SLE & QUANTUM GRAVITY

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Service de Physique Théorique de Saclay

Thematic program

SLE 06

KITP

University of California at Santa Barbara

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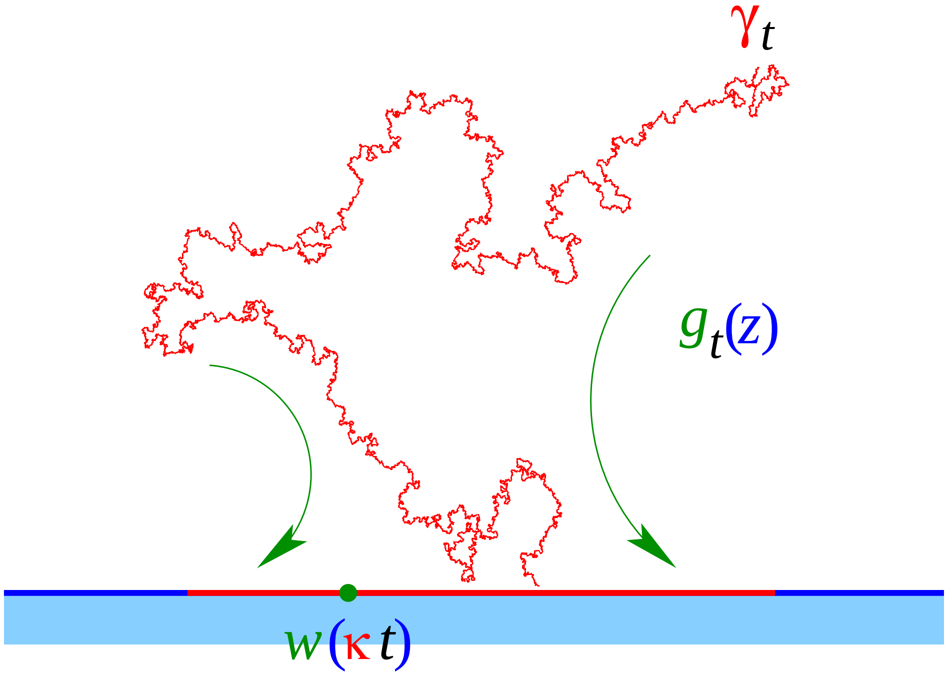
Stochastic Löwner Evolution (SLE$_\kappa$) (SCHRAMM)

SAW in half plane - 1,000,000 steps



$$\partial_t g_t(z) = \frac{2}{g_t(z) - w(\kappa t)}$$

A Stochastic Löwner Evolution?

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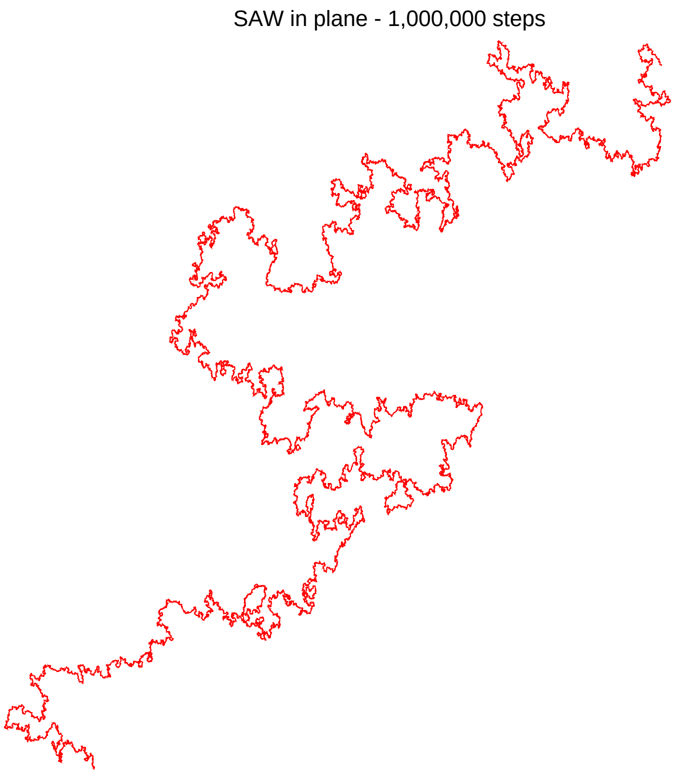
$$\partial_t g_t(z) = \frac{2}{g_t(z) - w(\kappa t)}$$

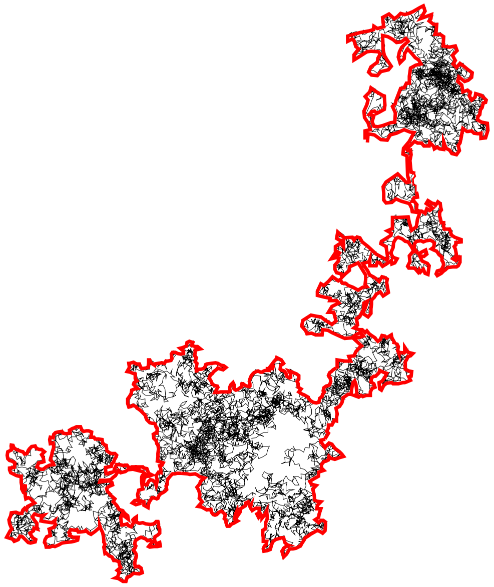
Self-Avoiding Walk & SLE(8/3)



(Courtesy of T. Kennedy)

B. Nienhuis (1982): $D = \frac{4}{3}$, $\langle r^2 \rangle \propto N^{3/2} a^2$

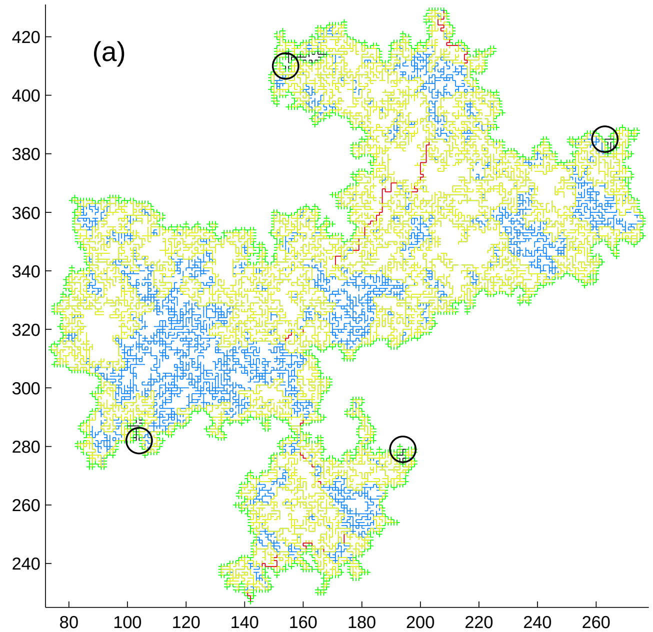
Planar Brownian Frontier



Mandelbrot conjecture (1982): Hausdorff dimension $D = \frac{4}{3}$.
In the plane, the Brownian frontier is the scaling limit of a self-avoiding walk.

B. D., 1998 (quantum gravity); G. F. Lawler, O. Schramm & W. Werner, 2000 (SLE). [Percolation External Perimeter: *M. Aizenman, B. D. & A. Aharony, 1999; S. Smirnov; LSW, 2001; V. Beffara, 2002.*]

Percolation Cluster Hull & Frontier



(J. Asikainen et al., 2003)

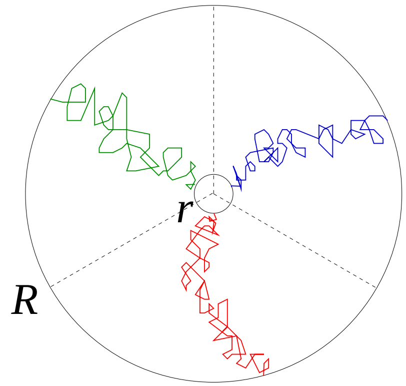
Hull: $D_{\mathrm{Hull}} = \frac{7}{4}$ *(B. D. & H. Saleur, 1987; S. Smirnov, 2001; V. Beffara, 2002; F. Camia & C. M. Newman, 2003, 2005);*
External Perimeter: $D_{\mathrm{EP}} = \frac{4}{3}$ *(M. Aizenman, B. D. & A. Aharony, 1999; LSW, 2001; Beffara, 2002;)* [DUALITY].

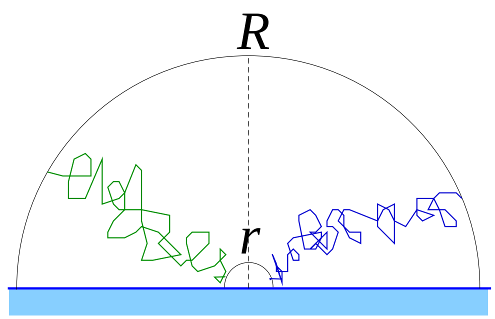
INTERSECTIONS OF BROWNIAN PATHS

A question by Michael Aizenman (1984)...

...and the promess of a good bottle of wine!

Intersections of Random Walks





Probability that L non-intersecting Brownian paths altogether traverse the (half-) annulus $\mathbb{D}(r,R)$ in $\mathbb{C}$ ($\mathbb{H}$) from the inner boundary circle of radius r to the outer one at distance R

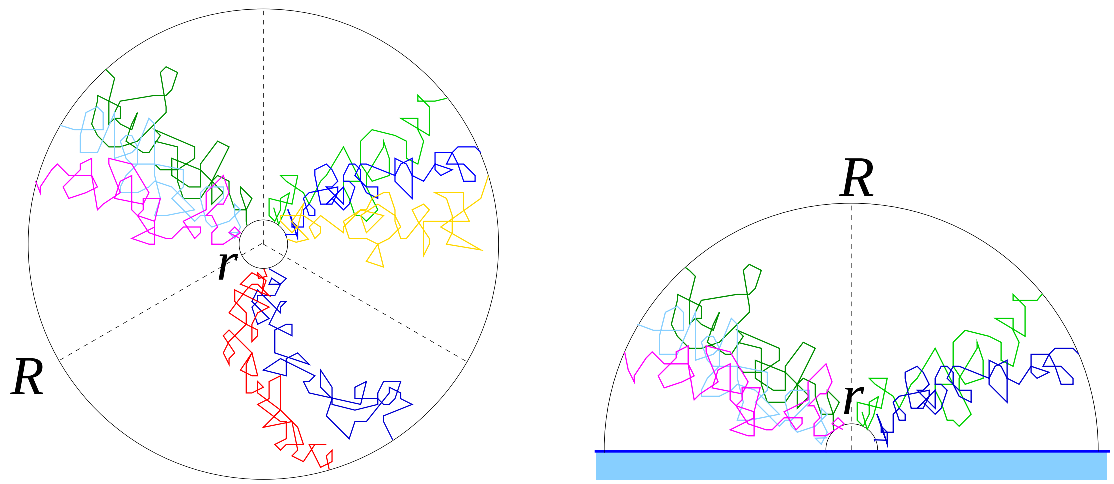
$$P_L(R) \approx (r/R)^{2\zeta_L} \quad , \quad \tilde{P}_L(R) \approx (r/R)^{\tilde{\zeta}_L}$$

$$\zeta_L = \frac{1}{24}\left(4L^2 - 1\right) \quad , \quad \tilde{\zeta}_L = \frac{1}{3}L(2L+1)$$

D. & Kwon, 1988

Intersections of Packets of Paths

(W. Werner, 97)



Non-intersection probability of packets $\ell = 1, \cdots, L$, of n_ℓ independent Brownian paths :

$$P_{n_1,\cdots,n_L}(r) \approx (r/R)^{2\zeta(n_1,\cdots,n_L)}, \ \tilde{P}_{n_1,\cdots,n_L}(r) \approx (r/R)^{\tilde{\zeta}(n_1,\cdots,n_L)}$$

L simple paths : $n_1 = \cdots = n_L = 1$.

