Non Euclidean Plates under Loading- The Minimal Spring

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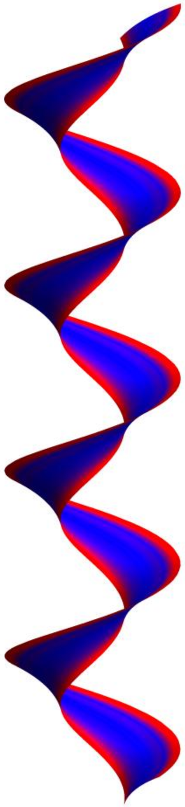
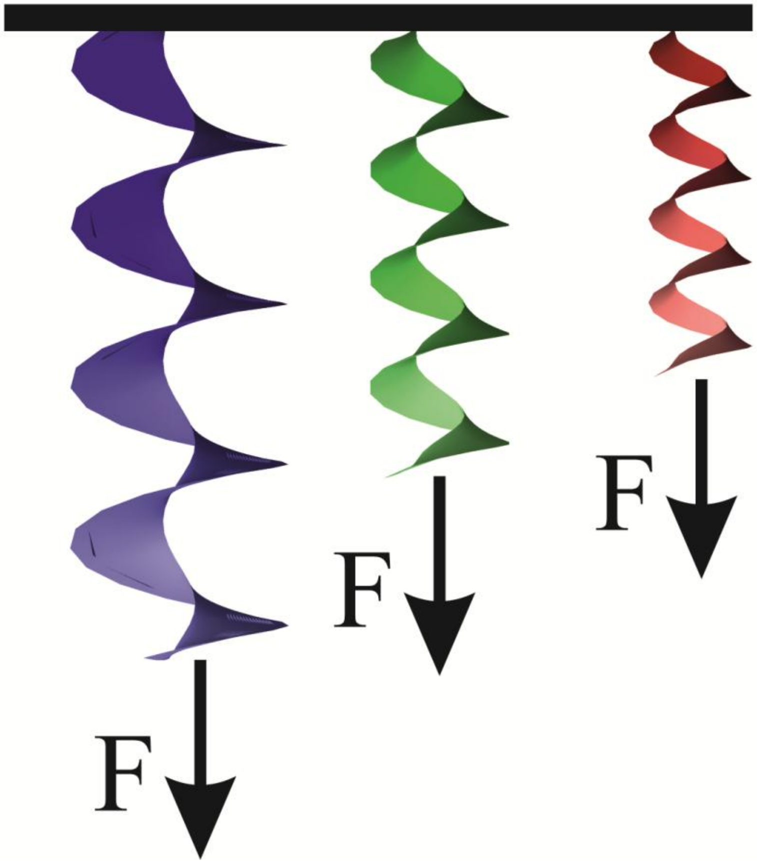
N. Ori (plants)
S. Venkataramani (math)
R. Elbaum (plants)
D. Danino (self assembly)



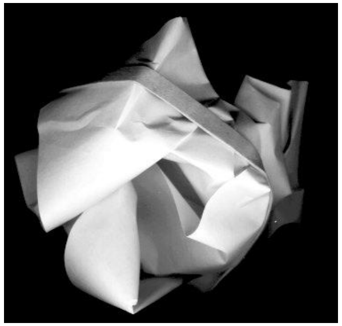


Outline:

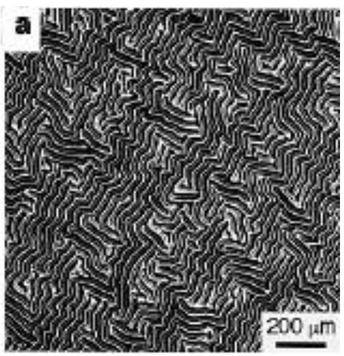
1. Background – frustrated sheets
2. Interesting directions and challenges:
   Stat. Mech. of frustrated sheets
   Growth and plasticity
   “Self actuating” frustrated sheets

3. The minimal spring

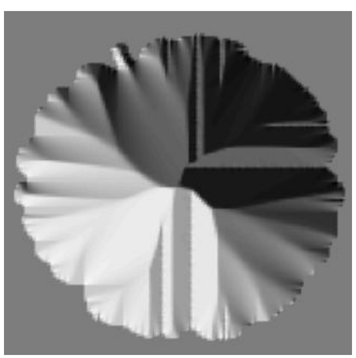
Confined thin elastic sheets



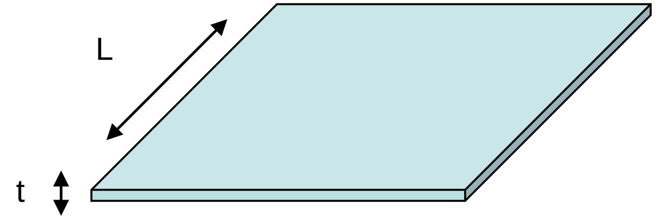
Crumpling (Lobkovsky 95)



Wrinkling (Bowden 98)



Blistering (Ortiz and Gioia 94)



$$E = \int (energy\_density)\,ds$$

For thin plates the elastic energy density can be approximated:

stretching — bending

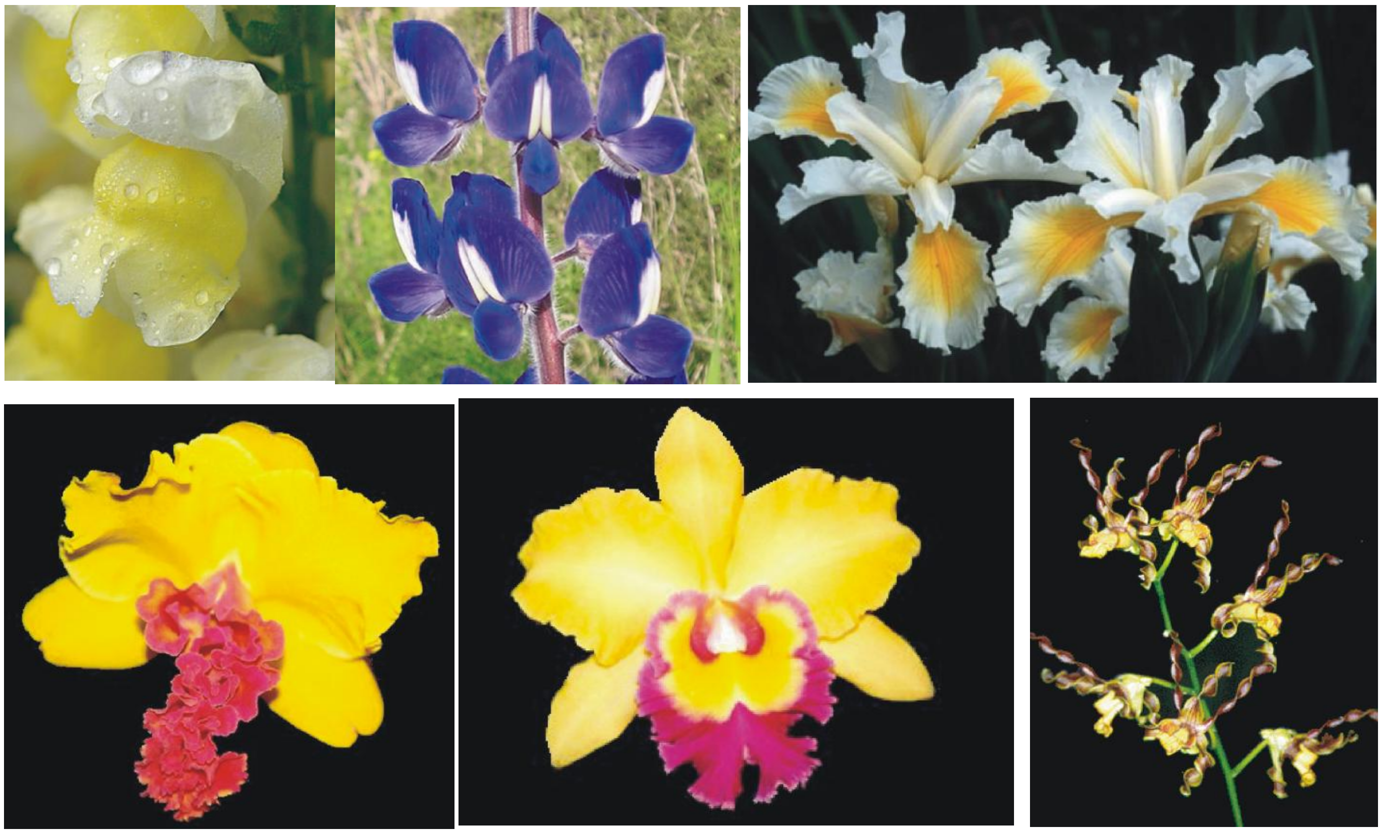
$$\varepsilon \sim t\,(\text{in plane strain})^2 + t^3\,(\text{curvature})^2$$

Dominant — "Floppy", but not zero

competition

Selection – Shape, length scale, type of singularities

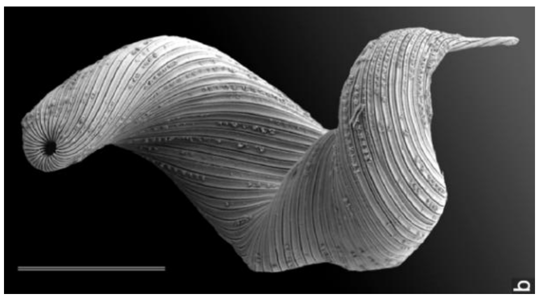
“Self Shaping” of growing sheets



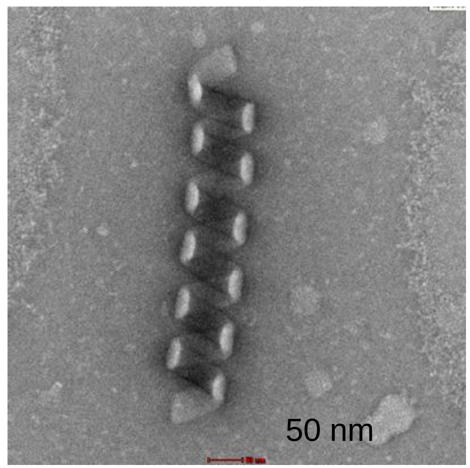
Nath et.al. 2003

D.Taimina

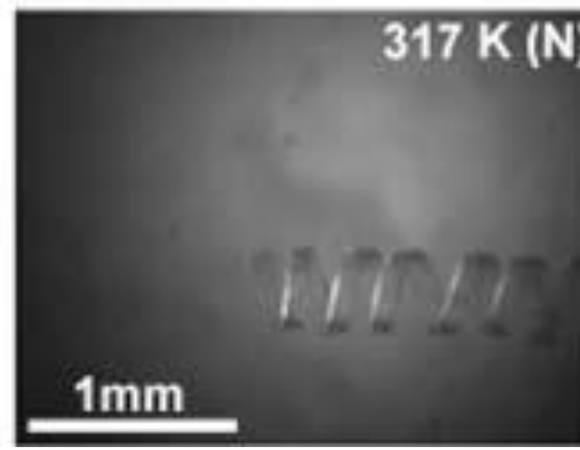
Aharoni et.al 2012



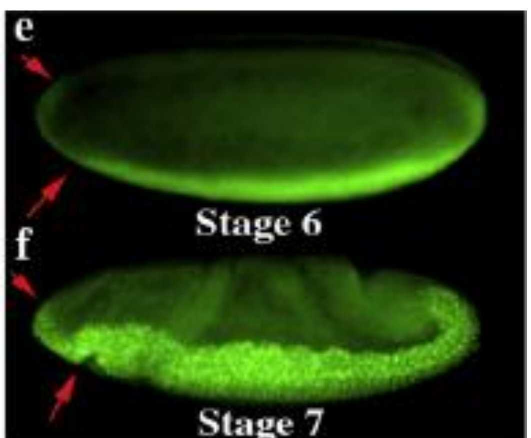
Arroyo and DeSimone 2013



M. Zhang



Sawa et.al. 2013



Desprat 2008

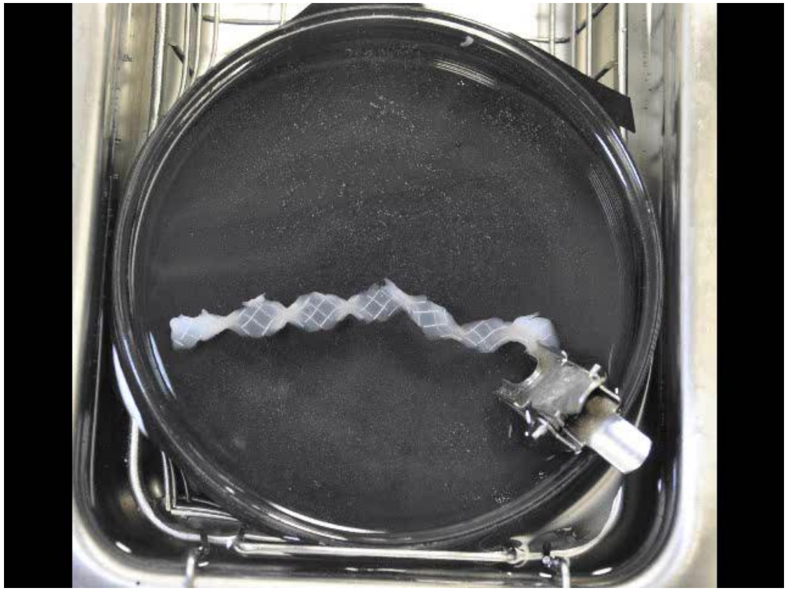
There is a field, in addition to elasticity, that encodes
Some internal geometry of the sheet

Shaping via “local Active deformation”

Global shape changes due to distribution of local active deformation of the tissue

Extensively used in nature – different mechanisms, different time scales

Hardly used in manmade structures



Need suitable theoretical framework

Techniques and materials

Gauss theorem – a link between metrics and shapes

<u>Input: Growth/swelling/reorientation/connectivity</u>

= Equilibrium distances across a surface.

Metric field- locally expresses distances across the surface.

