Non-singular behavior in loop quantum gravity

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Plan: (i) Origin of difference equations.
(ii) Solvable bounce model and effective equations.
(iii) Difference equations and space-like singularities.

Isotropy, spatially flat

Connection $A_a^i = \tilde{c}\delta_a^i$, densitized triad $E_i^a = \tilde{p}\delta_i^a$; conjugate pair $(\tilde{c}, \tilde{p})$ with $|\tilde{p}| = \frac{1}{4}a^2$ ($\mathrm{sgn}(\tilde{p})$: orientation), $\tilde{c} = \frac{1}{2}\gamma\dot{a}$.

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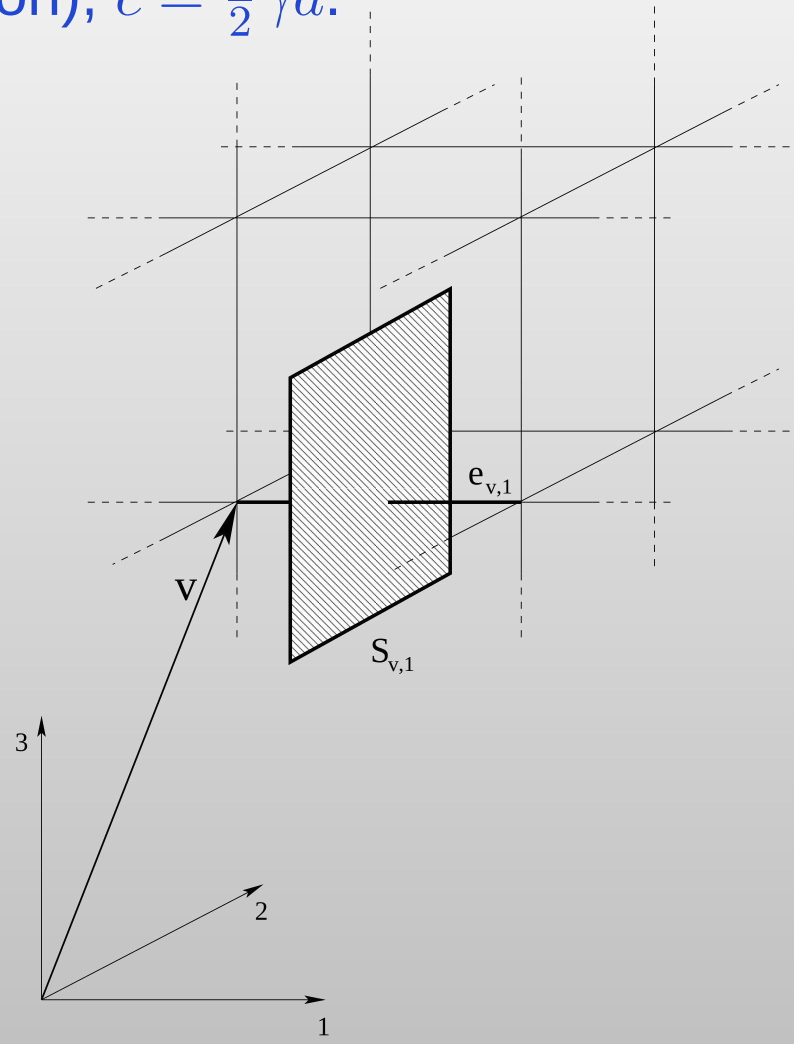
Holonomies along symmetry generators:

$$h_e = \mathcal{P}\exp\int_e A_a^i \tau_i \dot{e}^a \mathrm{d}t = \cos\left(\tfrac{1}{2}\int_e \tilde{c}\right) + 2\tau_i \dot{e}^i \sin\left(\tfrac{1}{2}\int_e \tilde{c}\right)$$



Basic lattice states

$$\langle c|\{\mu_{v,I}\}\rangle = \prod_{v,I} \exp\left(\tfrac{1}{2} i \mu_{v,I} \int_{e_{v,I}} \tilde{c}\right)$$

with edge labels $\mu_{v,I} \in \mathbb{Z}$.