Cascade Relations

$$\begin{cases} \tilde{\zeta}(n_1, \cdots, n_L) = U\left(\sum_{\ell=1}^{L} U^{-1}(n_\ell)\right) \\ \zeta(n_1, \cdots, n_L) = V\left(\sum_{\ell=1}^{L} U^{-1}(n_\ell)\right) \end{cases}$$

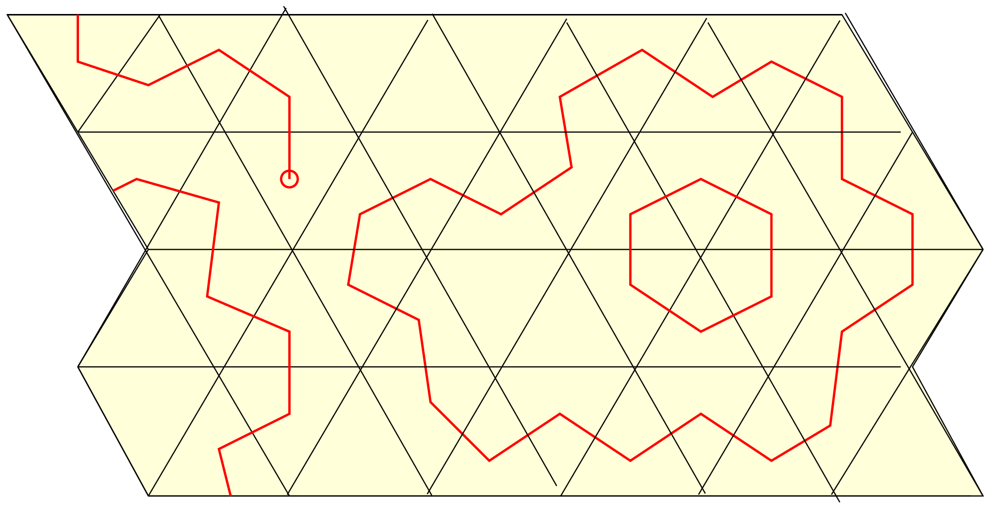
- *Lawler & Werner (98) :* Conformal invariance of Brownian motion

$$\begin{cases} U(L) = \tilde{\zeta}_L \quad \left\{= \frac{1}{3}L(1+2L)\right\} \\ V(L) = \zeta_L \quad \left\{= \frac{1}{24}(4L^2-1)\right\} \\ U^{-1}(n) = \frac{1}{4}(\sqrt{24n+1}-1) \end{cases}$$

- *B.D. (98) :* Interpretation and calculation in terms of "quantum gravity"
- *LSW (01) :* SLE_6

2D Quantum Gravity

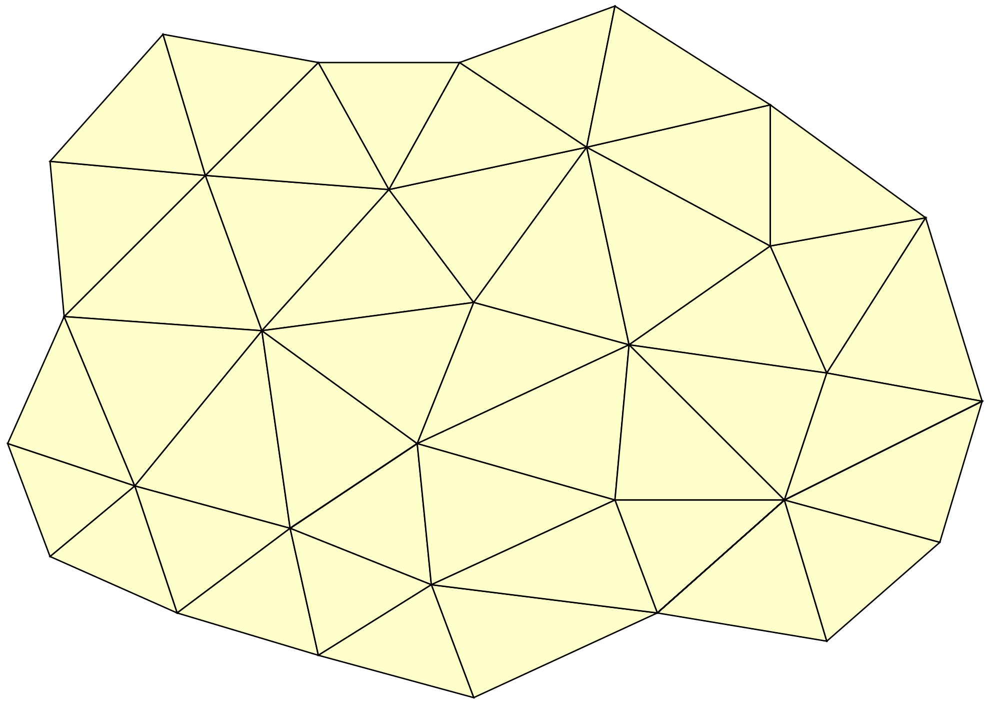
Statistical Mechanics on a Regular Lattice



(Courtesy of I. Kostov)

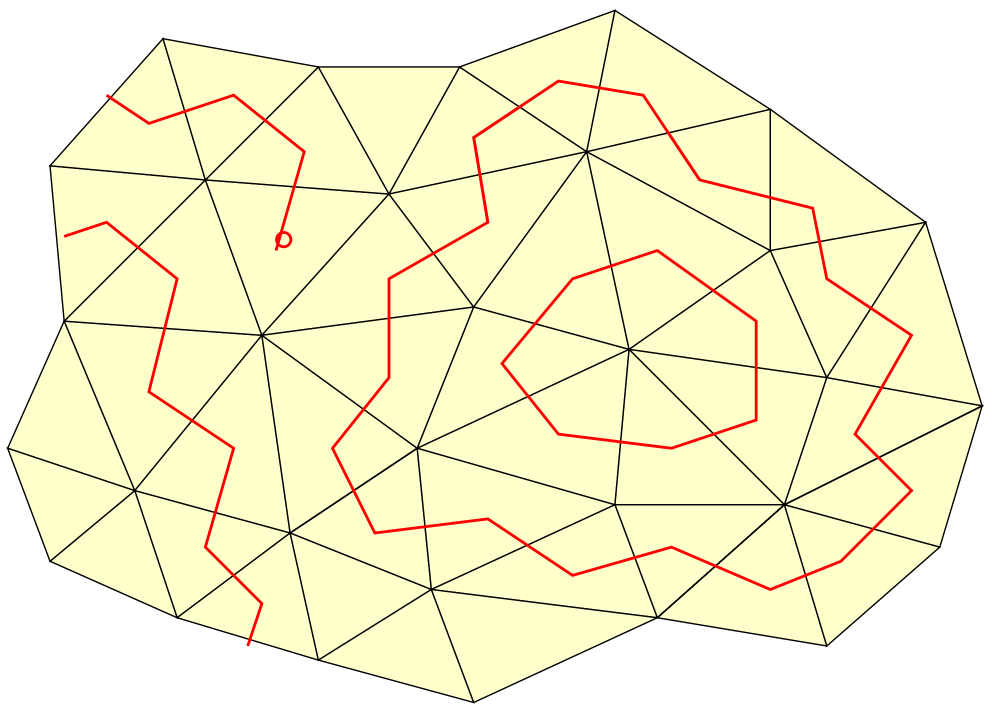
Random lines on the (dual of) a regular triangular lattice.

Randomly Triangulated Lattice



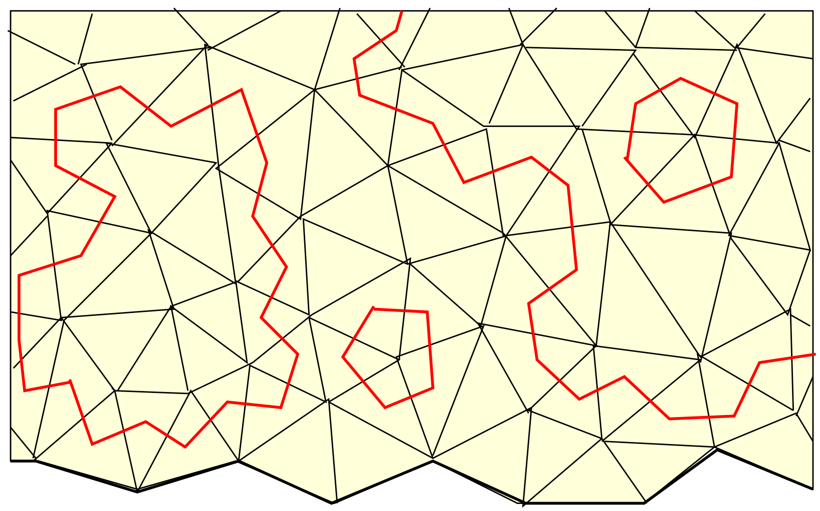
A random planar triangular lattice.

Statistical Mechanics on a Random Lattice



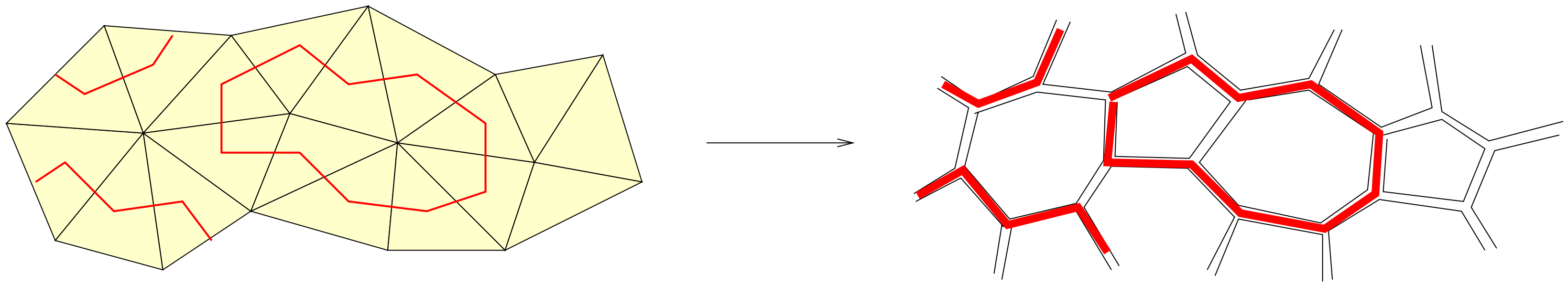
Statistical model on a random planar triangular lattice.

Boundary Effects



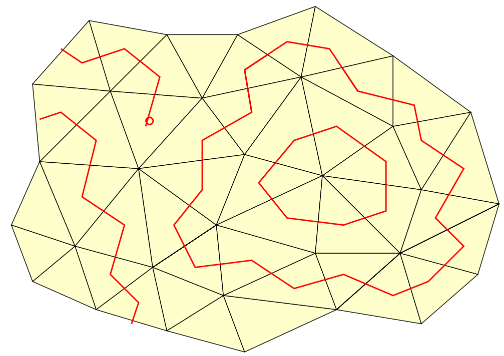
Dirichlet boundary conditions on a random disk.

Dual Lattice



Random loops on the dual random lattice made of "φ^3" trivalent vertices

Partition Function on a Random Lattice



Statistical model $\mathcal{M}$ on random lattice G.

$$Z(\beta) = \sum_{\text{planar } G} e^{-\beta|G|} \, \mathcal{Z}_G$$

$\mathcal{Z}_G$: partition function of the statistical model $\mathcal{M}$ on G.