<u>Output: Shape</u>

Shape - is characterized by curvatures $\kappa 1$, $\kappa 2$

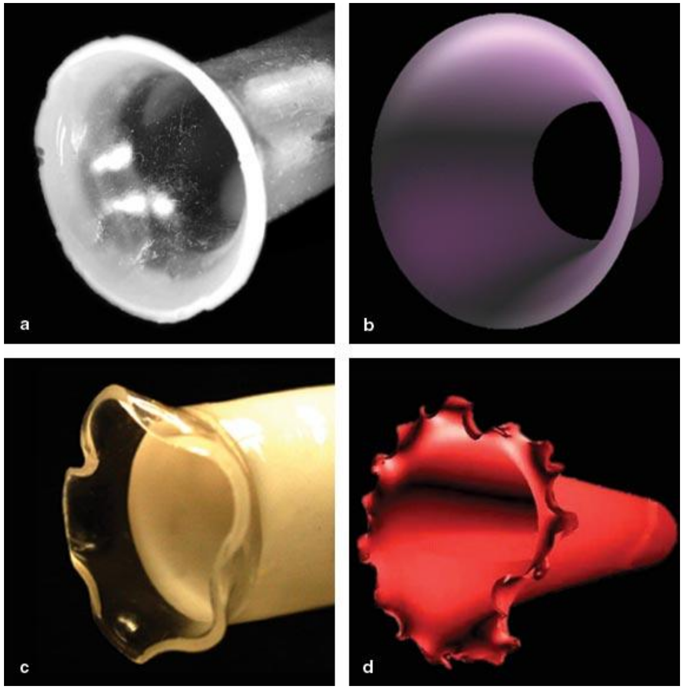
The Gaussian curvature: $K=\kappa 1 \kappa 2$

<u>The connection</u>

Gauss: K is completely determined by the metric

Or: Distances define (to some level) Shapes

The 3D Euclidean space is a non trivial constraint



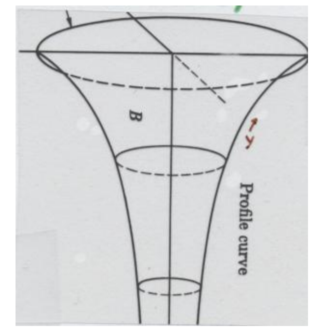
Example

“Exponential metric”- $f(y)=Ae^{-by}$

(Can result from a very simple growth law: dn/dy~n(y))

The result is a “pseudo-sphere”

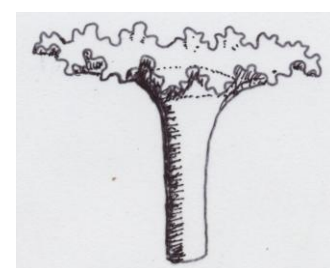
(Constant negative Gaussian curvature)



But!

It has a “cutoff” beyond which it does not exist

Buckling cascade



All from a constant simple growth law

Growing Thin Sheets – Theoretical framework - Elasticity

Energy density has to account for**:**

- growth
- elastic response
- Thin sheet approximation: 2D surface(mid-plane) + thickness(t) :

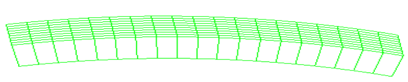
We express the elastic energy density in terms of the fundamental forms of a surface (rather than the displacement field)

In-plane strain

curvature difference

$$\varepsilon \approx t(a-\bar{a})^2 + t^3(b-\bar{b})^2$$

stretching **bending**



t – thickness
ā – dictated 2D metric tensor
a – actual 2D metric tensor
$\bar{b}$ – dictated curvature tensor
b – actual curvature tensor

For isotropic material

$$\approx Yt\int\left[(1-\nu)Tr\left((a-\bar{a})^2\right)+\nu Tr^2(a-\bar{a})\right]dS + Yt^3\int\left[(1-\nu)Tr\left((b-\bar{b})^2\right)+\nu Tr^2(b-\bar{b})\right]dS$$

E. Efrati, Y. Klein, H. Aharoni and ES, (2007), "Spontaneous Buckling of Elastic Sheets with a Prescribed Non-Euclidean Metric" *Phys. **D**.* 235, 29-32
E. Efrati, R. Kupferman and ES, (2009) "Elastic Theory of unconstrained non-Euclidean plates*". JMPS,* 57, 762-775
E. Efrati, R. Kupferman and ES, (2009) "Buckling transitions and boundary layers in non-Euclidean plates", Phys. Rev. E, 80, 016602.
E. S. and E. Efrati "(2010) The Mechanics of Non-Euclidean Plates", *Soft Matt.*,
E. Efrati, R. Kupferman and ES, (2010) "Non-Euclidean Plates and Shells"

Incompatible sheets

$$\varepsilon \approx \underbrace{t(a-\bar{a})^2}_{\textbf{stretching}} + \underbrace{t^3(b-\bar{b})^2}_{\textbf{bending}}$$

growth profile dictates $\bar{a},\bar{b}$ system chooses a,b

Why not simply choose $a=\bar{a}, b=\bar{b}$ and have zero energy?

Because ***a*** and ***b*** are not independent:

Gauss Theorema Egregium:

$$K = \frac{\det(b)}{\det(a)} = f(a)$$

Gaussian curvature

The interesting behavior occurs for incompatible surfaces:

dictated $\bar{a},\bar{b}$ do not satisfy Gauss theorem (thus allow no zero energy configuration).

Very likely to happen in biological tissues and locally growing systems.

Limits

$$\varepsilon = t(a-\bar{a})^2 + t^3(b-\bar{b})^2$$

stretching — bending

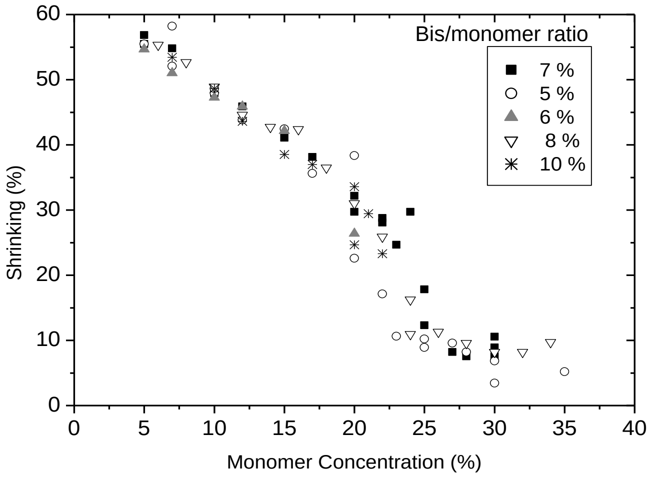
- **Thin limit** – body will obey $\bar{a}$, because the bending is "cheap".
- **Thick limit** – body will obey $\bar{b}$ and will pay in stretching.

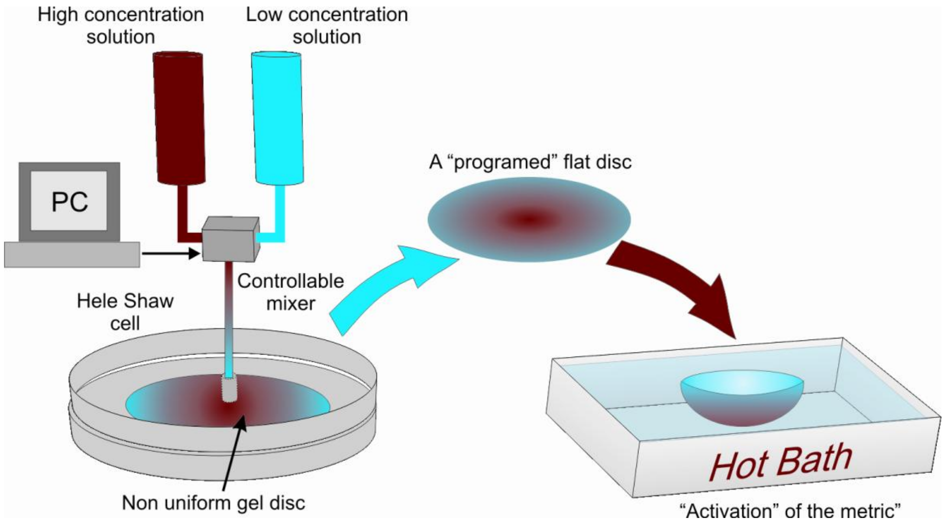
Many "interesting" **equilibrium configurations** were discovered and studied

B. Roman, M. Marder E. Efrati, S. Venkataramani, B. Adouly, C. Santangelo, M. Ben Amar, M. Muller, R. Kohn, L. Mahadevan, H. Aharoni, A. De Simone…

Experimental system: "Engineered" responsive non-Euclidean plates (Yael Klein, Hillel Aharoni)

N-Isopropylacrylamide gel - A volume reduction transition at a 32C^0 that strongly depends on monomer concentration

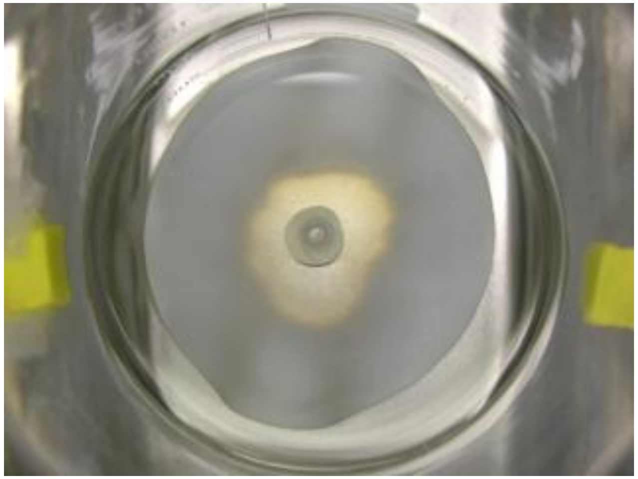




The Axi-symmetric reference metric in polar coordinates:

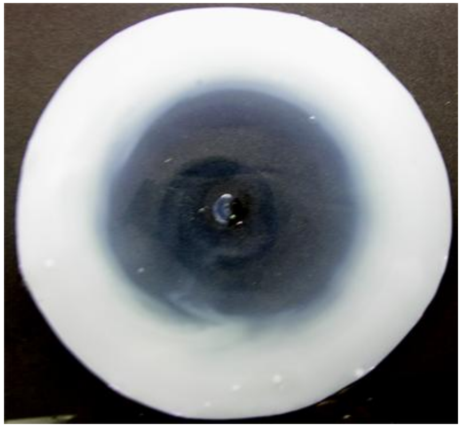
$$dl^2 = d\rho^2 + f^2(\rho)d\theta^2$$

$f(\rho)$ is determined by the concentration profile

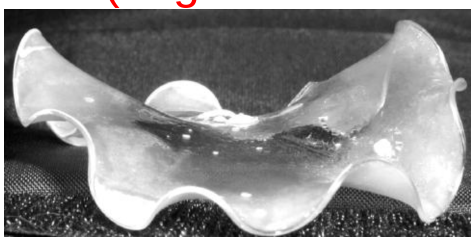


Many possible geometries

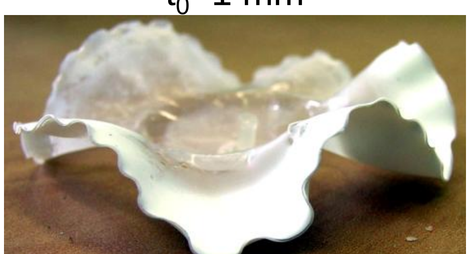
Cold - Flat



Warm (negative curvature)



t_0=1 mm



t_0=0.3 mm

Positive (no symmetry breaking)

Positive + flat

Positive + negative

Tubes

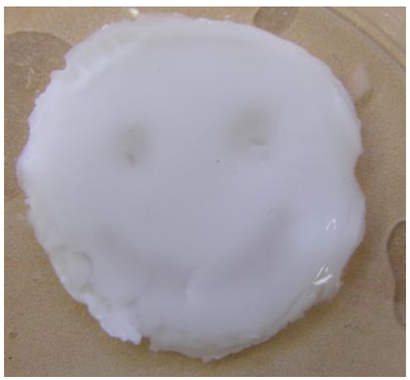
warm

cold

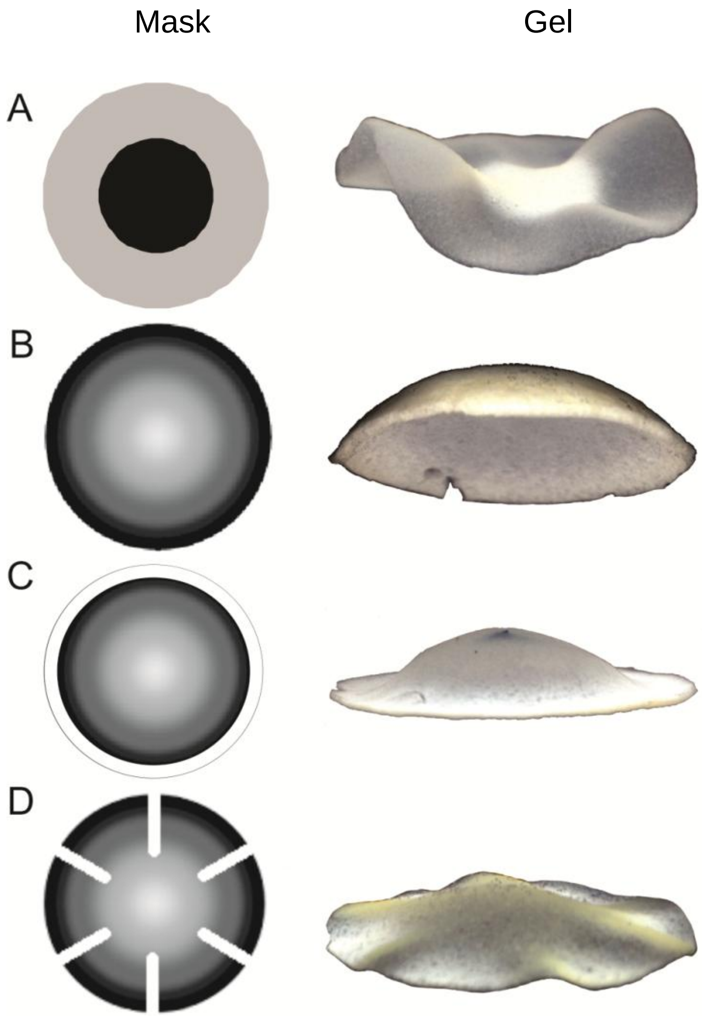
Y. Klein, E. Efrati and ES, *Science*. **315**, 1116 (2007).