Exact isotropy

Orthonormal states $\langle c|\mu\rangle = e^{i\mu c/2}$, $\mu = \sum \mu_{v,I}$ and basic operators

$$\begin{aligned} \widehat{e^{i\mu' c/2}}|\mu\rangle &= |\mu+\mu'\rangle \\ \hat{p}|\mu\rangle &= \tfrac{1}{6}\gamma\ell_{\mathrm{P}}^2\mu|\mu\rangle \end{aligned}$$

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Operator algebra follows from full holonomy-flux algebra: suitable lattice operators preserve distributional states supported only on isotropic connections.
Unique representation in full theory distinguishes representation of reduced models, *inequivalent* to Wheeler–DeWitt representation.
($\widehat{e^{i\mu' c/2}}$ not weakly continuous in μ'; $\hat{p}$ with discrete spectrum.)

Hamiltonian constraint

$$C[N] = \frac{1}{16\pi\gamma G}\int_\Sigma \mathrm{d}^3x N\left(\epsilon_{ijk}F^i_{ab}\frac{E^a_jE^b_k}{\sqrt{|\det E|}} - 2(1+\gamma^{-2})(A^i_a-\Gamma^i_a)(A^j_b-\Gamma^j_b)\frac{E^{[a}_iE^{b]}_j}{\sqrt{|\det E|}}\right)$$

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Requires inverse determinant, from relation

$$\left\{A^i_a, \int \sqrt{|\det E|}\mathrm{d}^3x\right\} = 2\pi\gamma G\epsilon^{ijk}\epsilon_{abc}\frac{E^b_j E^c_k}{\sqrt{|\det E|}}$$

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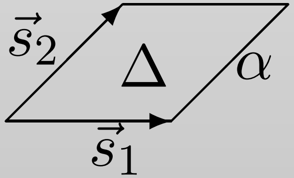
For curvature: $s^a_1 s^b_2 F^i_{ab}\tau_i = \Delta^{-1}(h_\alpha - 1) + O(\Delta)$



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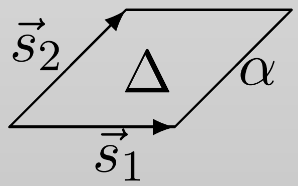

Extrinsic curvature:

$$K^i_a = \gamma^{-1}(A^i_a - \Gamma^i_a) \propto \left\{A^i_a, \left\{\int \mathrm{d}^3x F^i_{ab}\frac{\epsilon^{ijk}E^a_jE^b_k}{\sqrt{|\det E|}}, \int\sqrt{|\det E|}\mathrm{d}^3x\right\}\right\}$$

“Reduced” Hamiltonian constraint

Schematically: $\sum_v \sum_{IJK} \epsilon_{IJK} \mathrm{tr}(h_{\alpha_{IJ}} h_K [h_K^{-1}, \hat{V}])$,

$$\frac{i}{8\pi\gamma G\hbar} \mathrm{tr}(h_{v,I} h_{v+I,J} h_{v+J,I}^{-1} h_{v,J}^{-1} h_{v,K} [h_{v,K}^{-1}, \hat{V}_v])$$

$$= -\epsilon_{IJK} \Big\{ [(c_{v,I} c_{v+J,I} + s_{v,I} s_{v+J,I}) c_{v,J} c_{v+I,J} + (c_{v,I} c_{v+J,I} - s_{v,I} s_{v+J,I}) s_{v,J} s_{v+I,J}] \hat{A}_{v,K} \Big\}$$
$$+ \epsilon_{IJK}^2 \Big\{ [(c_{v,I} s_{v+J,I} - s_{v,I} c_{v+J,I}) s_{v,J} c_{v+I,J} + (s_{v,I} c_{v+J,I} + c_{v,I} s_{v+J,I}) c_{v,J} s_{v+I,J}] \hat{B}_{v,K} \Big\}$$

where $\hat{A}_{v,K} := \frac{1}{4\pi i\gamma G\hbar} \left(\hat{V}_v - c_{v,K} \hat{V}_v c_{v,K} - s_{v,K} \hat{V}_v s_{v,K} \right)$ and