DOUBLE CRITICAL POINT of $\mathcal{M}$ & G

$$Z(\beta) \sim (\beta - \beta_c)^{2-\gamma(c)}$$

The string susceptibility exponent γ depends on $\mathcal{M}$ through c

Double Critical Behavior

$\gamma(c)$ is related to the "central charge" c of the CFT describing the statistical model by

$$c = 1 - 6\gamma^2/(1-\gamma)\,,\ \gamma \leqslant 0$$

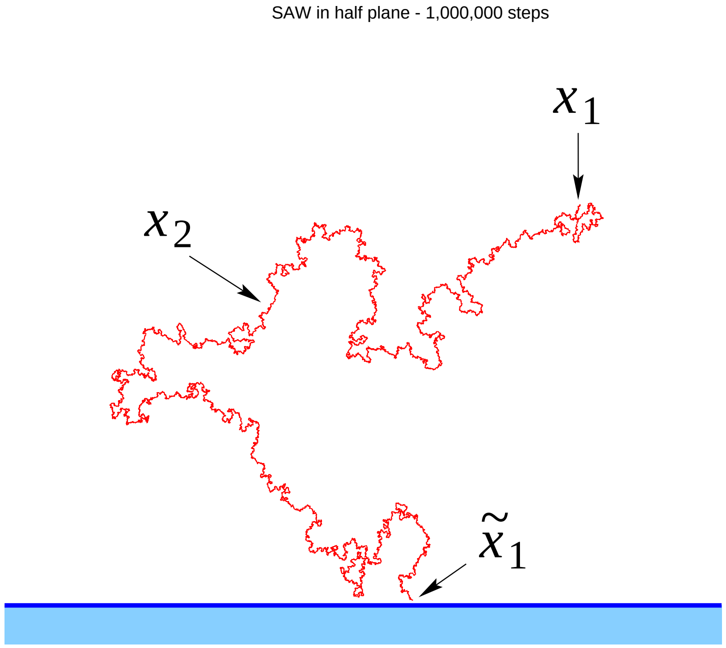
SLE_κ, $0 \leqslant \kappa \leqslant +\infty$

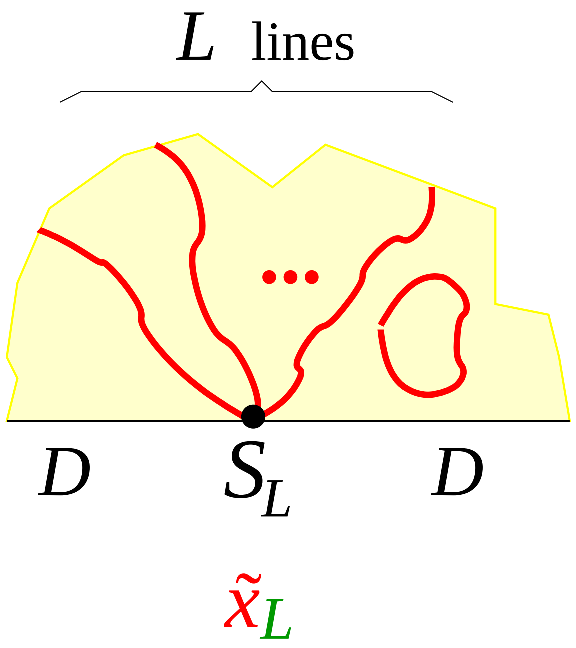
$$c = \frac{1}{4}(6-\kappa)\left(6 - \frac{16}{\kappa}\right)$$

$$\gamma = 1 - \frac{4}{\kappa},\ \kappa \leqslant 4\,, \quad \gamma = 1 - \frac{\kappa}{4},\ 4 \leqslant \kappa$$

Symmetric under *duality:* $\kappa \to \kappa' = 16/\kappa$

Conformal Weights of a Random Path in $\mathbb{C}$ or $\mathbb{H}$





Critical Behavior

Partition functions in $\mathbb{C}$ or $\mathbb{H}$

$$\mathcal{Z} \propto \left(\frac{r}{R}\right)^{2x}, \quad \tilde{\mathcal{Z}} \propto \left(\frac{r}{R}\right)^{\tilde{x}}$$

Partition functions in QG

$$Z \propto \langle |G| \rangle^{-\Delta}, \quad \tilde{Z} \propto \langle |\partial G| \rangle^{-\tilde{\Delta}}$$

Typical lattice area $\langle |G| \rangle$, boundary length $\langle |\partial G| \rangle$:

$$\langle |G| \rangle \sim \langle |\partial G| \rangle^2 \sim (\beta - \beta_c)^{-1}$$

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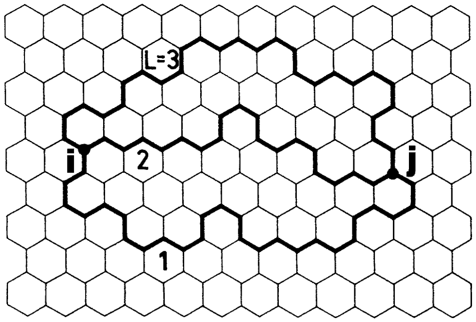
Example:

SAWs & Quantum Gravity

Infinite Measure on SAWs

B. Nienhuis (1982); B. D. & H. Saleur (1986)



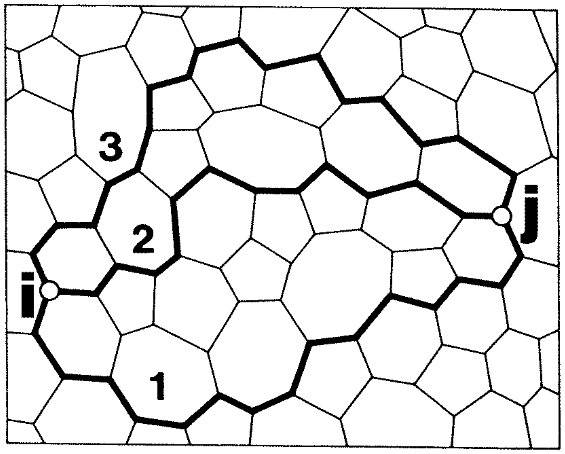
Multiple self-avoiding walks $\Gamma_{ij}^{(\ell)}, \ell = 1, .., L$ from i to j

$$\mathcal{Z}_L = \sum_{\substack{\Gamma_{ij}^{(\ell)} \\ \ell=1,\dots,L}} \mu_{\text{hex}}^{-|\Gamma|} \propto |i-j|^{-4x_L}$$

with a critical fugacity μ_{hex}^{-1} associated with the total length $|\Gamma|$ of the walks, and a conformal weight x_L.

SAWs on a Random Lattice

B. D. & I. Kostov (1988)

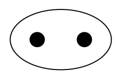


$L = 3$ *mutually- & self-avoiding walks*

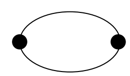
$$Z_L(\beta,\mu) = \sum_{\text{planar } G} e^{-\beta|G|}\left[e^{-\tilde{\beta}|\partial G|}\right] \sum_{i,j\in G[\partial G]} \sum_{\substack{\Gamma_{ij}^{(\ell)} \\ \ell=1,\dots,L}} \mu^{-|\Gamma|}$$

Conformal Weights

Partition function Z_L of a (doubly punctured) sphere with two *conformal operators* of conformal weights Δ_L (here sources of L mutually-avoiding walks)

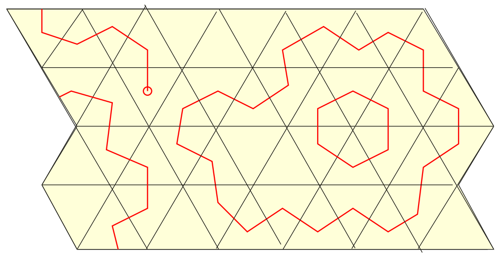
$$Z_L \sim Z[$$  $$] \star |G|^{-2\Delta_L} \sim |G|^{\gamma-2\Delta_L}$$

Boundary partition function $\tilde{Z}_L$ of a (doubly punctured) disk with two *boundary operators* of conformal weights $\tilde{\Delta}_L$

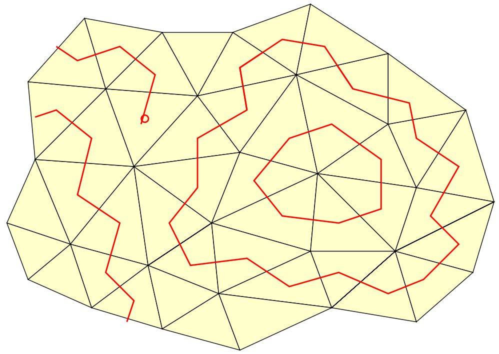
$$\tilde{Z}_L \sim Z($$  $$) \star |\partial G|^{-2\tilde{\Delta}_L}$$

$$\langle |G| \rangle \sim \langle |\partial G| \rangle^2 \sim (\beta - \beta_c)^{-1}$$

KPZ *Knizhnik, Polyakov, Zamolodchikov (88)*



A "conformal operator" O (e.g. creating a line extremity) has conformal weight $x = U(\Delta)$ in $\mathbb{C}$ (or $\tilde{x} = U(\tilde{\Delta})$ in $\mathbb{H}$)



where Δ (or $\tilde{\Delta}$) is the corresponding conformal weight in quantum gravity (or boundary Q. G.)

KPZ: Fundamental quadratic relation between the conformal dimensions Δ on a random planar surface and those x in $\mathbb{C}$ or $\mathbb{H}$

$$x = U(\Delta) = \Delta \frac{\Delta - \gamma}{1 - \gamma}$$

Inverse KPZ map

$$\Delta = U^{-1}(x) = \frac{1}{2}\left(\sqrt{4(1-\gamma)x + \gamma^2} + \gamma \right)$$

SLE & KPZ

Conformal dimensions Δ in (boundary) QG and x in $\mathbb{C}\,(\mathbb{H})$

$$x = U(\Delta) = \frac{1}{4}\Delta\left(\kappa\Delta + 4 - \kappa\right)$$

Inverse KPZ map

$$\Delta = U^{-1}(x) = \frac{1}{2\kappa}\left(\sqrt{16\kappa x + (\kappa - 4)^2} + \kappa - 4\right)$$

(duality $\kappa \to 16/\kappa$)

Duality & KPZ

Dual conformal dimensions Δ, Δ' in QG

$$x = U_\kappa(\Delta) = \Delta \times \frac{1}{4}(\kappa\Delta + 4 - \kappa) = \Delta \times \Delta'$$

Inverse KPZ κ-map

$$\begin{aligned} U_\kappa^{-1}(x) &= \frac{1}{2\kappa}\left(\sqrt{16\kappa x + (\kappa - 4)^2} + \kappa - 4\right) \\ &= \Delta \quad (\kappa \leqslant 4) \quad \text{or} \quad \Delta' \quad (\kappa \geqslant 4) \\ x &= U_\kappa^{-1}(x) \times U_{16/\kappa}^{-1}(x) \end{aligned}$$

Duality in Percolation

Hull & External Perimeter Dimensions

$$D_{\text{Hull}} = \frac{7}{4} \qquad \geqslant \frac{3}{2}$$

$$\left(D_{\text{Hull}} - 1\right)\left(D_{\text{EP}} - 1\right) = \frac{1}{4}$$

$$D_{\text{EP}} = \frac{4}{3} \qquad \leqslant \frac{3}{2}$$

SLE Duality

$$D_{\mathrm{EP}}(\kappa) \;=\; D_{\mathrm{H}}(\kappa),\ \kappa \leqslant 4$$

$$D_{\mathrm{EP}}(\kappa) \;=\; D_{\mathrm{H}}(\kappa' = 16/\kappa), \quad \kappa \geqslant 4$$

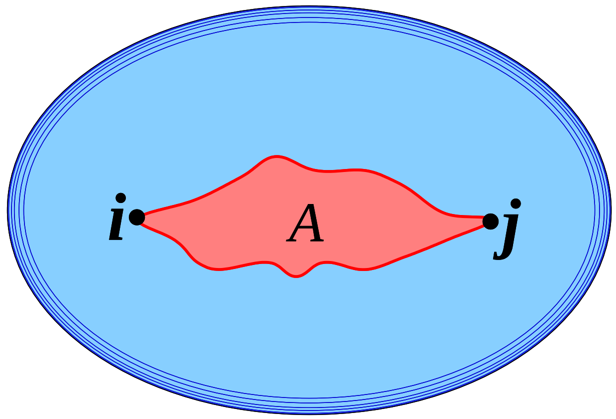
$$\frac{1}{4} \;=\; [D_{\mathrm{EP}}(\kappa) - 1][D_{\mathrm{H}}(\kappa) - 1]$$

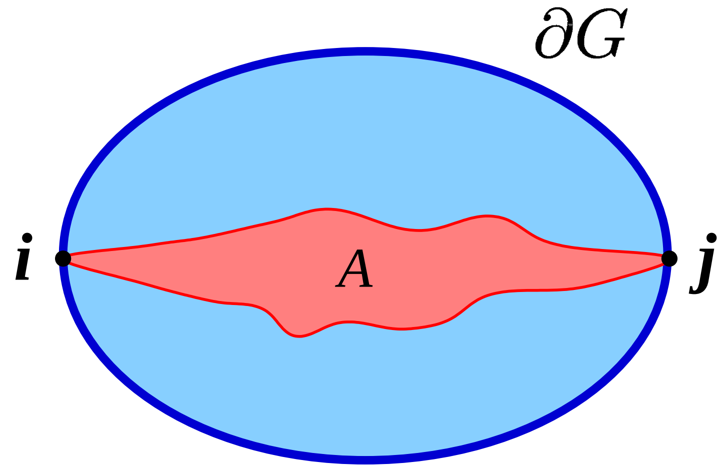
Duality: the external perimeter of $\mathrm{SLE}_{\kappa\geqslant 4}$ is the simple path of $\mathrm{SLE}_{[(16/\kappa)\leqslant 4]}$

Life is Easier in QG

...and in Santa Barbara!

