Matrix Models

in

the Quantum Hall Effect

Outline

- Introduction: the Laughlin wave function
- Jain's idea & the Gauss law
- Maxwell-Chern-Simons matrix theory

  → two regimes:

  $g = 0$ "matrix QHE"

  $g = \infty$ real QHE
- A conjecture

work with M. Riccardi, I. Rodriguez
(Florence)

Landau levels: one-body states

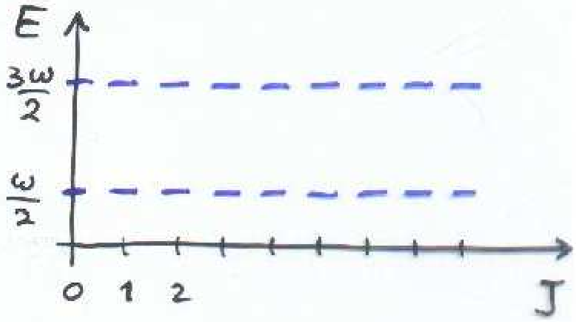
$$H = \frac{1}{2m}\left(\vec{P} - e\vec{A}\right)^2, \qquad A_i = \frac{B}{2}\varepsilon_{ij}x_j$$

$$z = x_1 + i x_2, \qquad \partial = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)$$

$\ell = \sqrt{\frac{2\hbar c}{eB}}$ magnetic length $\ell \to 1$

$$H = \omega\left(a^+ a + \tfrac{1}{2}\right)$$

$$J = \vec{x} \wedge \vec{p} = b^+ b - a^+ a$$



$$\begin{cases} a = \frac{z}{2} + \bar{\partial} \\ a^+ = \frac{\bar{z}}{2} - \partial \end{cases} \qquad \begin{cases} b = \frac{\bar{z}}{2} + \partial \\ b^+ = \frac{z}{2} - \bar{\partial} \end{cases}$$

$$[a, a^+] = 1, \quad [b, b^+] = 1$$

$$[a, b] = [a, b^+] = 0$$

- orbits have quantized radii $\pi r_n^2 B = n\phi_0$, $\phi_0 = \frac{hc}{e}$
- degeneracy $D_A = \frac{BA}{\phi_0} = \frac{\phi}{\phi_0} = \#$ fluxes unit flux
- Filling fraction $\nu = \frac{N}{D_A}$
- Lowest Landau level: $\omega = \frac{eB}{mc} \gg kT$

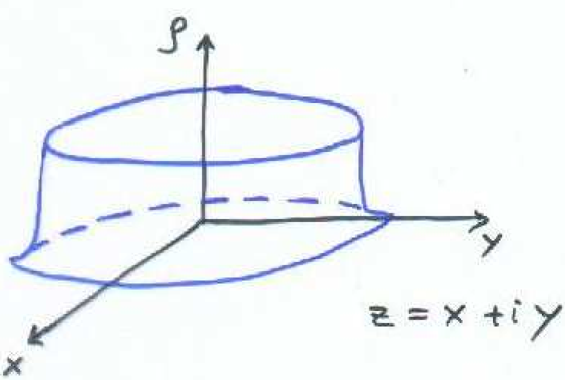
$$0 = a\Psi_0 = \left(\frac{z}{2} + \bar{\partial}\right)\Psi_0(z, \bar{z}), \quad \Psi_0 = e^{-\frac{1}{2}|z|^2}\varphi(z) \text{ analytic}$$

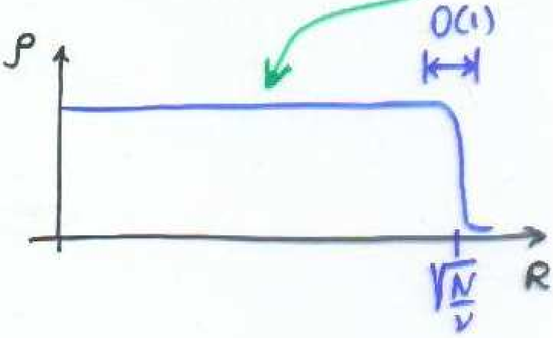
- projection to LLL: $\begin{cases} a = \frac{z}{2} + i\bar{p} = 0 \\ a^+ = \frac{\bar{z}}{2} - ip = 0 \end{cases}$