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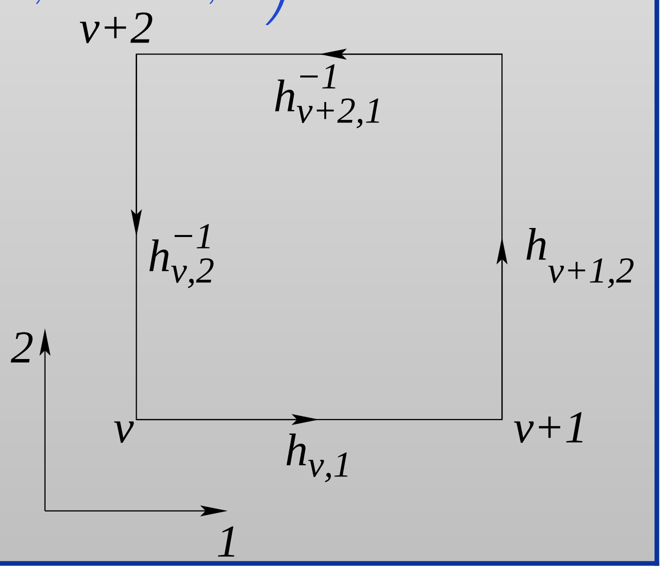
using $c_{v,I}$ and $s_{v,I}$ for cosines and sines.



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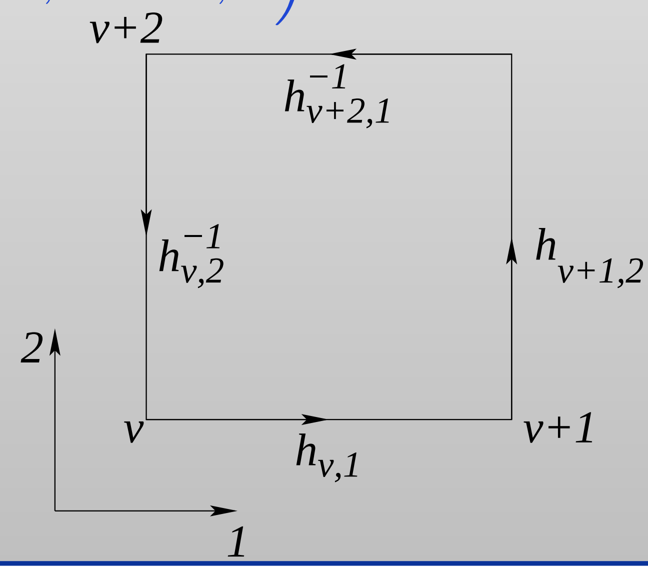

Exact isotropy: gravitational constraint

$\hat{C} \propto -\delta^{-3}\sin^2(\delta c)\hat{B} \to -\tilde{c}^2\sqrt{p} + \cdots$

edge length δ, thus $\mathcal{N} \propto \delta^{-3}$ vertices.

Difference equation

Action:

$$\hat{C}|\mu\rangle = \frac{3}{16\pi G\delta^3\gamma^3\ell_{\mathrm{P}}^2}(V_{\mu+\delta} - V_{\mu-\delta})(|\mu+4\delta\rangle - 2|\mu\rangle + |\mu-4\delta\rangle)$$

Operator equation $(\hat{C} + \hat{H}_{\mathrm{matter}})|\psi\rangle = 0$ to be solved for states $|\psi\rangle = \sum_\mu \psi_\mu(\phi)|\mu\rangle$ where $\psi_\mu(\phi)$ represents the state in the *triad representation.*

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Results in *difference equation* for $\psi_\mu(\phi)$:

$$(V_{\mu+5\delta} - V_{\mu+3\delta})\psi_{\mu+4\delta}(\phi) - 2(V_{\mu+\delta} - V_{\mu-\delta})\psi_\mu(\phi)$$
$$+(V_{\mu-3\delta} - V_{\mu-5\delta})\psi_{\mu-4\delta}(\phi) = -\tfrac{4}{3}\pi G\delta^3\gamma^3\ell_{\rm P}^2\hat{H}_{\rm matter}(\mu)\psi_\mu(\phi)$$

with volume eigenvalues $V_\mu = (\gamma\ell_{\rm P}^2|\mu|/6)^{3/2}$.