“Lithography of curvature”

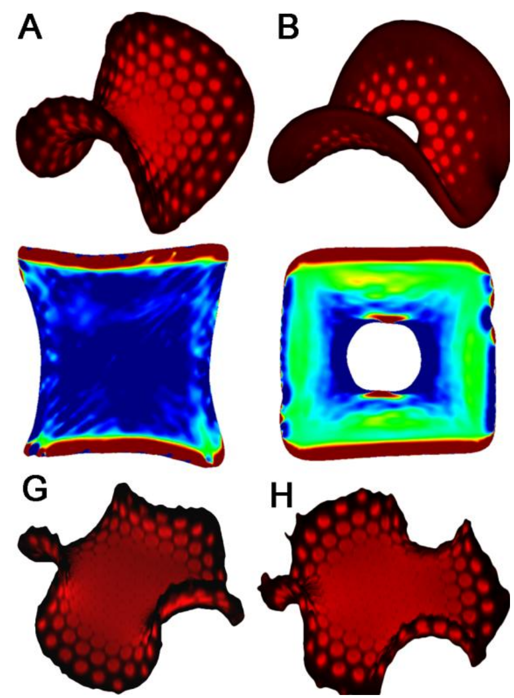
Selective UV crosslinking of the gel, via a mask



ES and E. Efrati Soft Matter 2010



Ido Levin (2012)



Kim et al. *Science* 2012

Sheets made of nematic elastomers/glasses

See works by Modes and Warner, DeSimone, White…

The local deformation:

$$\bar{a}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \bar{a} = \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^{-2\nu} \end{pmatrix}$$

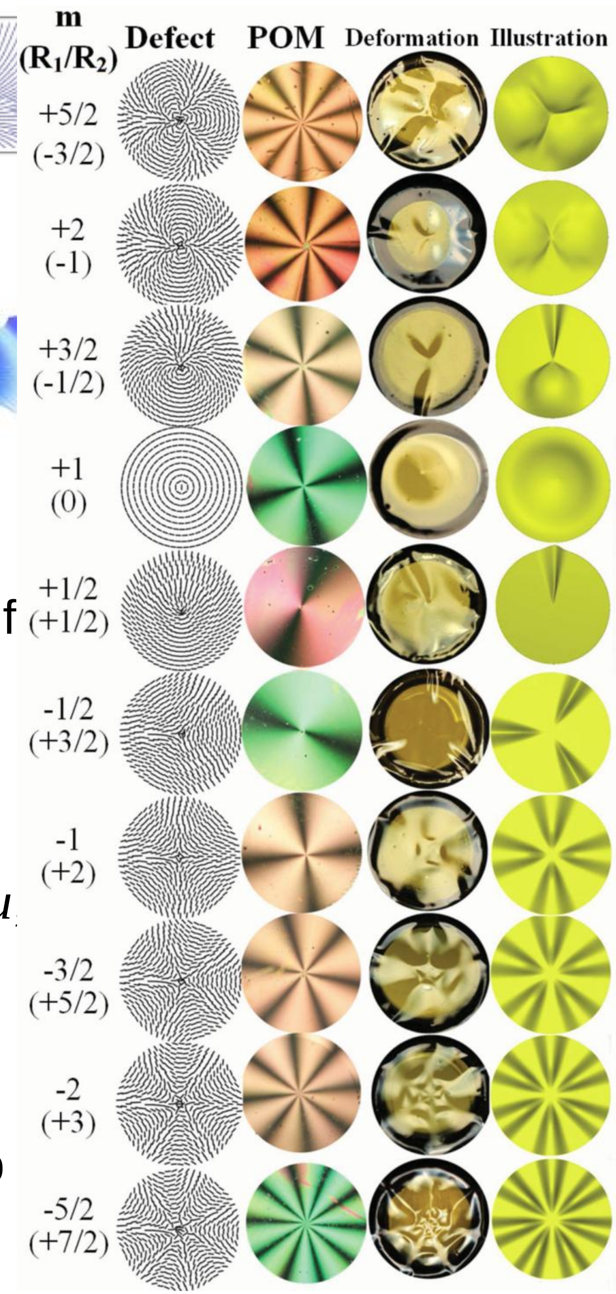
*In coordinates that are locally aligned with the director f

In fixed coordinates for a director field θ(u,v):

$$\bar{a}(u,v) = R[\theta(u$$

=> An expression for $\bar{K}[\theta(u,v)]$

Setting $\bar{b}$ by prescribing different director fields on top



H. Aharoni, R. Kupferman and E. S. (2014), “Geometry of Thin Nematic Elastomer Sheets”, *Phys. Rev. Lett.* **113**, 257801.

Tim White’s group – Adv. Matt 3013, Science 2014…. Hear more from Robin and Jonathan (?)

Integrating incompatible elasticity to Statistical mechanics

A flat-to-helical transition

Incompatible Hyperbolic shell

Using the principle directions of $\bar{b}$

$$\bar{a} = I$$

$$\bar{b} \neq 0 \longrightarrow \bar{b} = \begin{pmatrix} \bar{k}_1 > 0 & 0 \\ 0 & \bar{k}_2 < 0 \end{pmatrix}$$

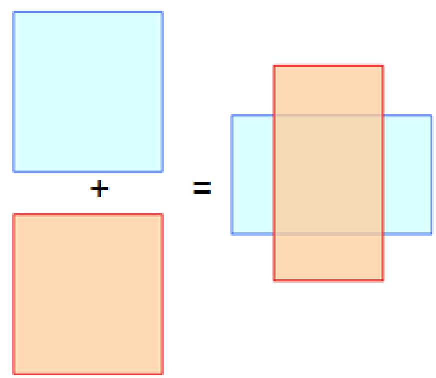
Euclidian

Hyperbolic

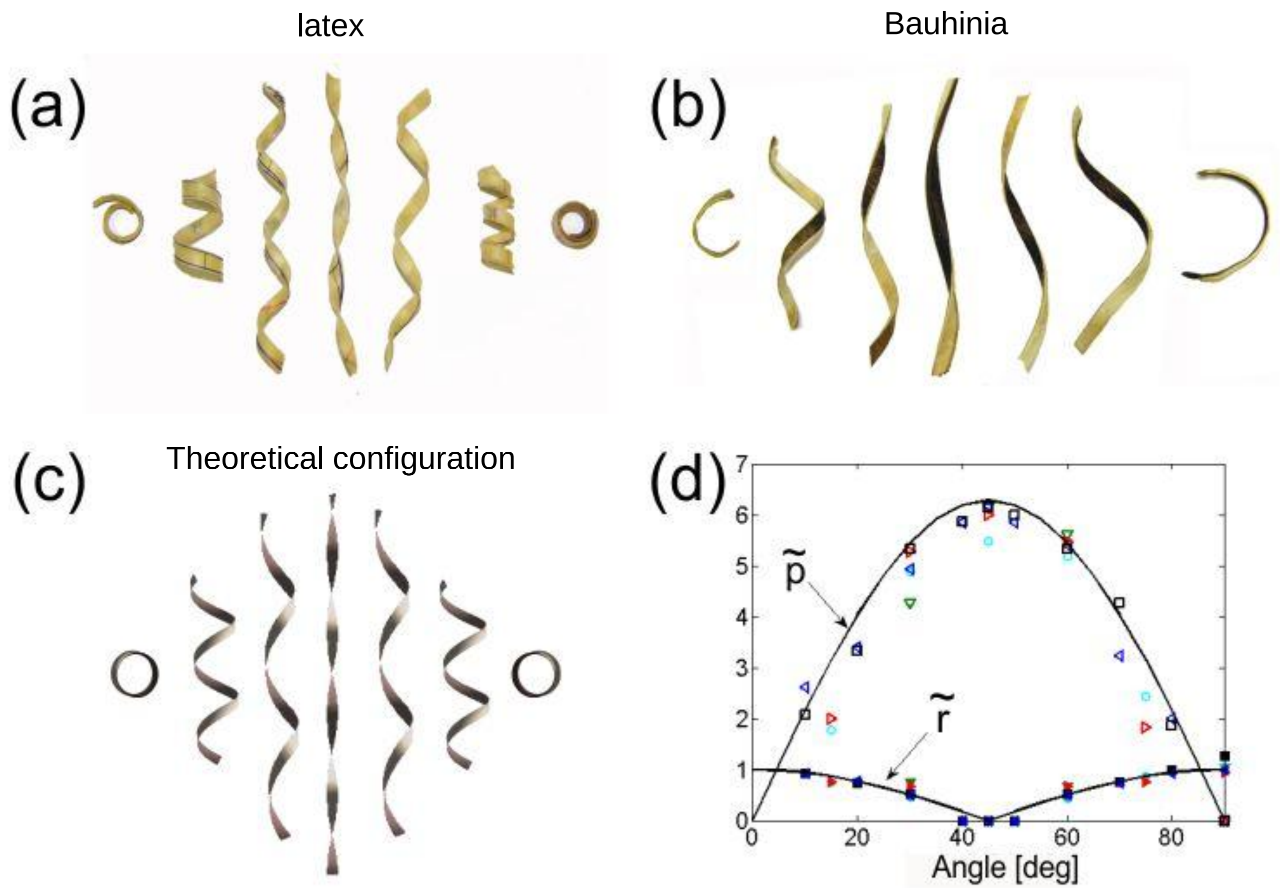
“Minimal“

Copying the pod tissue architecture

Latex model sheets



Narrow strips in different angles Bauhinias, strips & simulation:



Same material –different configurations, including handedness flipping

Armon et.al. *Science* (2011)

See also: Chen et.al. *Appl. Phys.Lett*. (2011), Sawa et.al. *PNAS* (2011)

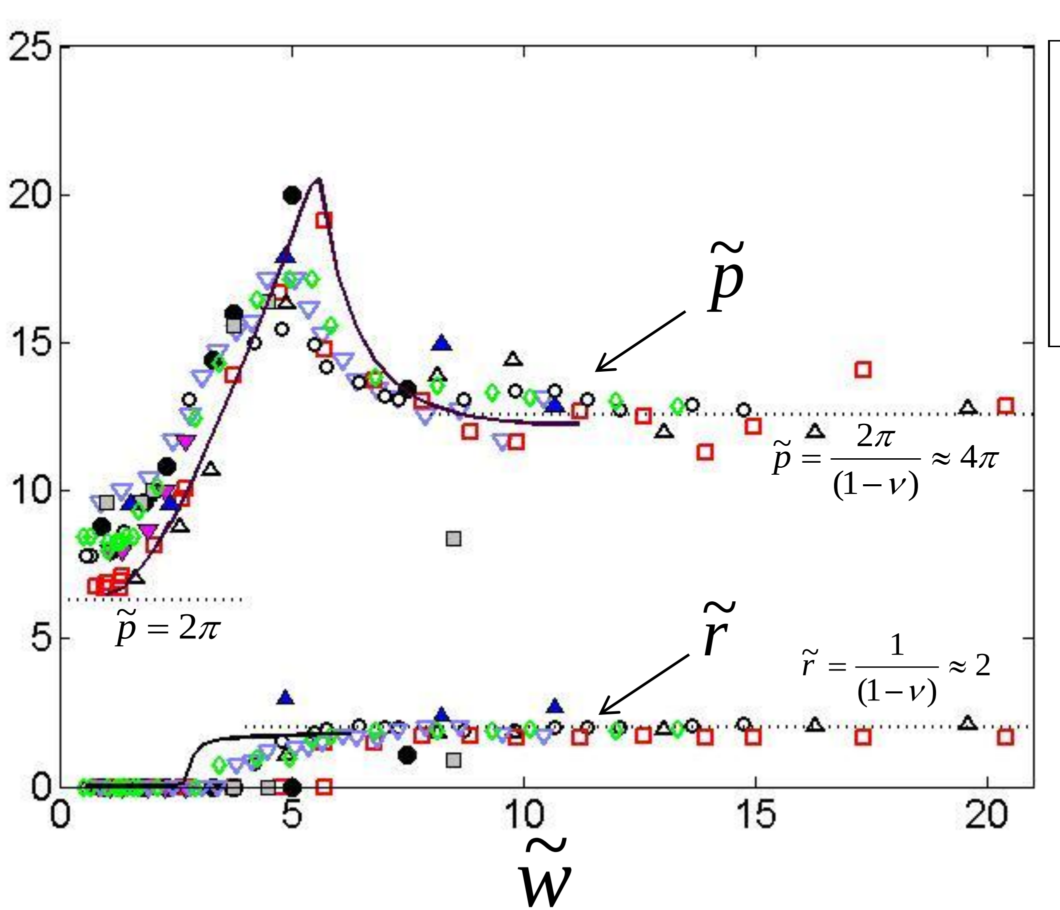
Twisted to Helical transition

Normalized pitch, radius and width

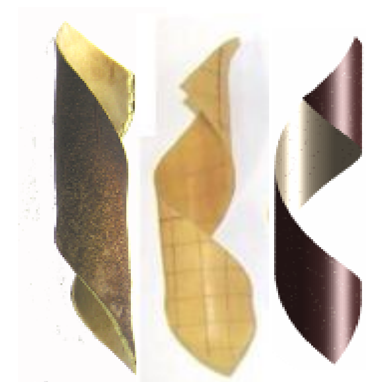
$$\tilde{p} = p\bar{k}$$
$$\tilde{r} = r\bar{k}$$
$$\tilde{w} = w\sqrt{t/\bar{k}}$$

narrow limit

Avoiding bending:
obeying b –

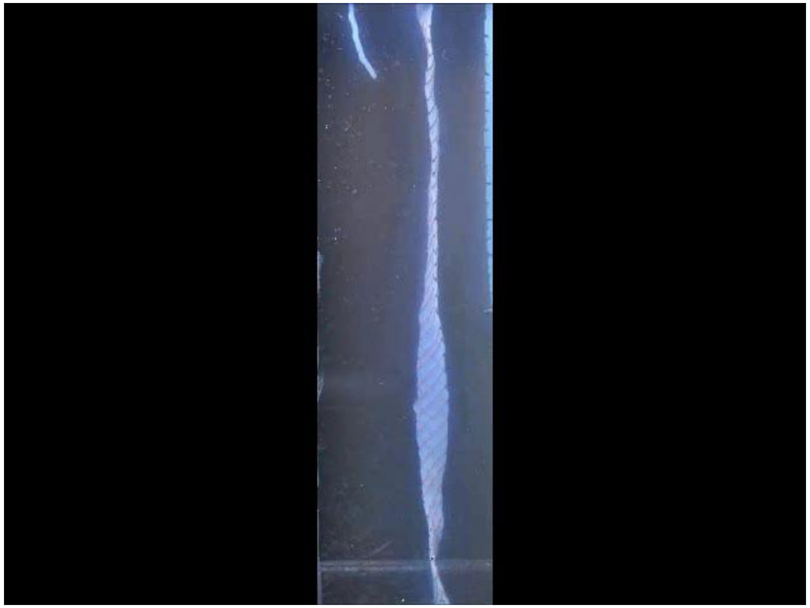


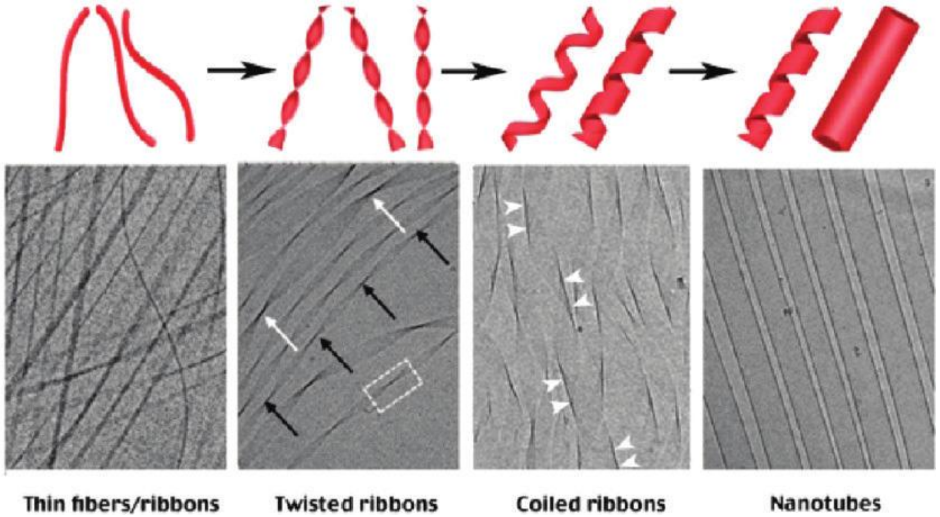
wide limit



avoiding stretching:
obeying a –

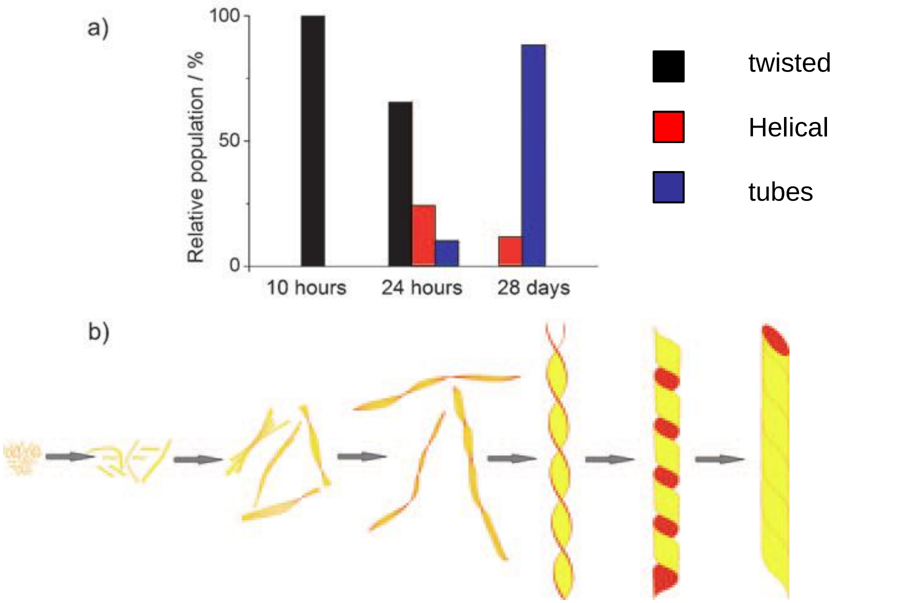
Observed flat->twisted->helical-> tube transitions in amphiphile aggregates (D. Danino):