Laughlin's quantum incompressible fluid

Electrons form a droplet of liquid without sound waves

$$\begin{cases} \text{Incompressible} \equiv \text{density waves have a gap} \\ \text{Fluid} \equiv \rho(\vec{x}) = \rho_0 = \text{const.} \end{cases}$$





A = area of the droplet

N = # of electrons

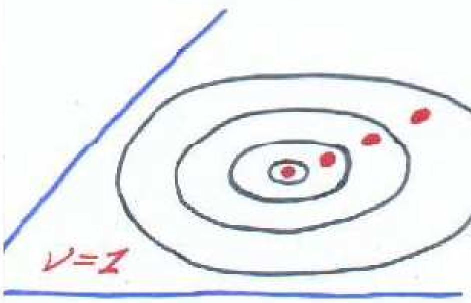
$\mathcal{D}_A = \dfrac{BA}{\frac{hc}{e}}$ = # of degenerate Landau orbitals

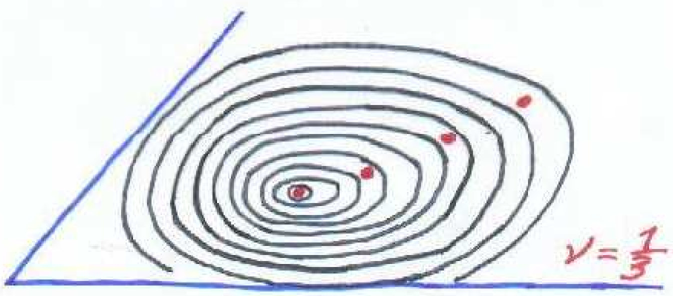
# of fluxes

$\rho = \dfrac{N}{A}$ = electron density

$\nu = \dfrac{N}{\mathcal{D}_A} = \dfrac{N}{BA/\Phi_0}$ = filling fraction = $1, \frac{1}{3}, \frac{1}{5}, \dots$

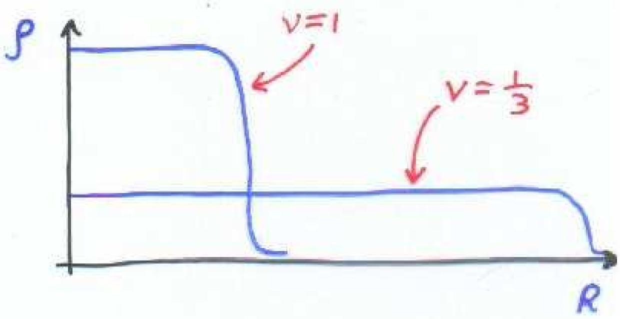
= density for quantum-mech. proble[m]





- Laughlin's trial wave function $\nu = \frac{1}{2k+1} = 1, \frac{1}{3}, \frac{1}{5}, \dots$

$$\Psi_{g.s.}(z_1, \dots, z_N) = \prod_{1 \le i < j}^{N} (z_i - z_j)^{2k+1} \; e^{-\sum |z_i|^2 / 2\ell^2}$$



- $\nu = 1$

  obvious gap for filled Landau level:

  $\text{gap} = \omega_c = \frac{eB}{mc} \gg k_B T$

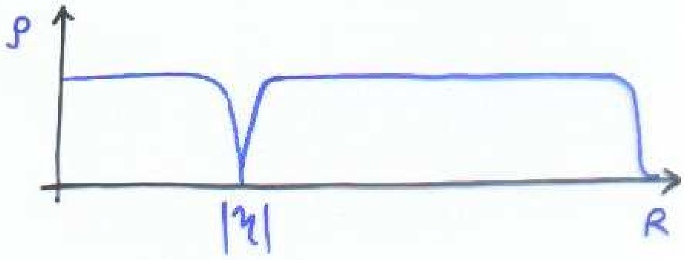
- $\nu = \frac{1}{3}$

  highly non-trivial gap due to repulsive electron-electron interaction:

  $\text{gap} = O\left(\frac{e^2}{\ell}\right)$ $\qquad \ell = \sqrt{\frac{2\hbar c}{eB}}$ "magnetic length"

- quasi-hole excitation ≈ vortex

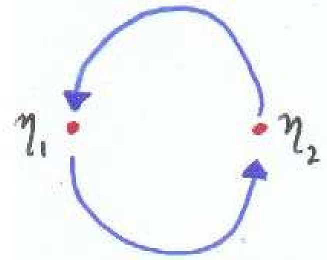
$$\Psi_{q\text{-}h}(\eta; z_1, \dots, z_N) = \prod_{i=1}^{N} (\eta - z_i) \prod_{i<j} (z_i - z_j)^{2k+1} \; e^{-\sum |z_i|^2 / 2\ell^2}$$



