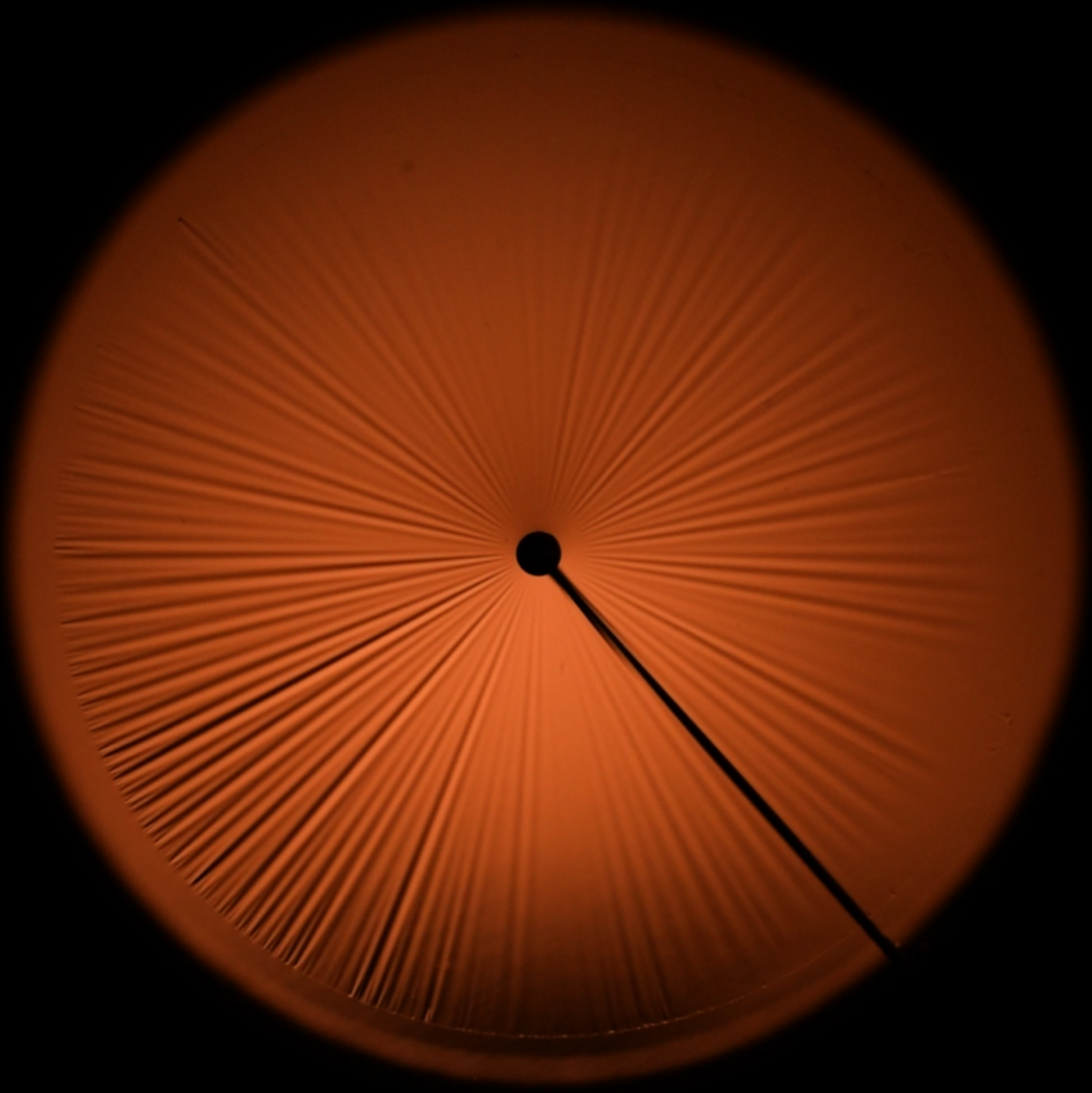
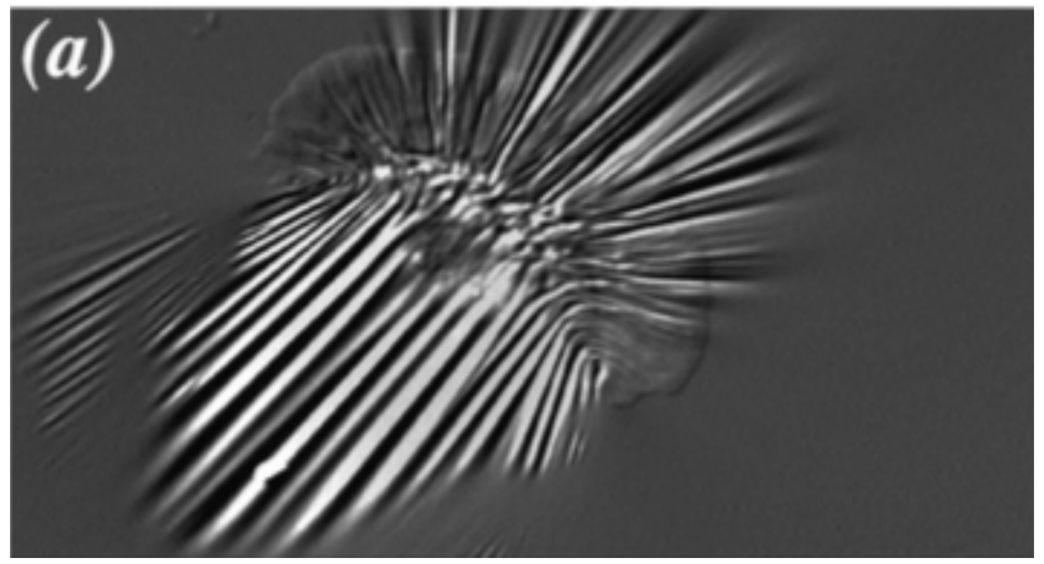
The wavelength of wrinkles in curved tensioned sheets



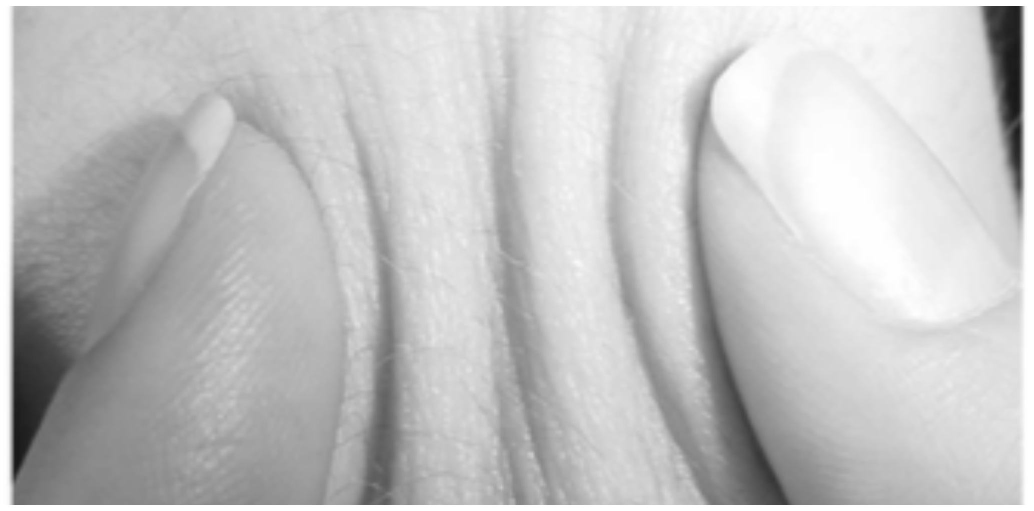
Joseph D Paulsen, **Syracuse University**