$\hat{H}_{\rm matter}$ fully quantized, including metric coefficients, e.g. $\hat{H}_\phi = -\frac{1}{2}\hbar^2\widehat{|p|^{-3/2}}\partial^2/\partial\phi^2 + |\hat{p}|^{3/2}W(\phi)$.

Matter Hamiltonian

$H_\phi = \frac{1}{2}|p|^{-3/2}p_\phi^2 + |p|^{3/2}W(\phi)$ contains p^{-1}, but $\hat{p}$ has discrete spectrum containing zero: no densely defined inverse.

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$$\mathrm{sgn}(p)|p|^{-3/2} = \left(\frac{1}{12\pi\delta\gamma G}\sum_{I=1}^{3}\mathrm{tr}(\tau_I h_I\{h_I^{-1}, |p|^{3/4}\})\right)^6$$

using only positive powers of p and “holonomies” $h_I = e^{\delta c\tau_I}$.

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Eigenvalues

$$(\widehat{\mathrm{sgn}(p)|p|^{-3/2}})_\mu = \left(\frac{4}{\delta\gamma\ell_{\mathrm{P}}^2}(|p_{\mu+\delta}|^{3/4} - |p_{\mu-\delta}|^{3/4})\right)^6$$

finite, vanish at $\mu = 0$.

Properties

$\longrightarrow \psi_\mu(\phi)$ replaces $\psi(a,\phi)$ in Wheeler–DeWitt quantization; Wheeler–DeWitt equation is good approximation when $\psi_\mu(\phi)$ does not vary strongly on small scales $\mu \pm 4\delta$.

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$\longrightarrow$ Two sides provided by basic variables (densitized triad); extendability consequence of Hamiltonian constraint. Would have been singular in other ordering choices.
Scenario testable in more complicated models: e.g. Kasner where one metric component diverges at the classical singularity; more complicated difference equations in less symmetric models.

Solvable model

Consider a free, massless scalar in a flat isotropic geometry: $\hat{H}_\phi = -\frac{1}{2}\hbar^2 \widehat{|p|^{-3/2}} \frac{\partial^2}{\partial\phi^2}$. For large p, ignoring $\widehat{|p|^{-3/2}} \neq$ "$|\hat{p}|^{-3/2}$":

$$-\frac{\partial^2}{\partial\phi^2}\psi(p,\phi) \propto (\widehat{\sin(c)p})^2 \psi(p,\phi) =: \hat{H}^2\psi(p,\phi)$$

(ordering to be specified).

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Solve for $\partial\psi/\partial\phi$, use $\phi = t$ as internal time: $i\hbar\dot{\psi} = \hat{H}\psi = |(\widehat{\sin(c)p})|\psi = \frac{1}{2}|i(\hat{J} - \hat{J}^\dagger)|\psi$ with $\hat{J} := \hat{p}\widehat{\exp(ic)}$.

Hamiltonian linear except for norm, linear algebra ($\mathrm{sl}(2,\mathbb{R})$) with basic operators:

$$[\hat{p},\hat{J}] = \hbar\hat{J} \quad , \quad [\hat{p},\hat{J}^\dagger] = -\hbar\hat{J}^\dagger \quad , \quad [\hat{J},\hat{J}^\dagger] = -2\hbar\hat{p} - \hbar^2$$