I • Bulk-Boundary Conformal Weights Relation

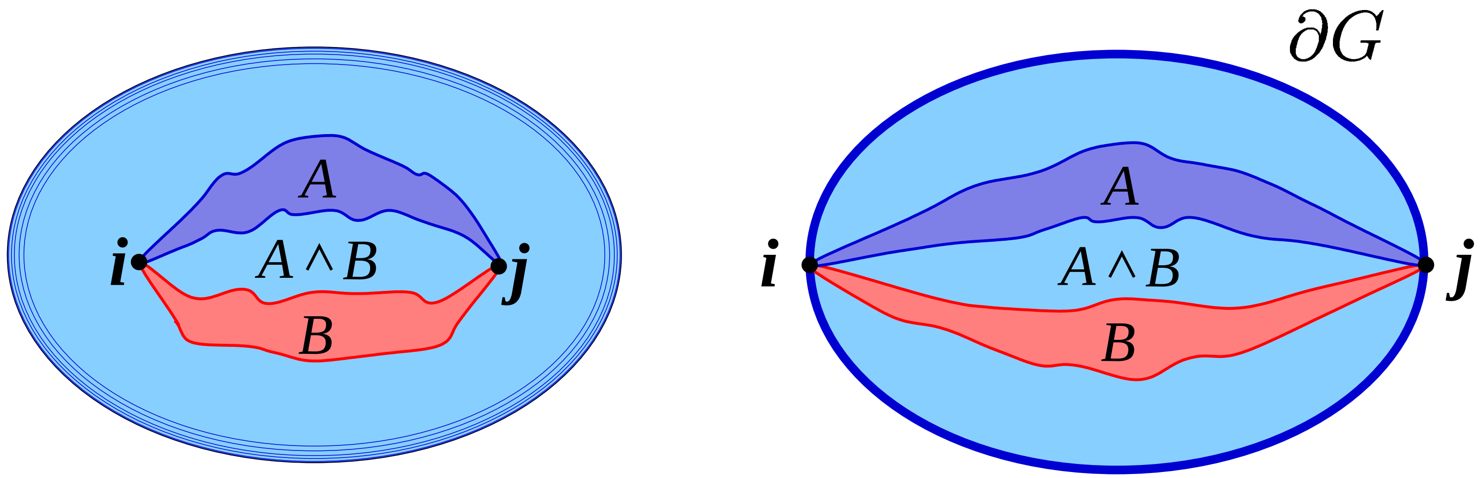




Δ_A $\tilde{\Delta}_A$

$$2\Delta_A - \gamma = \tilde{\Delta}_A$$

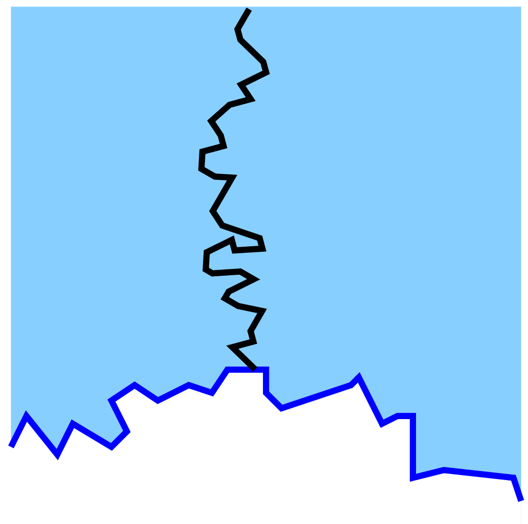
II • QG Boundary Additivity & Mutual Avoidance



$A \wedge B$: random sets A & B avoid each other

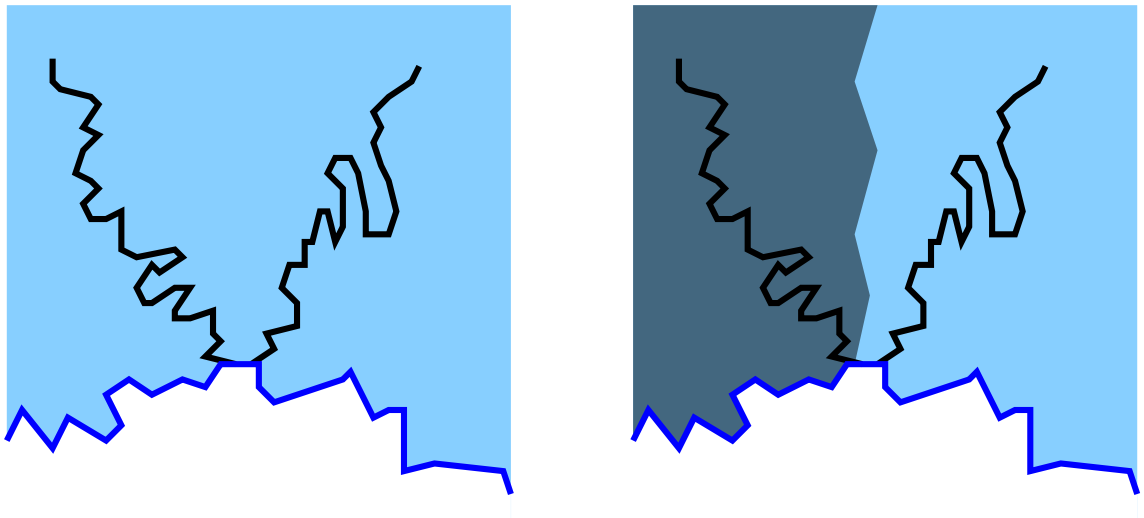
$$2\Delta_{A\wedge B} - \gamma = \tilde{\Delta}_{A\wedge B} = \tilde{\Delta}_A + \tilde{\Delta}_B$$

SLE & Boundary QG

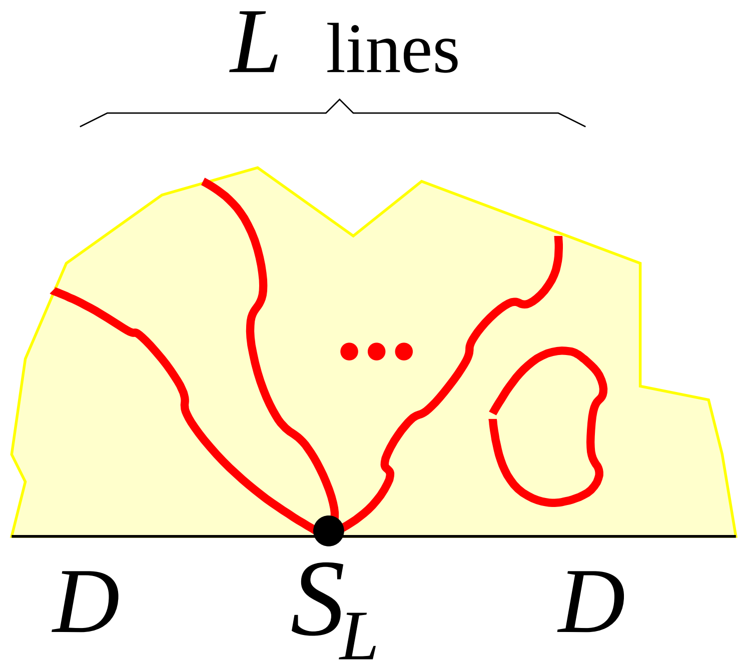


$$\tilde{\Delta}_1 = U_\kappa^{-1}(\tilde{x}_1) = \frac{2}{\kappa}$$

Boundary Quantum Gravity is Additive



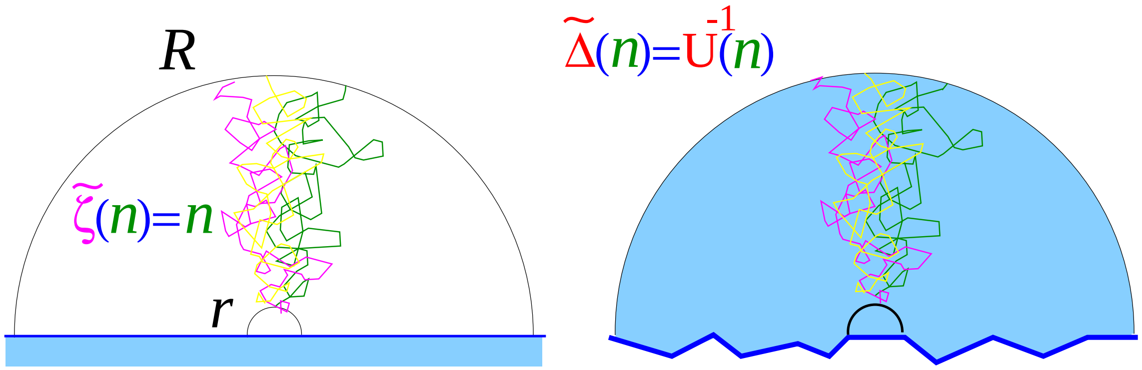
$$U_\kappa^{-1}(\tilde{x}_2) = \tilde{\Delta}_2 = 2\tilde{\Delta}_1 = 2U_\kappa^{-1}(\tilde{x}_1) = \frac{4}{\kappa}$$



An L-star vertex at the Dirichlet boundary, with conformal weight $\tilde{\Delta}_L$ in QG & $\tilde{x}_L = U_\kappa(\tilde{\Delta}_L)$ in $\mathbb{H}$

$$U_\kappa^{-1}(\tilde{x}_L) \;=\; \tilde{\Delta}_L = L\tilde{\Delta}_1 = LU_\kappa^{-1}(\tilde{x}_1) = \frac{2L}{\kappa}$$

Brownian Packet in QG

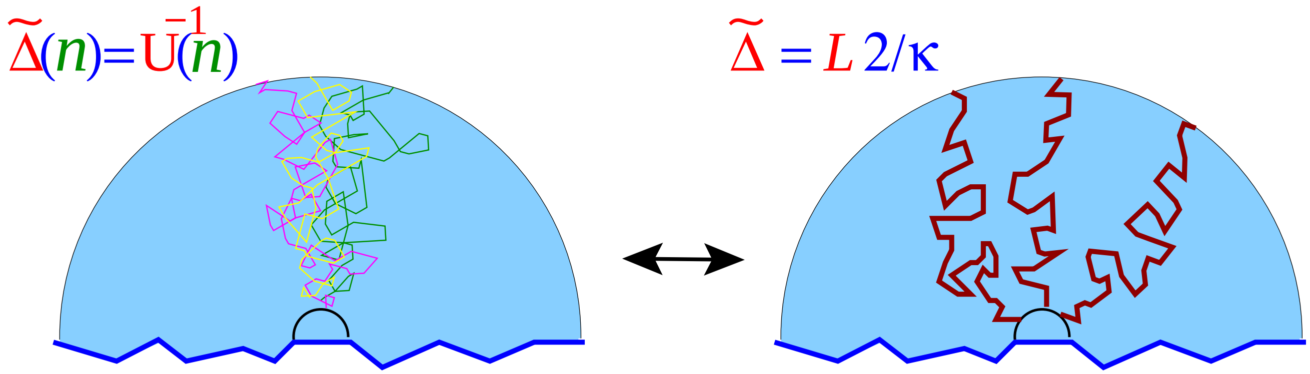


Left: Dirichlet boundary conditions for a packet of n independent Brownian paths in $\mathbb{H}$; *right:* its conformal weight $\tilde{\Delta}$ in boundary QG

$$\tilde{\Delta}(n) = U_{\kappa}^{-1}(n) = \frac{1}{2\kappa}\left(\sqrt{16\kappa n + (\kappa - 4)^2} + \kappa - 4\right)$$

The Brownian paths, independent in a fixed metric, are strongly coupled by the metric fluctuations in quantum gravity.