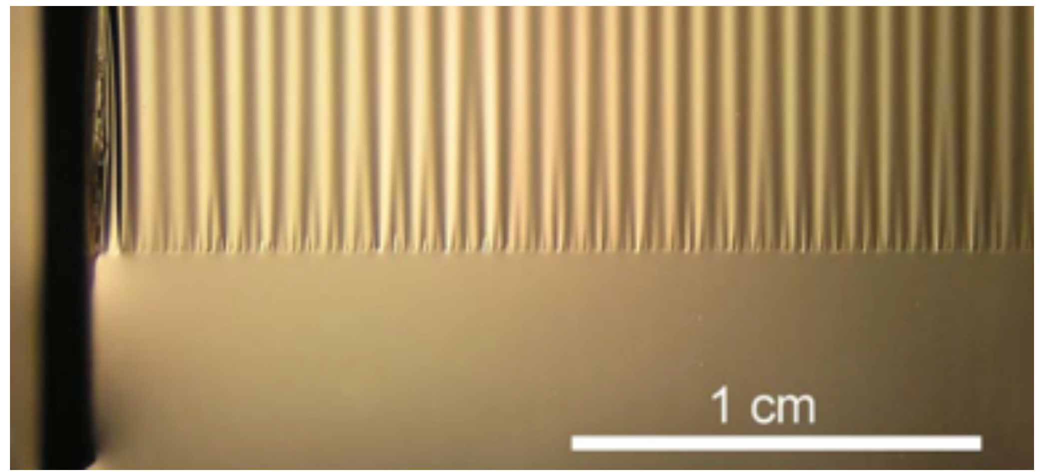
Thin sheets can **wrinkle**

N. Bowden et al., 1998



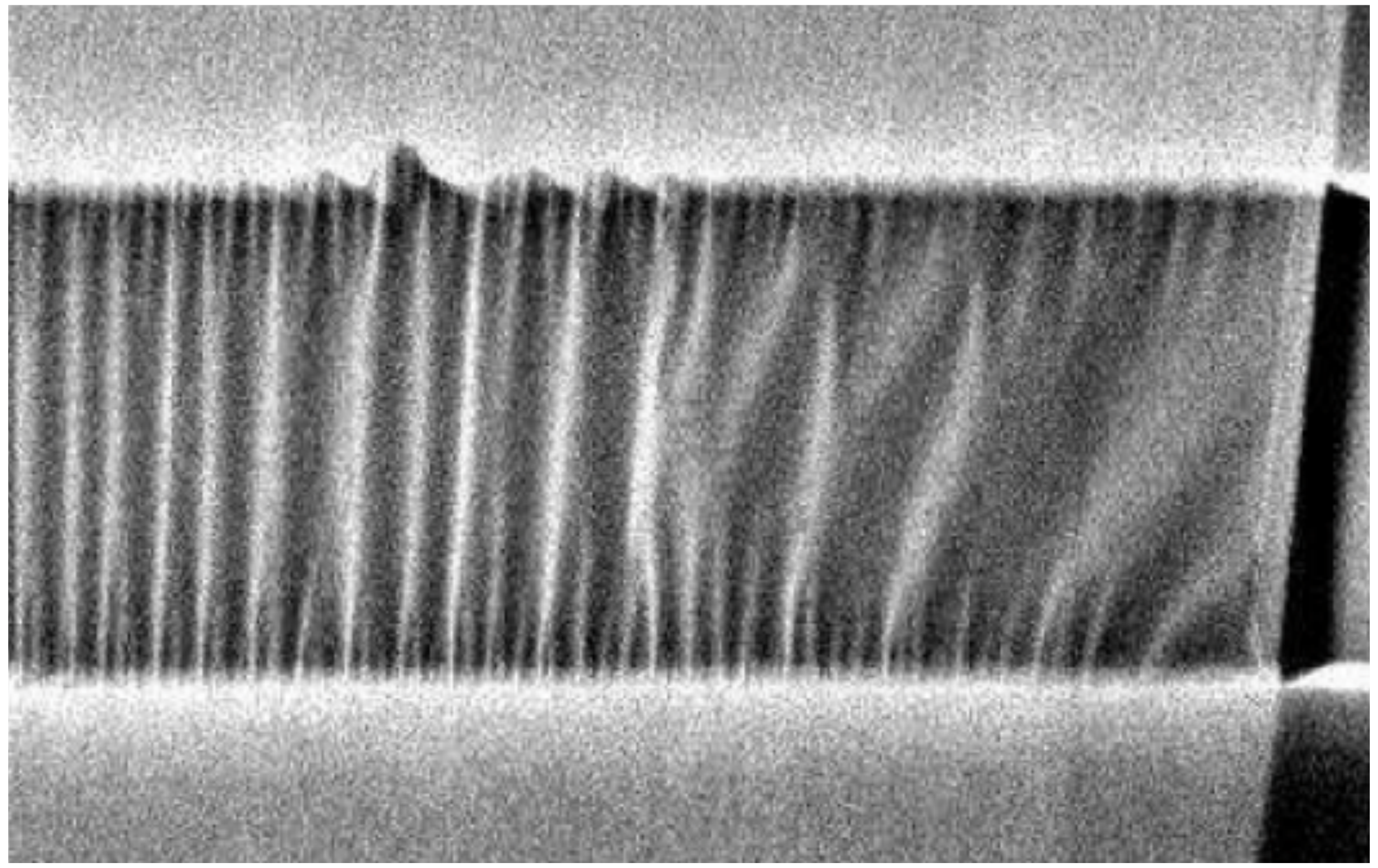
K. Burton et al., 1999

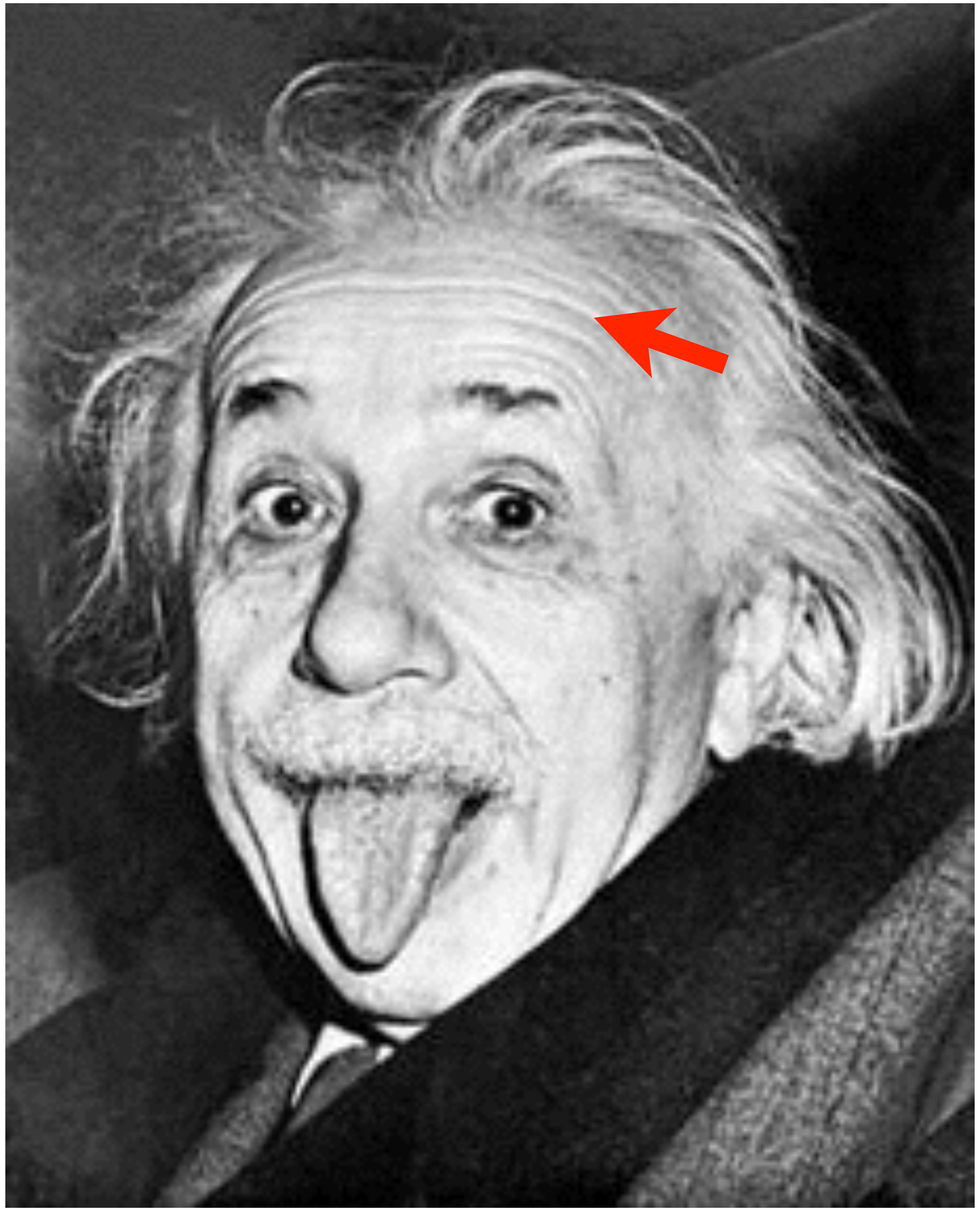

Mahadevan and Cerda, 2003



J. Huang et al., 2010

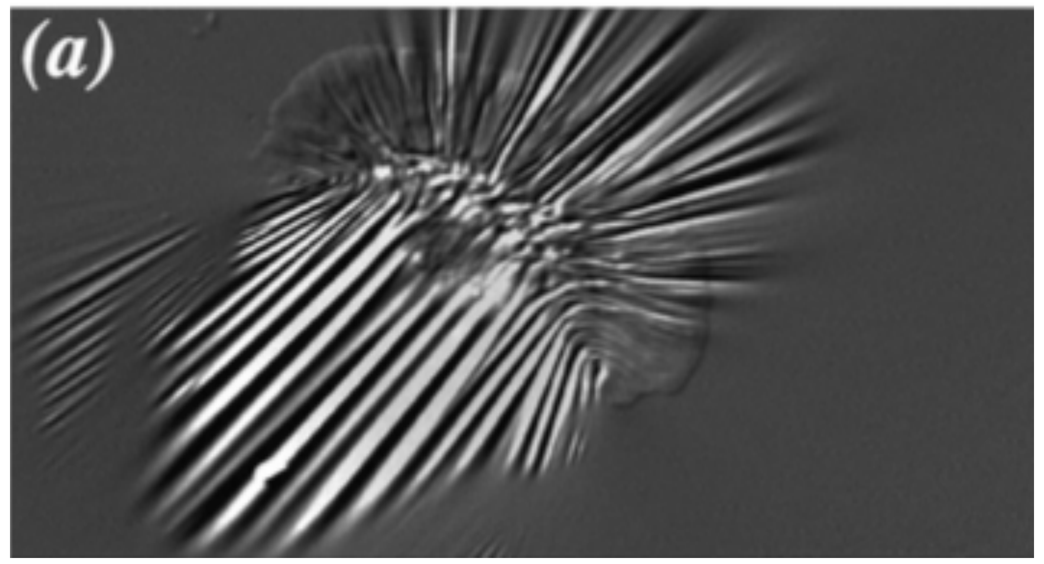

infobarrel.com


Vandeparre et al., 2011

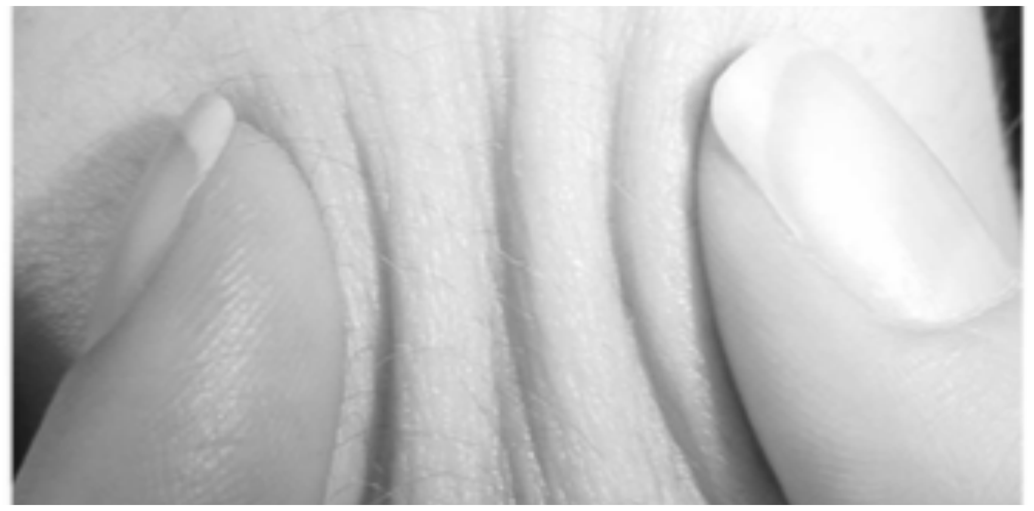

wikipedia.org

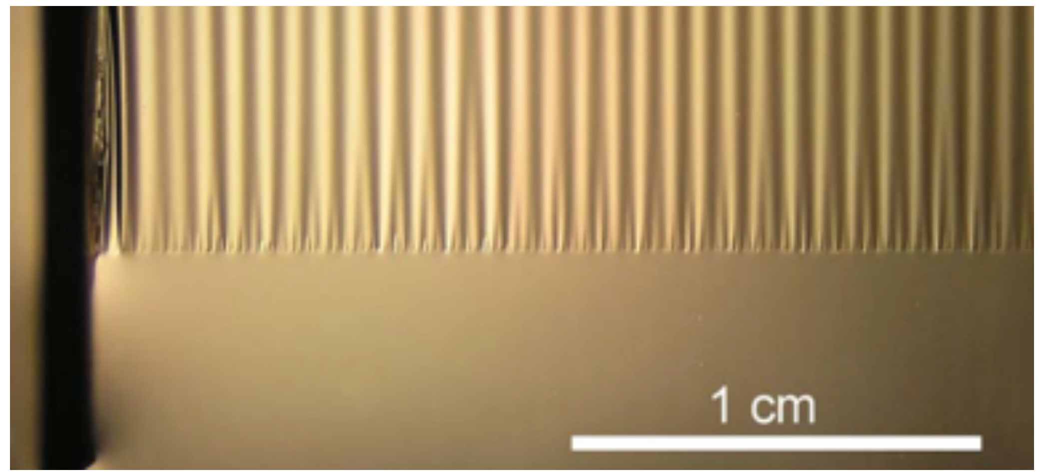
Why do we **care?**

N. Bowden et al., 1998



K. Burton et al., 1999


Mahadevan and Cerda, 2003



J. Huang et al., 2010

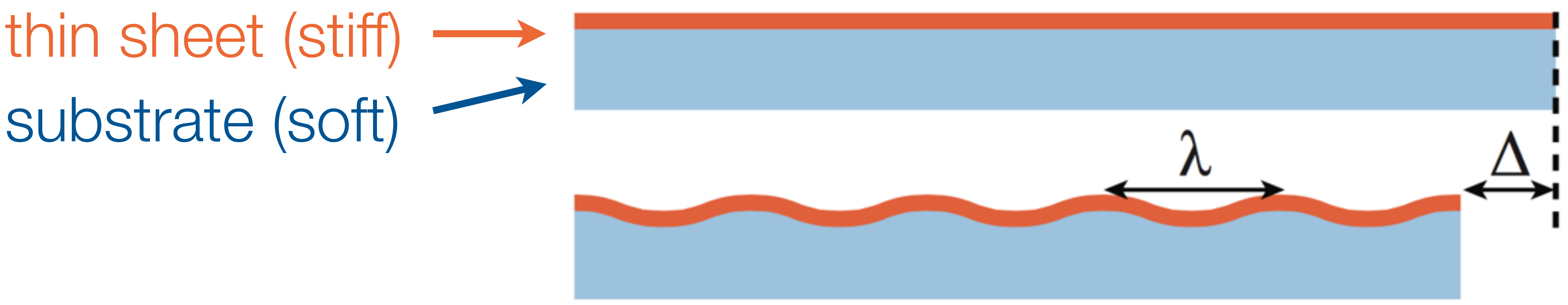
Applications

- Micropatterning
- **Non-invasive** probe of local environment
- Metrology for modulus/thickness of thin films

Fundamental Questions

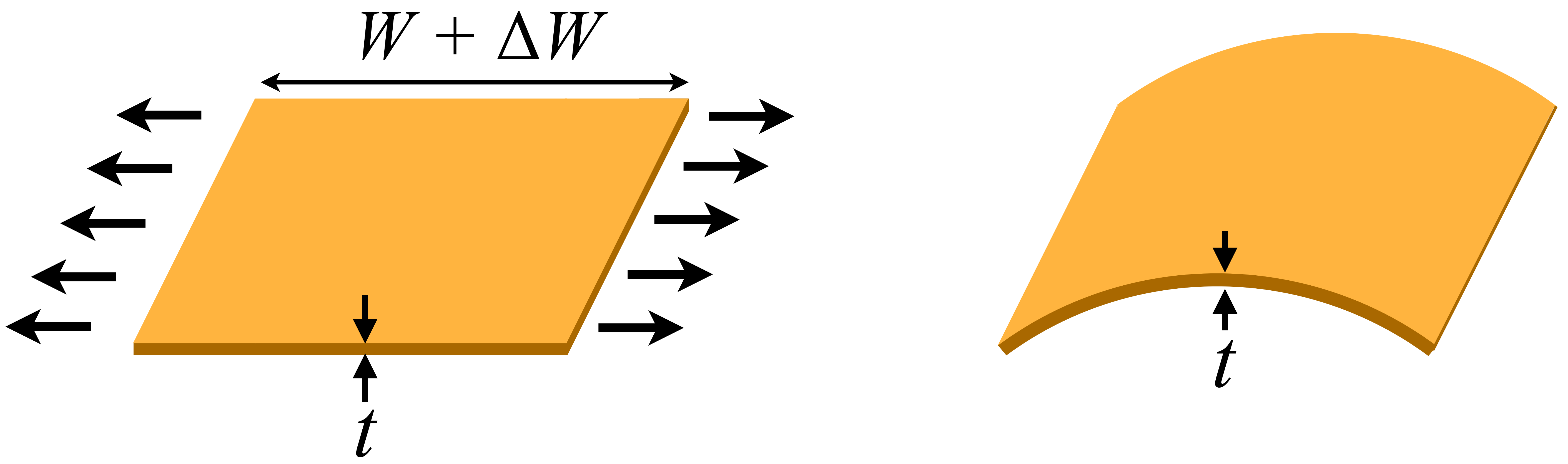
- Threshold?
- Wavelength?
- How is stress/strain distributed in thin materials?

What do we **know?**



Wrinkles **“hide”** excess material

Bending is cheaper than stretching

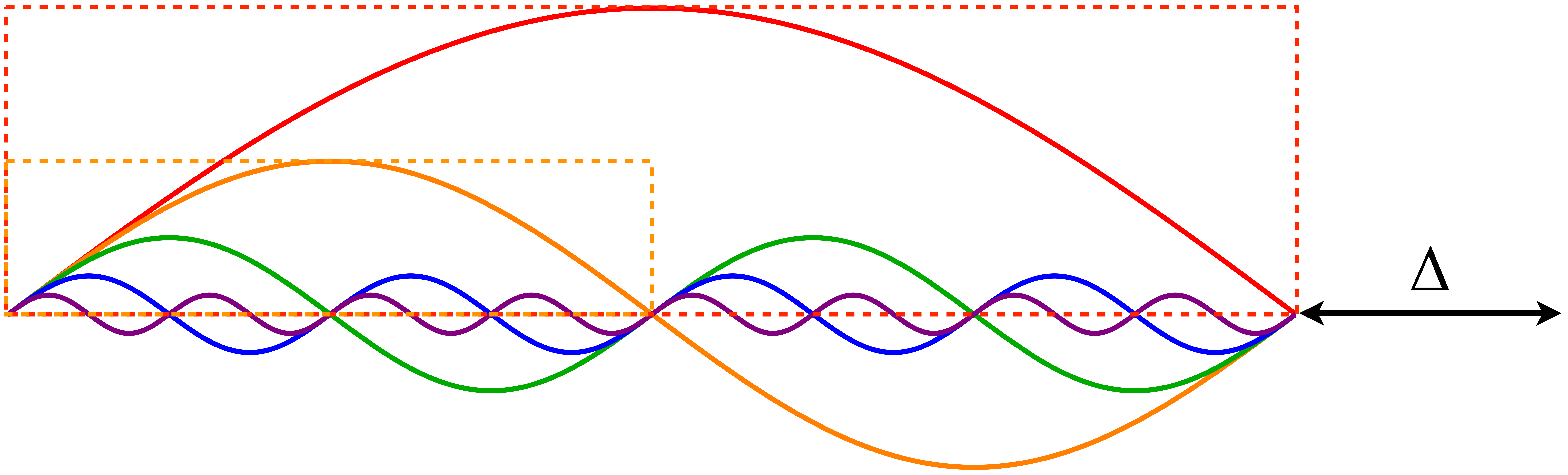


$$\frac{U_{\text{bend}}}{U_{\text{stretch}}} \sim \left(\frac{W}{\Delta W}\right)^2 \left(\frac{t}{W}\right)^2$$

$\leftarrow \dfrac{1}{\text{von Kármán \#}}$

small t/W: bending energy ≪ stretching energy

How are wrinkle wavelength & amplitude related?