Equations of motion

Look at states for which $\Delta H \ll \langle\hat{H}\rangle$ (conserved); ignore norm of $\hat{H} = \frac{1}{2i}(\hat{J} - \hat{J}^\dagger)$. Equations of motion for expectation values

$$\frac{\mathrm{d}\langle\hat{p}\rangle}{\mathrm{d}t} = \frac{1}{i\hbar}\langle[\hat{p},\hat{H}]\rangle = -\tfrac{1}{2}(\langle\hat{J}\rangle + \langle\hat{J}\rangle^*)$$

$$\frac{\mathrm{d}\langle\hat{J}\rangle}{\mathrm{d}t} = \frac{1}{i\hbar}\langle[\hat{J},\hat{H}]\rangle = -\tfrac{1}{2}(\langle\hat{p}\rangle + \hbar) = \frac{\mathrm{d}\langle\hat{J}\rangle^*}{\mathrm{d}t}$$

and fluctuations

$$\dot{G}^{pp} = \frac{1}{i\hbar}\langle[\hat{p}^2,\hat{H}]\rangle - 2\langle\hat{p}\rangle\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{p}\rangle = -G^{pJ} - G^{p\bar{J}}$$

$$\dot{G}^{JJ} = -2G^{pJ} \quad , \quad \dot{G}^{\bar{J}\bar{J}} = -2G^{p\bar{J}}$$

$$\dot{G}^{pJ} = -\frac{1}{2}G^{JJ} - \frac{1}{2}G^{J\bar{J}} - G^{pp} \quad , \quad \dot{G}^{p\bar{J}} = -\frac{1}{2}G^{\bar{J}\bar{J}} - \frac{1}{2}G^{J\bar{J}} - G^{pp}$$

$$\dot{G}^{J\bar{J}} = -G^{pJ} - G^{p\bar{J}}$$

where $G^{AB} = \frac{1}{2}\langle\hat{A}\hat{B} + \hat{B}\hat{A}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle$, and higher moments.

Solution

$$\langle\hat{p}\rangle(t) = \tfrac{1}{2}(c_1 e^{-t} + c_2 e^{t}) - \tfrac{1}{2}\hbar \quad , \quad \langle\hat{J}\rangle(t) = \tfrac{1}{2}(c_1 e^{-t} - c_2 e^{t}) + iH$$

to be restricted to satisfy $\hat{J}\hat{J}^\dagger = \hat{p}^2$,

$$|\langle\hat{J}\rangle|^2 = (\langle\hat{p}\rangle + \tfrac{1}{2}\hbar)^2 - G^{pp} + G^{J\bar{J}} + \tfrac{1}{4}\hbar^2 = (\langle\hat{p}\rangle + \tfrac{1}{2}\hbar)^2 + c_3$$

$|\langle\hat{J}\rangle|^2 - (\langle\hat{p}\rangle + \frac{1}{2}\hbar)^2 = -c_1c_2 + H^2$ implies $c_1c_2 = H^2 + c_3$; *only bouncing solutions for $-c_3 < H^2$ (conserved).*

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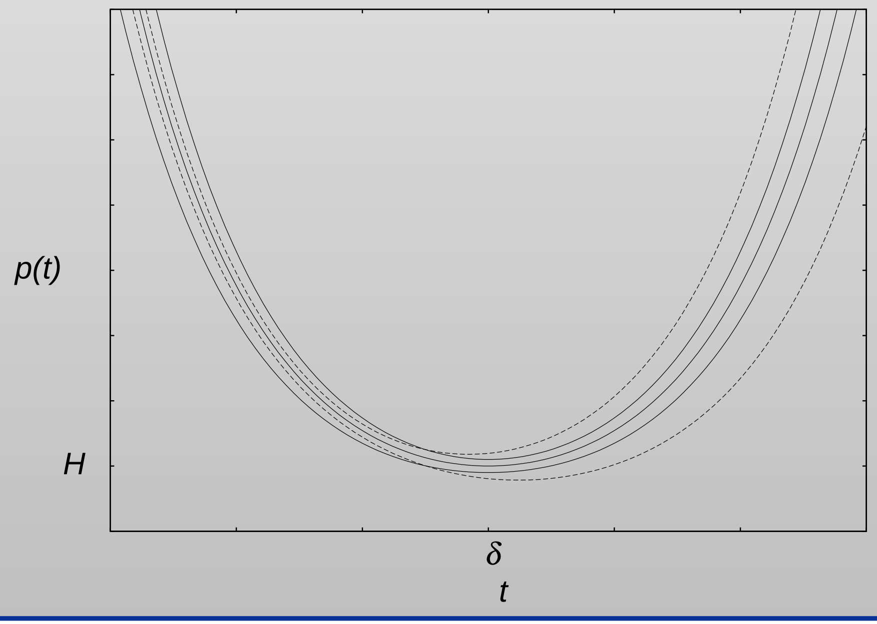
Solutions for G^{AB} show behavior of dispersions.
Basis for derivation of *effective equations* and perturbation theory around solvable model (without explicit construction of states).