- $\nu = \frac{1}{2k+1}$ it has fractional charge $\qquad Q = \frac{e}{2k+1}$

  and fractional statistics $\qquad \frac{\theta}{\pi} = \frac{1}{2k+1}$

$$\Psi_{2q\text{-}h}(\eta_1,\eta_2;z_1\dots z_N) = (\eta_1-\eta_2)^{\frac{1}{2k+1}} \prod_i(\eta_1-z_i)\prod_i(\eta_2-z_i)\,\Psi_{g.s.}$$



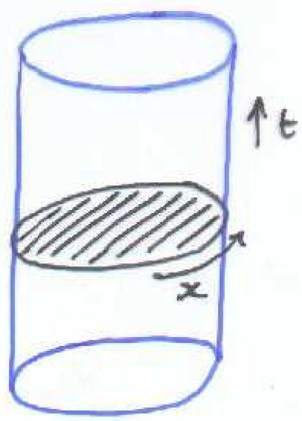
$$\Psi_{2qh}\left(\eta_1-\eta_2 \to e^{i\pi}(\eta_1-\eta_2)\right) = e^{i\frac{\pi}{2k+1}}\,\Psi_{2qh}(\eta_1,\eta_2)$$

fractional statistics $\frac{\theta}{\pi} = \frac{1}{2k+1} = \frac{1}{3}, \frac{1}{5}$

- fractional statistics is a long-range correlation of vortices, independent of distance, i.e. "topological"
- long-distance physics of incompressible fluid is universal, e.g. independent of type of repulsive interaction
- Laughlin's wave function is a "good representative" of the universality class

→ low-energy effective field theory
→ conformal field theory of massless edge excitations

- lot of nice work
- experimental confirmations



Non-relativistic effective field theories

- CFT description is very nice, also practical
- CFT almost completely determined by symmetries

BUT:

- cannot describe how the gapful ground states of incompressible fluids are formed
- cannot prove Laughlin's theory

$\rightarrow$ need a non-relativistic theory

- dynamical gap is <u>nonperturbative</u>

$\rightarrow$ try effective interactions & theories

- <u>Jain's idea</u> : the role of fluxes

$$\frac{1}{\nu} = \frac{\#\ \text{1-p states}}{\#\ \text{electrons}} = \frac{\#\ \text{Fluxes}}{N} = \frac{1}{m} + 2k = \frac{B}{2\pi\rho_0}$$

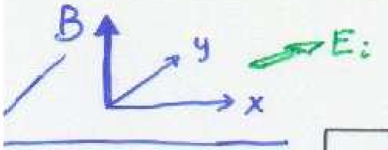
Removing $2k$ fluxes per electron would give