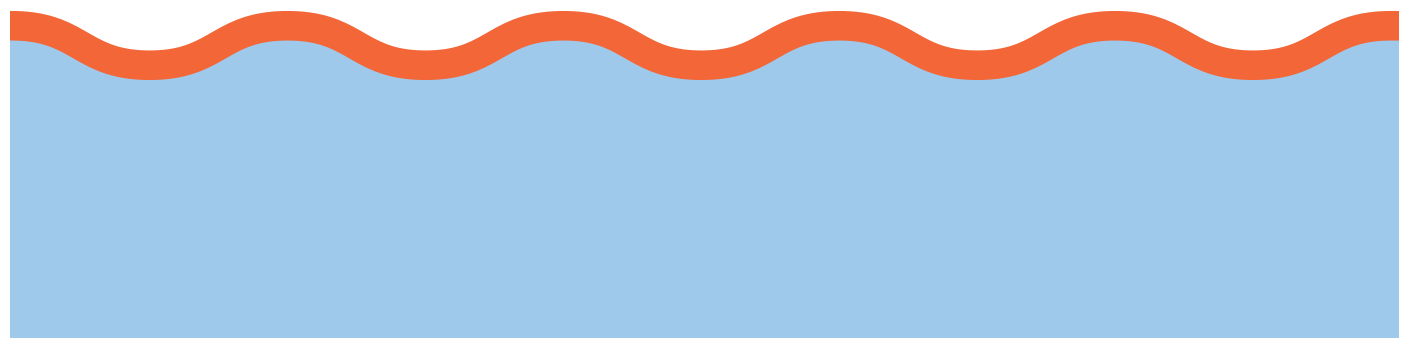
amplitude wavelength

“Slaving condition”: fixed compression ⇔ A/λ constant

Large amplitude, large λ
Small amplitude, small λ

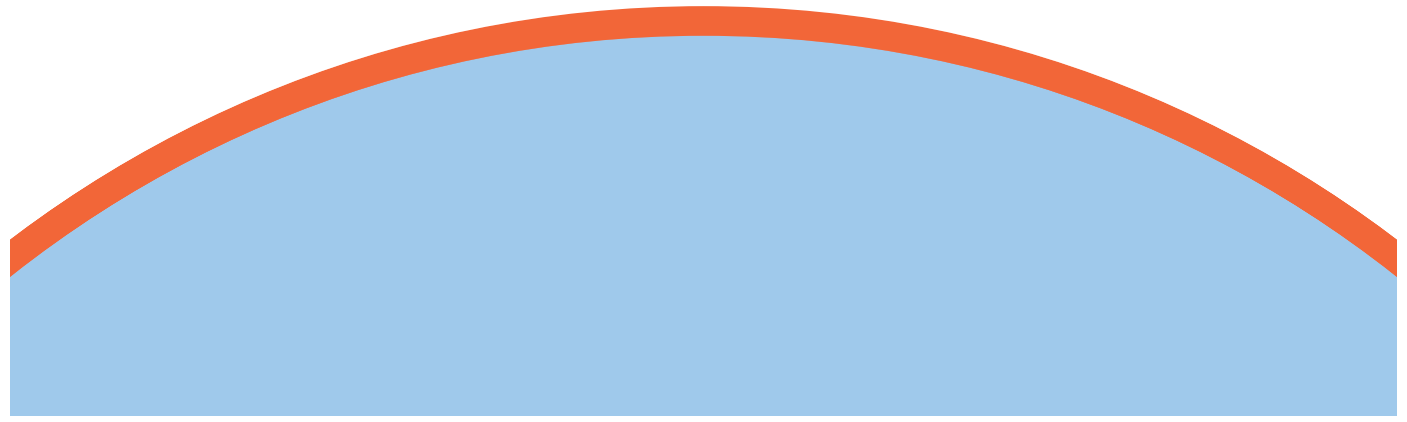
Wavelength selection. Example: Sheet floating on liquid surface

Short wavelength: good for liquid



$$U_{\text{gravity}} \sim \rho g \lambda^2$$

Long wavelength: good for sheet



$$U_{\text{bend}} \sim \frac{B}{\lambda^2}$$

where $B = \frac{Et^3}{12(1-\Lambda^2)}$

Balance energies: compromise

$$\lambda = 2\pi \left(\frac{B}{\rho g}\right)^{1/4}$$

Another type of stiffness: **Tension**

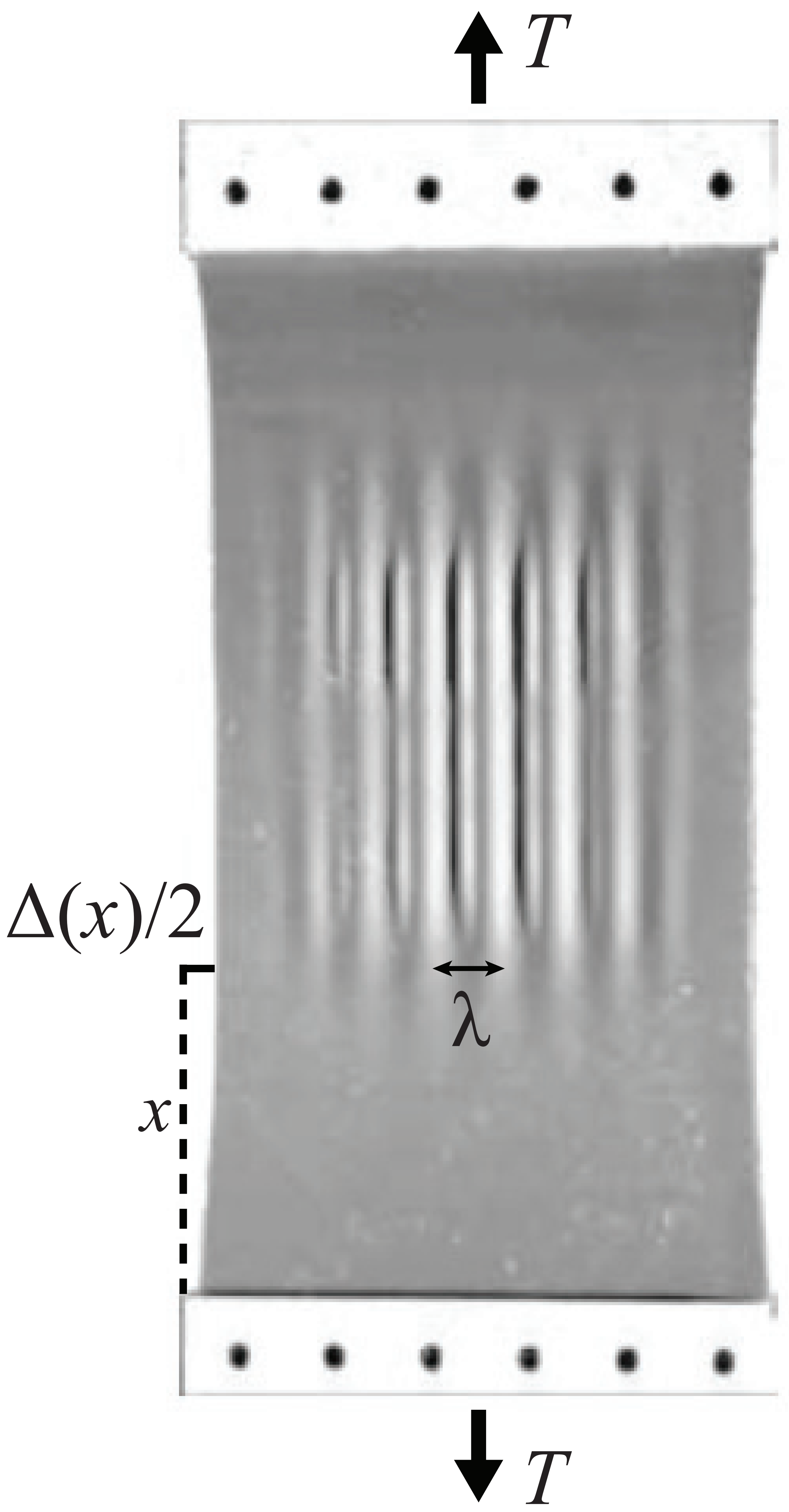


Image: K. Ravi-Chandar

Tension penalizes long wavelengths

Long wavelength
⇒ Large amplitude
⇒ Stretching along wrinkles (expensive)

Balance with bending: $\lambda = 2\pi \left(\frac{B}{K}\right)^{1/4}$

with $K_{\mathrm{tens}} = T\left(\frac{\Phi'(x)}{\Phi(x)}\right)^2$, where $\Phi = A/\lambda$

Universal law: just insert relevant stiffness, K

Cerda & Mahadevan, PRL 2003

Another type of stiffness: **Tension**

Image: K. Ravi-Chandar

What is wavelength for **non-uniform** and/or **curved** wrinkles?

bending: $\lambda = 2\pi \left(\frac{B}{K}\right)^{1/4}$

with $K_{\text{tens}} = T \left(\frac{\Phi'(x)}{\Phi(x)}\right)^2$, where $\Phi = A/\lambda$

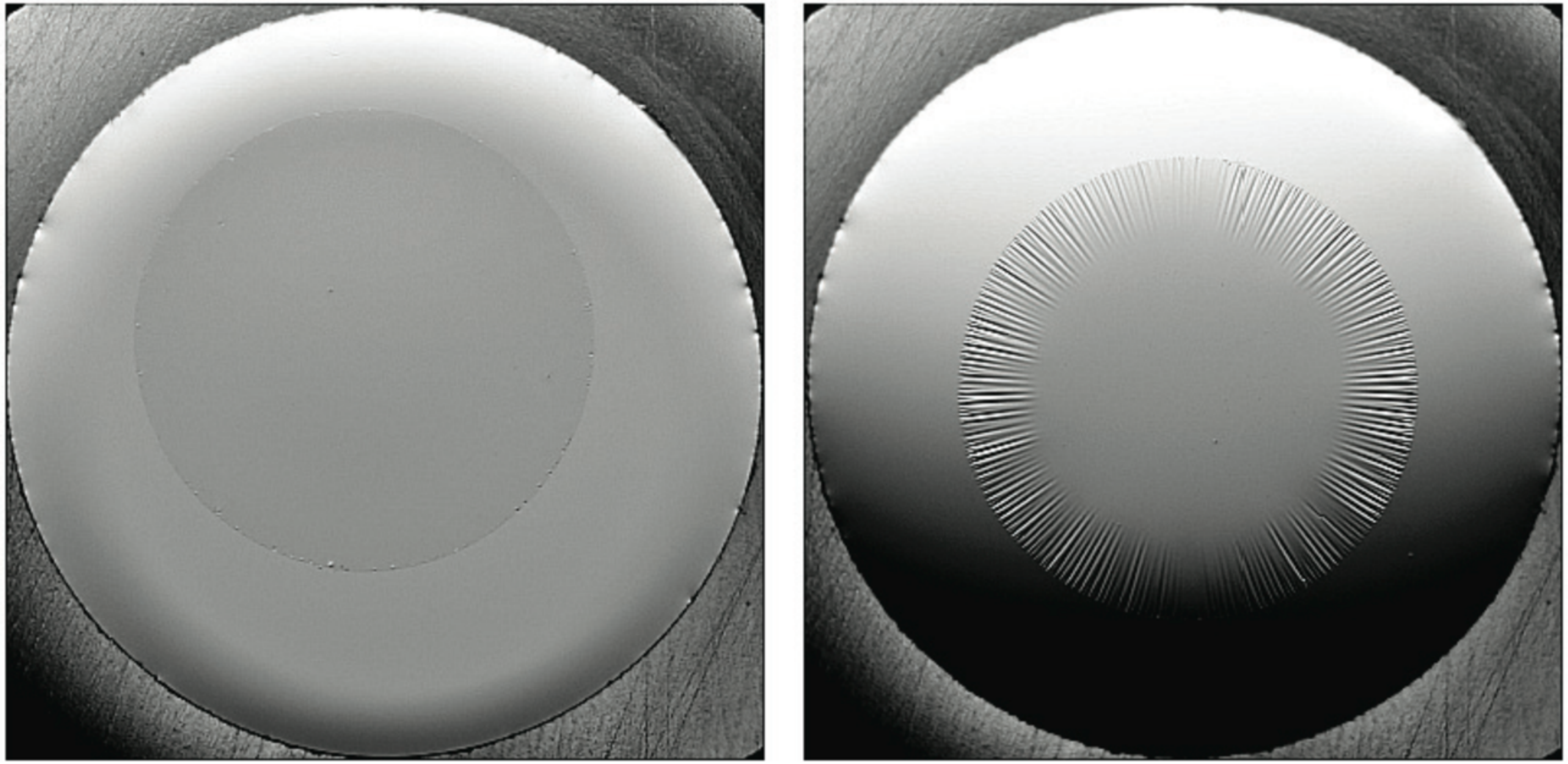
Universal law: just insert relevant stiffness, K

Cerda & Mahadevan, PRL 2003

Initially flat polymer film on a curved water drop

King, Schroll, Davidovitch, & Menon, PNAS 2012
Hunter King, PhD Thesis, 2013

top:



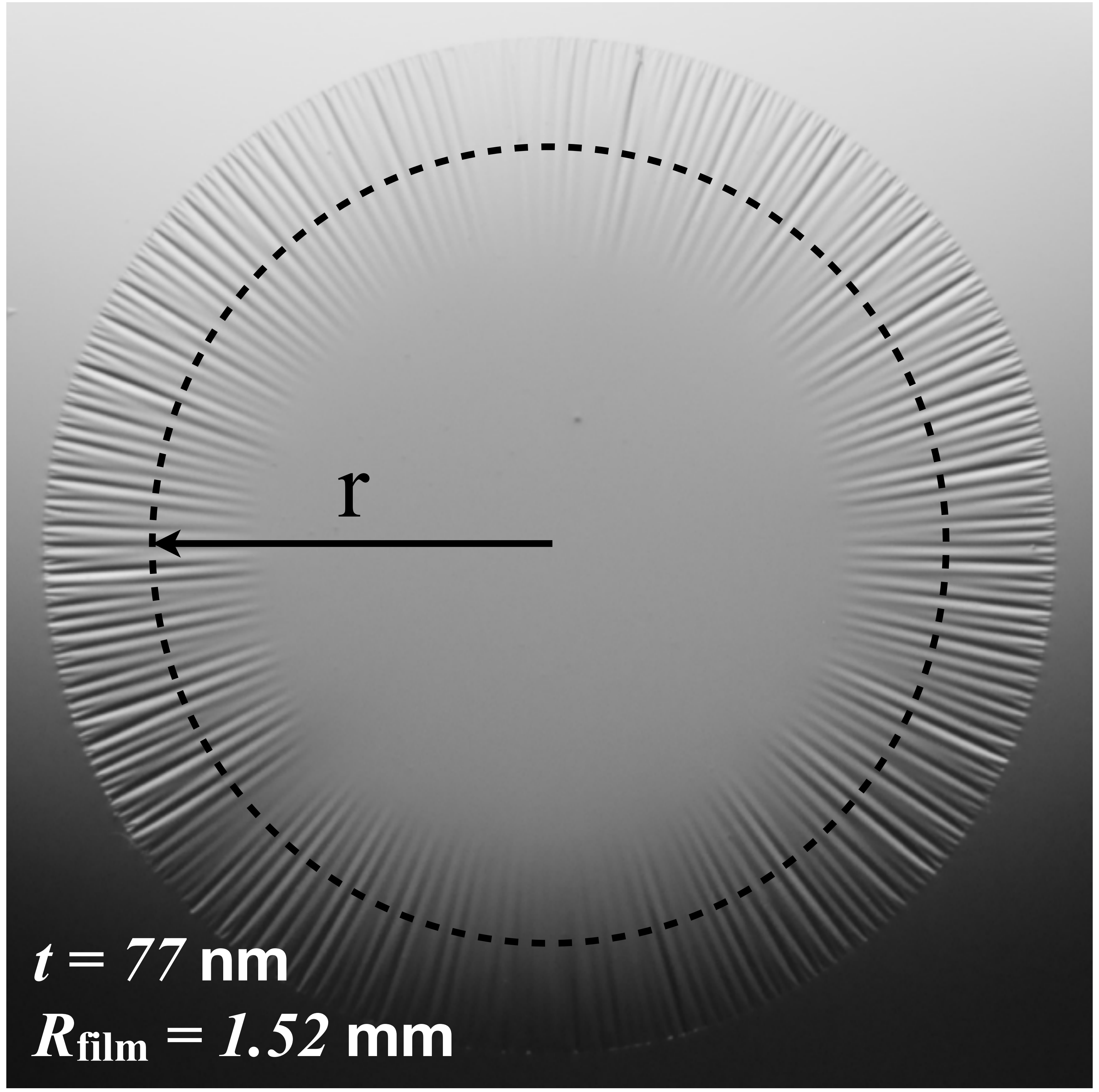
side:

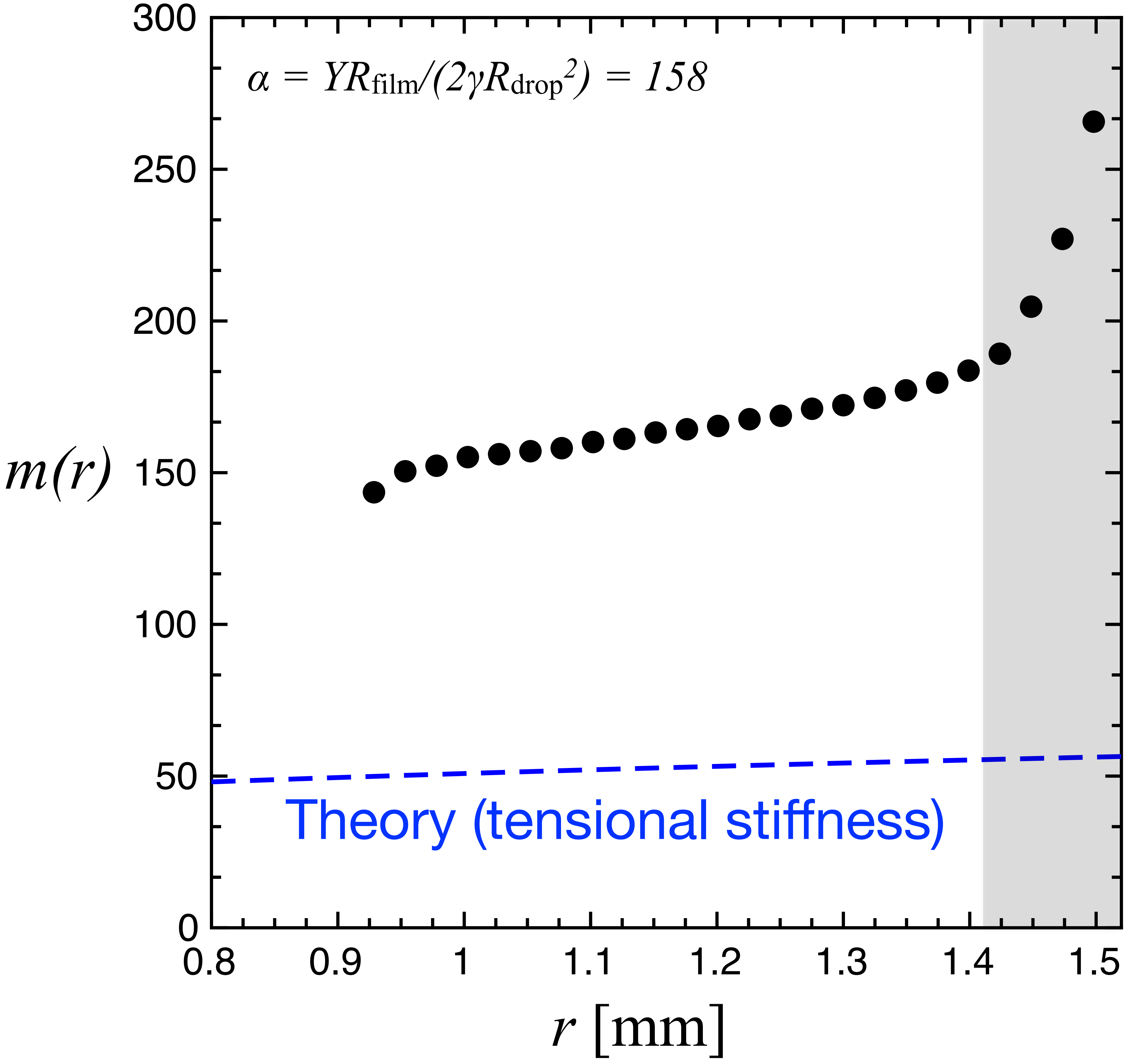
Curvature playing two roles:

Gaussian curvature ***causes*** wrinkles

Wrinkles ***live*** in curved environment

Polymer film on curved surface: **Number of wrinkles, m(r)**



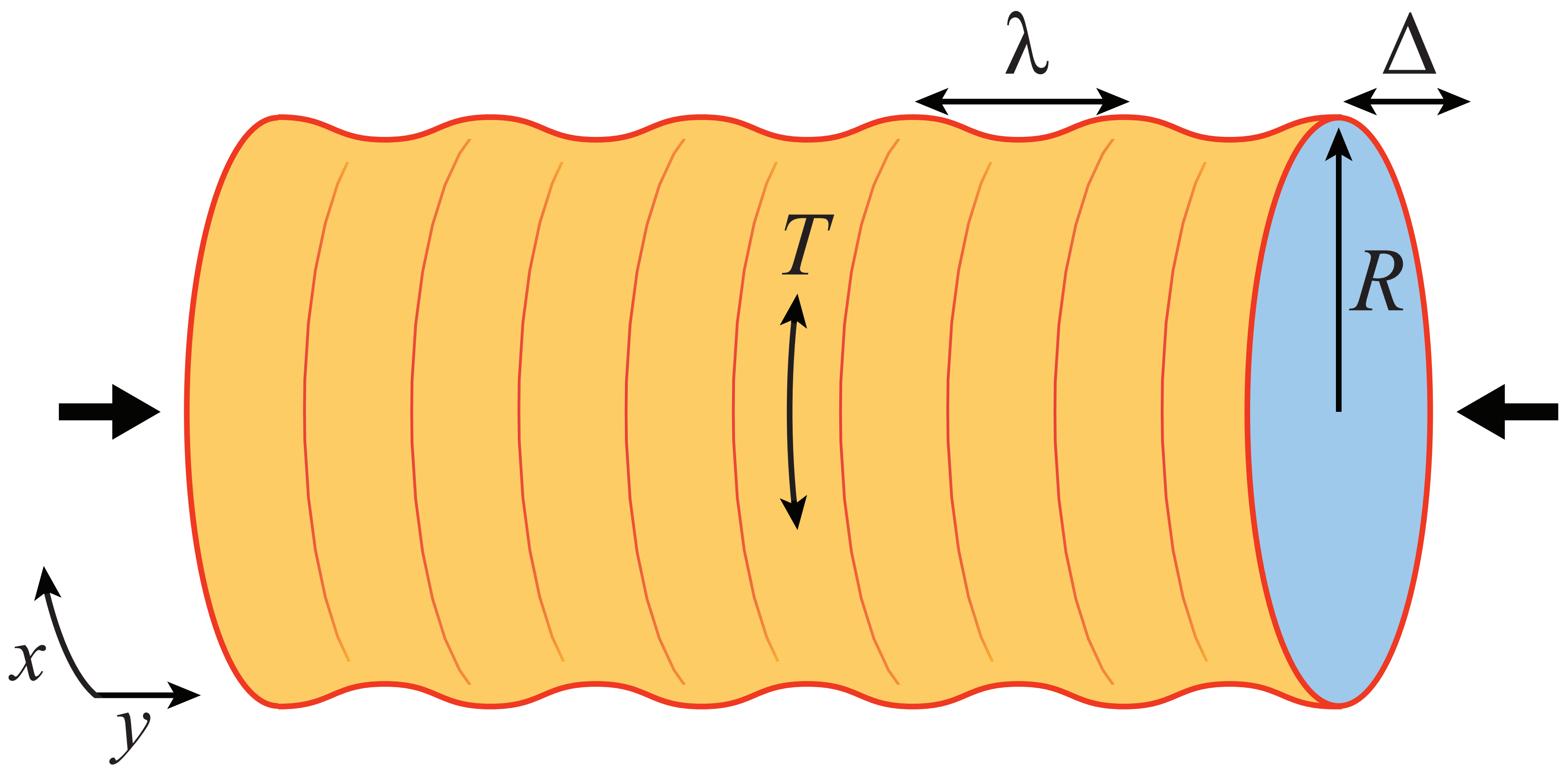


Environment ***stiffer*** than expected

Motivates new stiffness: $K_{curv} = ?$

Another type of stiffness: **Curvature**

Evan Hohlfeld, Dominic Vella, Benny Davidovitch



1. **Normal force balance** (1st FvK eqn.)

2. **Assume cylindrical shape + sinusoidal wrinkles**

3. **Far-from-threshold expansion:** find first order correction to stress along wrinkle direction

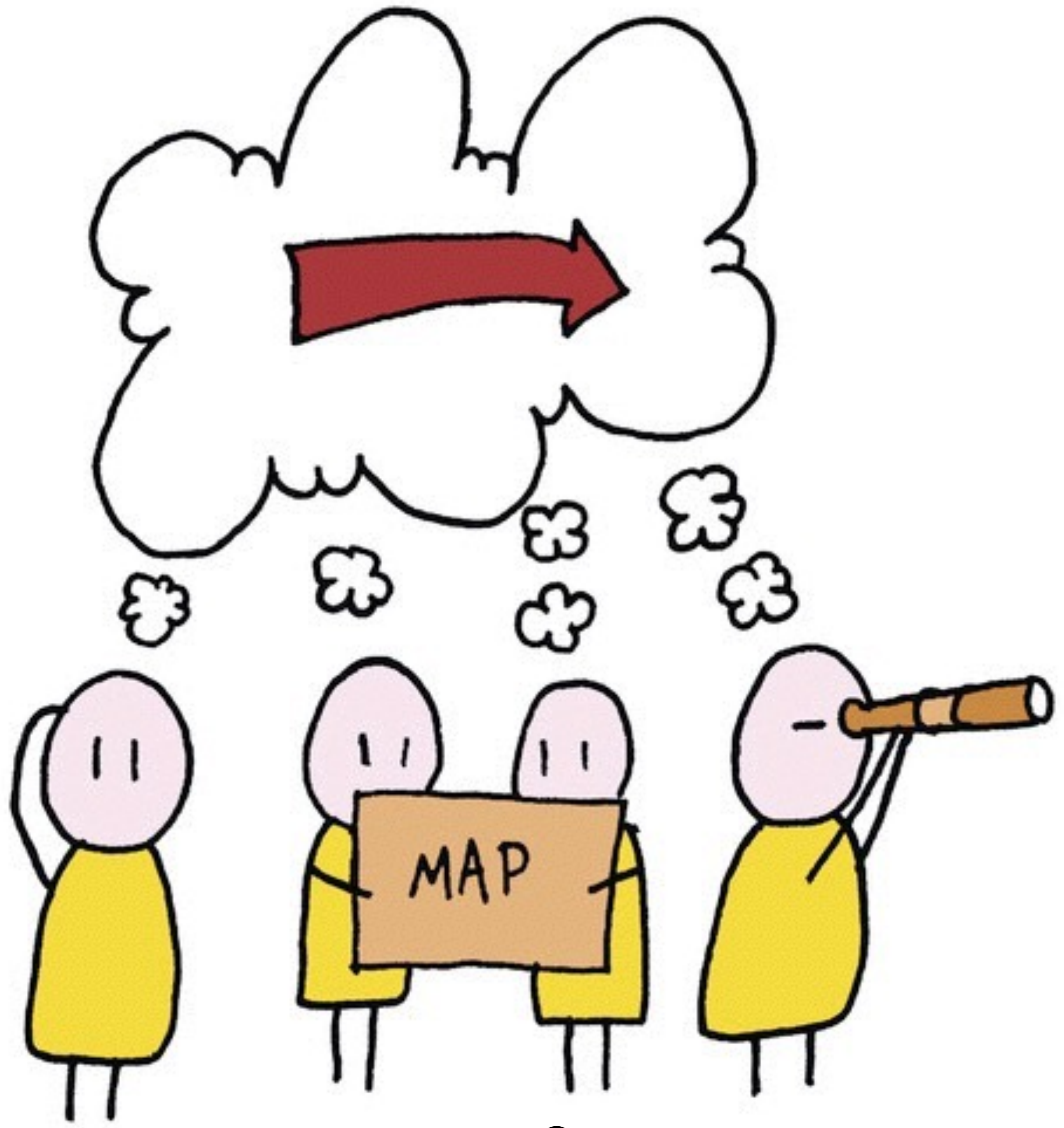
4. **Plug correction into normal force balance:**