L. Ziserman, A. Mor, D. Harries, and **D. Danino**, (2011) PRL 106, 238105.

Evolution of Amyloid fibrils (R. Mezzenga)

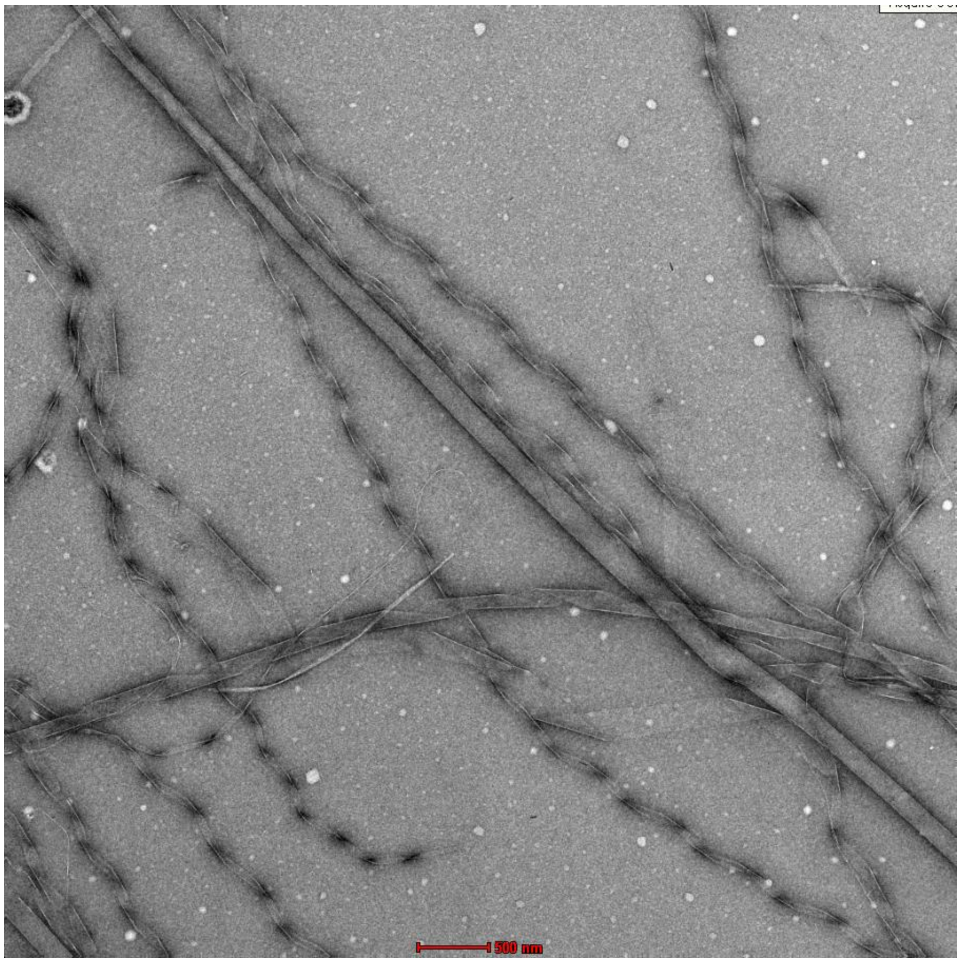


Adamcik et.al., 2011, Angew. Chem. Int. Ed. , 50, 5495.

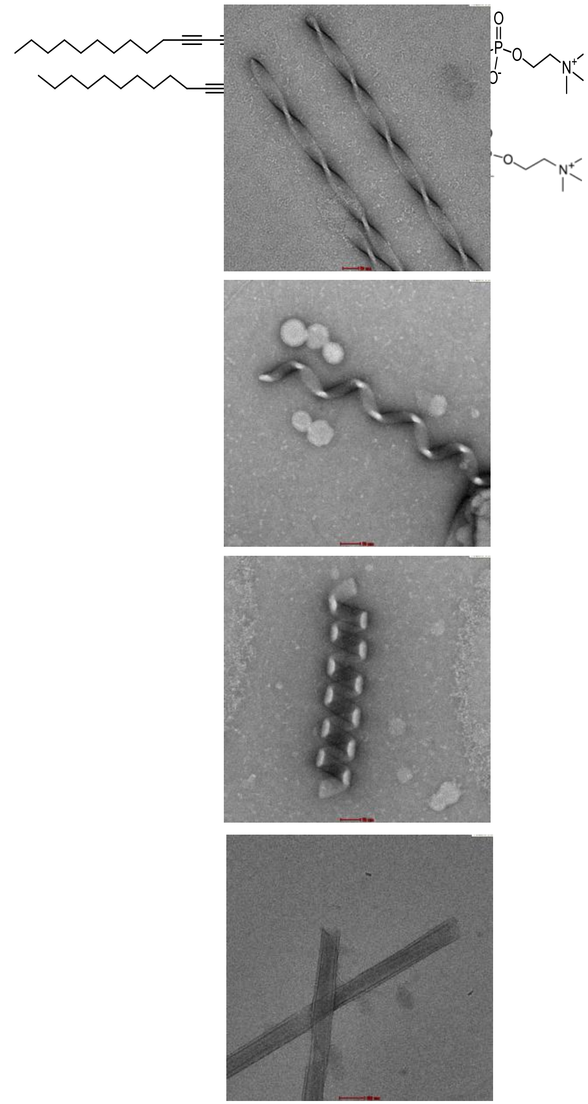
Application to self assembled chemical systems - Mingming Zhang in collaboration with D. Danino

Quantitative predictions, connecting the "large scale" geometry to the molecular structure

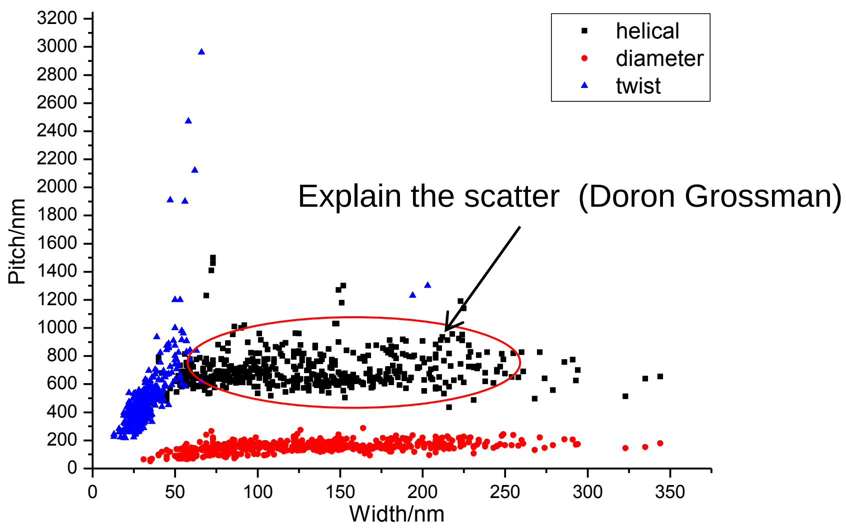
C_{12}-β_{12} Amphiphils



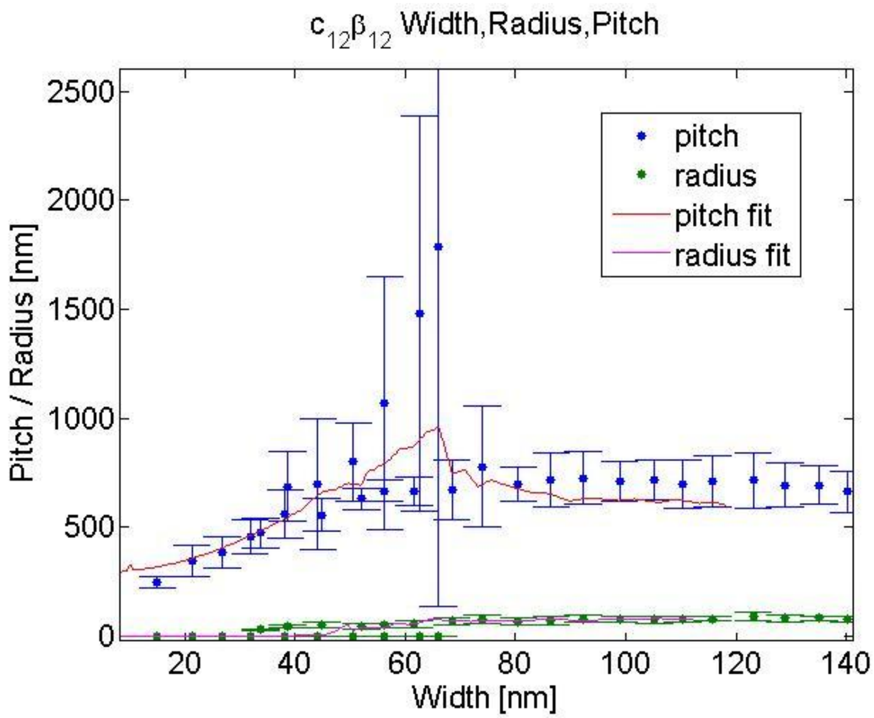
DC89PC, DNPC lipids 1:1 mixture



Results (C_{12}-β_{12} system)



The transition is observed



Best fit leads to:

Effective thickness: 3.53 nm (real thickness =3.4 nm)

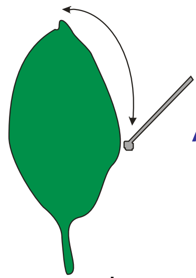
$\kappa_0 = 0.02 nm^{-1} \Longrightarrow$ "bonding angle"= 4.3^0

Poisson's ratio = 0.5 => $\kappa_G = \frac{\kappa_{ee}}{2}$

Plates mechanics is relevant to plants morphogenesis

Using Auxin to change leaf geometry

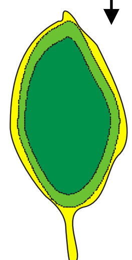
Application of Auxin
on the edge of the leaves



Auxin

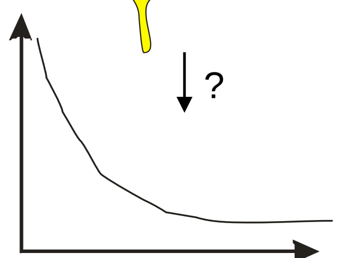
?

Generation of gradient
of Auxin in the leaf



?

Generation of a hyperbolic
metric on the leaf



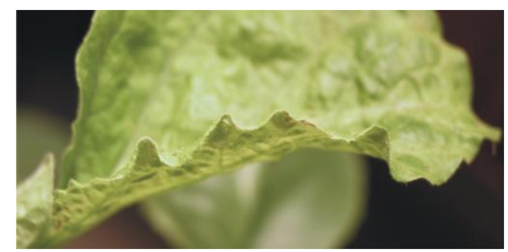
?

