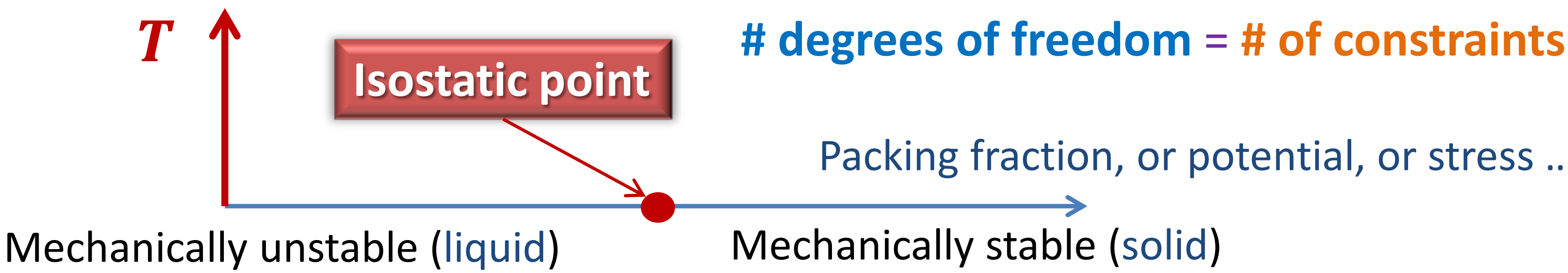
  - Zhang and Mao, arXiv:1503.05274 (2015).

- Mechanisms, topological phase transitions and fluctuation effects

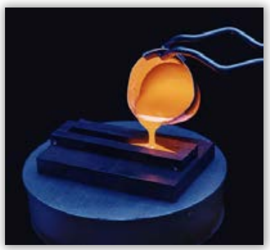
Zeb Rocklin
(Michigan)

Vincenzo Vitelli
(Leiden)

# Motivation of The Original “Finite-T Instability” Project

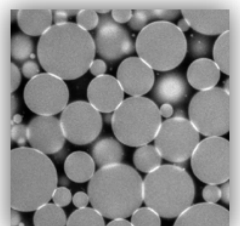



Examples of soft matter systems:

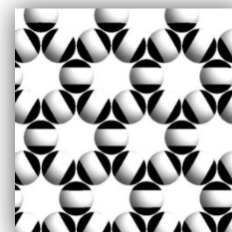


Metallic glasses

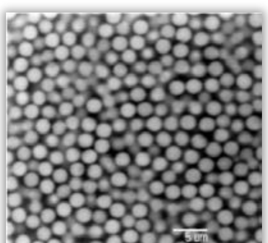
Granular matter



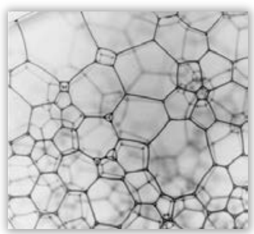
Emulsions



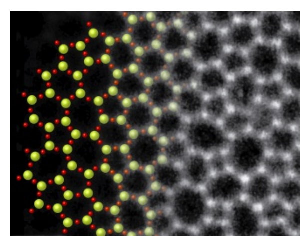
Self-assembled open lattices



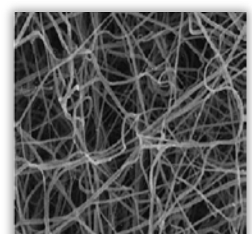
Colloids



Foams



Network glasses



Fiber networks

# Finite-T Transitions in Critical Lattices

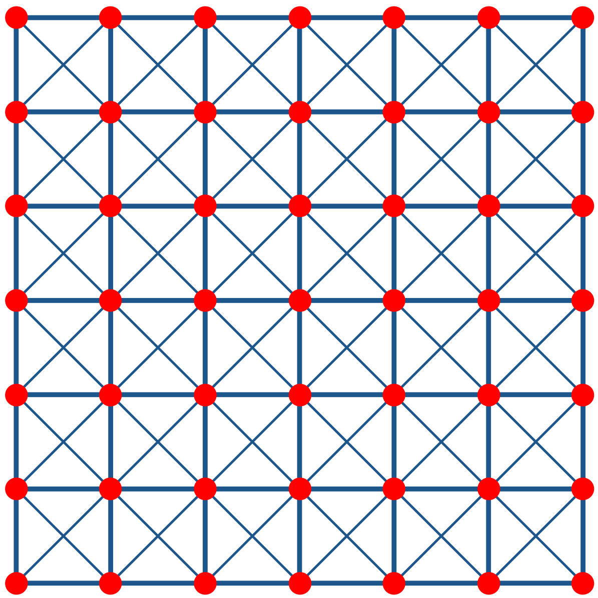
Ordinary continuous phase transitions:


Critical mechanical structures:

Competition between **a large number of instability**

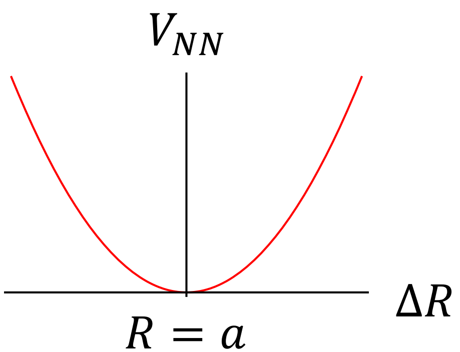
# Square Lattice Model

A simple lattice model with a controlled mechanical instability



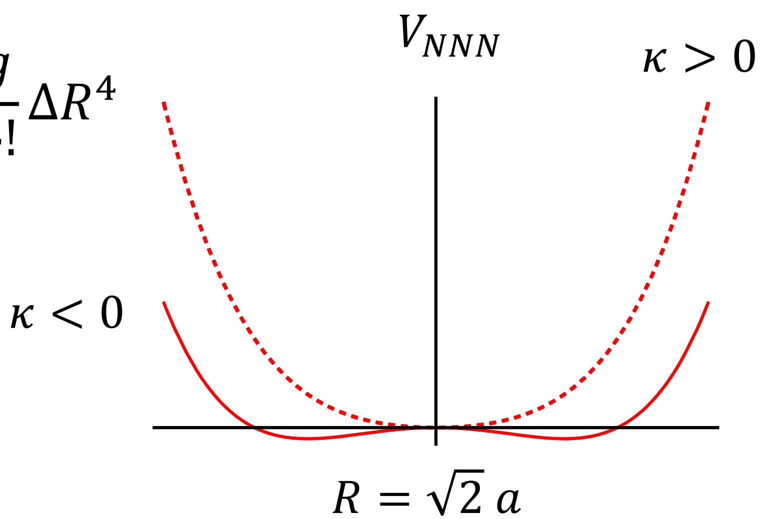
- NN bond: harmonic spring

$$V_{NN} = \frac{k}{2}\Delta R^2$$

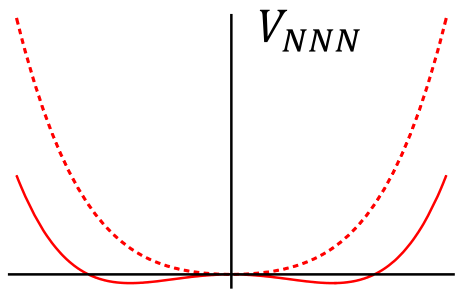



- NNN bond: anharmonic spring
  - Harmonic $\kappa$
  - Quartic $g$

$$V = \frac{\kappa}{2}\Delta R^2 + \frac{g}{4!}\Delta R^4$$

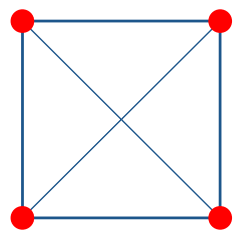


# The $T = 0$ Mechanical Instability

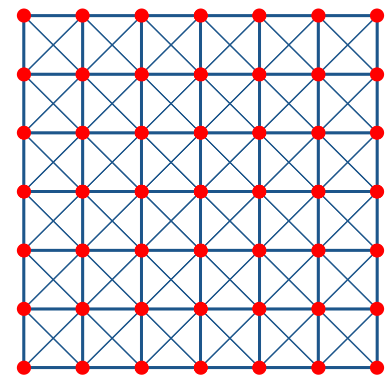



Consider one plaquette:

- When $\kappa > 0$ stable state ($T = 0$): square

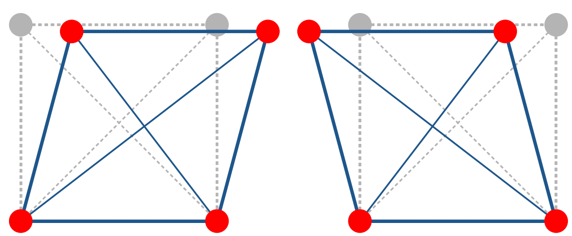


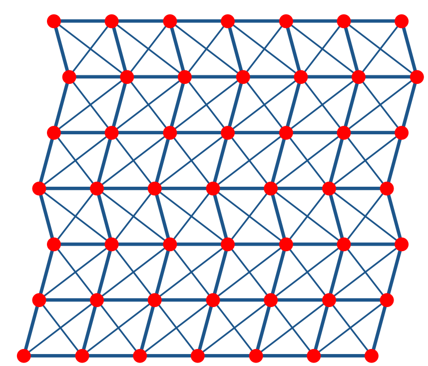
System $T = 0$ stable state



Unique ground state

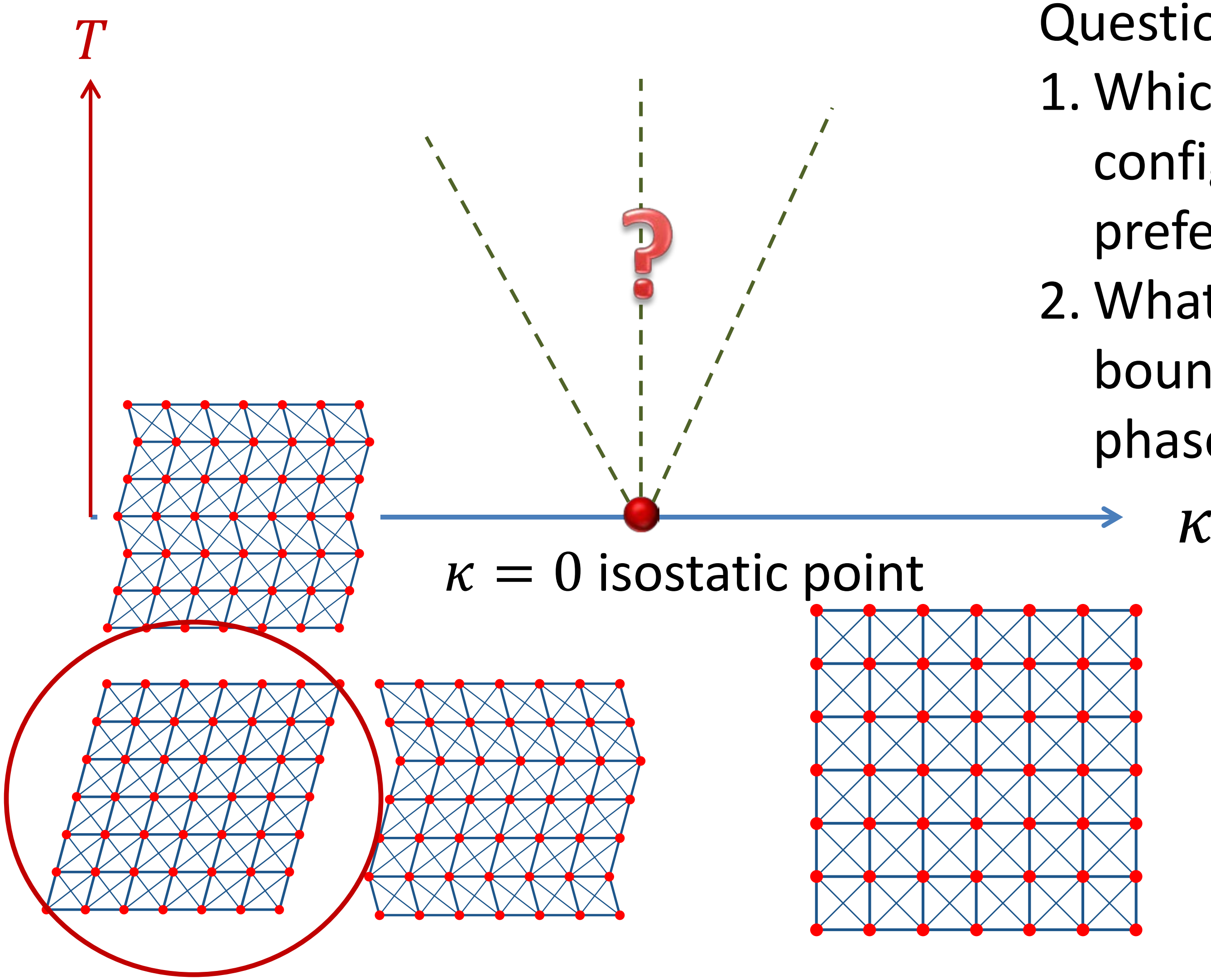
- When $\kappa < 0$ stable state ($T = 0$): rhombus





Multiple degenerate ground state: $e^{\sqrt{N}}$

# Effect of Temperature?




Favored: order-by-disorder

Questions:

1. Which "zig-zagging configuration" is more preferred? ✓ $(F = U - TS)$
2. What is the stability boundary of the square phase?