$$\left[B\left(\frac{2\pi}{\lambda}\right)^4 - T\frac{d^2}{dx^2} + Y\zeta_0''^2 + K_{sub}\right]f = -\sigma_{yy}\left(\frac{2\pi}{\lambda}\right)^2 f$$

bending · tension-induced stiffness · curvature-induced stiffness · substrate stiffness · compression

$$K_{curv} = Y/R_{\|}^2$$

Paulsen, Hohlfeld et al., PNAS 2016

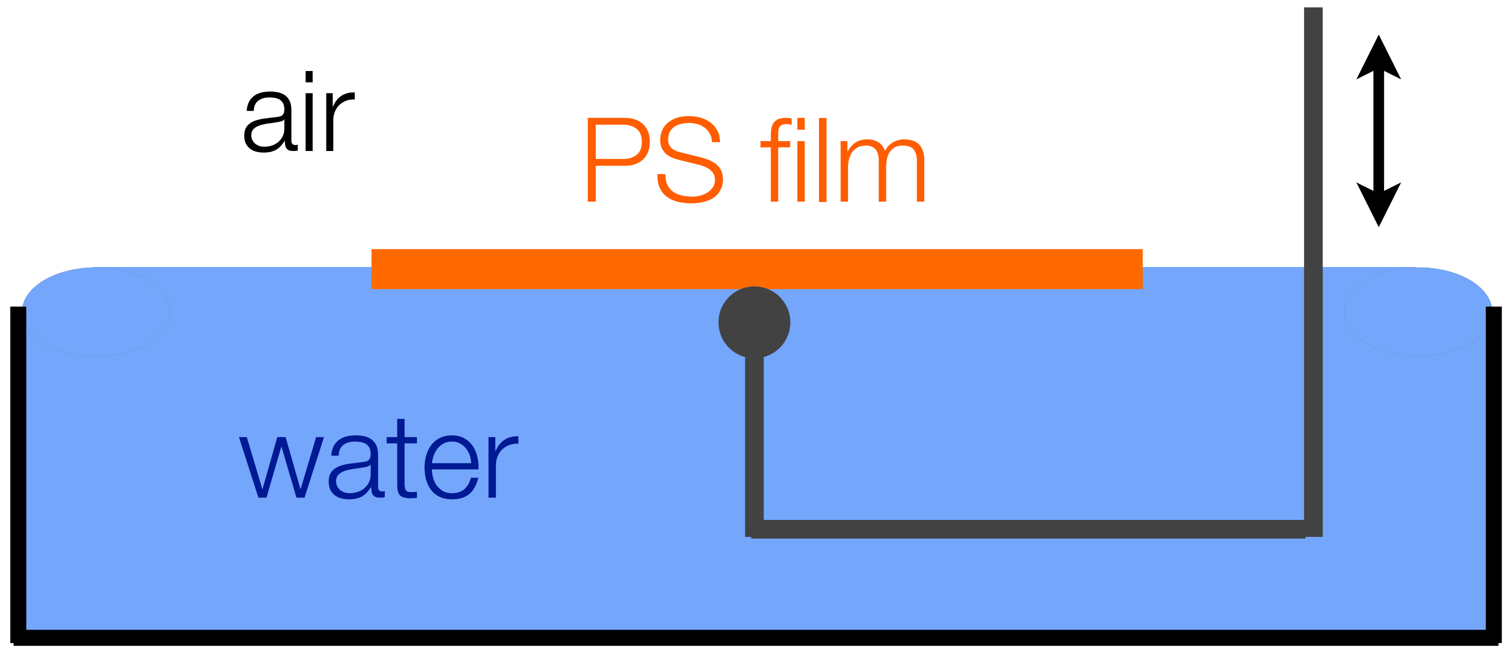


Shown: Must account for curvature

Prediction: Curvature-induced stiffness, $K_{curv} = Y/R_{\|}^2$

Now: Test in two experimental settings

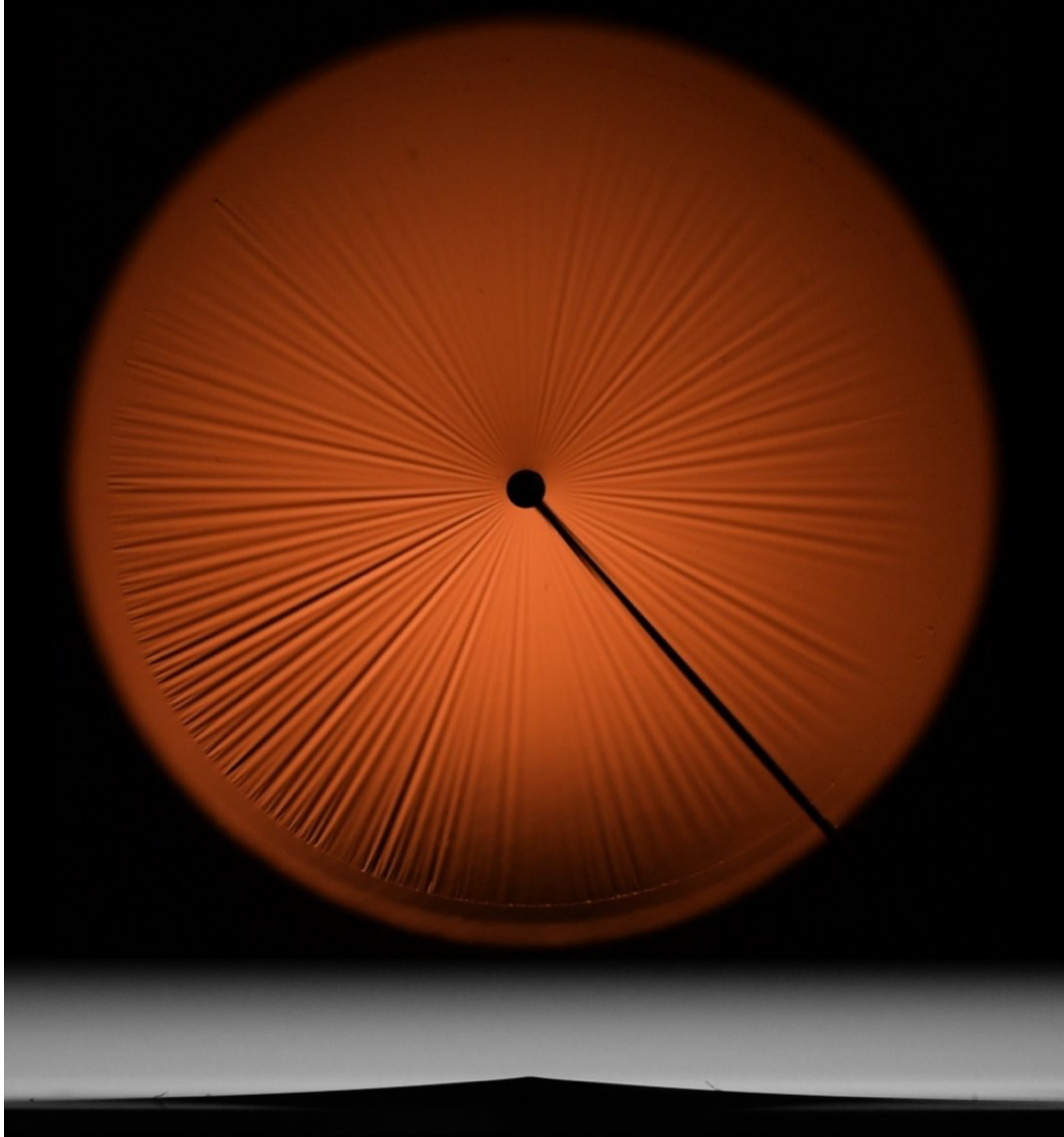
Experiment: Poking



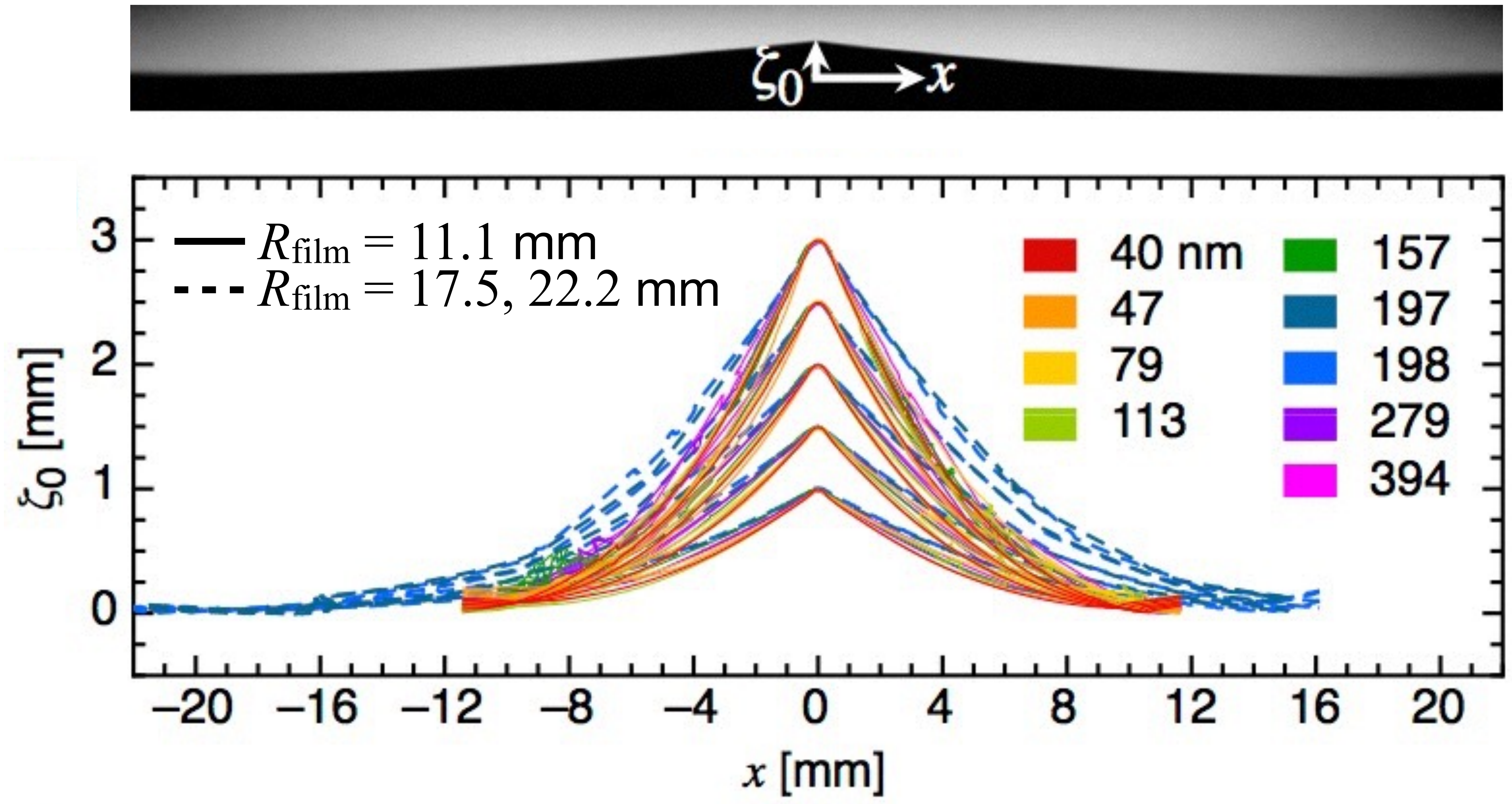
Circular film: R_{film} = 11 to 22 mm
Thickness: t = 40 to 400 nm

von Kármán number, $(W/t)^2$:

10^9 to 10^{11}



Side profile: measure local curvature

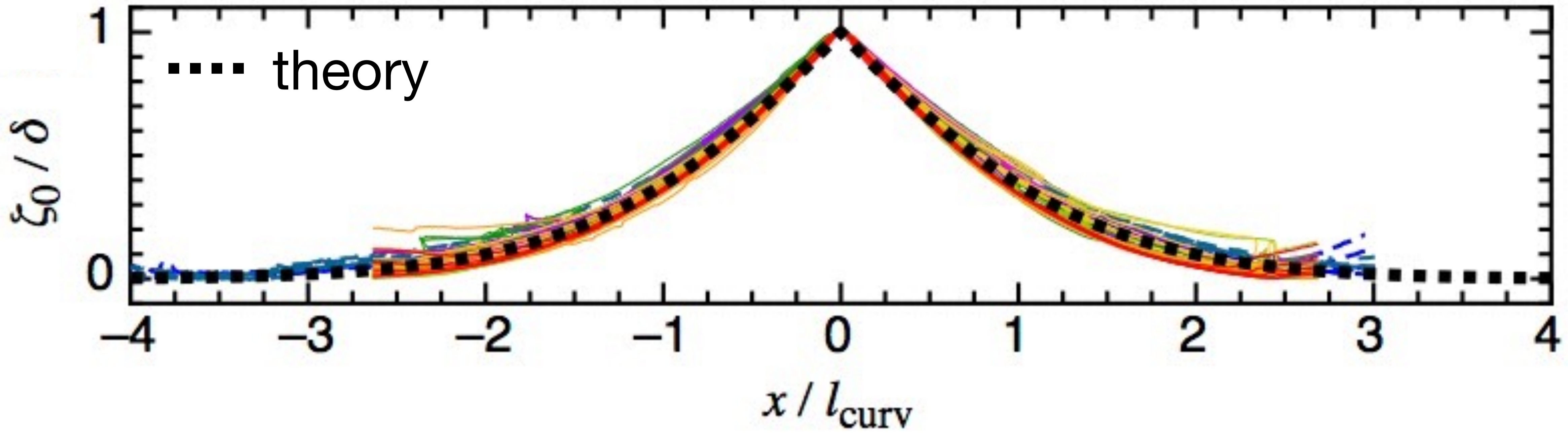


Theoretical prediction:

Vella, Huang, Menon, Russell, & Davidovitch, PRL 2015

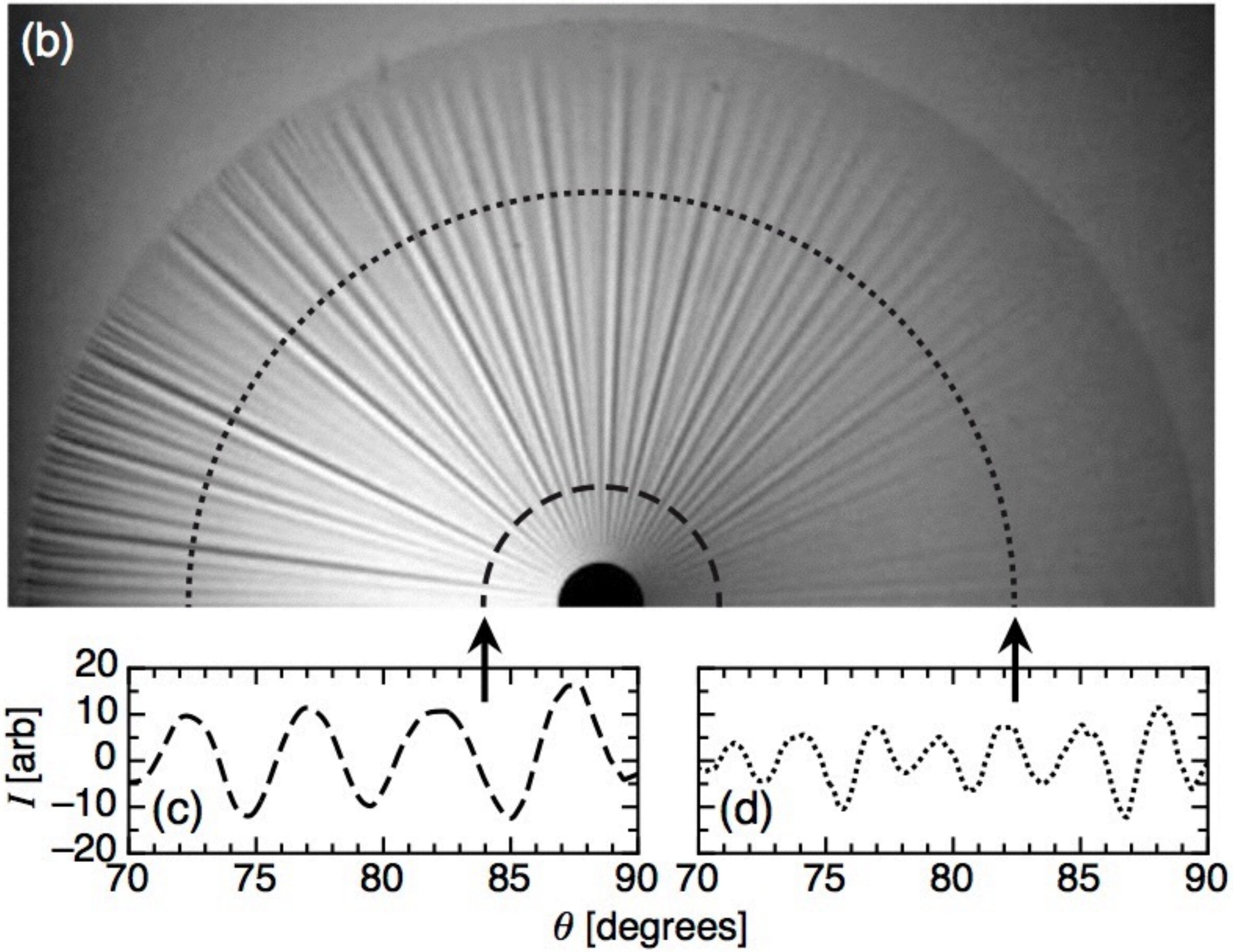
$$\zeta_0(r) \approx \delta \mathrm{Ai}(r/\ell_{\mathrm{curv}})/\mathrm{Ai}(0)$$

where $\ell_{\mathrm{curv}} = R_{\mathrm{film}}^{1/3} \ell_c^{2/3}$ and $\ell_c = \sqrt{\gamma/\rho g}$



Paulsen et al., PNAS 2016

Number of wrinkles varies radially



Simplest theoretical assumption: Minimize energy ***locally*** at every radius (***neglect*** energetic cost of $\mathbf{d\lambda/dr}$)

Paulsen, Hohlfeld et al., PNAS 2016

Wrinkle wavelength at $r = l_{\mathrm{curv}}$, versus poking amplitude, δ

Expectation:

δ increases

⇒ curvature increases

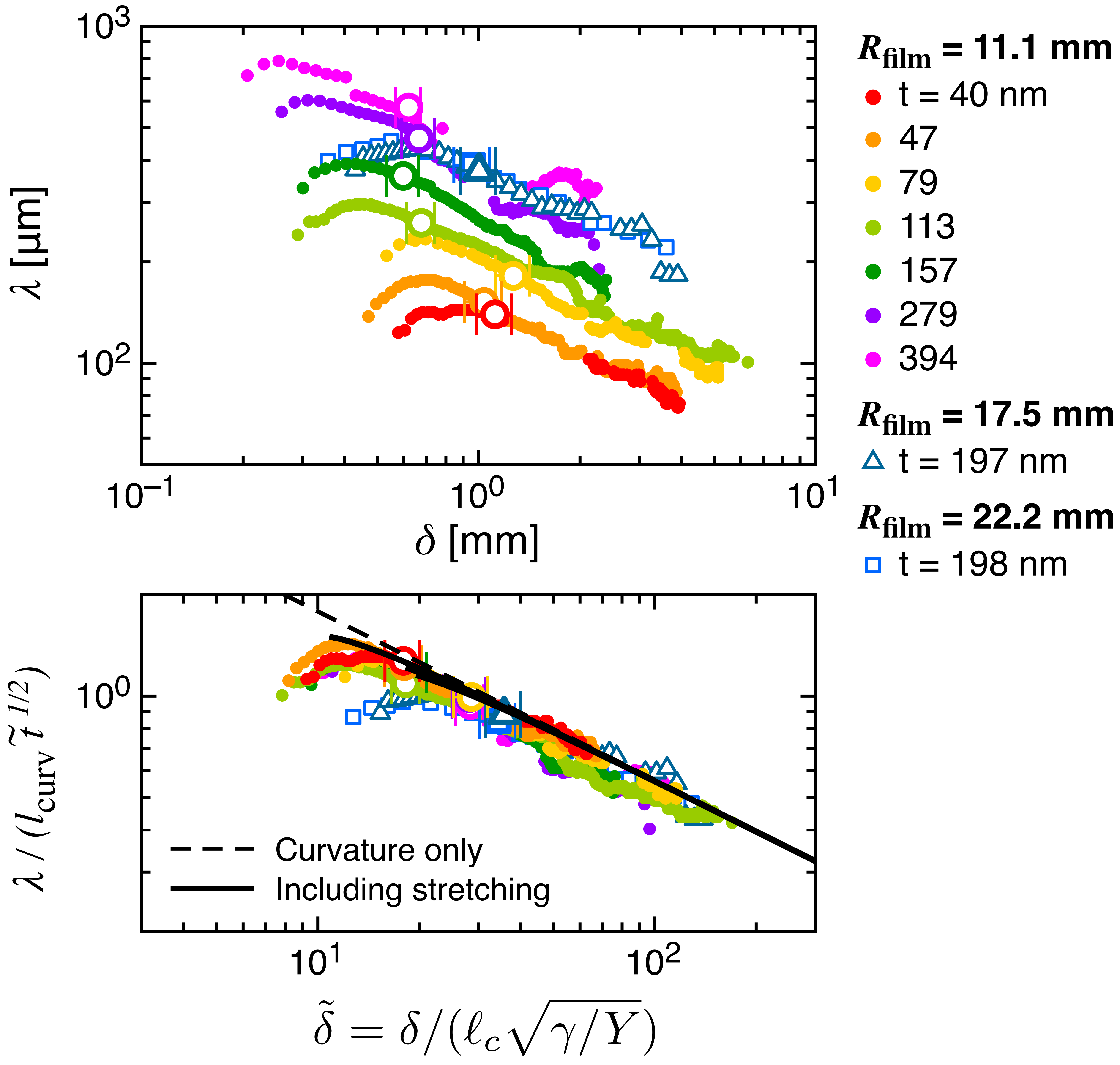
⇒ stiffness increases

⇒ wavelength decreases

Quantitatively,

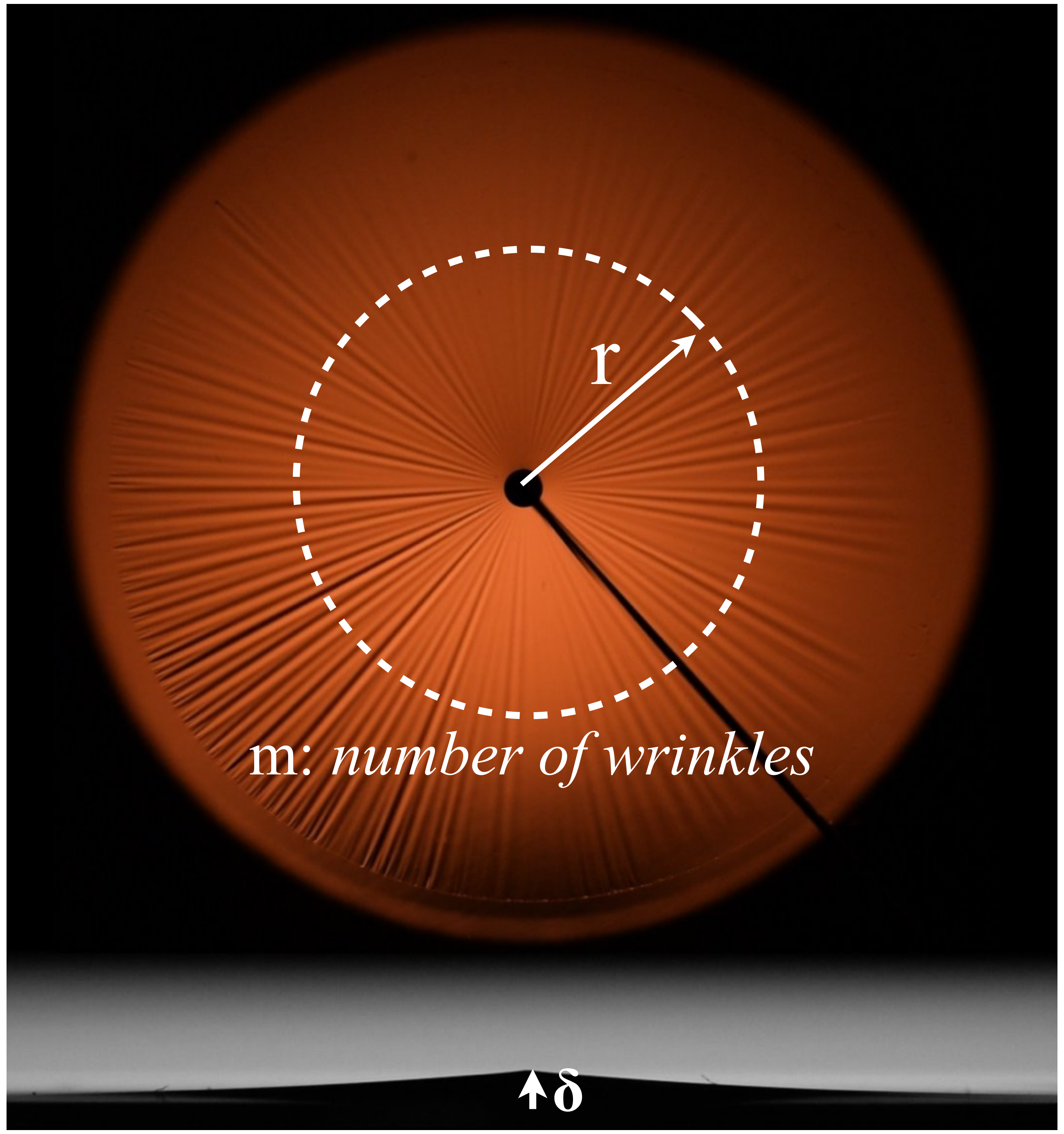
$$\lambda \sim (B/K_{\mathrm{curv}})^{1/4} \sim (t R_{\|})^{-1/2} \sim (t/\delta)^{-1/2}$$

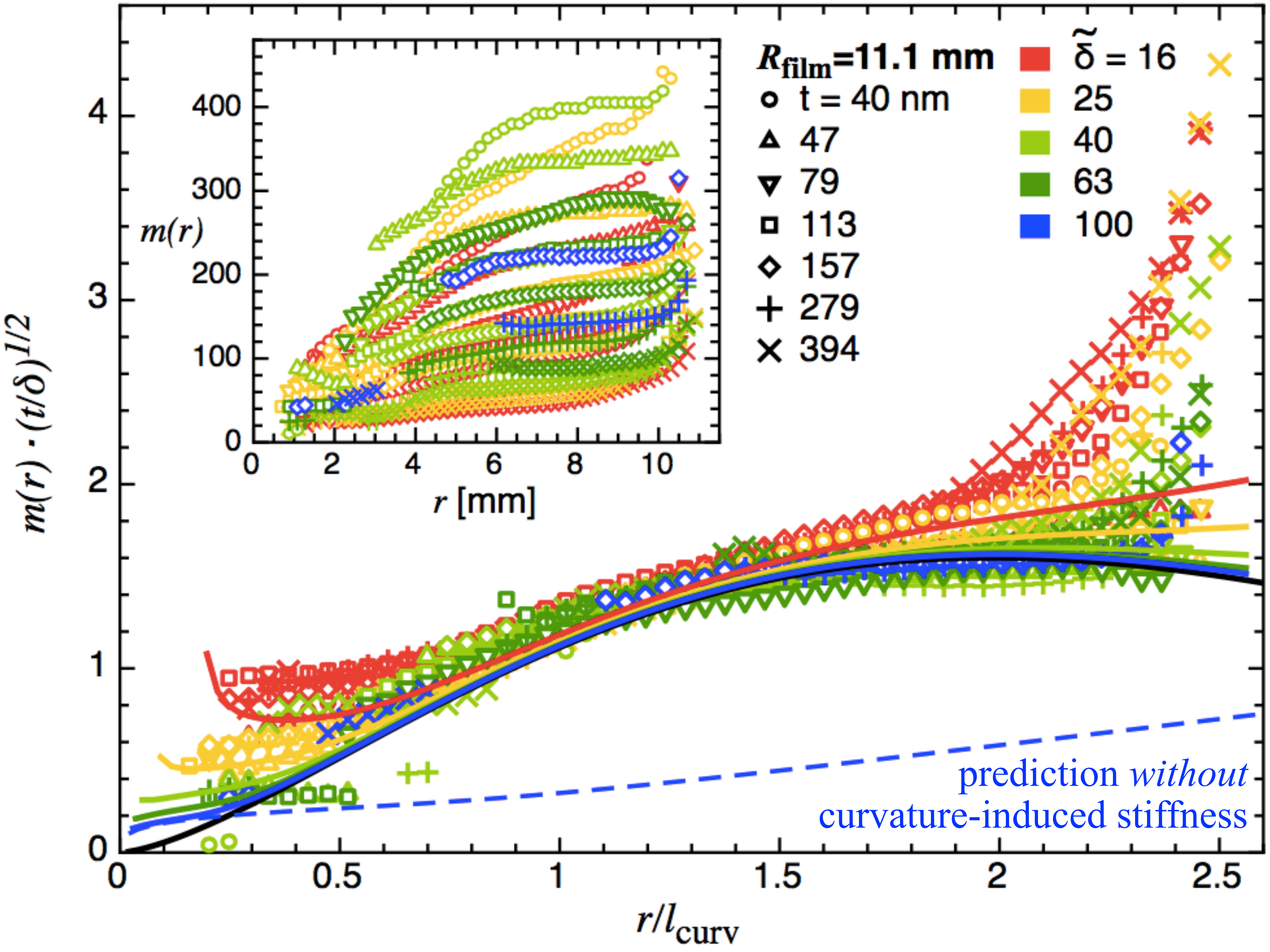
(numerical prefactors also predicted)