The leaf should turn
wavy

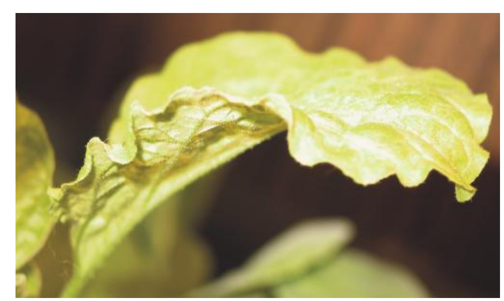
No Auxin

After 1 week

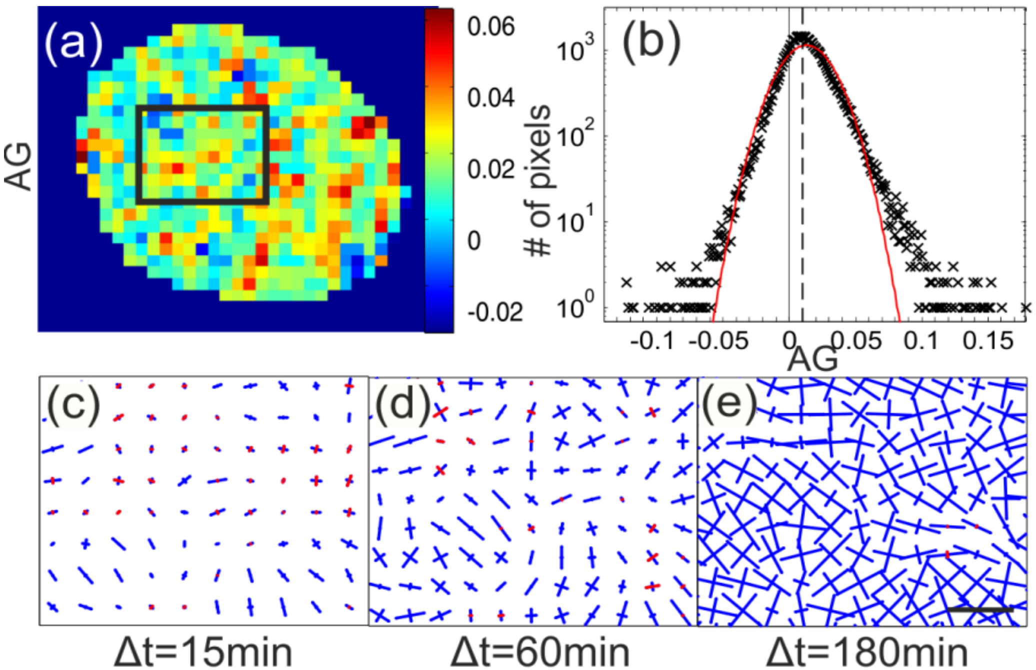
After 12 days



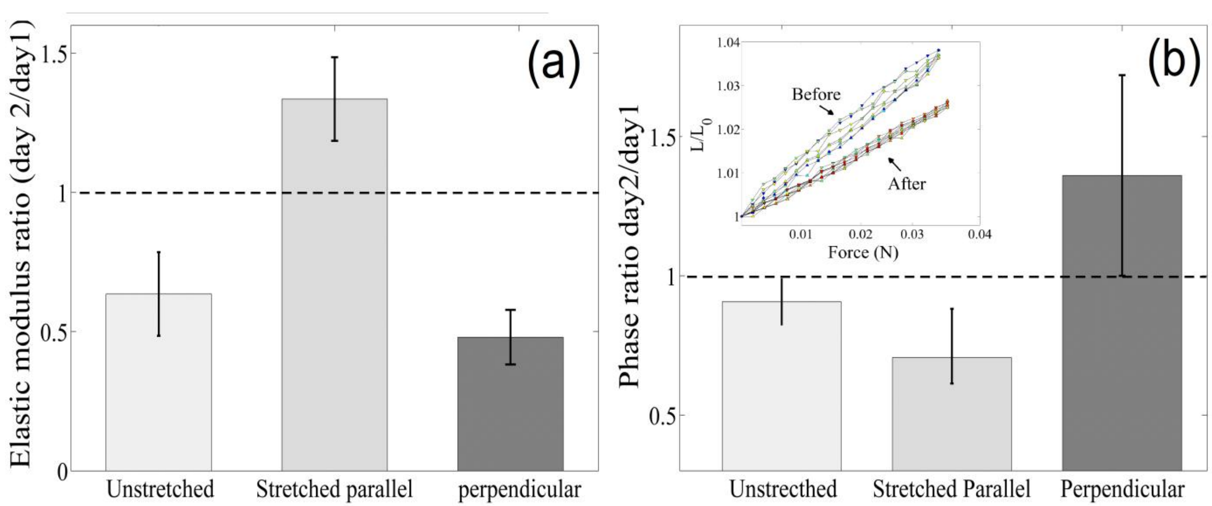
After 2 weeks



Noisy growth field with non trivial “rheology” of a leaf



Shahaf Armon



Michal Sahaf

NEP under load – The minimal spring

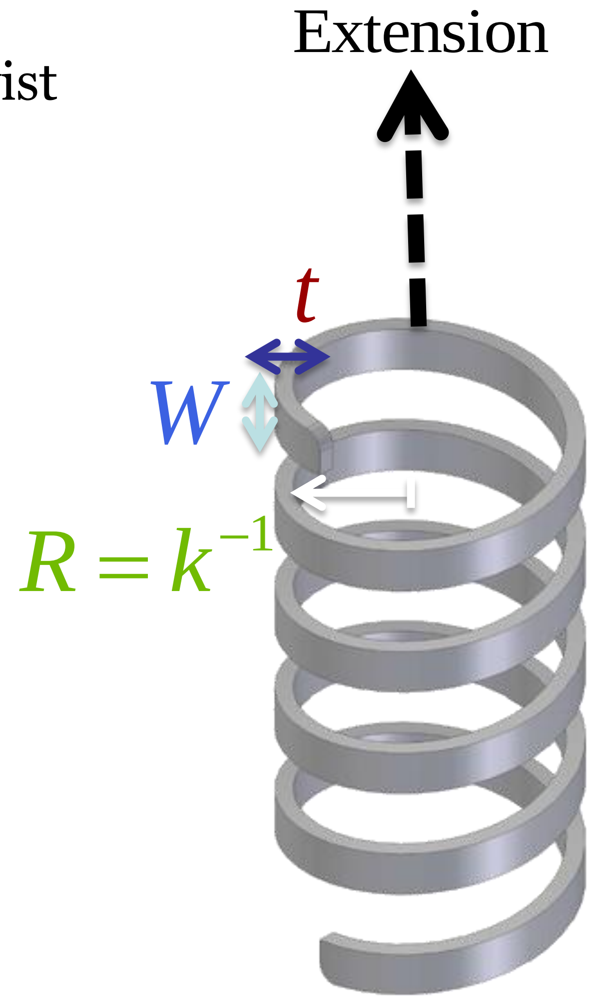
Ribbon springs

Characterized by their reference curvature or twist

Deforms by bending ⟶ $\kappa \propto t^3 W k^2$





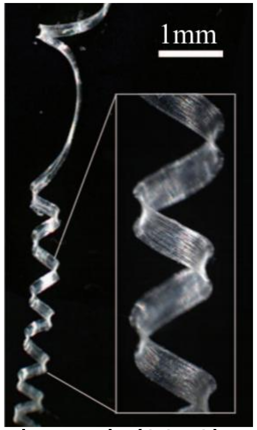


Non trivial behavior. See: E. Starostin and G. van der Heijden **(2008)**, J. Chopin, V. D´emery, and B. Davidovitch, **(2014).**

Incompatible ribbon springs

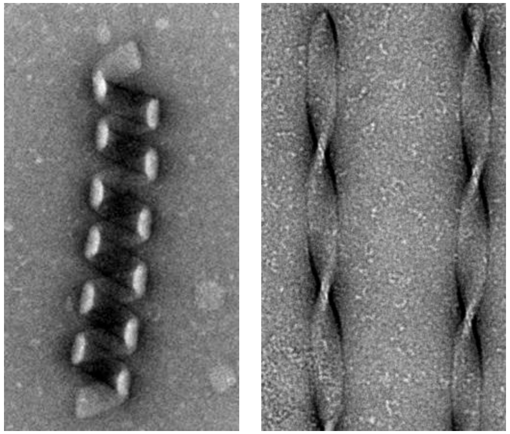
Armon et al. (2014)

Pancratium sickenbergeri



Gerbode et al. (2012)

M. Zhang (our lab)

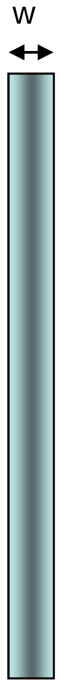


length scale

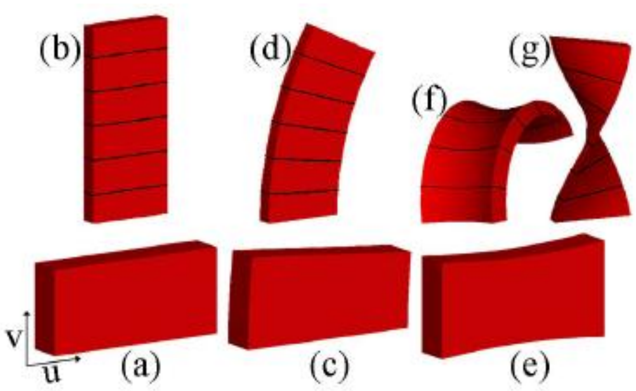
cm µm nm

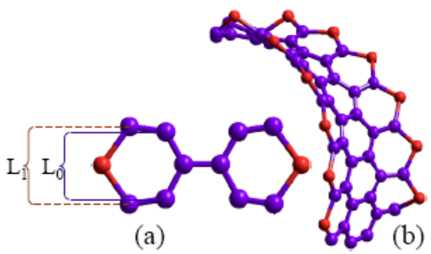
No stress-free configuration → Non trivial energy landscape at minimum

Almost minimal Non-Euclidean strips (Efrati, Kupferman, ES 2011)



Consider a thin narrow strip with an imposed negative curvature (invariant along the strip)





In this case there is an infinite number of exact embeddings. What will be the configuration?

Find the embedding with smallest bending (a proof by Lewicka and Pakzad)

$$\varepsilon_b \propto 4H^2 - K$$

Fixed: $K=K_{tar}$

Would like to have H=0 everywhere: **A minimal surface**

Though **it is impossible for an arbitrary K** , it can be shown that for any K<0 we can find an exact embedding with $\varepsilon_b \sim w^5$ - very floppy.

Experimental Results

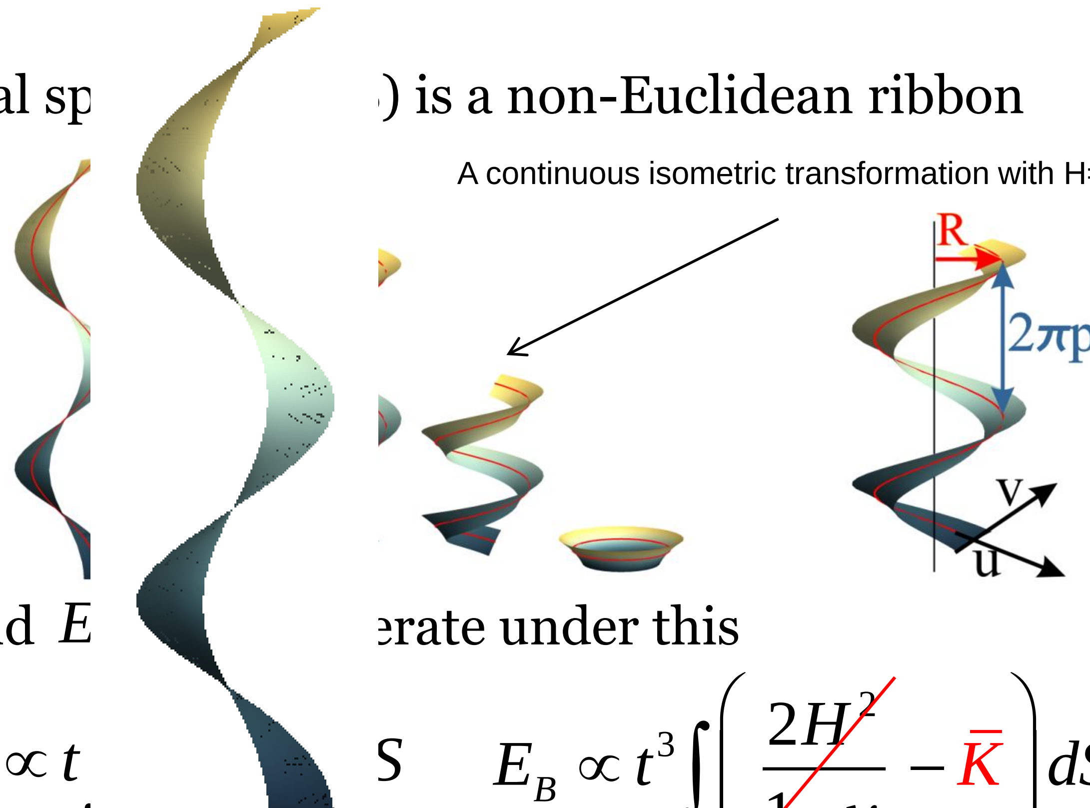
But what is the reference metric IS that of a minimal surface?

Non-Euclidean minimal springs

- Non-Euclidean minimal sp …) is a non-Euclidean ribbon with $\bar{a}$ of a helicoid

$$a(p) = \bar{a}$$

$$H(p) = 0$$



- In that case both E_S and E … erate under this transformation:

$E_s \propto t$ … S

$$E_B \propto t^3 \int \left(\frac{2H^2}{1-\nu} - \bar{K} \right) dS$$

- **An Apparent total degeneracy of the elastic energy**

Mechanical properties of the minimal springs

Degeneracy is removed only by boundary (layers) effects (see Efrati et.al. 2009)

$k_n \propto \sqrt{1-p^2}$ → Boundary layers are less effective for lower pitch values

↓

the degeneracy is removed $\quad W_b \propto \sqrt{t/k_n} \quad E_B \propto t^3 \int dS \left(\frac{2H^2}{1-\nu} - \bar{K} \right)$

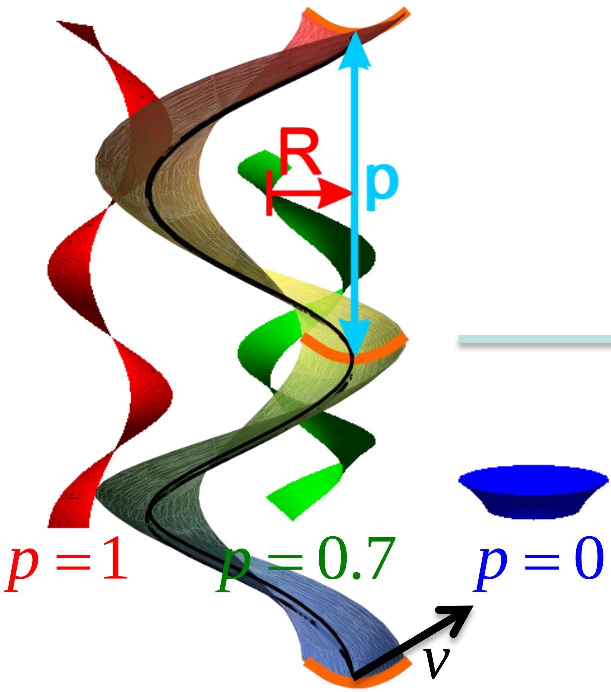
$$E(p) \propto t^{7/2} k_{\perp}^{3/2} \left[(1-\nu) p^2 + 1 + \nu \right]^{5/4} + E_0$$

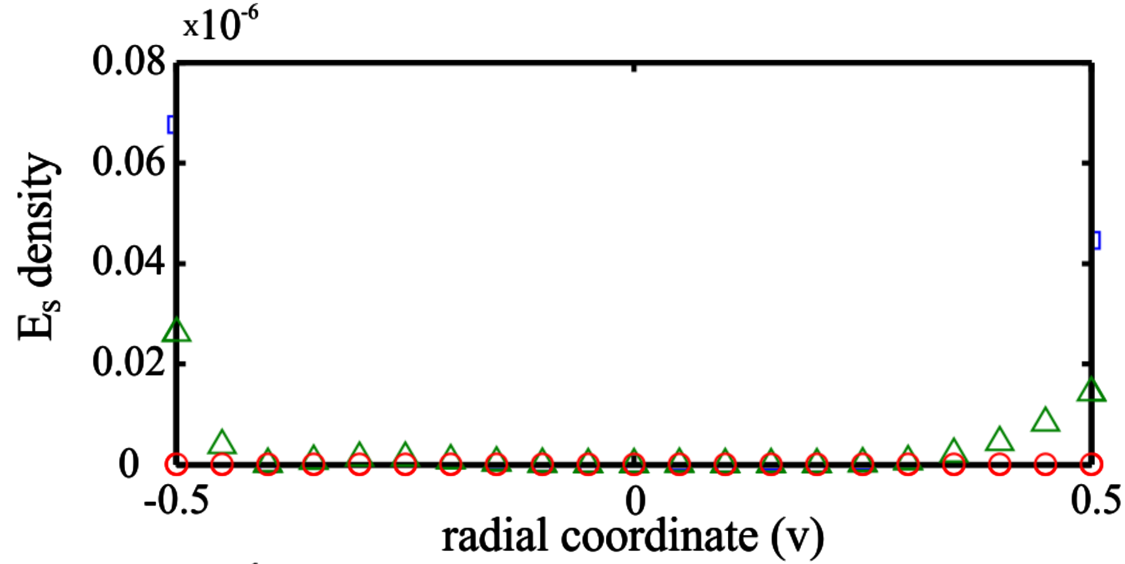
We predict three unique properties of the NEMS: $\left(\begin{array}{c} E_0 \text{ - degenerate bulk energy} \\ p \text{ - pitch} \end{array} \right)$