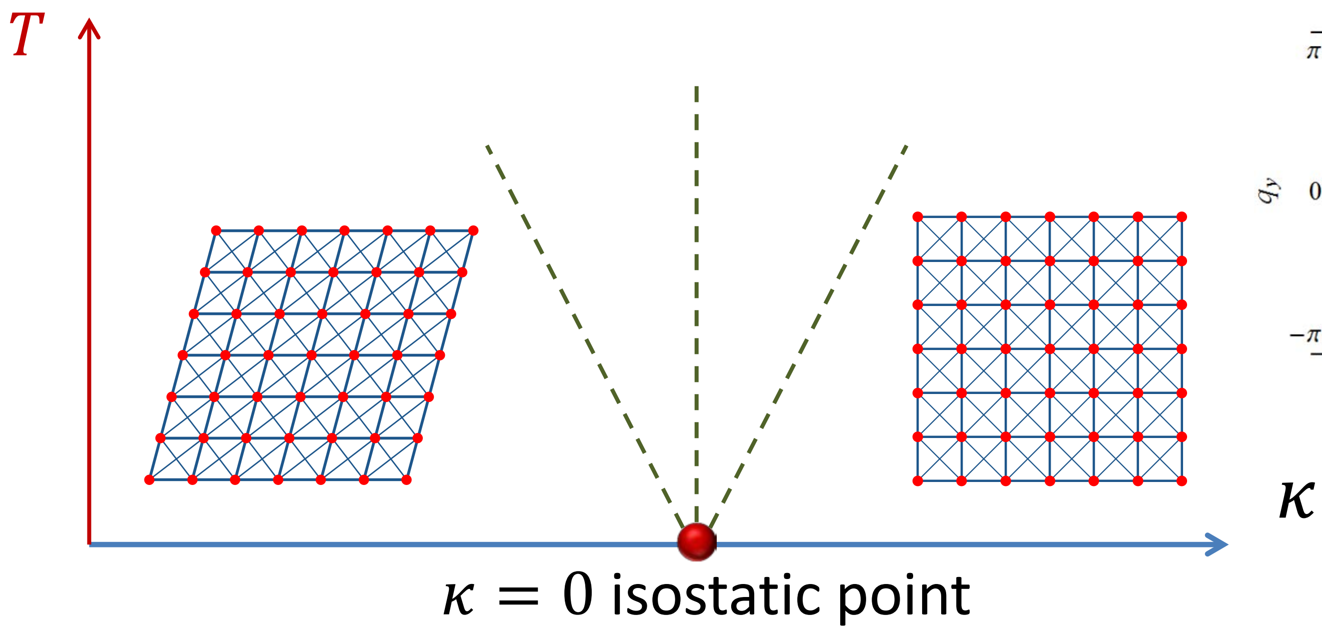
# Entropic Stabilization of the Square Lattice

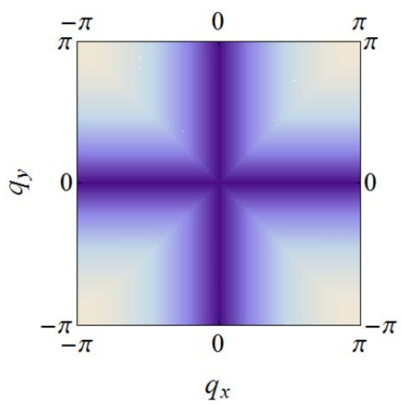
2. What is the stability boundary of the square phase?

Method: Calculate fluctuation corrected shear rigidity (integrate out finite wavelength fluctuations)