Paulsen, Hohlfeld et al., PNAS 2016

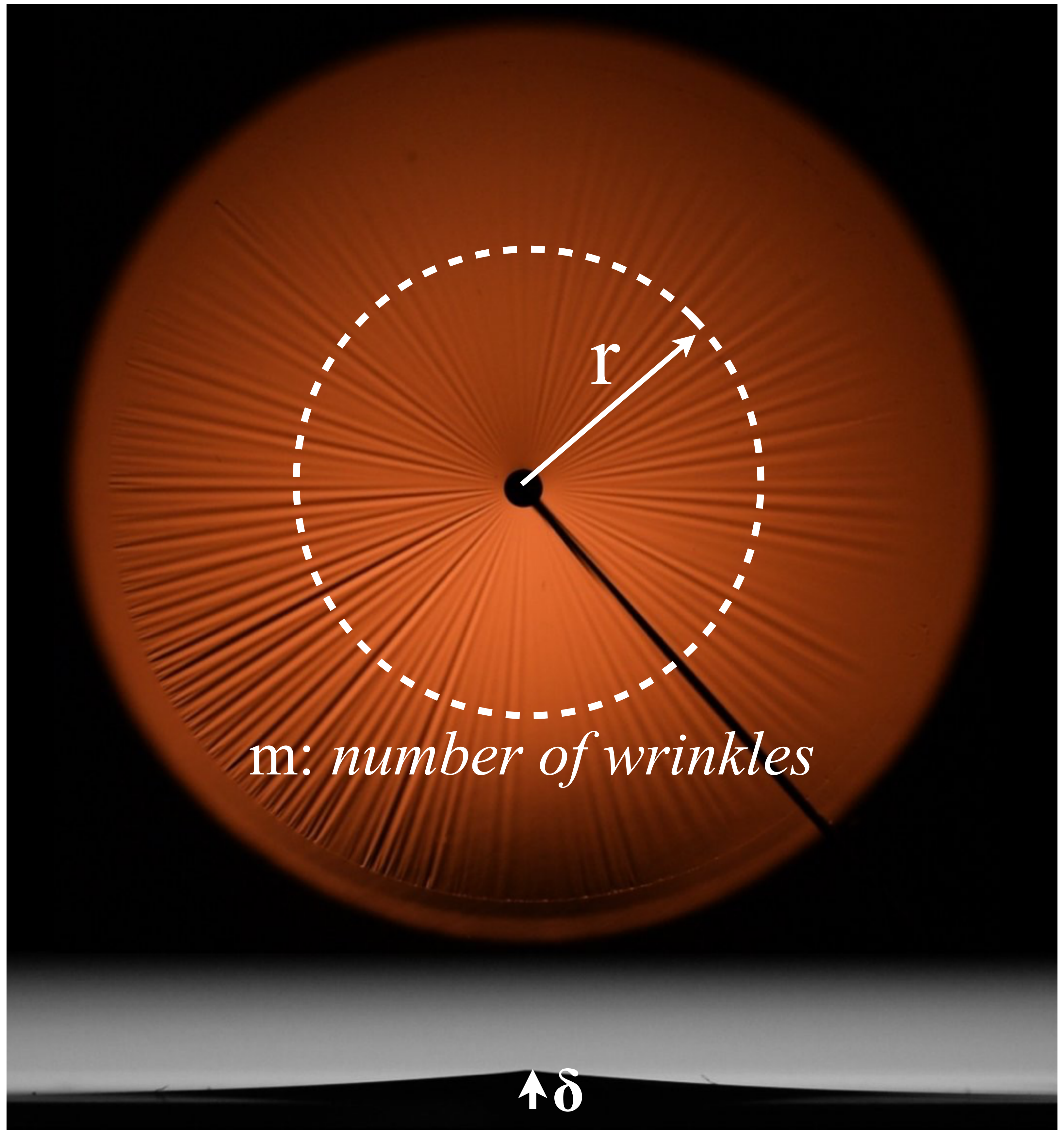
Wrinkle number versus poking amplitude, δ, and radius

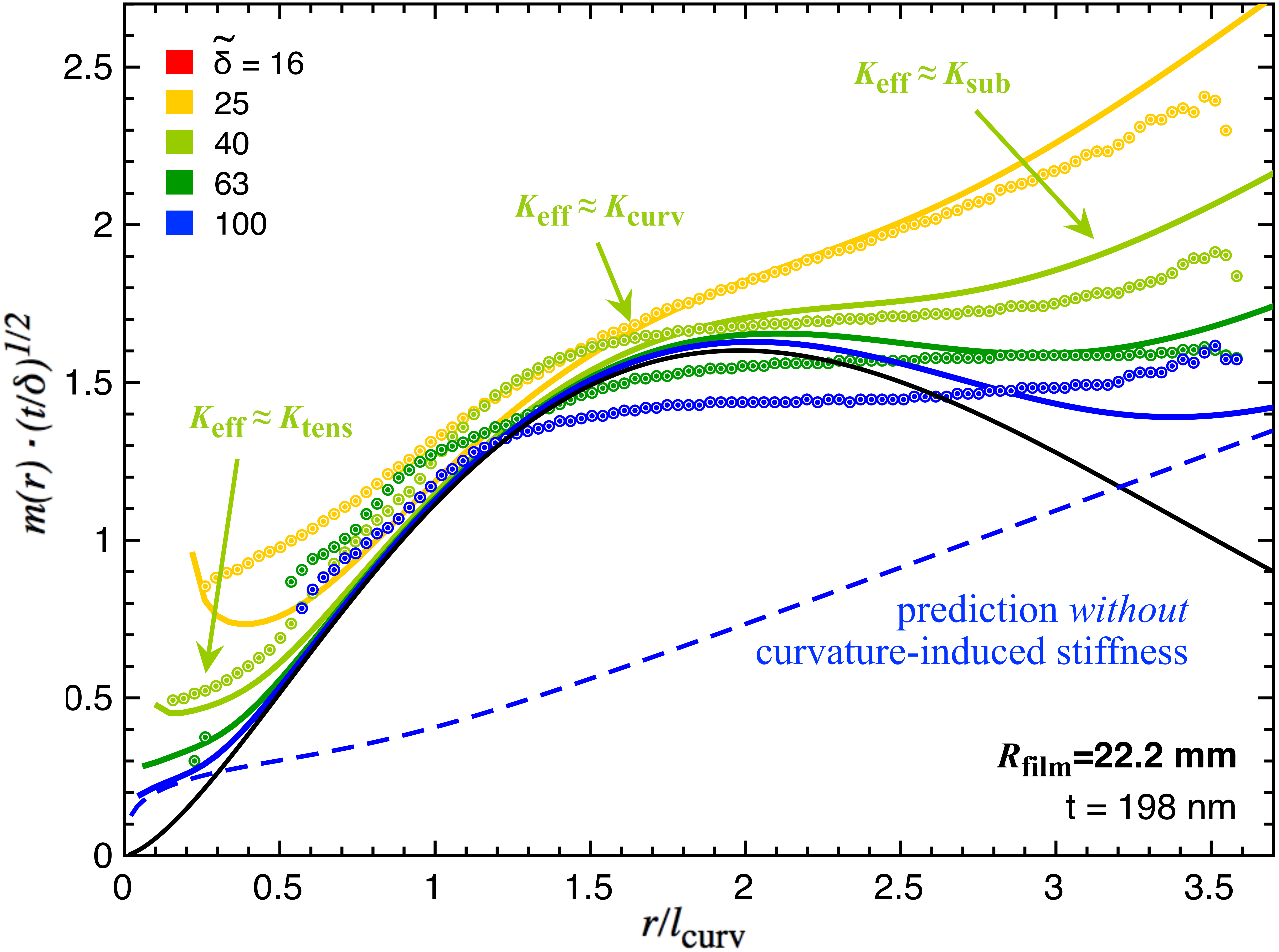




Paulsen, Hohlfeld et al., PNAS 2016

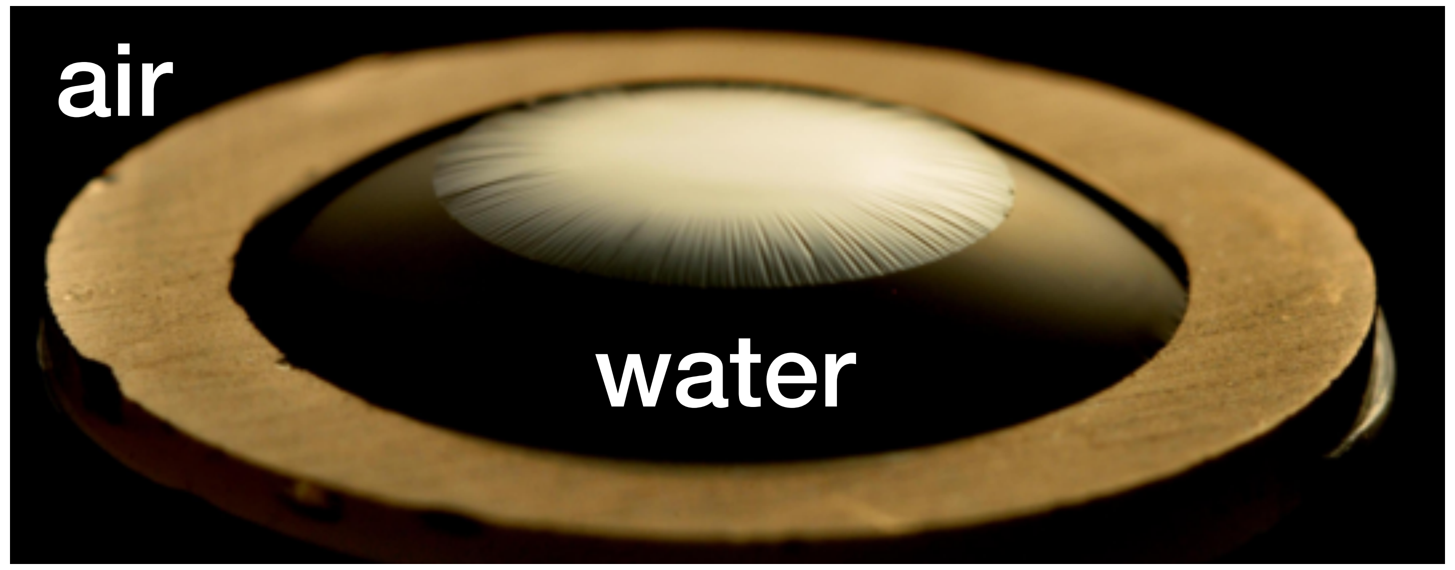
Wrinkle number versus poking amplitude, δ, and radius