Singularity resolution

Bounce to new semiclassical space-time region in select models so far. More generally, states will spread, and dispersions (and higher moments) couple to expectation values. Detailed geometrical pictures more difficult to develop in such cases.

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More basic statement using difference equations: *any* quantum state extended across classical singularity in superspace (through deep quantum regime). New region beyond singularity, but no general statements yet about re-emergence of semiclassical parts. Available in several situations, non-trivial consistency checks:

$\longrightarrow$ Kasner singularity
$\longrightarrow$ Breakdown for perturbative anisotropy
$\longrightarrow$ Inhomogeneous situations, BKL spirit?

Kasner

Diagonal densitized triad $E^a_i = p^{(i)}\delta^a_i$, relation to diagonal metric components a_I by $p^1 = a_2 a_3$ and cyclic.
Kasner solution: $a_I(t) \propto t^{\alpha_I}$ with $\sum_I \alpha_I = 1 = \sum_I \alpha_I^2$. One component always diverges and $-1 < \alpha_I \leq 1$.

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Difference equation (restricted anisotropy, black hole interior):

$$
\begin{aligned}
&2(\sqrt{|\nu+2\delta|}+\sqrt{|\nu|})\psi_{\mu+2\delta,\nu+2\delta}-2(\sqrt{|\nu+2\delta|}+\sqrt{|\nu|})\psi_{\mu-2\delta,\nu+2\delta}\\
&+(\sqrt{|\nu+\delta|}-\sqrt{|\nu-\delta|})((\mu+2\delta)\psi_{\mu+4\delta,\nu}-(1+2\gamma^2\delta^2)\mu\psi_{\mu,\nu}+(\mu-2\delta)\psi_{\mu-4\delta,\nu})\\
&+2(\sqrt{|\nu-2\delta|}+\sqrt{|\nu|})\psi_{\mu+2\delta,\nu-2\delta}-2(\sqrt{|\nu-2\delta|}+\sqrt{|\nu|})\psi_{\mu-2\delta,\nu-2\delta}=0
\end{aligned}
$$

Non-singular behavior

Two independent triad variables μ and ν, corresponding to metric

$$\mathrm{d}s^2 = \frac{\mu(t)^2}{|\nu(t)|}\mathrm{d}x^2 + |\nu(t)|\mathrm{d}\Omega^2$$

Classical singularity at $\nu = 0$, wave function uniquely extended across.

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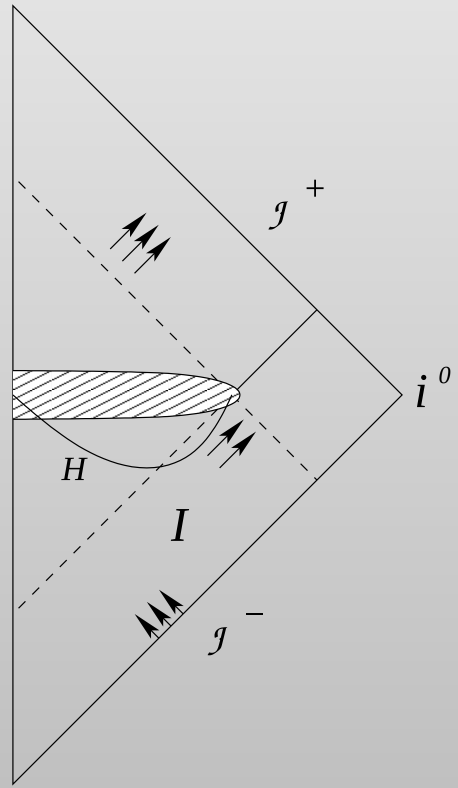

Non-singular behavior

Two independent triad variables μ and ν, corresponding to metric

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Classical singularity at $\nu = 0$, wave function uniquely extended across.