$$\frac{1}{\nu} \rightarrow \frac{1}{\nu^*} = \frac{1}{m}$$

integer Hall effect
obvious non-interacting theory with gap $\frac{B^*}{m}$

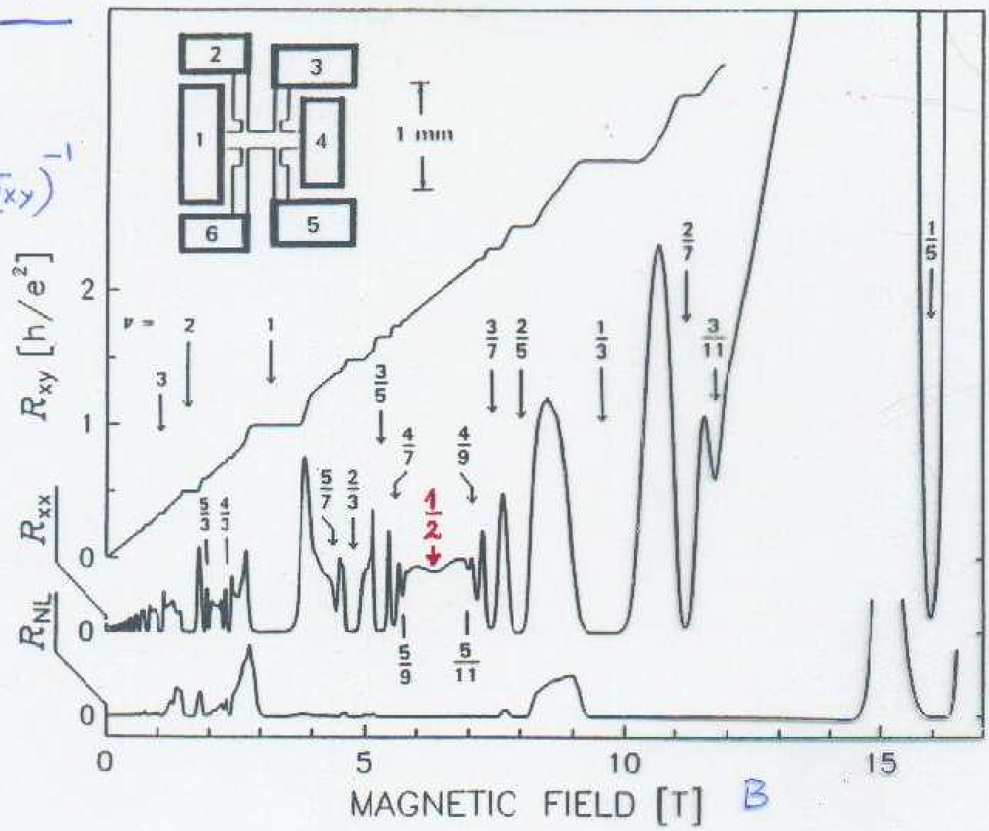
$$B \rightarrow B^* = B - \Delta B \ , \quad \underline{\Delta B = 2k\ 2\pi\rho_0}$$

eq. of motion of
<u>U(1) Chern-Simons gauge theory</u>



Theory of the quantum Hall effect

$R_{xy} = (\sigma_{xy})^{-1}$



- at plateaux $R_{xx} = \sigma_{xx} = 0 \rightarrow$ gap
- Laughlin's series $\nu = \frac{1}{2k+1} = 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ $k = 0, 1, 2, \dots$
- Jain's hierarchy $\nu = \frac{m}{2km+1}$, $m = 1, 2, 3$

Ex: $k=0$ integer QHE

$k=1$, $\nu = \frac{m}{2m+1} = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13} \rightarrow \left(\frac{1}{2}\right)^-$

$\nu = \frac{m}{2m-1} = \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \frac{6}{11}, \rightarrow \left(\frac{1}{2}\right)^+$ "charge conjugate"

→ Add Chern-Simons effective interaction

$$S_{NR} = S_{\substack{\text{LANDAU}\\ \text{LEVELS}}} + \frac{1}{4\pi k}\int d^3x\, A\,dA + JA$$

- each electron is given a magnetic charge $2k$
  → Jain's "composite Fermion"

- effective integer Hall effect → gap
  → mean field theory + fluctuations
  → very nice results
  → simple
  → difficult to improve

(Fradkin, Lopez; Halperin, Lee, Read; Shankar, ..)

- Another idea from strings (Susskind '01)

NR electrons → D0 branes

$\vec{x}_\alpha(t)$, $\alpha = 1, \ldots, N$ | $\vec{X}_{\alpha\beta}(t)$ $N \times N$ matrices

permutation symmetry
$\pi_{\alpha\beta}$: $\alpha \Leftrightarrow \beta$

U(N) gauge symmetry
$X \to U X U^+$

→ U(N) gauge theory in 0+1 dimensions of two Hermitean matrices $\vec{X} = (X_1, X_2)$

→ eigenvalues $\vec{\lambda}_\alpha \approx$ coordinates $\vec{x}_\alpha$
+ additional "angular" variables V, W

$$X_1 = V \Lambda_1 V^+ \quad , \quad X_2 = W \Lambda_2 W^+$$

Gauss law

$$G = i \sum_{i=1}^{2} [X_i, \Pi_i] = \mathbb{1}_N B\theta \qquad \mathbb{1}_N = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & {}_N \end{pmatrix}$$

const. background

→ gauge invariant states should satisfy it;

for $\theta = \frac{1}{2\pi\rho_0}$, it amounts to Jain's relation
flux ↔ density

$$B\theta = k \in \mathbb{Z}, \qquad B = k\, 2\pi \rho_0$$

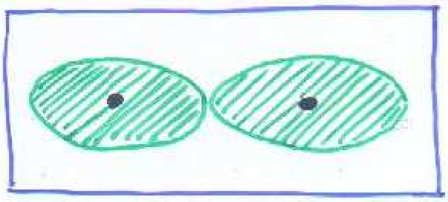
• Two ways to satisfy Gauss law

I. matrix angular variables V, W are constrained and induce a two-body repulsion among eigenvalues of Calogero type

$$V = \sum_{\alpha \neq \beta} \frac{(B\theta)^2}{(\vec{x}_\alpha - \vec{x}_\beta)^2}$$