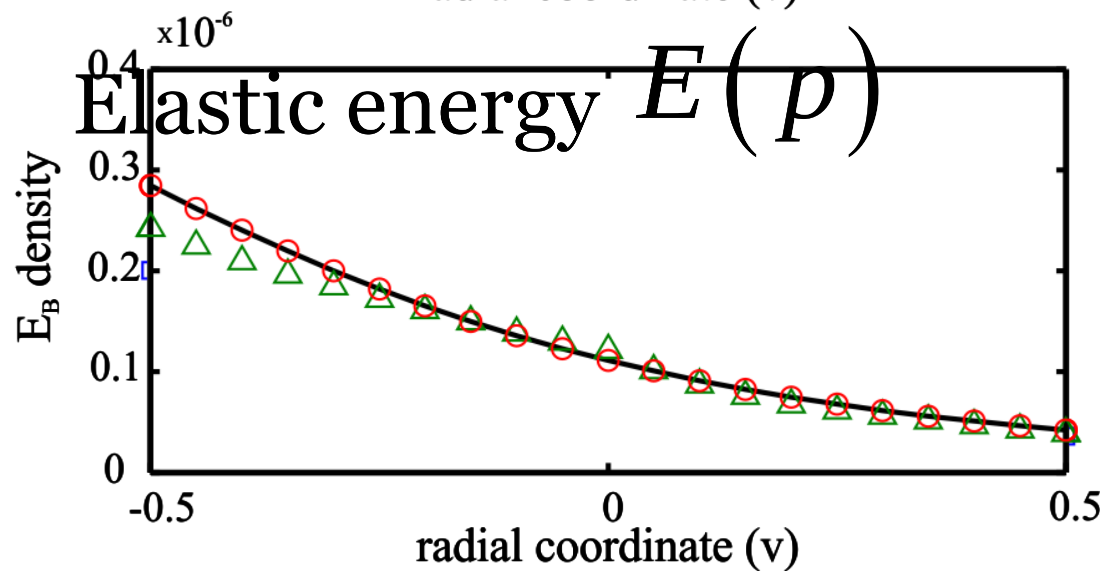
- **Anomalous softness :** $\kappa = \frac{d^2E}{dp^2} \propto t^{7/2}$
- **Rigidity does not depend explicitly on the width**
- **Extended linearity (small quartic term)**

Numerical results – Only boundary effects

1D numerical simulations

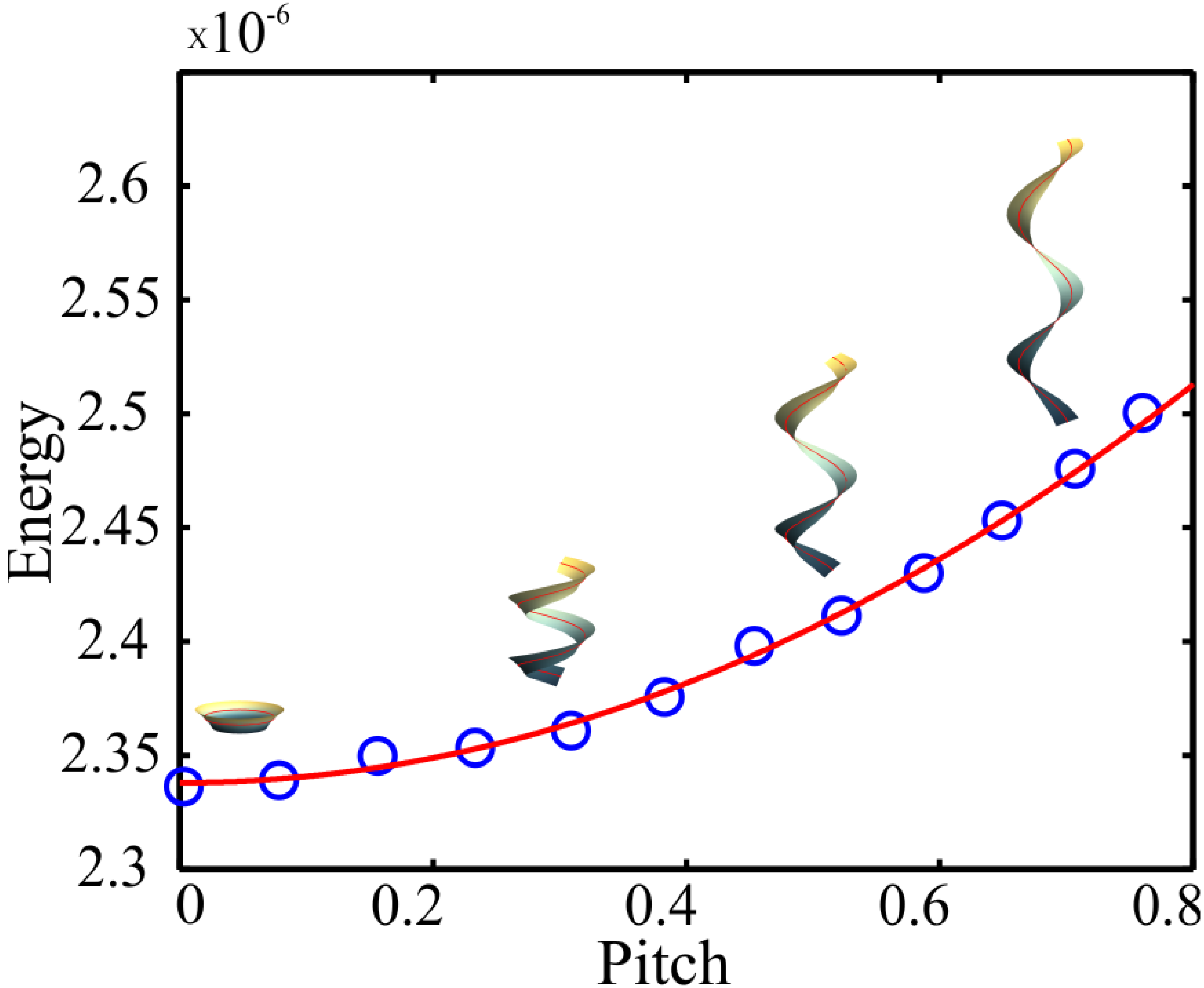




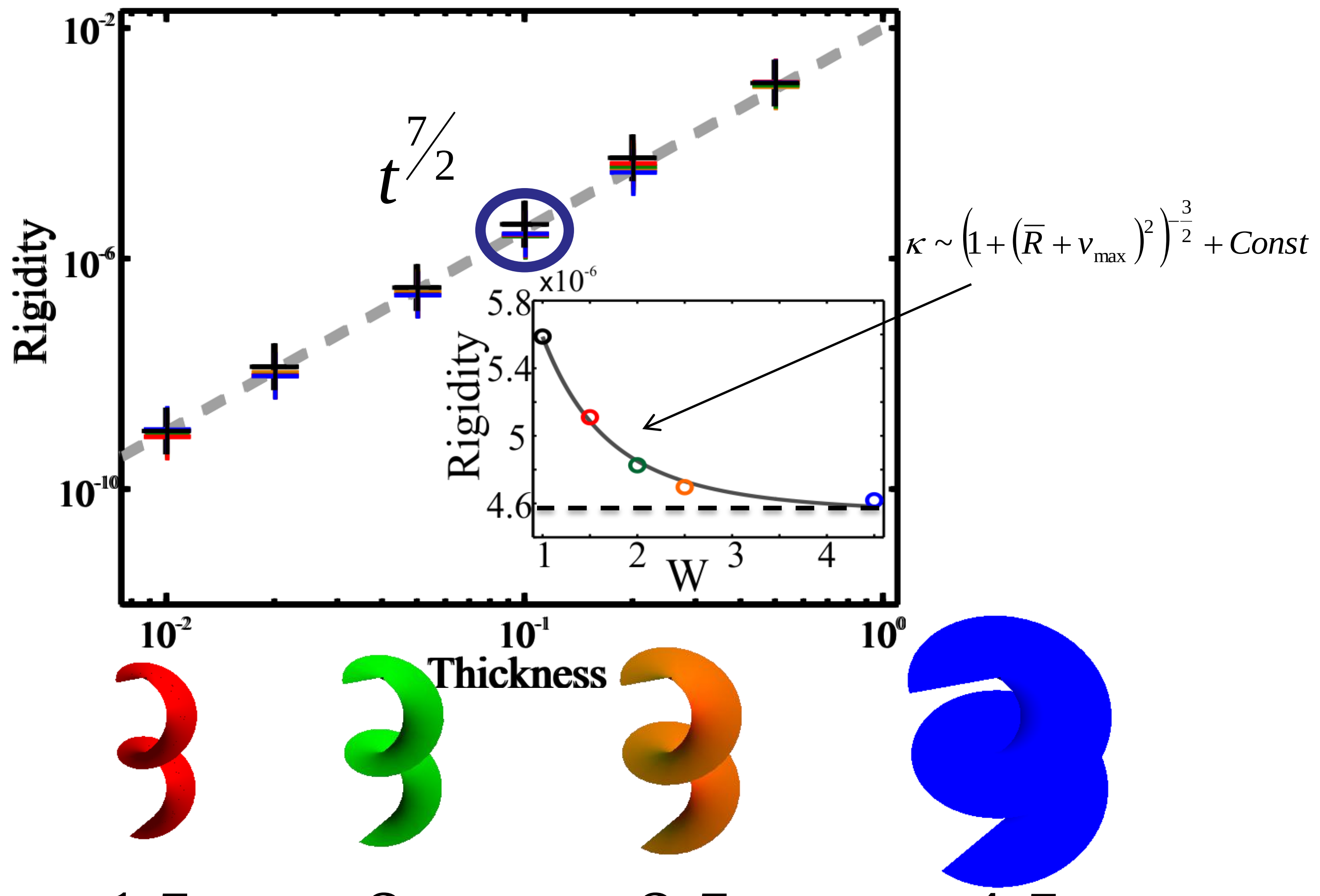


Elastic energy $E(p)$

Linearity



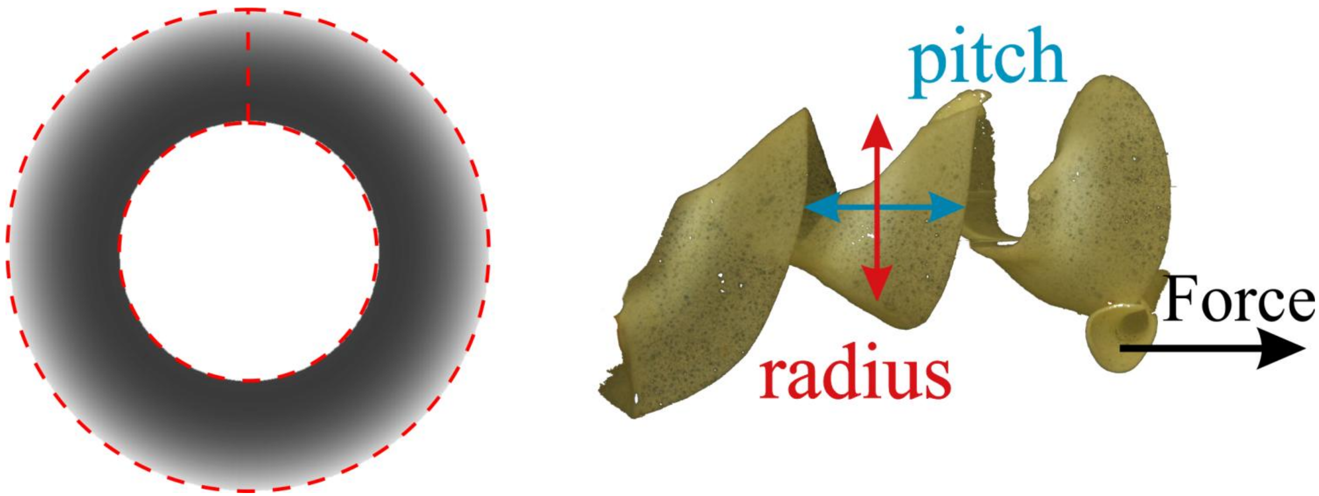
Ultra soft + Softening with increasing width

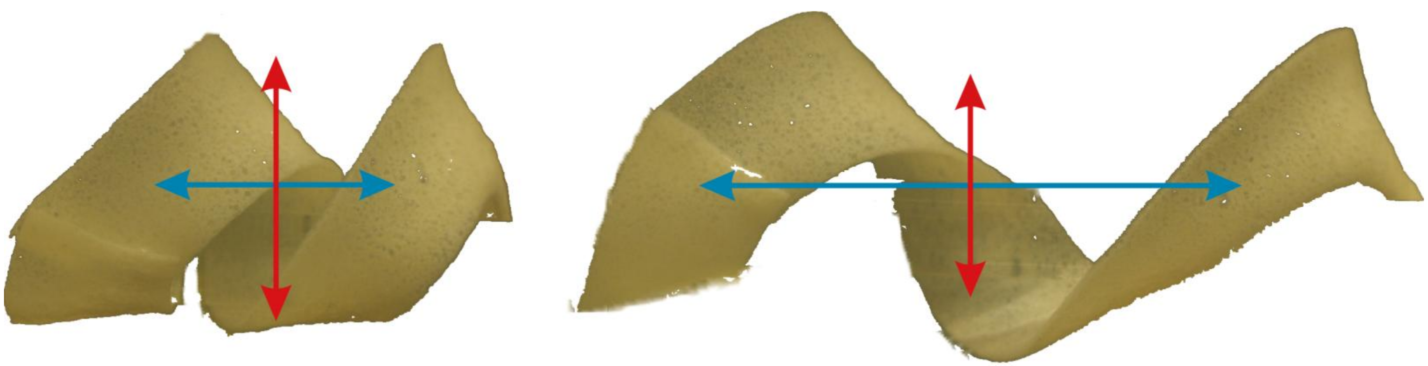


W 1 1.5 2 2.5 4.5

I. Levin and ES, prl 2016

Experimental realization





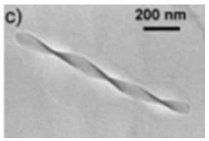
$\Delta E \approx 1$ erg

$F \approx 1$ dyne

Shaping via Active deformation

Chemistry

Macromolecules

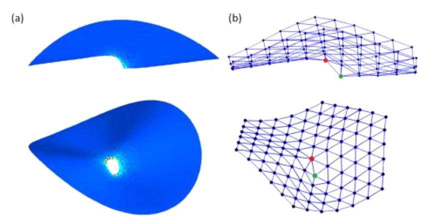


Math-geometry

"Embedding experiments"

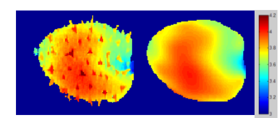
Physics

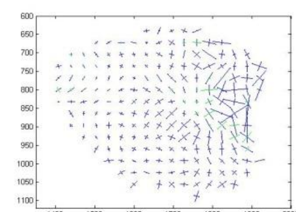
Dynamics
Defects in amorphous material
Plasticity – equations of Motion for the target metric
Other materials