All anisotropies quantized non-perturbatively, discreteness of all triad components considered.
Alternatively: Treat anisotropy as small parameter, perturbation on isotropic “background”

$$\mu = \bar{\nu} + \epsilon \quad , \quad \nu = \bar{\nu} - 2\epsilon$$

Expand wave function and coefficients in difference equation in anisotropy ϵ and its momentum P (Wheeler–DeWitt limit for anisotropy but not for isotropic variable).

Perturbative singularity

Matrix elements of Hamiltonian in perturbative basis,
$|s >= \sum_{\bar\nu} s_{\bar\nu}(\epsilon)|\bar\nu >$

$$< \bar\nu, \epsilon|\hat H|s > \propto \left\{ \hat A_{\bar\nu+4\delta} s_{\bar\nu+4\delta}(\epsilon) + \hat B_{\bar\nu} s_{\bar\nu}(\epsilon) + \hat C_{\bar\nu-4\delta} s_{\bar\nu-4\delta}(\epsilon) \right\}$$

with coefficients, using $\Delta_3 f(\bar\nu) := \frac{1}{6}(f(\bar\nu+3\delta) - f(\bar\nu-3\delta))$,

$$\begin{aligned} \hat C_{\bar\nu} \quad = \quad & -\tfrac{2}{3}(|\bar\nu|^{1/2} + |\bar\nu|\Delta_3|\bar\nu|^{1/2}) + \tfrac{2}{3}(|\bar\nu|^{1/2} - 2|\bar\nu|\Delta_3|\bar\nu|^{1/2})i\epsilon \\ & + \left(\tfrac{2}{3}|\bar\nu|^{-1/2} - \Delta_3|\bar\nu|^{1/2} + |\bar\nu|\Delta_3|\bar\nu|^{-1/2}\right)\hat P \\ & + \tfrac{3}{2}\left(\tfrac{1}{2}|\bar\nu|^{-3/2} + \Delta_3|\bar\nu|^{-1/2} + \tfrac{1}{2}|\bar\nu|\Delta_3|\bar\nu|^{-3/2}\right)\hat P^2 \\ & - \left(|\bar\nu|^{-1/2} + 2\Delta_3|\bar\nu|^{1/2} - 2|\bar\nu|\Delta_3|\bar\nu|^{-1/2}\right) i\epsilon\hat P \\ & + \tfrac{1}{3}\left(|\bar\nu|^{1/2} + 4|\bar\nu|\Delta_3|\bar\nu|^{1/2}\right)\epsilon^2 + O(3) \end{aligned}$$

PENNSTATE

Spherical Symmetry

Coupled equations in inhomogeneous models. States:

$|\psi\rangle = \sum_{\vec{k},\vec{\mu}} \psi(\vec{k},\vec{\mu}) \; \overset{\cdots}{} \underset{\mu_-}{\bullet} \overset{k_-}{\text{———}} \underset{\mu}{\bullet} \overset{k_+}{\text{———}} \underset{\mu_+}{\bullet} \overset{\cdots}{}$ ($k_e \in \mathbb{Z}$, $0 \leq \mu_v \in \mathbb{R}$)

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Again, extension *non-singular.* Depends crucially on form (possible zeros) of coefficients $\hat{C}_{R\pm}(\vec{k})$, much more non-trivial than in isotropic models. Structure qualitatively different from homogeneous models: k_+- and k_--terms crucial.
No BKL-type argument for decoupling so far.

Summary

Basic, possibly general mechanism of non-singular behavior: *extended Hilbert space* (compared to Wheeler–DeWitt quantization) provided by loop quantization (orientations of densitized triad, connected by difference equation). Non-trivially generalized to anisotropic and midi-superspace models.

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Development of explicit bounce pictures more involved, but possible by *perturbation* around solvable model based on effective equations. Testable whether perturbation theory remains valid throughout, especially for inhomogeneities.