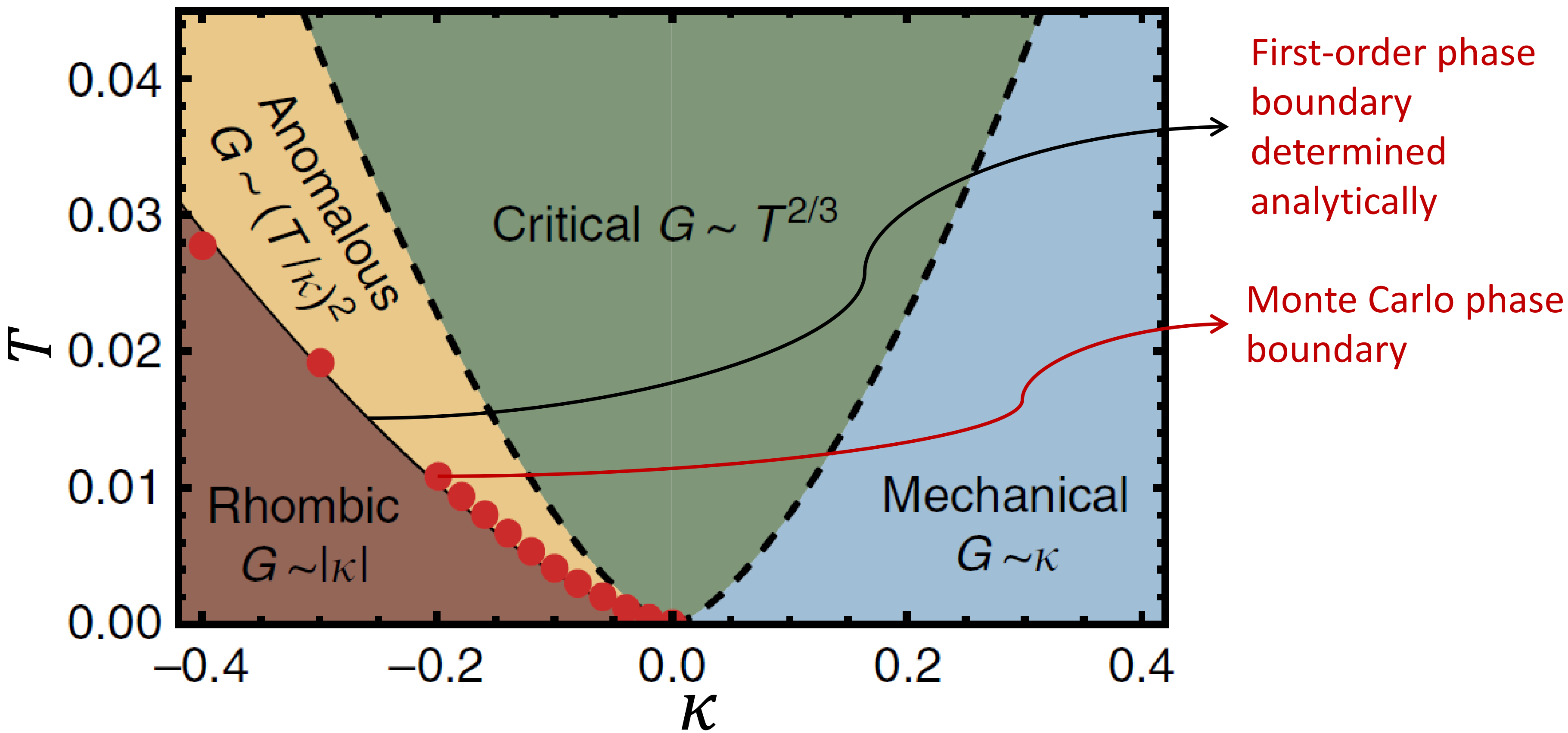
What's special: floppy modes living on a 1D manifold in *p*-space and give a divergent contribution to shear modulus → fluctuation driven first order transition

(similar to Brazovskii '75 theory for finite wavelength instability in pattern formation)



# Phase Diagram

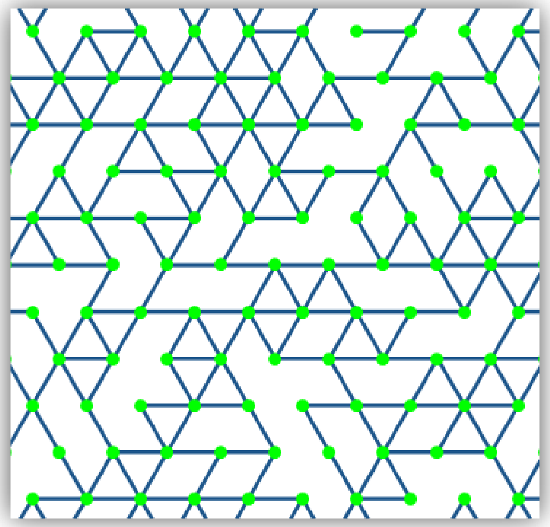



- Transitions near mechanical instability can become first-order if a large number of modes **simultaneously** become critical at mechanical instability (connection to glass transitions)
- This is a mechanical transition in an ordered lattice, which can be seen as a special type of structural phase transition (crystal - crystal), what about liquid – (disordered) solid?