Biology

Application to morphogenesis





design

Soft machinery-
Coupling to an energy source

Assa Ashoach

Thank you

The relevance to other plants and to self assembly of macromolecules

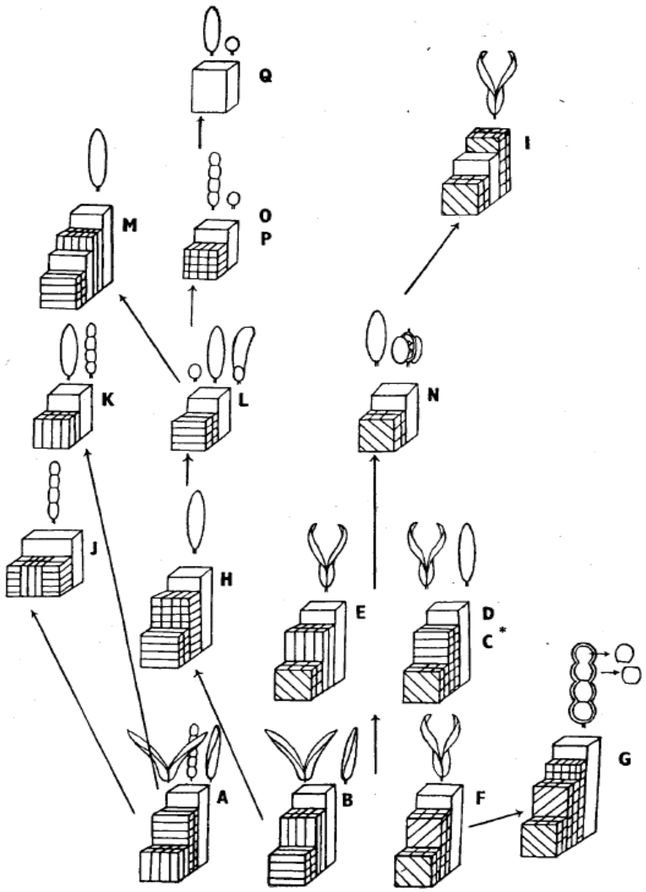
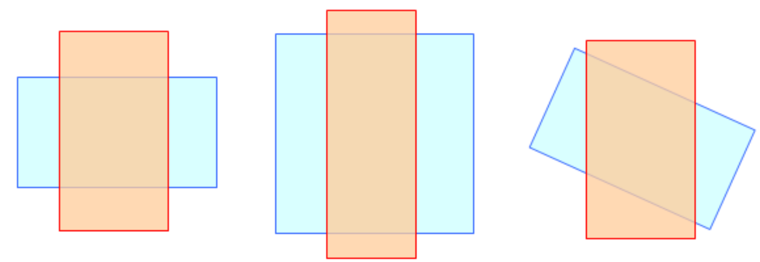


Fig. 21.

different growth strategies may lead to the same result:



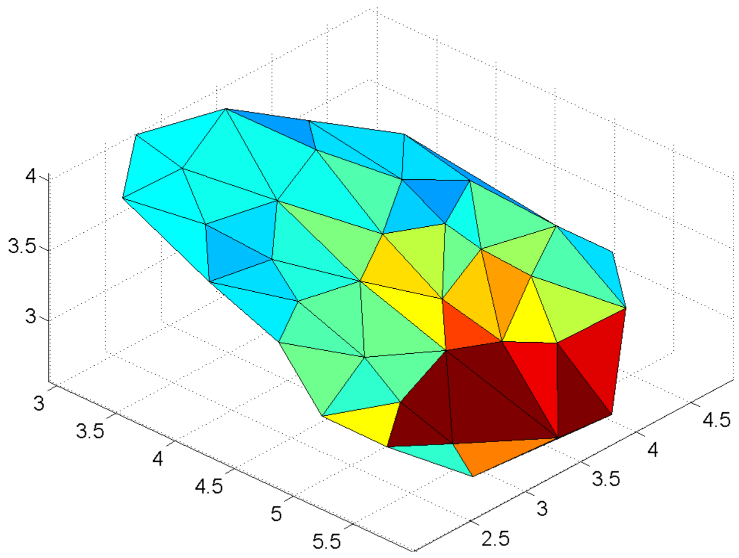
$$\bar{b} = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix} \qquad \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix} \qquad \begin{pmatrix} k\cos\theta & 0 \\ 0 & -k\cos\theta \end{pmatrix}$$

Fahn and Zohari, “on the Pericarpial Structure of the Legumen, its Evolution and Relation To Dehiscence” (1955)

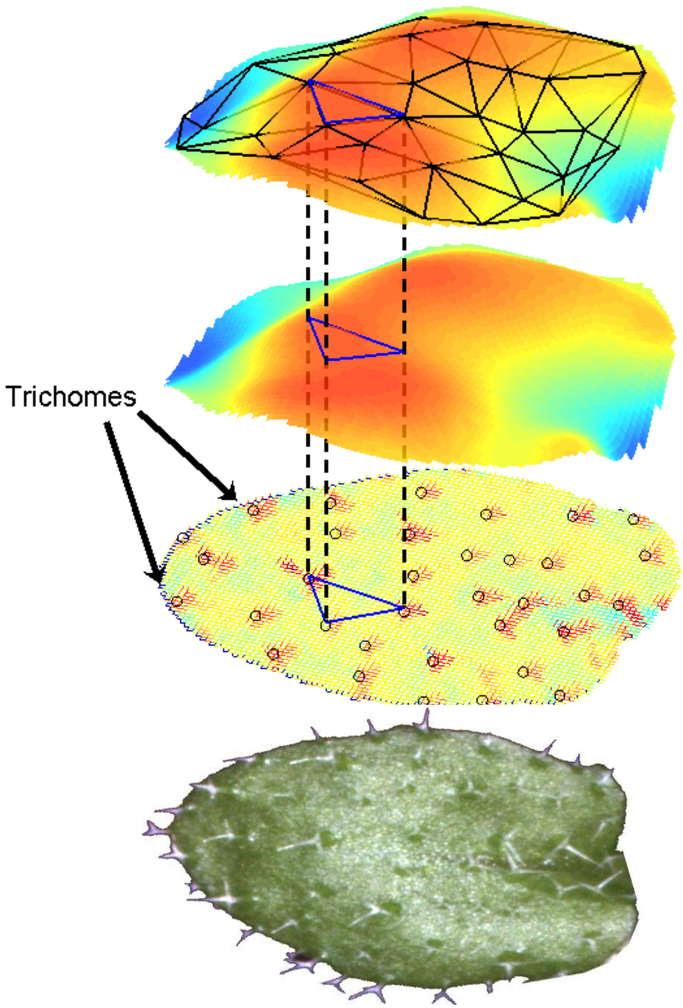
Some leaves seems to be “mechanically wrinkled”

E. S., M. Marder and H. L. Swinney , *American Scientist,* **92,** 254, (2004).

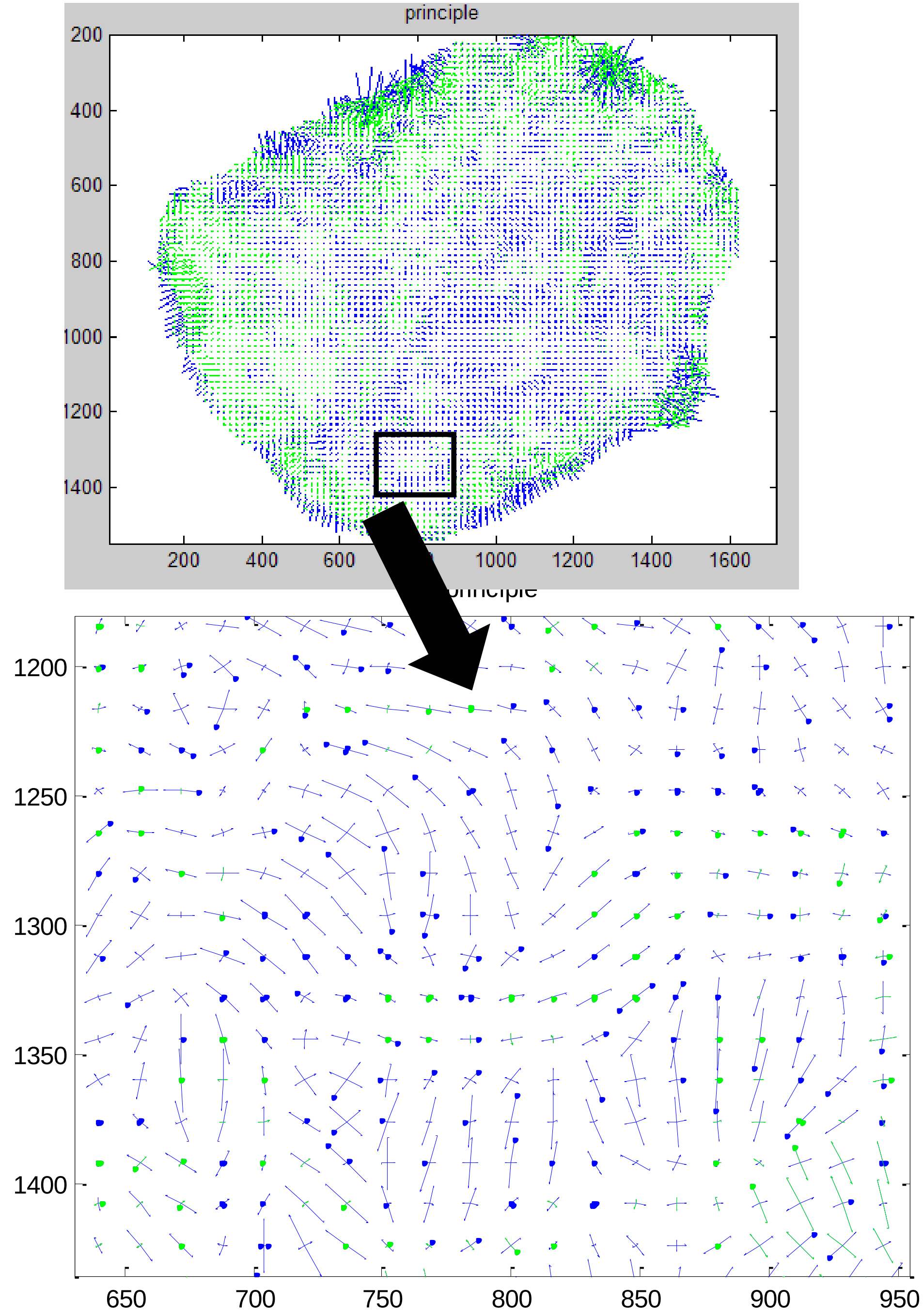
Lagrangian Measurements (Particle Tracking)