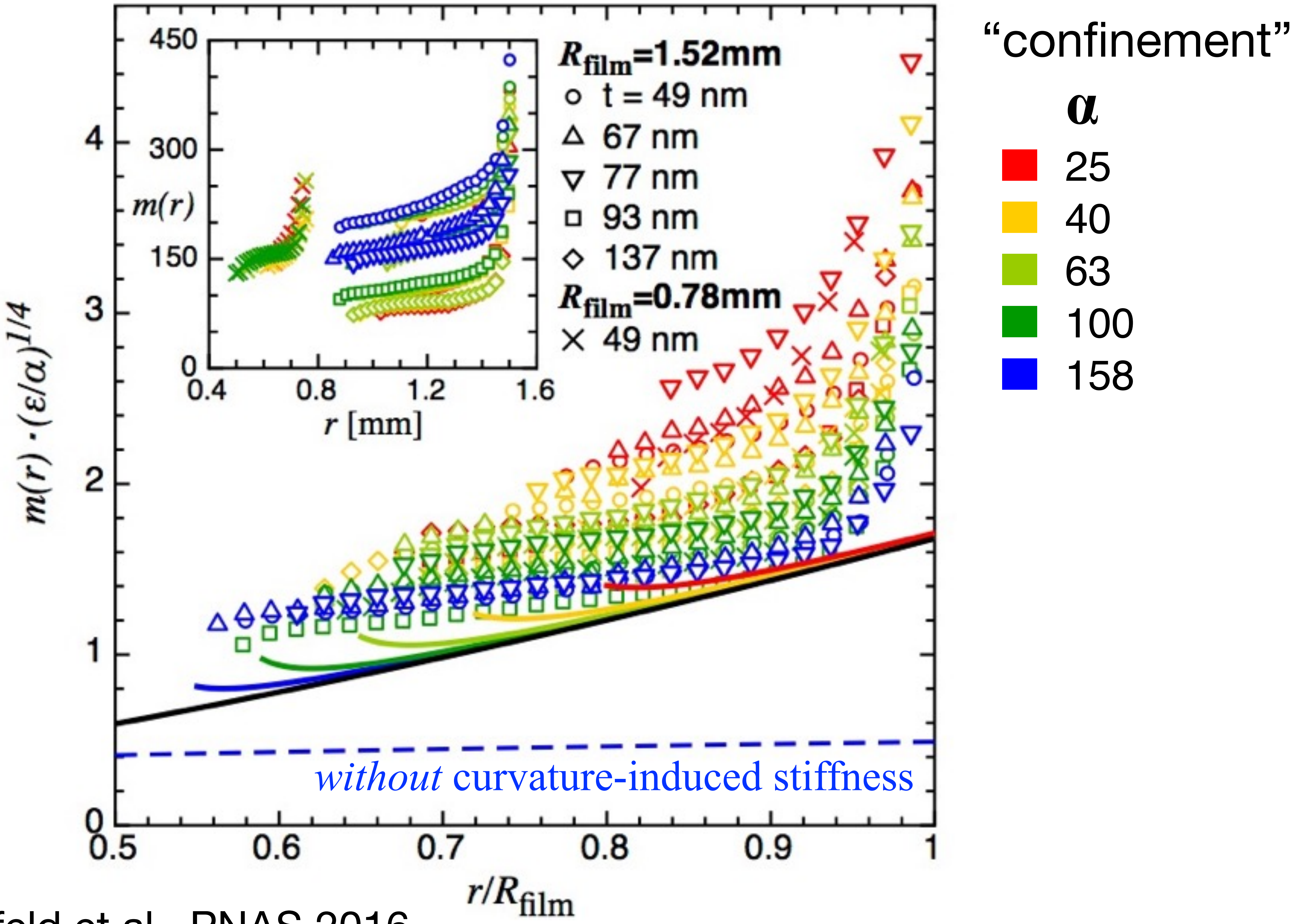
Paulsen, Hohlfeld et al., PNAS 2016

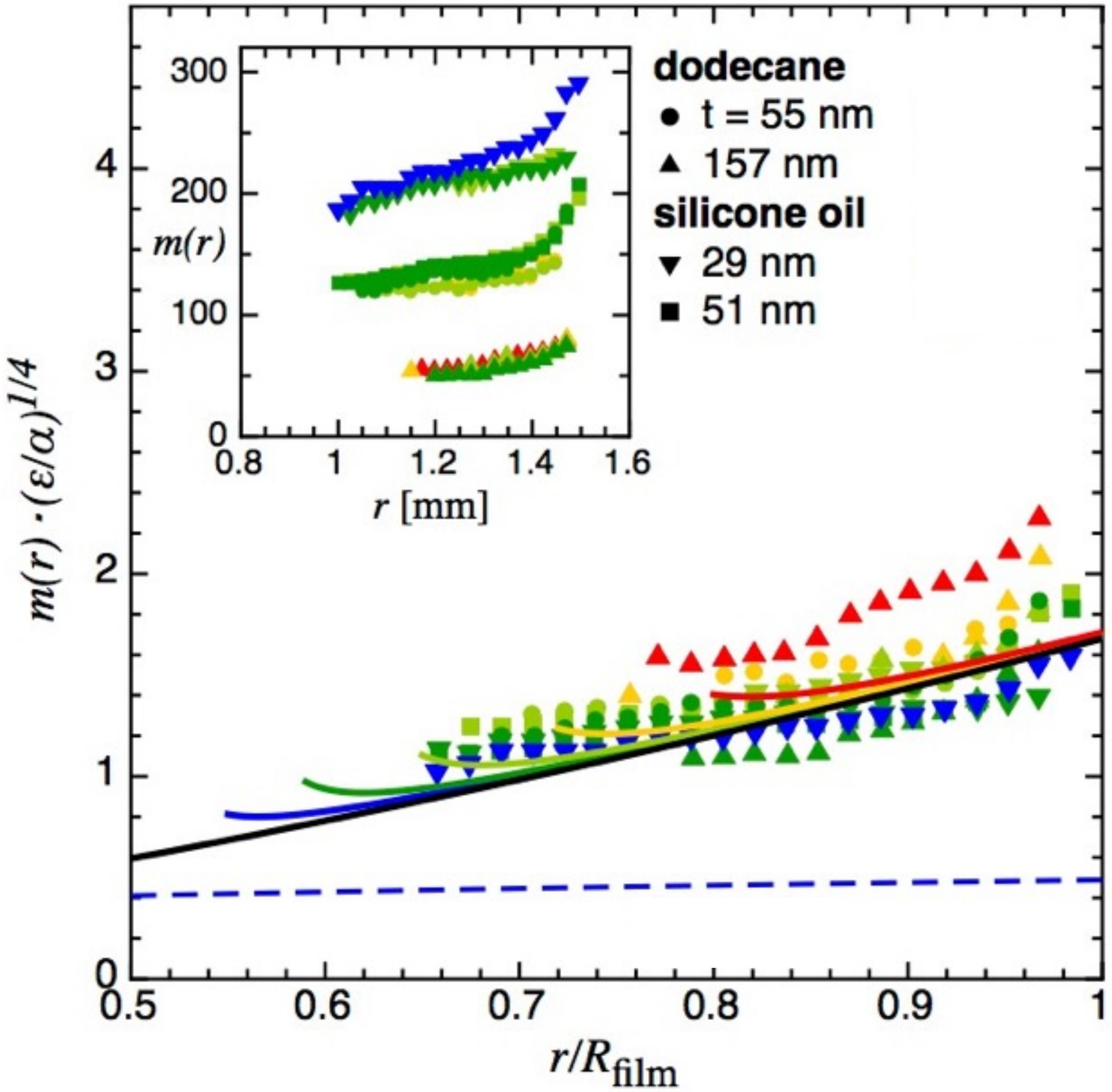
Back to where we started: **a sheet on a drop**



Air/water: Hunter King

Oil/water: Me





Paulsen, Hohlfeld et al., PNAS 2016

Take-home message: *curvature* can be crucial in wrinkle wavelength selection

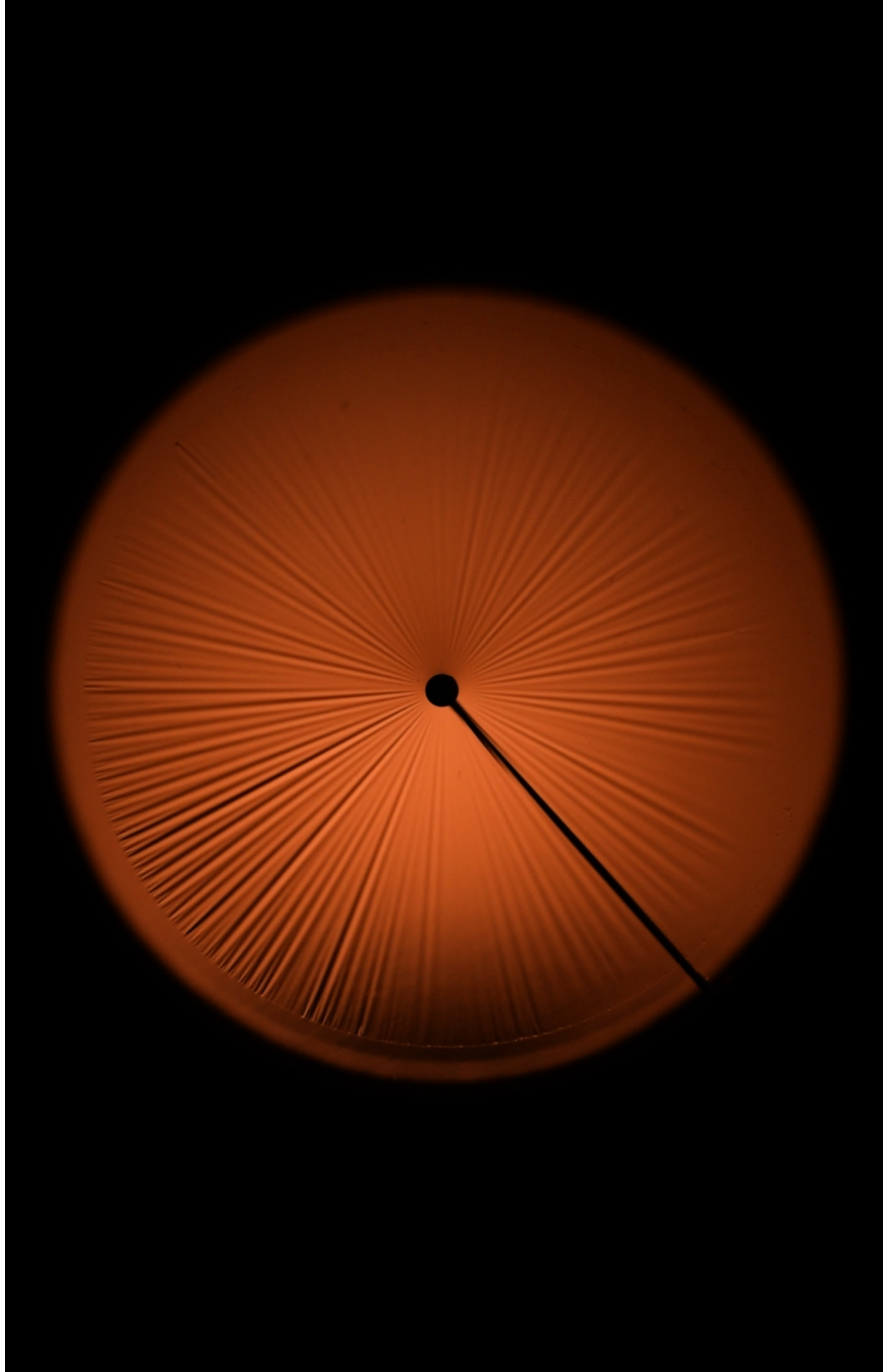
Conclusions

"Local-lambda law"

- Curvature-induced stiffness, $K_{curv} = Y/R_{||}^2$
- $K_{eff} = K_{tens} + K_{curv} + K_{subs}$
- Wavelength selection: local energy minimization

Open questions

- When can we neglect $d\lambda/dx$?
- How does edge cascade depend on curvature?
- What happens without tension?

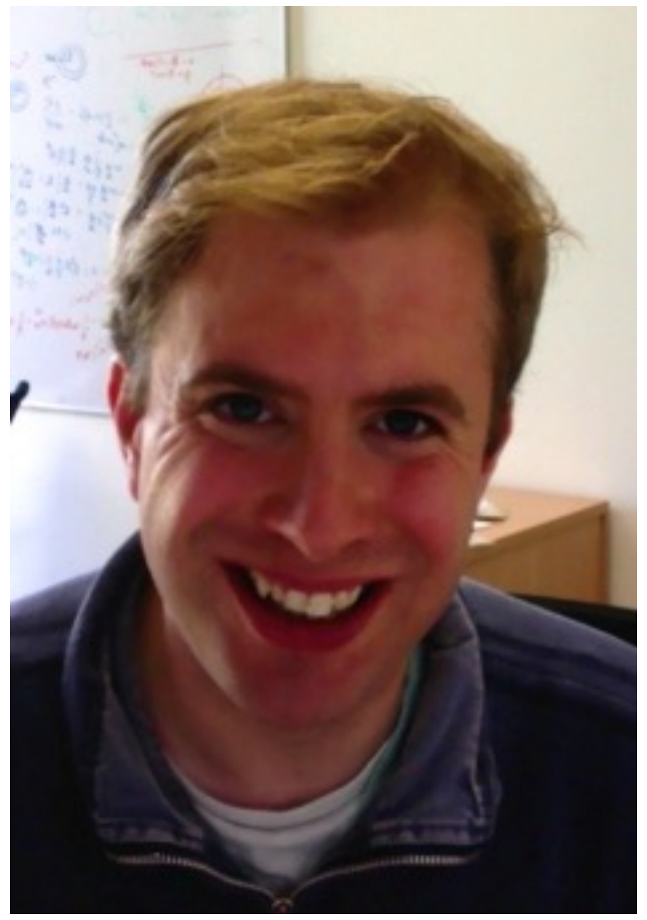


Curvature-induced stiffness and the spatial variation of wavelength in wrinkled sheets.
Paulsen, Hohlfeld, King, Huang, Qiu, Russell, Menon, Vella, and Davidovitch, PNAS 2016

Theory:


Evan Hohlfeld


Zhanlong Qui

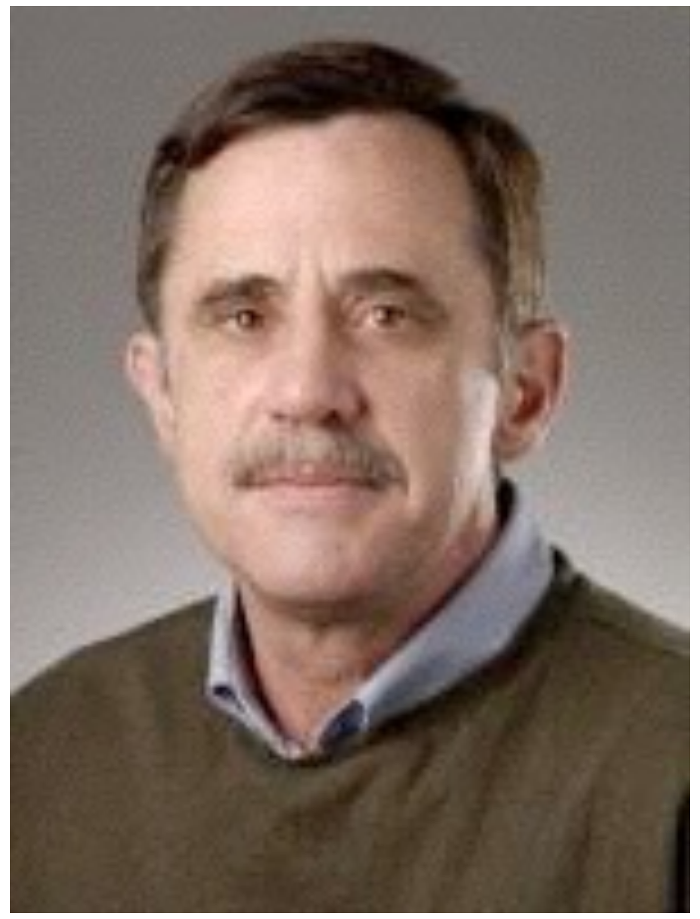

Dominic Vella


Benny Davidovitch

Experiment:


Hunter King


Jiangshui Huang


Thomas Russell


Narayanan Menon

$$$:

W. M. Keck Foundation, NSF-DMR 120778, NSF-DMR-11-51780
ERC StG 637334, Simons Foundation Award 305306