SLE Transmutation



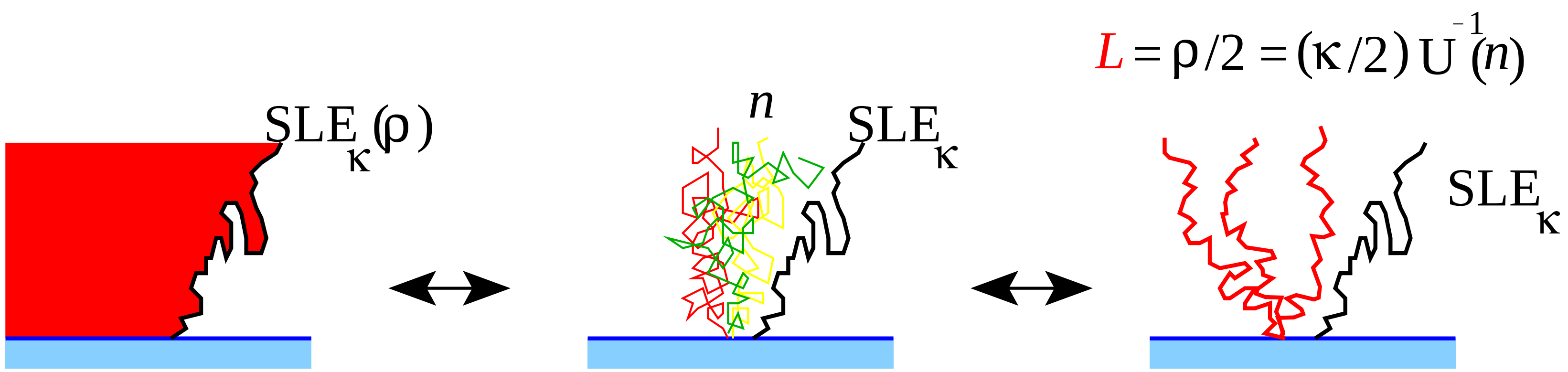
n independent Brownian paths $\Longleftrightarrow$ L mutually-avoiding

SLE paths:

$$L = \frac{U_\kappa^{-1}(n)}{U_\kappa^{-1}(\tilde{x}_1)} = \frac{\kappa}{2} U_\kappa^{-1}(n)$$

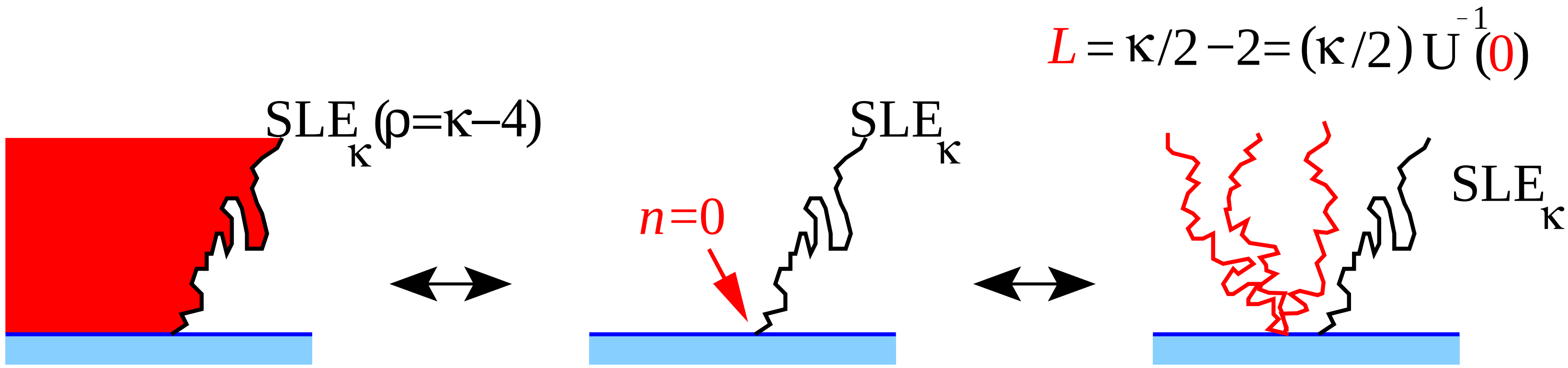
from ADDITIVITY OF BOUNDARY QG

SLE(κ,ρ) & QG



Left: SLE(κ,ρ) *in* $\mathbb{H}$ *(LSW, 2003, Werner, 2004); middle:* SLE(κ) *and its counterpart of* n *independent Brownian paths; right: the counterpart as* L *equivalent* SLE(κ)*s from QG*

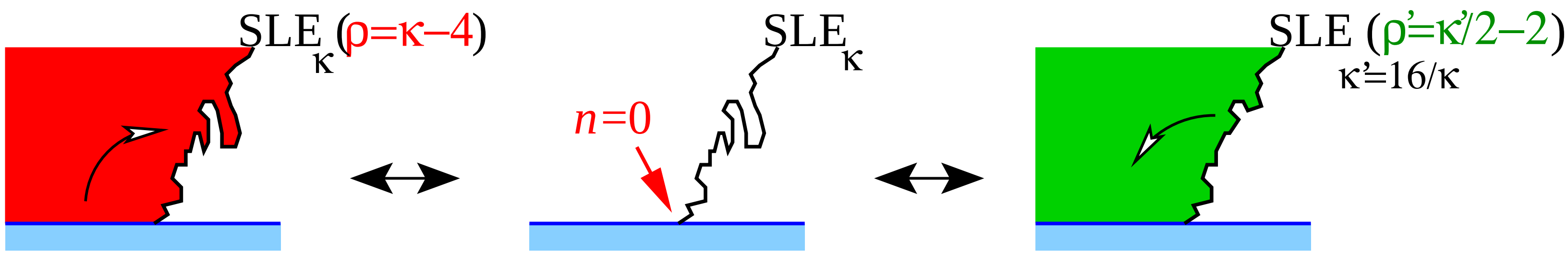
$\mathrm{SLE}(\kappa \geqslant 4, \rho = \kappa - 4)$ *(J. Dubédat, 2005)*



Left: $\mathrm{SLE}(\kappa, \rho = \kappa - 4)$ *conditioned to avoid* $\mathbb{R}^-$; *middle:* $\mathrm{SLE}(\kappa)$ *and* $n = 0$ *Brownian paths (root accessibility); right: the counterpart as* $L = \kappa/2 - 2$ *equivalent* $\mathrm{SLE}(\kappa)$*s from QG*

$$\rho = \kappa U_\kappa^{-1}(0), \quad U_\kappa^{-1}(0) = \theta(\kappa - 4)\left(1 - \frac{4}{\kappa}\right)$$

$\mathrm{SLE}(\kappa \geqslant 4, \rho = \kappa - 4)$ &
$\mathrm{SLE}(\kappa' = 16/\kappa, \rho' = \kappa'/2 - 2)$ *(Dubédat, 2005)*



Left: $\mathrm{SLE}(\kappa \geqslant 4, \rho = \kappa - 4)$ *avoids* $\mathbb{R}^-$*; right: equivalent to* $\mathrm{SLE}(\kappa', \rho' = \kappa'/2 - 2 \leqslant 0)$

$$\mathrm{SLE}(\kappa, \rho = \kappa - 4) \ \& \ \mathrm{SLE}(\kappa' = 16/\kappa, \rho' = \kappa'/2 - 2)$$

$$\rho = \kappa U_\kappa^{-1}(0), \quad U_\kappa^{-1}(0) = \theta(\kappa - 4)\left(1 - \frac{4}{\kappa}\right)$$

$$\tilde{\Delta} = \frac{1}{\kappa}(2 + \rho) = \frac{2}{\kappa} + U_\kappa^{-1}(0) = 1 - \frac{2}{\kappa}$$

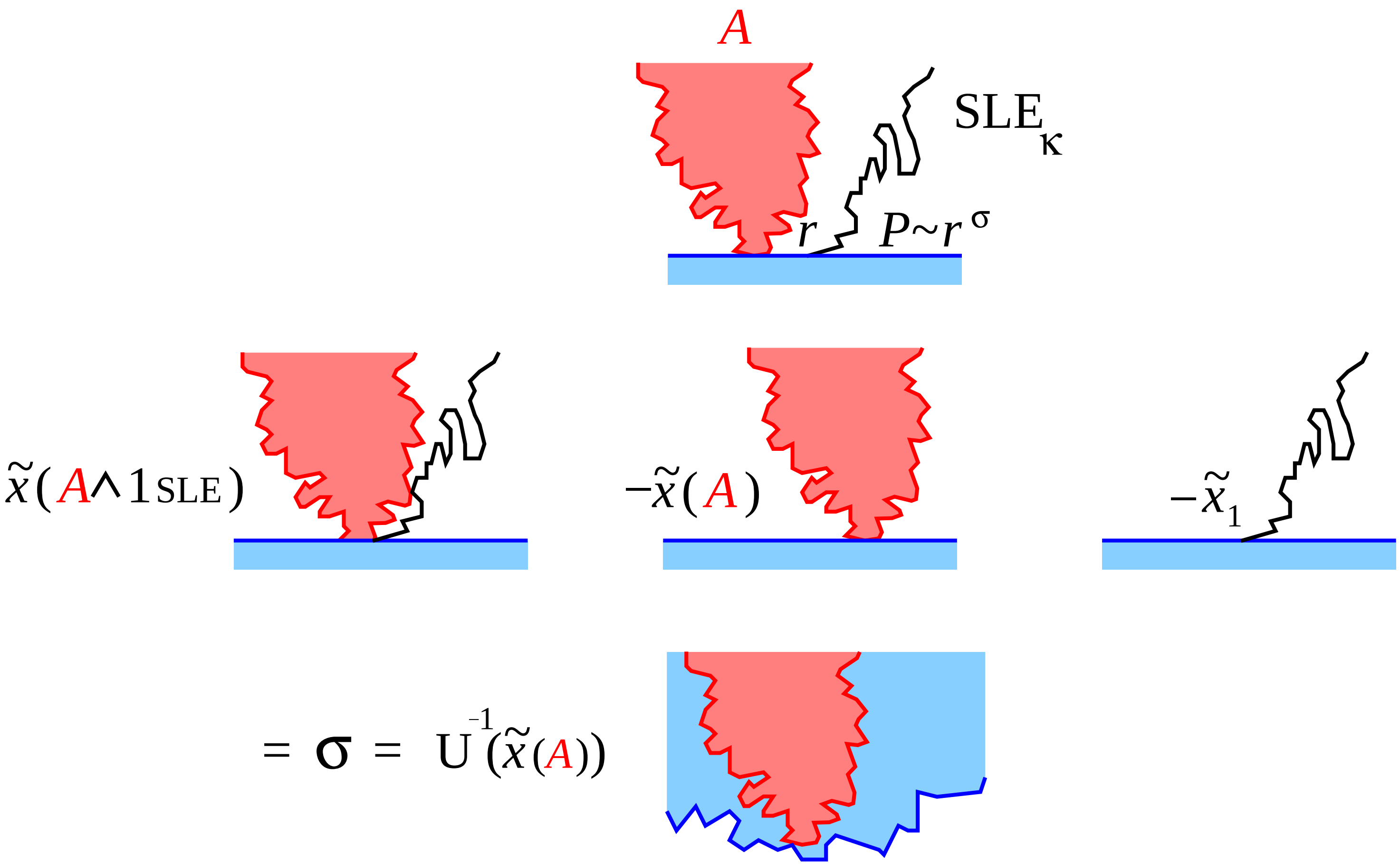
$$\rho' = \frac{\kappa'}{2} - 2, \quad \tilde{\Delta}' = \frac{1}{\kappa'}(2 + \rho') = \frac{1}{2}$$