II. upon projection to Lowest Landau Level

$$\Pi_1 = -\frac{B}{2} X_2, \quad \Pi_2 = \frac{B}{2} X_1 \quad \rightarrow \quad [X_1, X_2] = i\theta$$



noncommutative fields

$$\rho_0 = \frac{N}{A} = \frac{1}{2\pi\theta}$$

Maxwell-Chern-Simons Matrix Theory

$$S = \int dt \, \mathrm{Tr}\left[\frac{m}{2}(D_t X_i)^2 + \frac{B}{2}\varepsilon_{ij} X^i D_t X^j + g\,[X_1, X_2]^2 \right]$$
$$+ \mathrm{Tr}\left[B\theta A_0 \;\; - i\psi^+ D_t \psi \right]$$

- $D_t X_i = \dot{X}_i - i[A_0, X_i]$, $\quad \psi$: auxiliary vector $\psi(t) = \psi_0 = \mathrm{const.}$
- $U(N)$ gauge invariance $\quad X_i \to U X_i U^+$
$$S \to S \; - i B\theta \int dt \, \mathrm{Tr}\left[U^+ \dot{U} \right] \qquad B\theta = k \in \mathbb{Z}$$
(Nair, Polychronakos
- dimensional reduction from 2+1 dim. to 0+1 : D0 branes
- Hamiltonian + Gauss law

$$H = \mathrm{Tr}\left[\frac{B}{m} a^+ a \; - g\,[X_1, X_2]^2 \right],$$
$$a = \frac{B}{4}(X_1 + i X_2) + \frac{i}{2}(\Pi_1 + i\Pi_2)$$
$$G = i\,[X_1, \Pi_1] + i\,[X_2, \Pi_2] - B\theta + \psi\psi^+ \approx 0$$
$$\mathrm{Tr}\, G = 0 \;\; \to \;\; \|\psi\|^2 = B\theta N = kN$$

- two solutions of $G \approx 0$ are realized for $g = 0$ and $g = \infty$, respectively
- parameters: $\frac{B}{m}$, $\frac{g}{m}$; $\quad \underset{\text{fixed}}{B\theta = k} \longleftrightarrow \frac{1}{\nu} = \frac{B}{2\pi\rho_0}$

$g = \infty$ Limit : back to electrons

$$H = \frac{B}{4m} \mathrm{Tr}\left[\left(\pi_1 + \frac{B}{2} X_2\right)^2 + \left(\pi_2 - \frac{B}{2} X_1\right)^2 \right] - g \, \mathrm{Tr}\left[X_1, X_2\right]^2$$

$g = \infty \quad \rightarrow \quad [X_1, X_2] = 0$ Normal Matrices

$$X_1 = U x_1 U^+ , \quad X_2 = U x_2 U^+$$

- $U \in U(N)$ gauge d.o.f. $\rightarrow$ $U = \mathbb{1}$
- $x^i = \mathrm{diag}\,(x^i_\alpha) \quad \alpha = 1, \dots, N$ eigenvalues ≈ coordinates
- $\Pi_i = U (p_i + \Gamma_i) U^+$
  $\nwarrow$ diagonal, conjugate to x^i
- Gauss law : $i[X_1, \Pi_1] + i[X_2, \Pi_2] = \kappa \mathbb{1} - \psi\psi^+$

$$(\Gamma_i)_{\alpha\beta} = i \frac{\kappa}{2} \frac{(x^i_\alpha - x^i_\beta)}{|\vec{x}_\alpha - \vec{x}_\beta|^2} , \quad \alpha \neq \beta \qquad \kappa = B\theta$$

$\rightarrow$ induced interaction $1/|\vec{x}|^2$ 2d Calogero

$$H = \frac{B}{4m} \mathrm{Tr}\left[\underbrace{\left(p_1 + \frac{B}{2} x_1\right)^2 + \left(p_2 - \frac{B}{2} x_2\right)^2}_{\text{diagonal}} + \Gamma_1^2 + \Gamma_2^2 \right] \qquad \sum_{\alpha\neq\beta} \frac{\kappa^2}{|\vec{x}_\alpha - \vec{x}_\beta|^2}$$