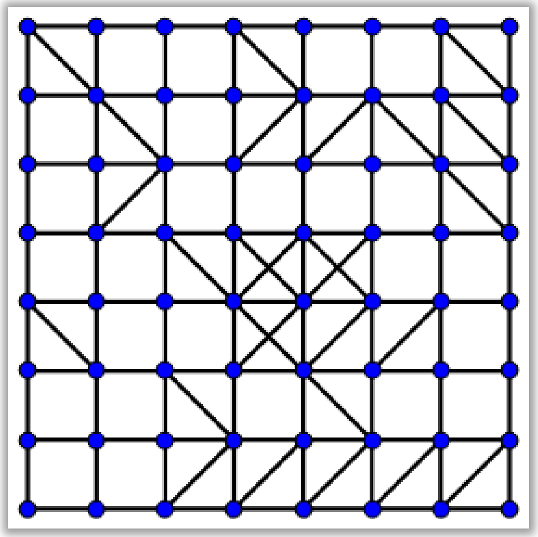
# Two CF Model Lattices

Finite T?

Rigidity percolation in a triangular lattice



Square lattice with random NNN bonds

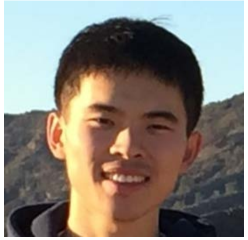


$T = 0$ transition

$p_c \approx 2/3$ ← Isostatic point → $p_c = 0$

$\xi \sim (p - p_c)^{-1.21}$

Non-meanfield



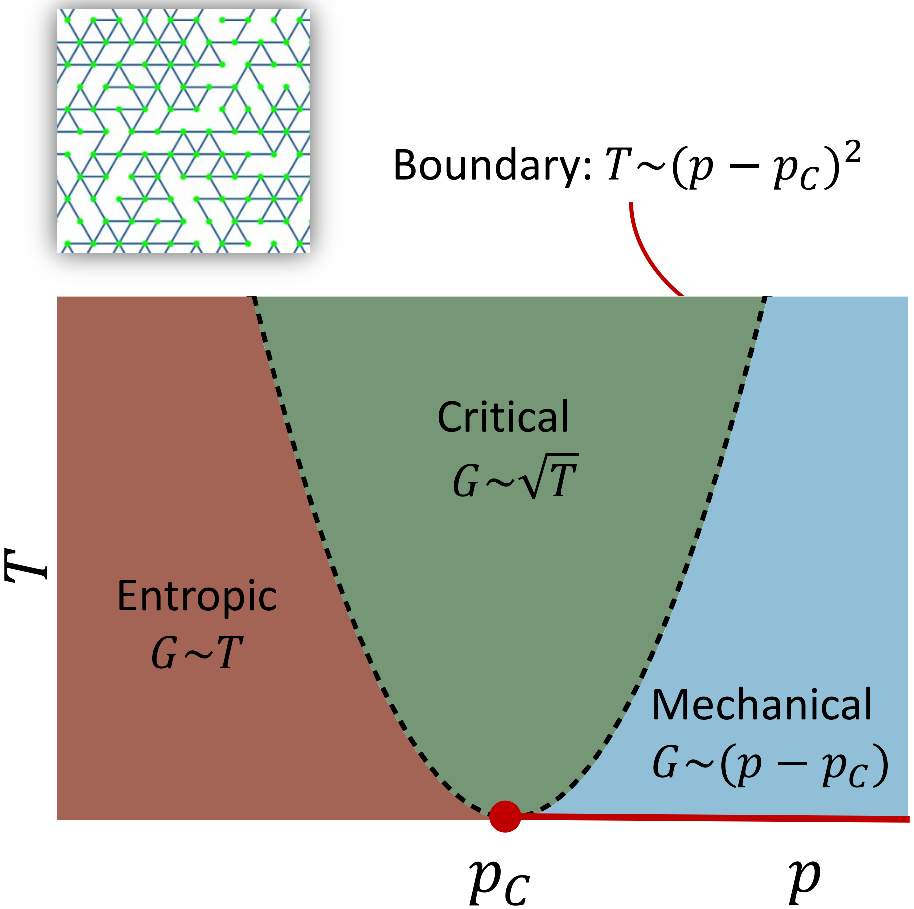
Leyou Zhang