DUALITY

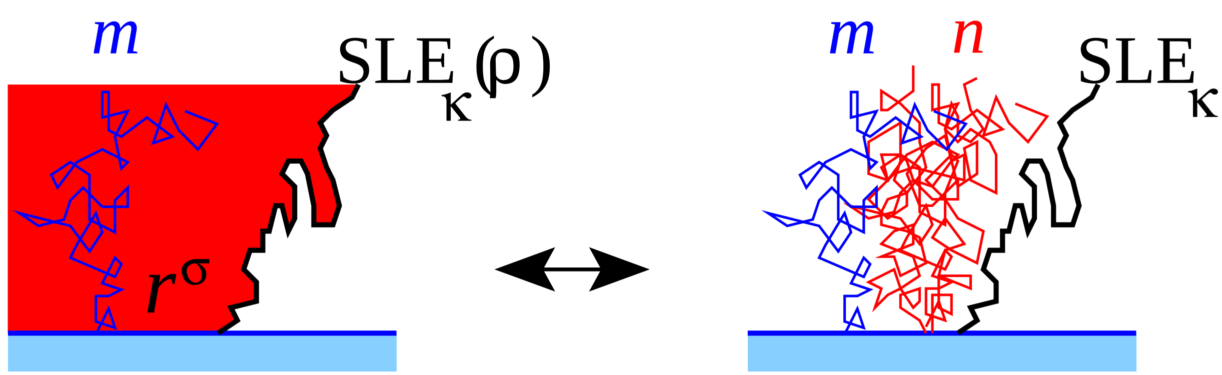
$$\tilde{x} = U_\kappa^{-1}(\tilde{x}) \times U_{16/\kappa}^{-1}(\tilde{x})$$

$$= \tilde{\Delta} \times \tilde{\Delta}' = \left(1 - \frac{2}{\kappa}\right) \times \frac{1}{2}$$

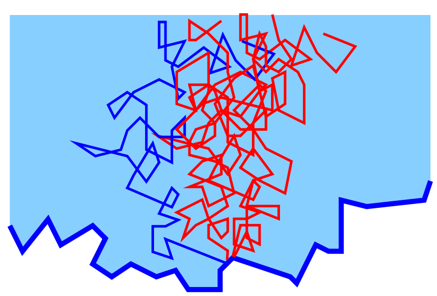
Contact Exponents and QG



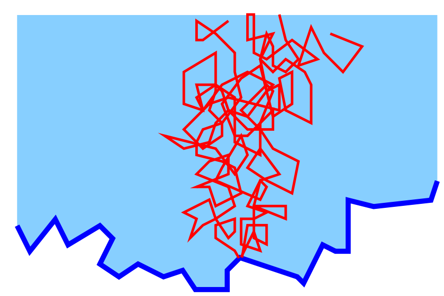
Contact Exponents of SLE(κ, ρ)



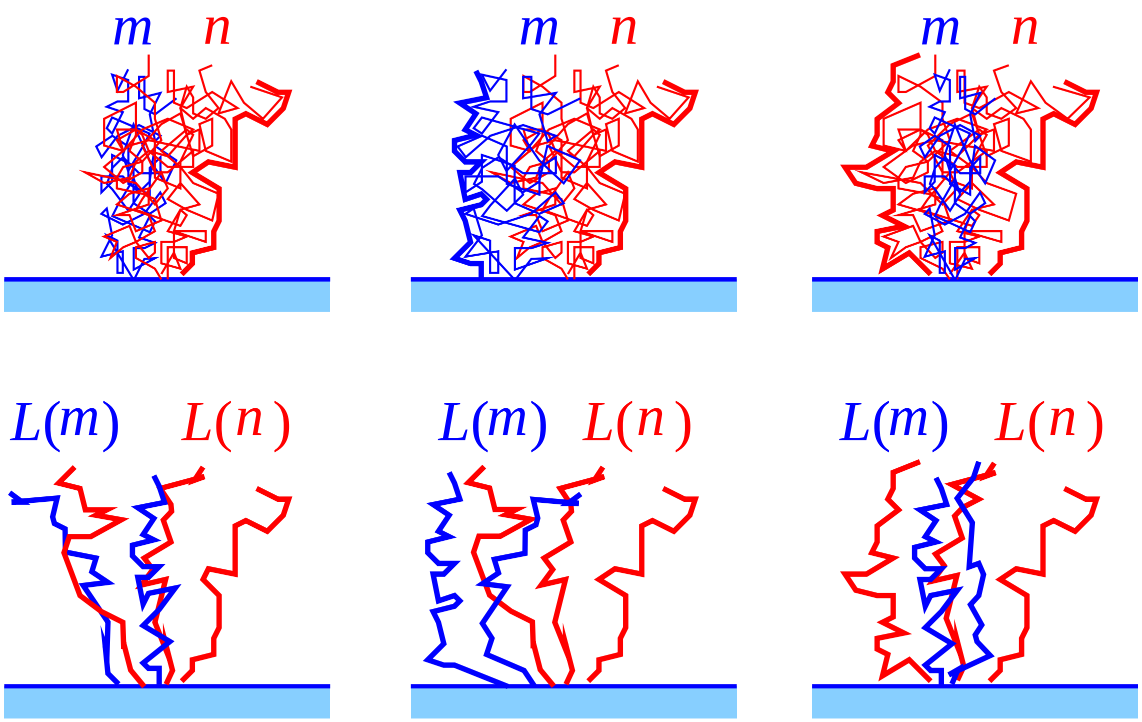
$\sigma = U^{-1}(m+n)$



$- \; U^{-1}(n)$



Brownian Hiding Exponents and SLE(8/3)



$$U^{-1}_{\kappa=8/3}(n) = \frac{2}{\kappa} L(n) = \frac{3}{4} L(n)$$

Hiding Exponents

Combining conformal dimensions $\tilde{\Delta}$ in boundary QG and $\tilde{x}$ in $\mathbb{H}$

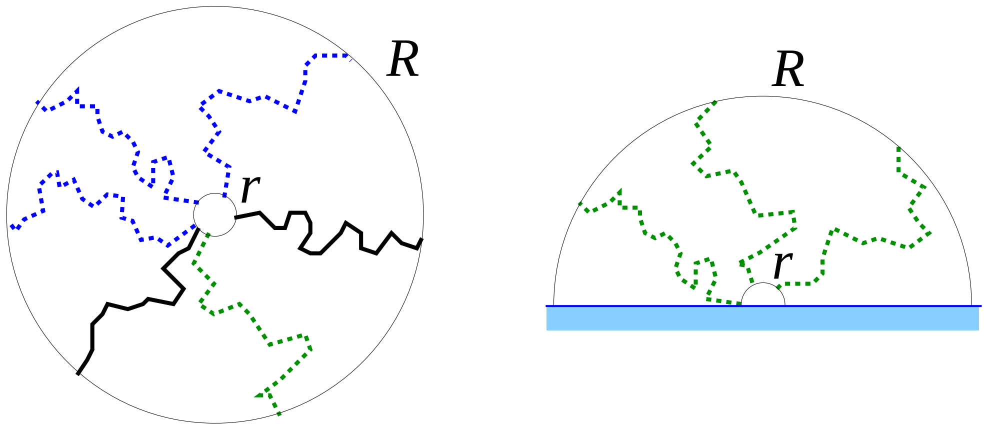
$$\tilde{x}_{m,n} = U\left[\frac{3}{4} + U^{-1}\left[m + U\left(U^{-1}(n) - \frac{3}{4}\right)\right]\right]$$

$$\tilde{x}_{m,n} = m + n + \frac{1}{4}\sqrt{24m + \left(\sqrt{1+24n} - 3\right)^2} - \frac{1}{4}\left(\sqrt{1+24n} - 3\right)$$

PATH CROSSINGS

Another question by Michael Aizenman (1997)...

Dual Paths *(ADA, 1999; Smirnov-Werner, 2001)*



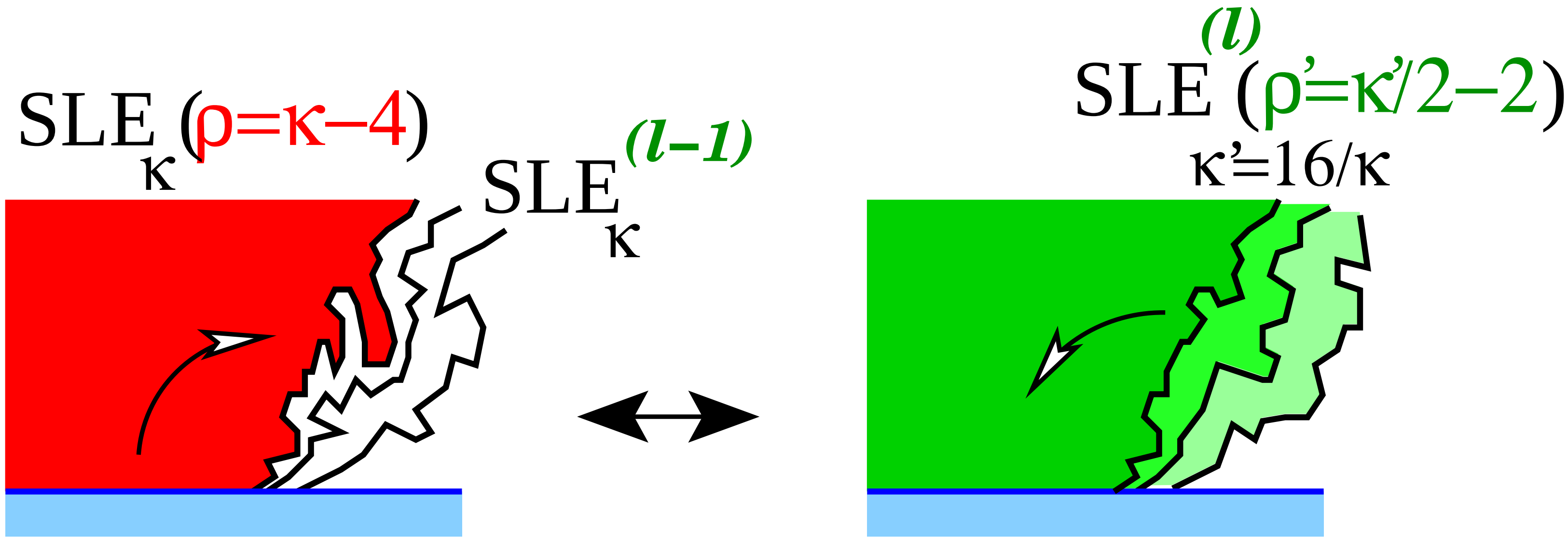
Probability that ℓ non-intersecting monochromatic paths altogether traverse the (half-) annulus $\mathbb{D}(r,R)$ in $\mathbb{C}$ ($\mathbb{H}$) from the inner boundary circle at r to the outer one at R

$$P_\ell(R) \approx (r/R)^{2x_\ell}, \ \tilde{P}_\ell(R) \approx (r/R)^{\tilde{x}_\ell}$$

$$x_\ell^{\mathcal{P}} = x_\ell^{O(N=1)} = \left(\ell^2 - 1\right)/24, \ \tilde{x}_\ell^{\mathcal{P}} = \tilde{x}_{\ell+1}^{O(N=1)} = \ell\left(\ell+1\right)/6$$

$$\mathrm{SLE}(\kappa, \rho = \kappa - 4) \times \mathrm{SLE}^{(\ell-1)}(\kappa)$$

$$\& \quad \mathrm{SLE}^{(\ell)}(\kappa' = 16/\kappa, \rho')$$



Left: $\mathrm{SLE}(\kappa \geqslant 4, \rho = \kappa - 4) \times \mathrm{SLE}^{(\ell-1)}(\kappa)$ *avoids* $\mathbb{R}^-$;
right: *equivalent to* $(\ell - 1)\mathrm{SLE}^{(\ell-1)}(\kappa', \rho' = \kappa'/2 - 2 \leqslant 0)$

$$\mathrm{SLE}(\kappa, \rho = \kappa - 4) \times$$
$$\mathrm{SLE}^{(\ell-1)}(\kappa) \;\&\; \mathrm{SLE}^{(\ell)}(\kappa' = 16/\kappa, \rho')$$

$$\tilde{\Delta}_\ell = \frac{1}{\kappa}(2\ell + \rho) = \ell\frac{2}{\kappa} + U_\kappa^{-1}(0) = \ell\frac{2}{\kappa} + 1 - \frac{4}{\kappa}$$
$$\rho' = \frac{\kappa'}{2} - 2, \quad \tilde{\Delta}'_\ell = \ell\frac{1}{\kappa'}(2 + \rho') = \ell\frac{1}{2}$$