$\rightarrow$ back to original problem with $1/|x| \rightarrow 1/|x|^2$

$\rightarrow$ gap is dynamical & nonperturbative

Physical states for $g=0$: Matrix LL

$$H = \frac{B}{m} \mathrm{Tr}[a^\dagger a] \qquad a = \frac{B}{4}(X_1 + iX_2) + \frac{i}{2}(\Pi_1 + i\Pi_2)$$

$$G = i[X_1, \Pi_1] + i[X_2, \Pi_2] + \psi\psi^\dagger - \kappa = 0, \qquad B\theta = \kappa$$

$$\mathrm{Tr}\, G = 0 \rightarrow \psi^\dagger \psi = N\kappa$$

- Landau levels of N^2 "particles" with coordinates $\vec{X}_{\alpha\beta}$
- physical states $G_{\alpha\beta} \Psi(X_1, X_2, \psi) = 0$

  $\rightarrow$ $U(N)$ singlets with $N\kappa$ components ψ_i

- **Claim:** allowed $\rho =$ const. ground states are (matrix extensions of) Laughlin and Jain states; they are all gapful

  $U(N)$ gauge symmetry $\longrightarrow$ "Kinematic" Fractional QHE

- start by filling the lowest Landau level

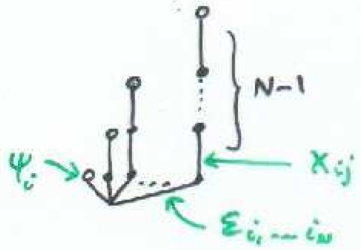
$$a_{\alpha\beta} \Psi(X_1, X_2, \psi) = 0, \qquad \Psi = e^{-\frac{1}{2}\mathrm{Tr}(X^\dagger X)} \varphi(X, \psi)$$

analytic of $X = X_1 + iX_2$

- Solution of Gauss law (Hellermann, Von Ramsdonk '01)

$$\varphi(X,\psi) = \left[\varepsilon_{i_1\dots i_N}\,\psi^{i_1}(X\psi)^{i_2}\dots(X^{N-1}\psi)^{i_N}\right]^k$$

represent it like a tree with different branches



- recover Laughlin wave function by diagonalizing

$$X = V\Lambda V^{-1} \qquad \Lambda = \mathrm{diag}(z_1,\dots,z_N)\,, \quad \psi = V\phi$$

$$\varphi = \Big[\det V \prod_{\alpha<\beta}(z_\alpha - z_\beta)\prod_\alpha \phi_\alpha\Big]^k \propto \prod_{\alpha<\beta}(z_\alpha - z_\beta)^k$$

- semiclassical limit of incompressible fluid with $\rho_0 = \frac{1}{2\pi\theta}$ $\quad \frac{1}{\nu} = \frac{B}{2\pi\rho_0} = B\theta + 1 = k+1$ (Susskind) NC Chern-Simons

- states with higher density (higher ν) in LLL are not physical $\quad [X_1, X_2] = i\theta$

  $\rightarrow$ "kinematic" repulsion

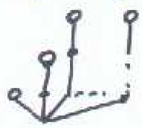
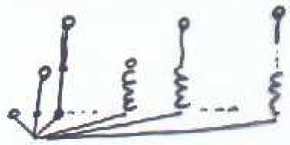
- quasi-particles have gap $\omega = \frac{B}{m}$

• Jain's ground states $\quad \frac{1}{\nu} = k + \frac{1}{m}$

start to fill higher Landau levels to achieve higher densities

• II LL : $\quad (a_{\alpha\beta})^2 \Psi = e^{-\frac{1}{2}\mathrm{Tr}(X^+X)} \left(\frac{\partial}{\partial X^+_{\alpha\beta}}\right)^2 \varphi = 0$

$\varphi(X, X^+, \psi)$ at most linear in $X^+_{\alpha\beta}$, $\forall \alpha, \beta$

$\varphi = \Big[$  $\Big]^{2K-1} \cdot$  $\leftarrow \frac{N}{2}$ matrices $X^+_{\alpha\beta}$

→ upon diagonalization, it is a Slater determinant of II LL filling as hypotized by Jain

$$\frac{1}{\nu} = 2k + \frac{1}{2} \quad , \quad E_0 = \frac{B}{m} \cdot \frac{N}{2} \quad , \quad \text{gap} = \frac{B}{m}$$