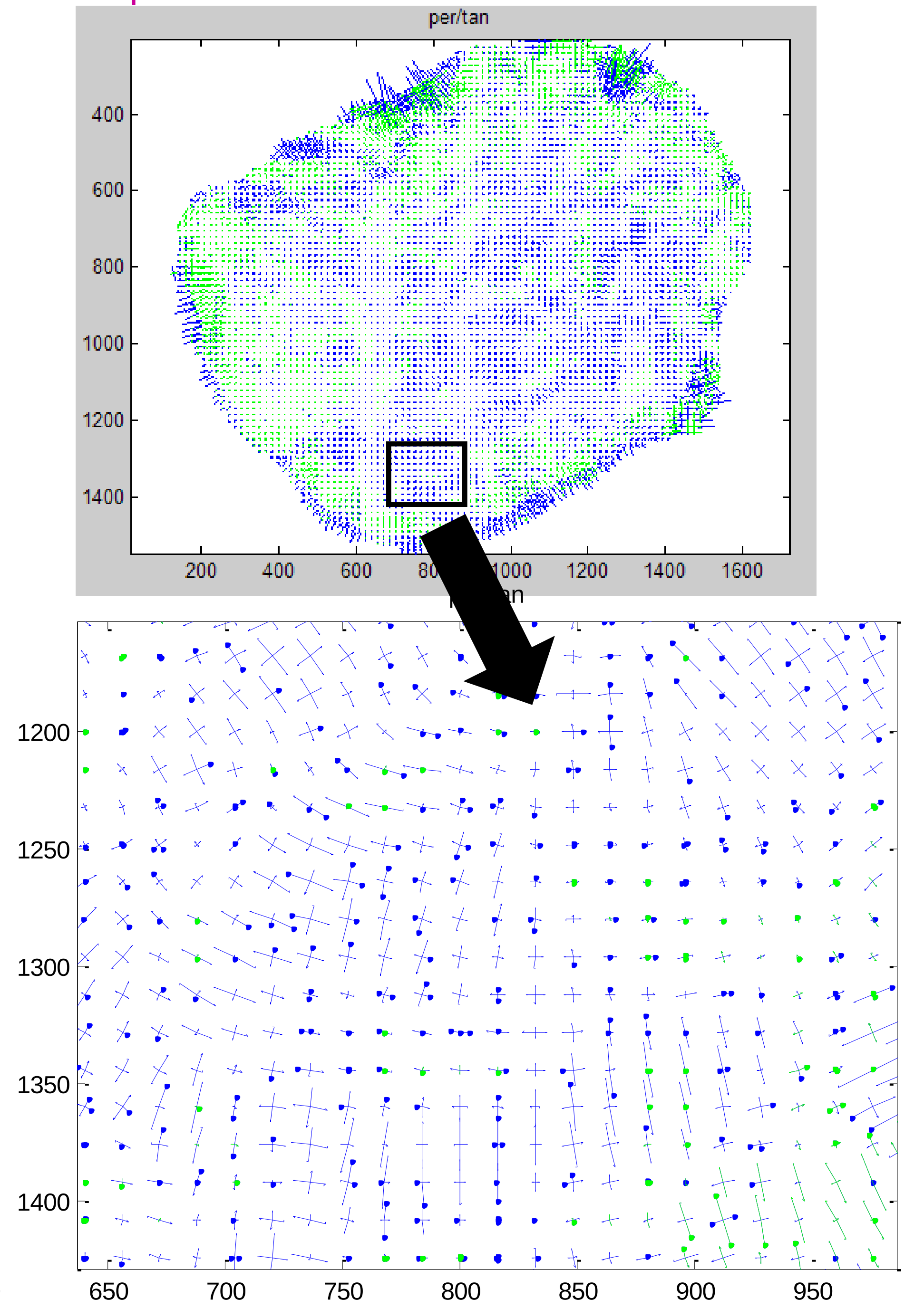
Surface growth within one week

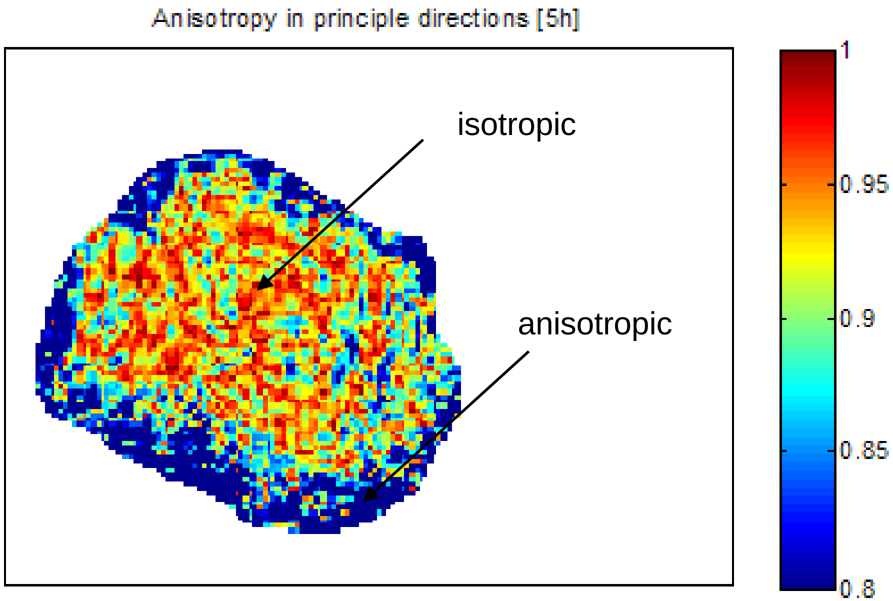


Decomposition to principle directions

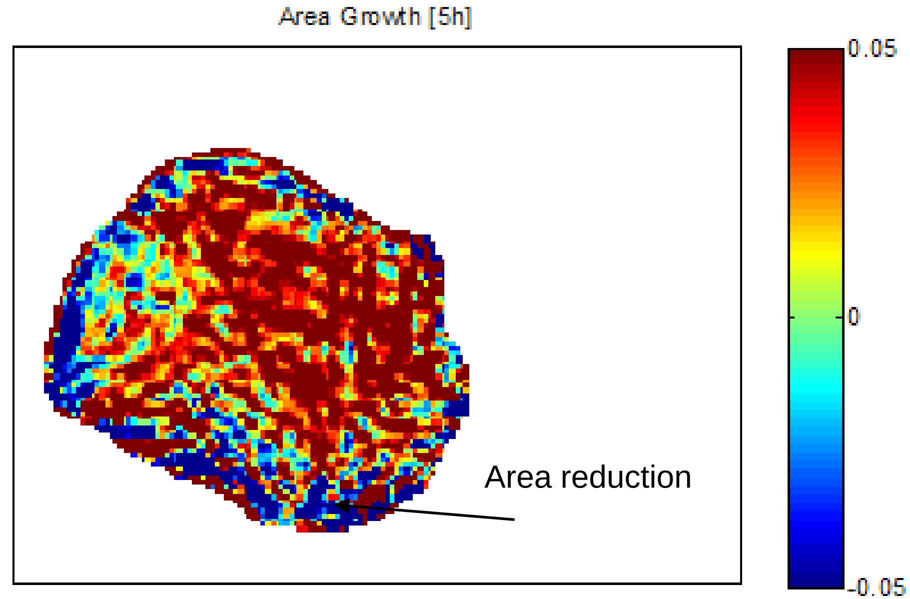


Decomposition to azimuthal and “radial” directions

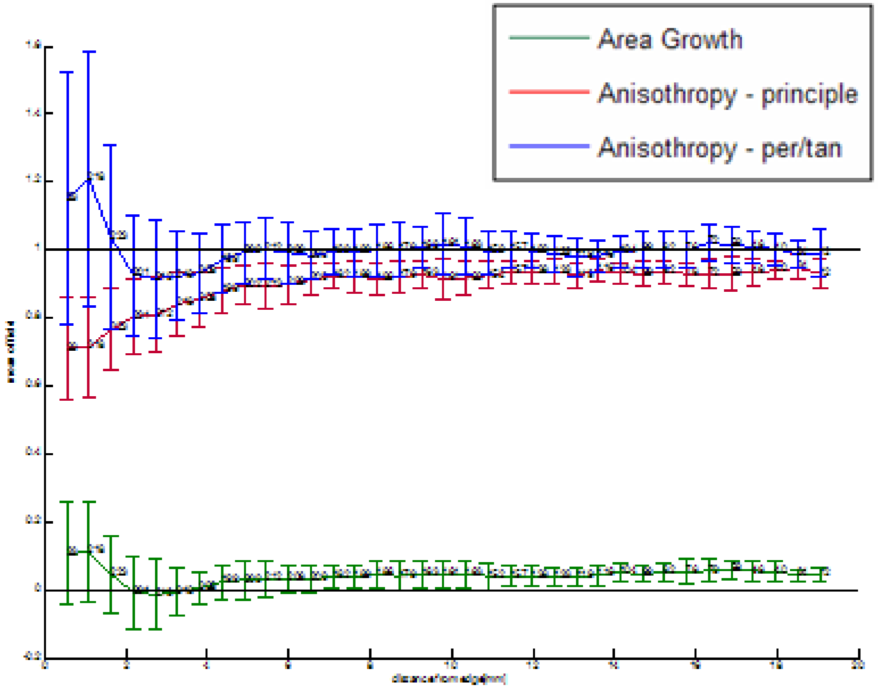




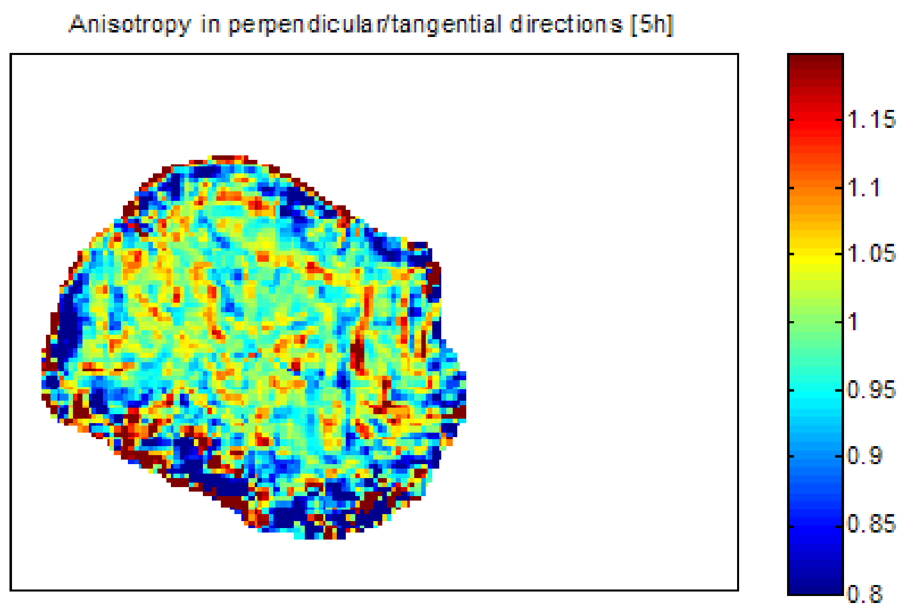
Anisotropy in principle directions [5h]
isotropic
anisotropic
1
0.95
0.9
0.85
0.8



Area Growth [5h]
Area reduction
0.05
0
-0.05

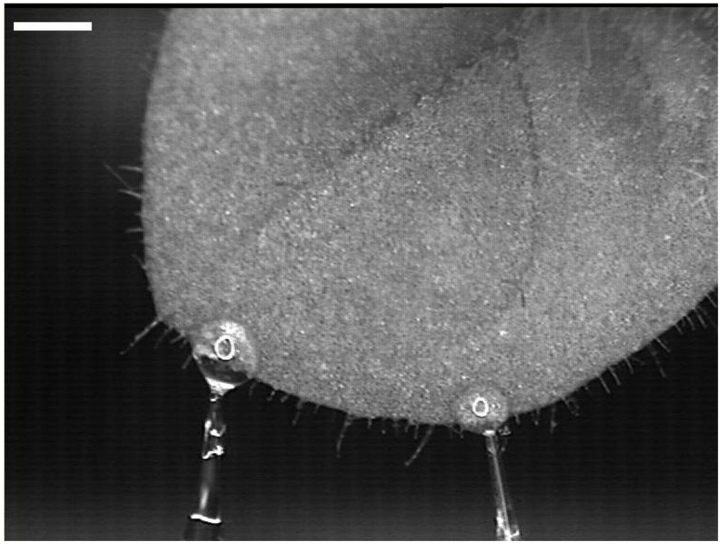


Area Growth
Anisothropy - principle
Anisothropy - per/tan

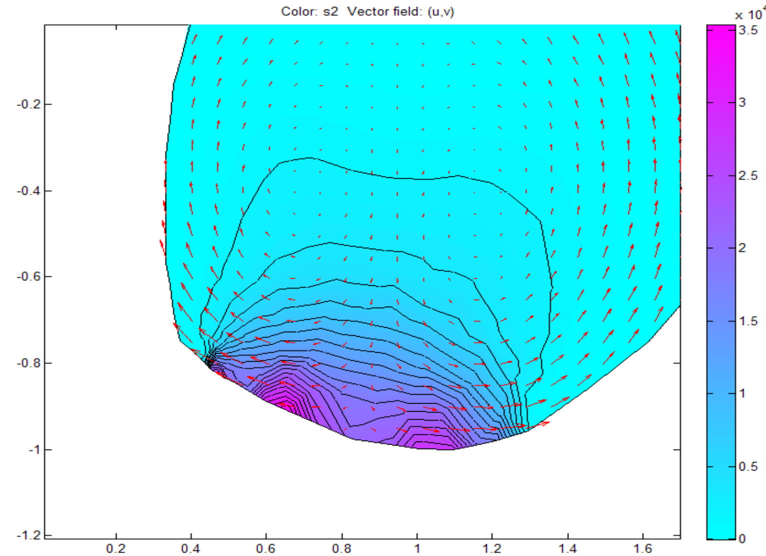


Anisotropy in perpendicular/tangential directions [5h]
1.15
1.1
1.05
1
0.95
0.9
0.85
0.8

Stress application and feedback on growth

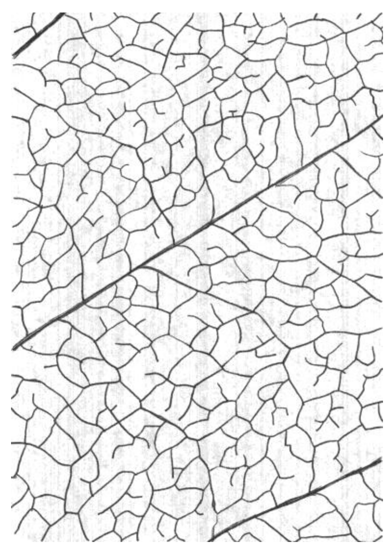


tension is applied between the needles

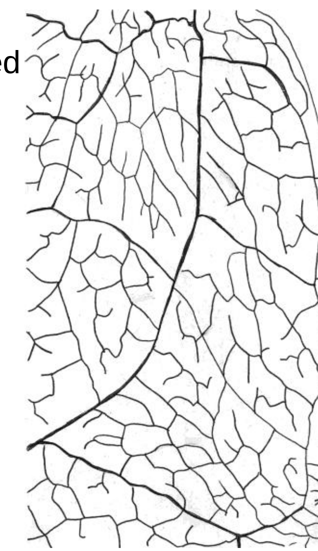


Numerical calculation of stress distribution

normal



stretched



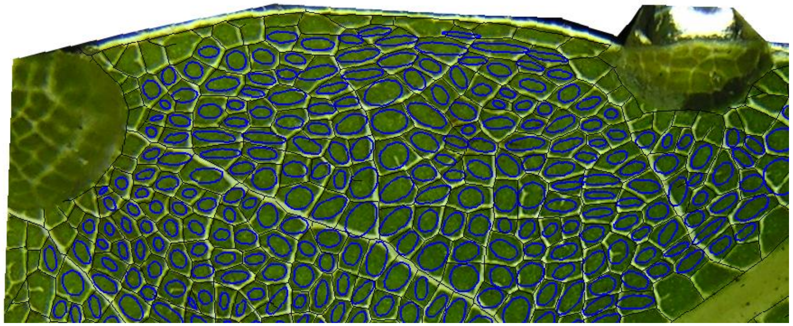
Possible feedback:

Cell shape (SEM)

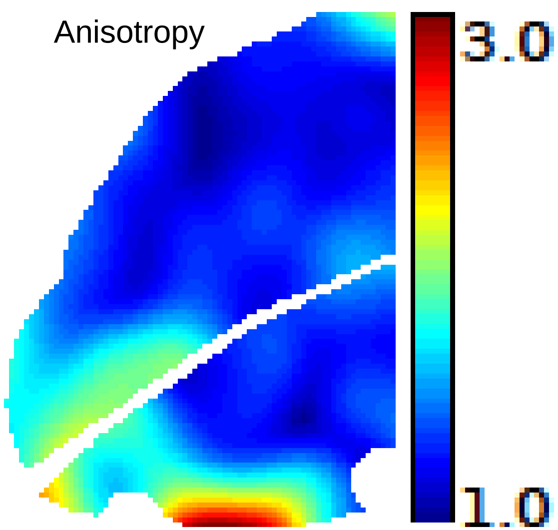
Gene expression (live GFP, MicroArray)

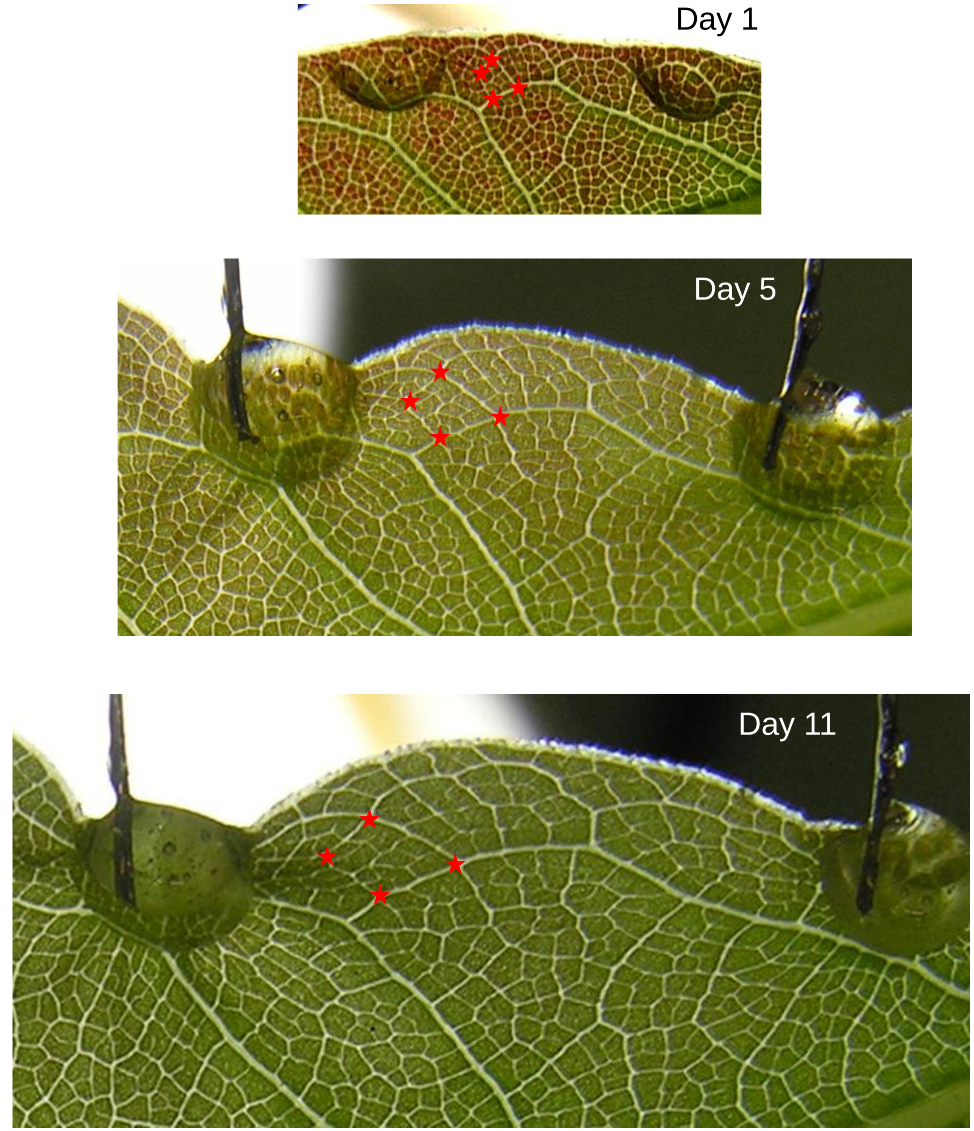
Venation network

Calculating the “texture tensor”



Anisotropy





What type of effect?

Passive “plasticity”?

Active biological response?