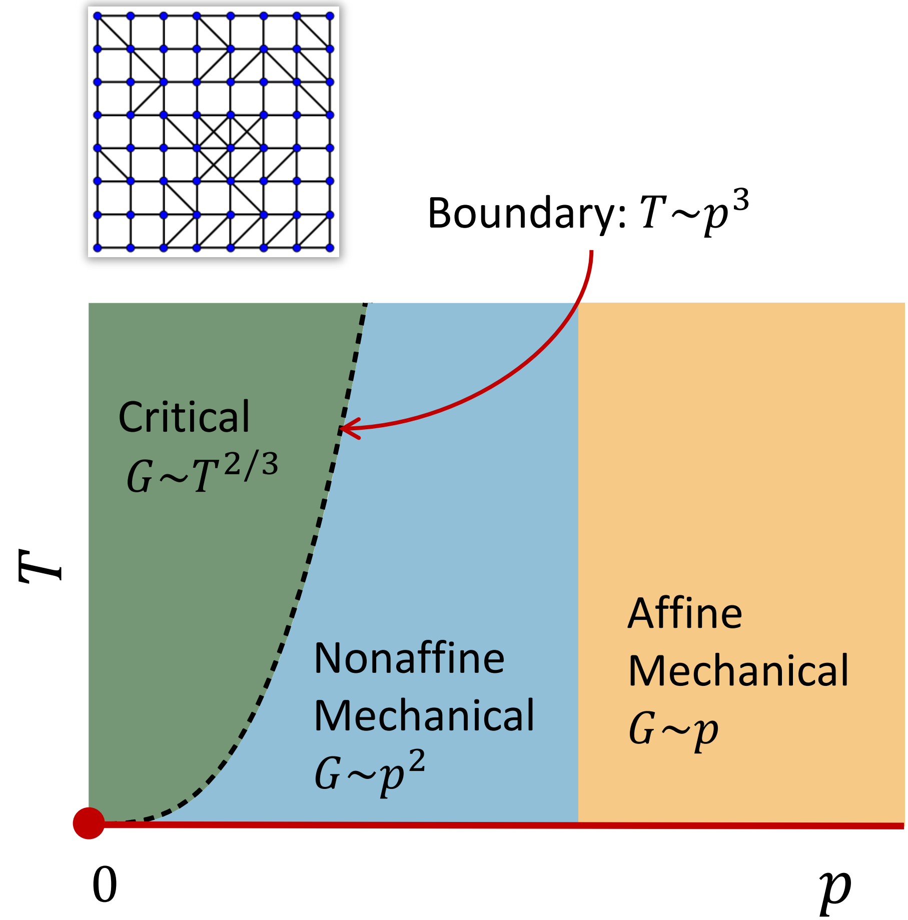
$\xi \sim (p - p_c)^{-1}$

Meanfield,
Agrees with **Jamming**

- Feng&Sen, PRL 52, 216 (1984).
- Jacobs&Thorpe, PRL 75, 4051 (1995).
- Zhang and Mao, arXiv:1503.05274 (2015).
- Mao, Xu, & Lubensky, PRL, 104, 085504 (2010)

# Finite-T EMT Phase Diagrams



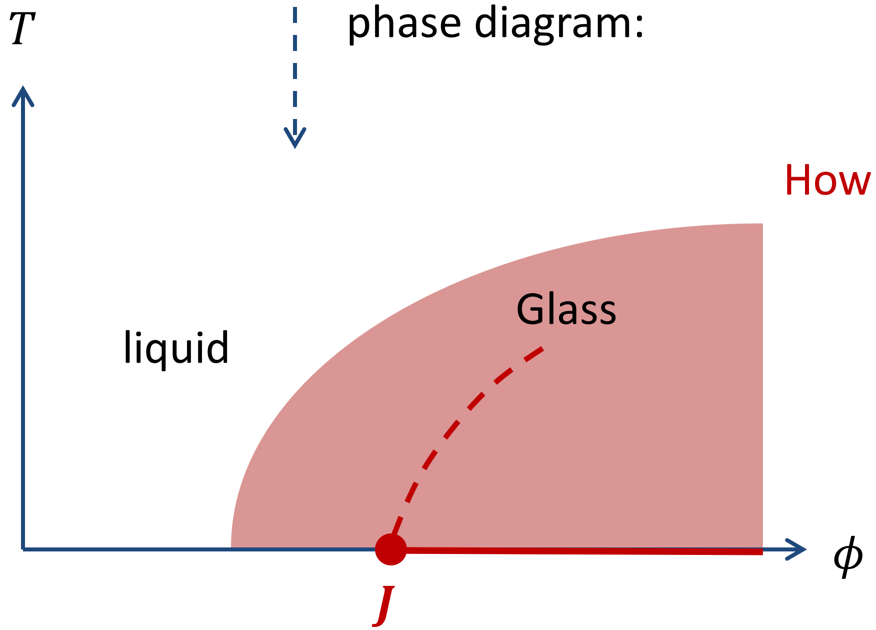

- Zhang and Mao, arXiv:1503.05274 (2015).