• analysis extends to filling III LL and higher

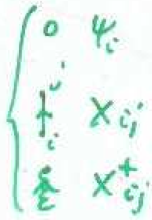
• full Maxwell-Chern-Simons theory is "weakly" non commutative

$$G = \frac{B}{2}[X, X^+] + [X^+, a] + [a^+, X] - B\theta + \psi\psi^+ = 0$$

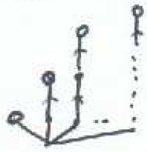
it can vanish on higher LL w. $a, a^+ \neq 0$

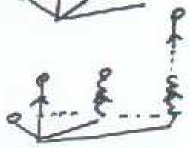
this happens for $g \to \infty$ that forces $[X, X^+] \to 0$

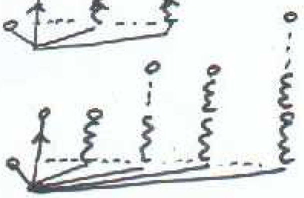
- "generalized" Jain hierarchy



$$\varphi_{\{p_1, p_2 \cdots\}} = \varphi_{p_1} \cdot \varphi_{p_2} \cdots\cdots$$

$\nu^* = 1 \quad \varphi_1 =$  $\rightarrow \prod (z_\alpha - z_\beta)$ filled 1st LL

$\nu^* = 2 \quad \varphi_2 =$  $\rightarrow$ filled 1st & 2nd

$\nu^* = 3 \quad \varphi_3 =$  $\rightarrow$ quadratic in X^+_{ij}

first 3 LL filled

$\vdots$

- Jain fillings $\frac{1}{\nu} = k + \frac{1}{m}$ are $\varphi_{\{k-1, m\}} = (\varphi_1)^{k-1} \varphi_m$
- any product of k blocks $\varphi_{p_1} \cdots \varphi_{p_k}$ is possible : it has increasingly higher energy E_0 and higher density

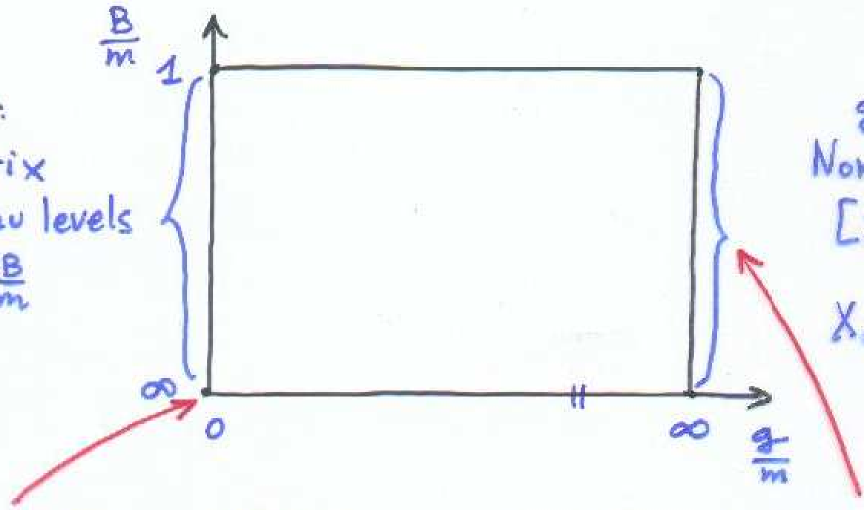
$$\frac{1}{\nu} = 1 + \sum_{i=1}^{k} \frac{1}{p_i} < k + \frac{1}{m}$$

- higher densities are far from semiclassical limit of incompressible fluids $\frac{1}{\nu} = k+1 = B\theta + 1$
  - $\rho \neq$ const ?
  - additional d.o.f. in the fluid?
  - . . . .