DUALITY

$$\begin{aligned} \tilde{x}_\ell &= U_\kappa^{-1}(\tilde{x}_\ell) \times U_{16/\kappa}^{-1}(\tilde{x}_\ell) \\ &= \tilde{\Delta}_\ell \times \tilde{\Delta}'_\ell = \left(\ell\frac{2}{\kappa} + 1 - \frac{4}{\kappa}\right) \times \ell\frac{1}{2} \end{aligned}$$

Percolation Path Crossings & Duality

Boundary exponents

$$\tilde{x}_\ell^{\mathcal{P}} = U^{-1}_{\kappa=6}(\tilde{x}_\ell) \times U^{-1}_{16/\kappa=8/3}(\tilde{x}_\ell)$$

$$= (\ell+1)\frac{1}{3} \times \ell\frac{1}{2}$$

Bulk exponents

$$x_\ell^{\mathcal{P}} = U^{-1}_{\kappa=6}(x_\ell) \times U^{-1}_{16/\kappa=8/3}(x_\ell)$$

$$= (\ell+1)\frac{1}{6} \times (\ell-1)\frac{1}{4}$$

Brownian Intersections & Duality

Boundary exponents

$$\tilde{\zeta}_L \equiv \tilde{x}^{\mathcal{P}}_{\ell=2L} = U^{-1}_{\kappa=6}\left(\tilde{x}_{2L}\right) \times U^{-1}_{16/\kappa=8/3}\left(\tilde{x}_{2L}\right)$$

$$= (2L+1)\frac{1}{3} \times L \times 1$$

Bulk exponents

$$\zeta_L \equiv x^{\mathcal{P}}_{\ell=2L} = U^{-1}_{\kappa=6}\left(x_{2L}\right) \times U^{-1}_{16/\kappa=8/3}\left(x_{2L}\right)$$

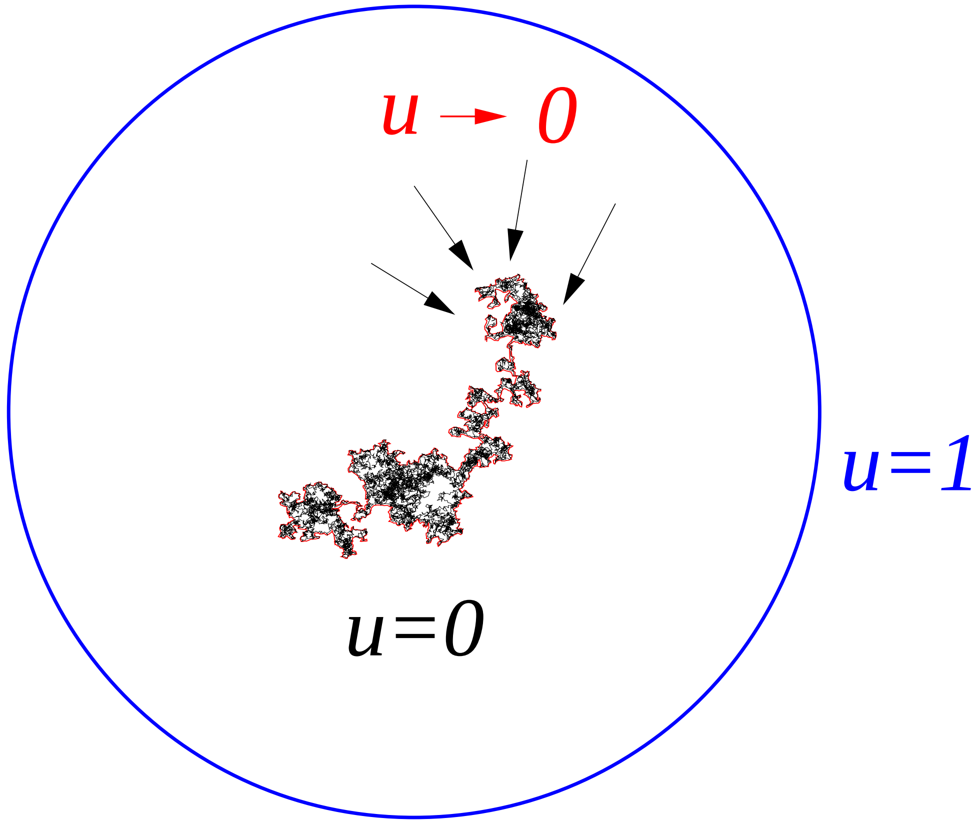
$$= (2L+1)\frac{1}{6} \times (2L-1)\frac{1}{4}$$

Potential Theory

&

Multifractality

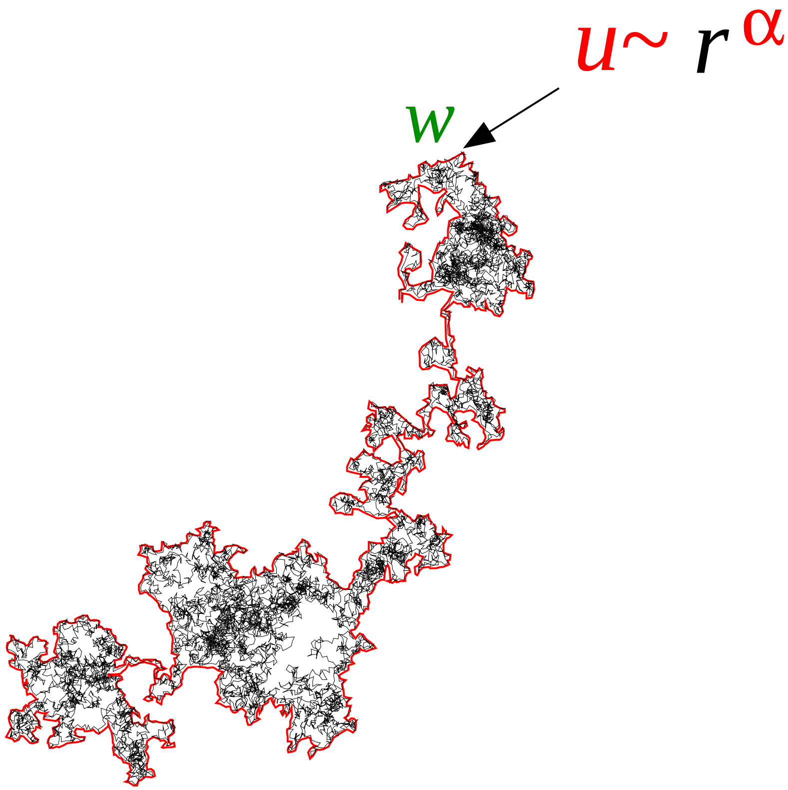
Potential Distribution



Laplace Eq.: $\Delta u = 0$

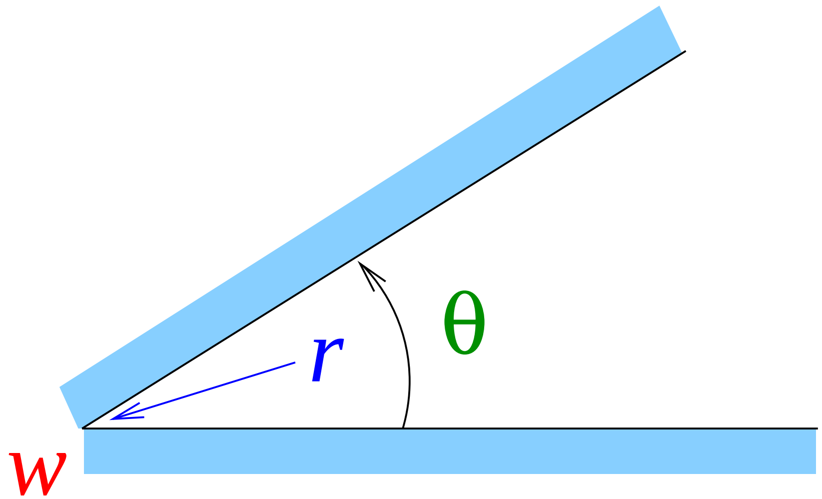
Multifractality

(T. C. Halsey & al., 1986)



$$w \in \mathcal{F}_\alpha : \quad \dim \mathcal{F}_\alpha = f(\alpha)$$

Electrostatic Angles



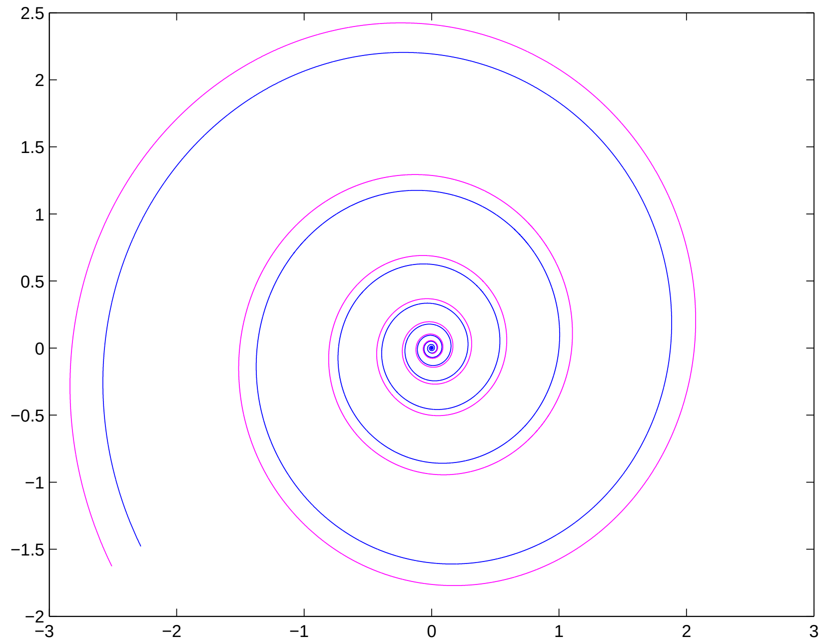
$$u(w,r) \sim r^{\alpha}, \quad \alpha = \pi/\theta$$

$$\theta \in (0,\ 2\pi\,],\quad \alpha \in \left[\frac{1}{2},\ +\infty\right)$$

Spikes, Fjords

Equipotentials

Logarithmic Spirals



A point w on the frontier with a double logarithmic spiral.