# Relation to Glasses?

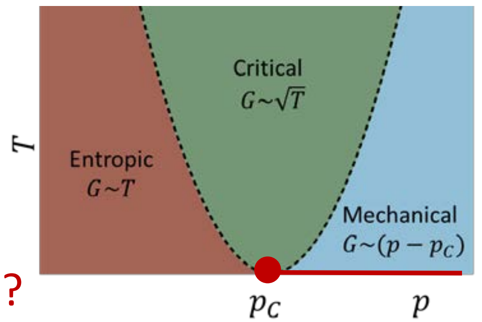
What happens when thermal fluctuations are introduced near point J?

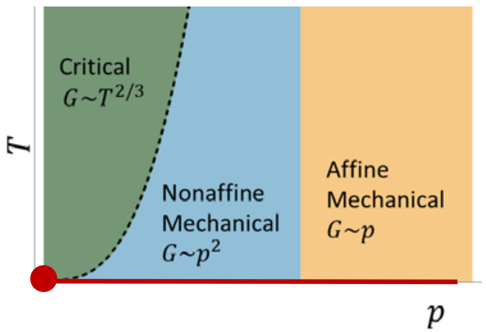
- Zhang et al, Nature 459, 230 (2009).
- Ikeda, Berthier, & Sollich, PRL 109, 018301 (2012).
- Parisi & Zamponi, RMP 82, 789 (2010).
- DeGiuli, Lerner, & Wyart, arXiv:1501.06995 (2015).

A sketch of suggested phase diagram:



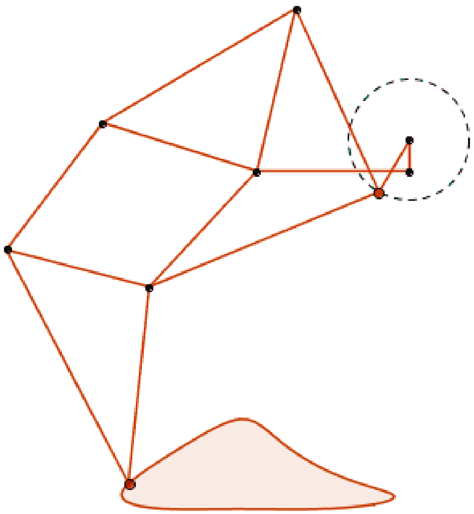

How are they related?



# Nonlinear Mechanisms At Finite T?

Macroscopic machines and robots

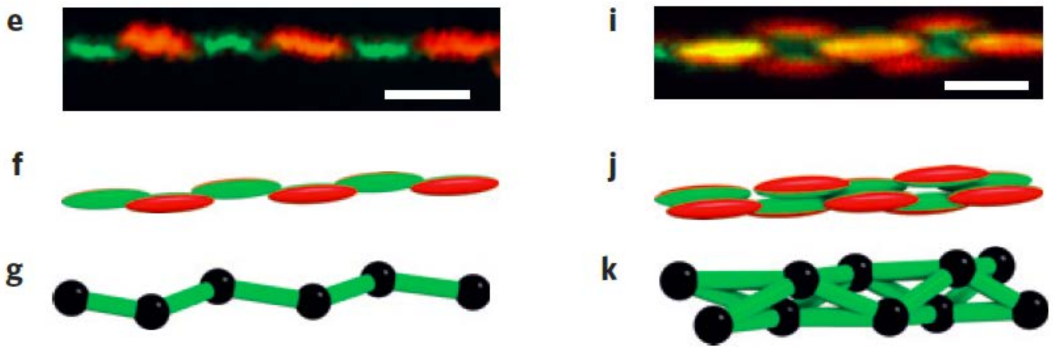


Zeb Rocklin
(Michigan)

Vincenzo Vitelli
(Leiden)

Theo Jansen's walking mechanism and "Strandbeest" (wind powered robot using this mechanism)

**How small can a mechanism go?**




Shah et. al., Nat. Mat. 14, 117 (2014).

# Summary

- Topological transitions in critical mechanical structures → dramatic change of mechanical properties → "transformable topological mechanical metamaterials" (TTMM)
- General classification of critical mechanical structures

  Floppy edge modes vs. Floppy bulk modes
- Interplay between entropy – floppy modes -- topology

THANK YOU!