Phase diagram

$$H = \mathrm{Tr}\left[\frac{B}{m} a^{+} a - g\,[X_1, X_2]^2\right]$$

g=0:
Matrix
Landau levels
$\omega = \frac{B}{m}$



$g = \infty$:
Normal matrices
$[X_1, X_2] = 0$

$X_i \to U x_i U^{+}$

LLL: $B \gg m$, $a \equiv 0$
Chern-Simons Matrix Model

$$S = \int \mathrm{Tr}\left[\frac{B}{2}\varepsilon_{ij} X^i D_t X^j + B\theta A_0\right]$$

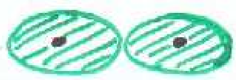
(Susskind, Polychronakos)

$G \approx 0 \quad [X_1, X_2] = i\theta$

$\rho_0 = \frac{1}{2\pi\theta}$ incompressible fluid

$\frac{1}{\nu} = \frac{B}{2\pi\rho_0} = B\theta = k$

Laughlin filling



"Kinematic" repulsion

complete reduction to eigenvalues

$$G = i[X_1, \Pi_1] + i[X_2, \Pi_2] - B\theta$$

$$(\Pi_i)_{\alpha\beta} = \frac{iB\theta(x^i_\alpha - x^i_\beta)}{|\vec{x}_\alpha - \vec{x}_\beta|^2}$$

Induced interaction

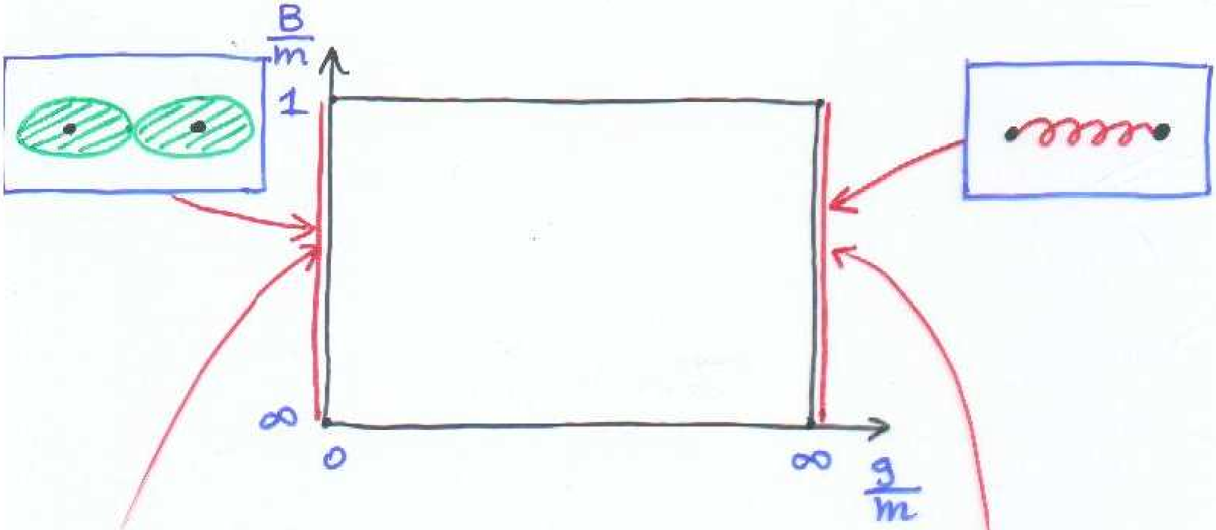
$$H = \mathrm{Tr}(\Pi_1^2 + \Pi_2^2) + \ldots$$
$$= \sum_{\alpha \neq \beta} \frac{(B\theta)^2}{|\vec{x}_\alpha - \vec{x}_\beta|^2} + \ldots$$

gap is dynamical

interaction

Conjecture on Maxwell-Chern-Simons MM

$$H = \mathrm{Tr}\left[\tfrac{B}{m} a^{+} a - g\,[X_1, X_2]^2\right]$$



g=0 Matrix QHE

- all expected states with $\rho = \text{const}$ & gap
- N^2 d.o.f.

g=∞ physical QHE

- $[X_1, X_2] = 0 \rightarrow$ eigenvalues
- Calogero interaction $\approx$ Coulomb inter.

- As $g: 0 \rightarrow \infty$, Kinematic repulsion is replaced by Calogero interaction; matrix angular d.o.f. projected out

Conjecture

As $g: 0 \rightarrow \infty$, gapful $\rho = \text{const.}$ ground states prepared at $g=0$ remain gapful for all g values and have smooth $g=\infty$ limit

Conjecture: no phase change for $0 < g \leq \infty$

for densities that admit gapful $\rho = \text{cost}$ ground states near $g \sim 0$

- Problem: find method to analyze interaction $g\,\mathrm{tr}([X_1, X_2]^2)$

- Maxwell-Chern-Simons matrix theory could provide another effective non-relativistic theory of fractional QHE

- It generalizes the Chern-Simons matrix theory (Susskind, Polychronakos, ...) that was too much constrained (in particular, g interaction is meaningless)