Winding angle: $\varphi(w, r) = \lambda \ln r$

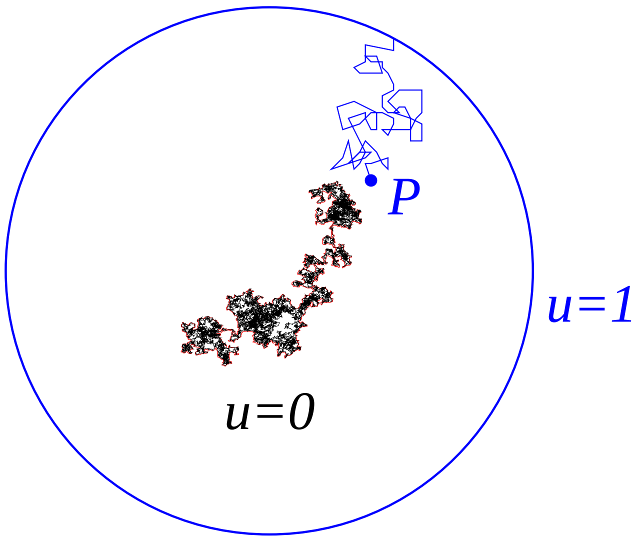
Mixed Multifractal Spectrum

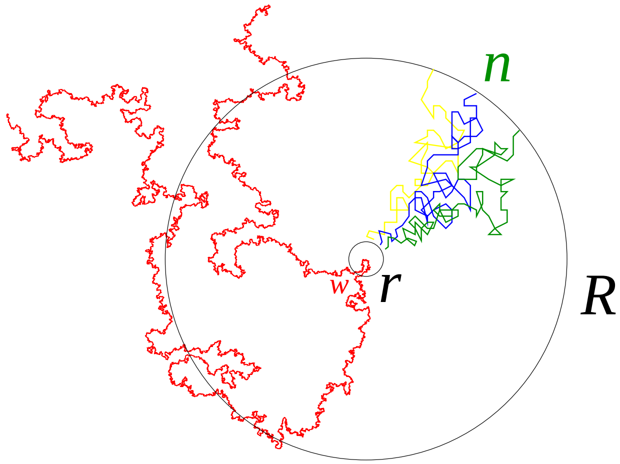
(I. Binder, 1996)

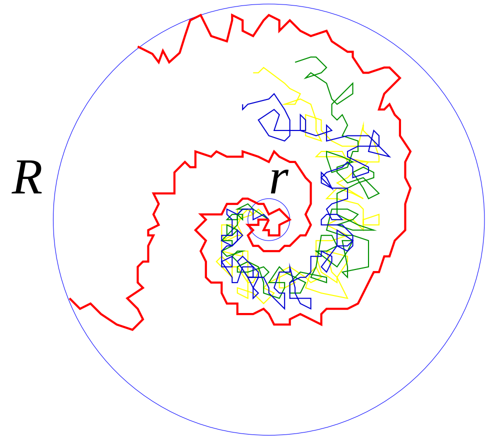
$$w \in \mathcal{F}_{\alpha,\lambda} \iff \begin{Bmatrix} u(w,r) \sim r^{\alpha} \\ \varphi(w,r) \sim \lambda \ln r \end{Bmatrix}$$

$$\dim \mathcal{F}_{\alpha,\lambda} = f(\alpha,\lambda)$$

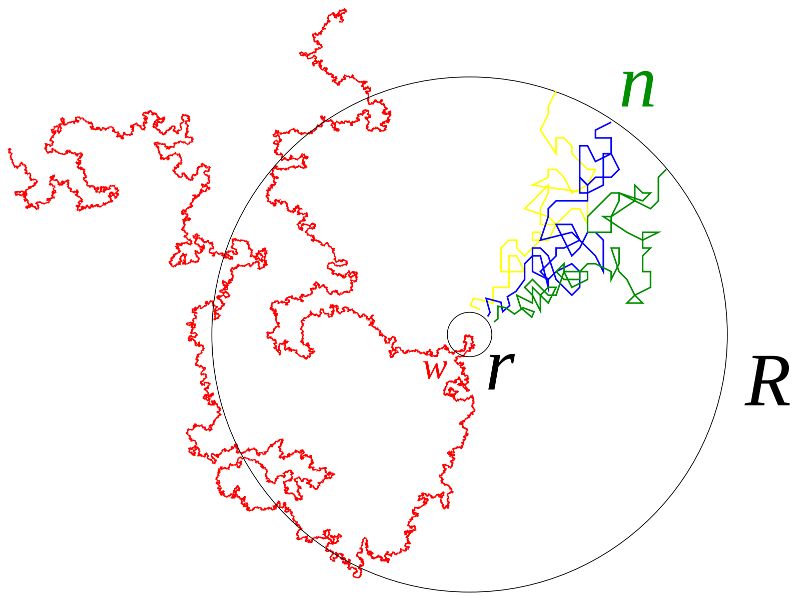
Potential & Brownian Paths







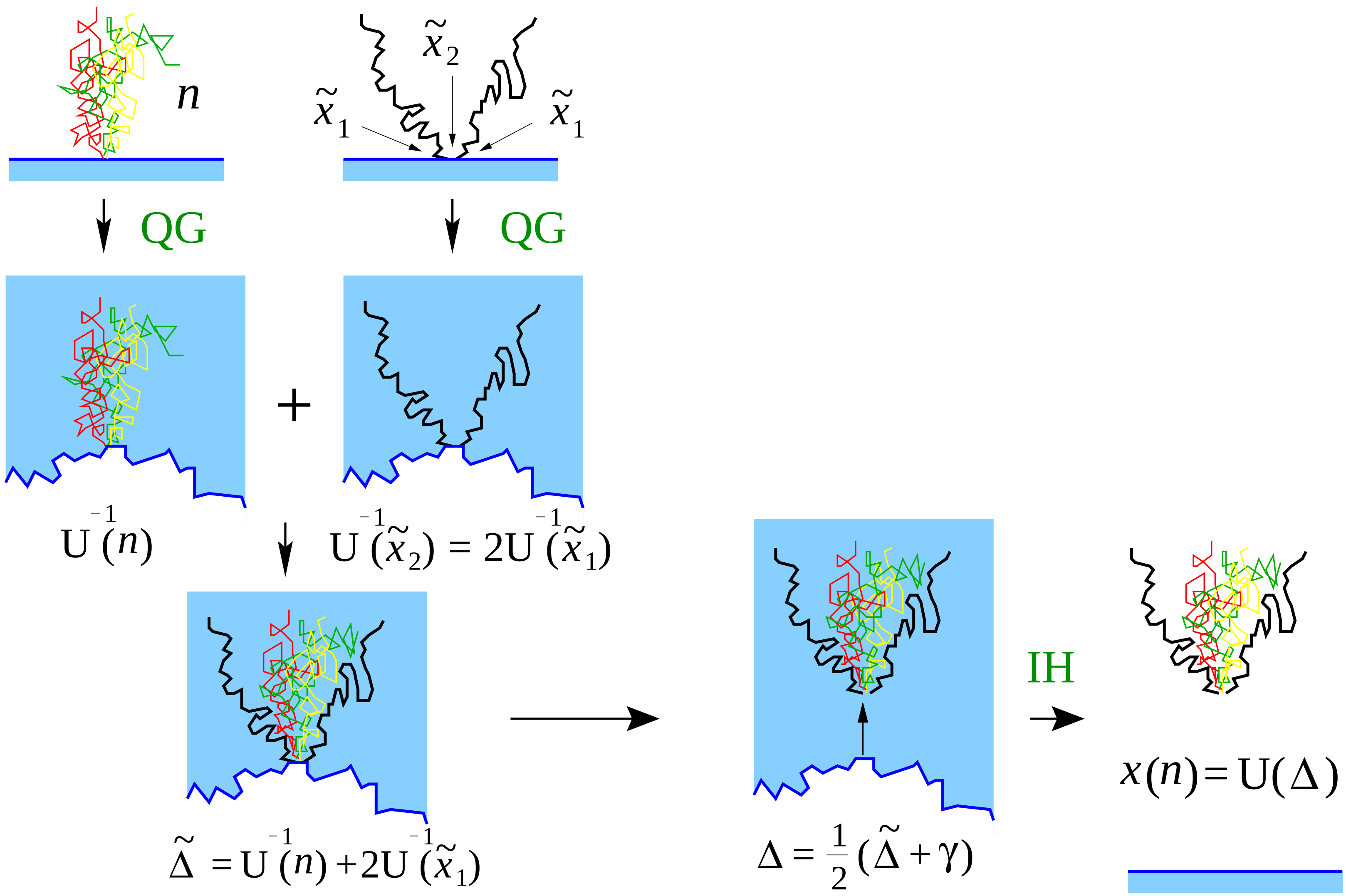
Moments & Brownian Paths



$$\sum_{w} H^{n}(w,r) \approx (r/R)^{2x(n)-2}$$

$H(w,r)$: harmonic measure in ball $B(w,r)$

Quantum Gravity Construction



Quantum Gravity Construction

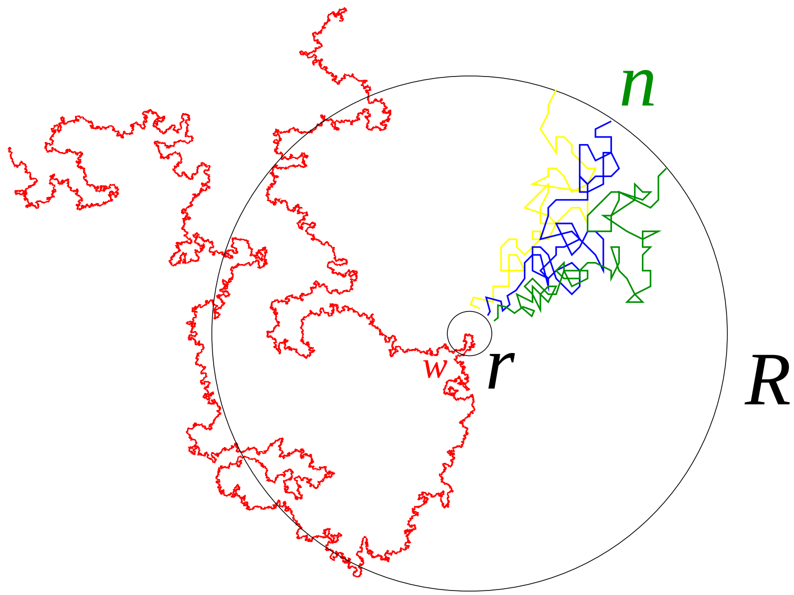
- Boundary

$$\begin{aligned}\tilde{\Delta} &= U^{-1}(n) + 2U^{-1}(\tilde{x}_1) \\ &= U^{-1}(n) + 1 - \gamma\end{aligned}$$

- Bulk

$$\begin{aligned}\Delta &= \frac{1}{2}\left(\tilde{\Delta} + \gamma\right) \\ &= \frac{1}{2}U^{-1}(n) + \frac{1}{2}\end{aligned}$$

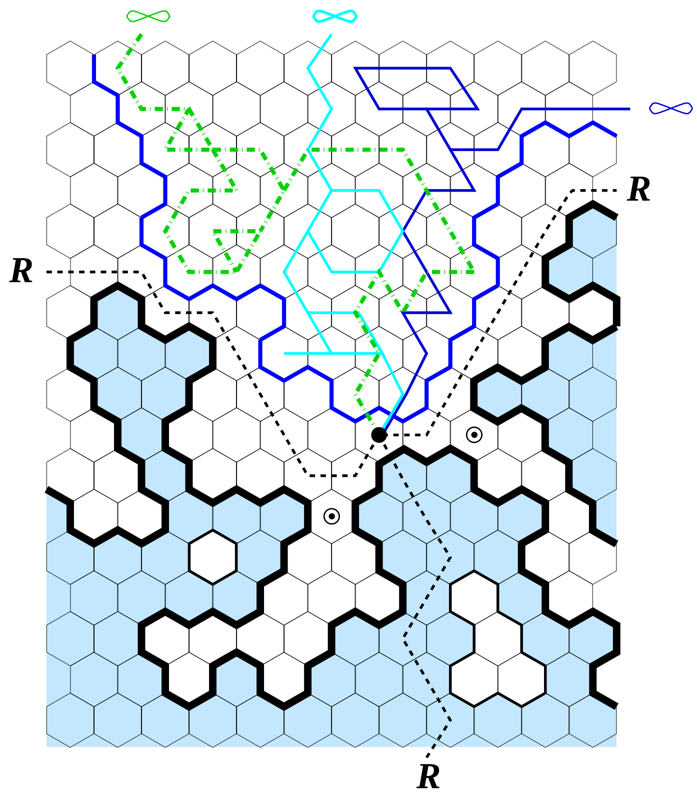
Multifractal Exponents & QG



$$x(n) = U\left(\frac{1}{2}U^{-1}(n) + \frac{1}{2}\right)$$

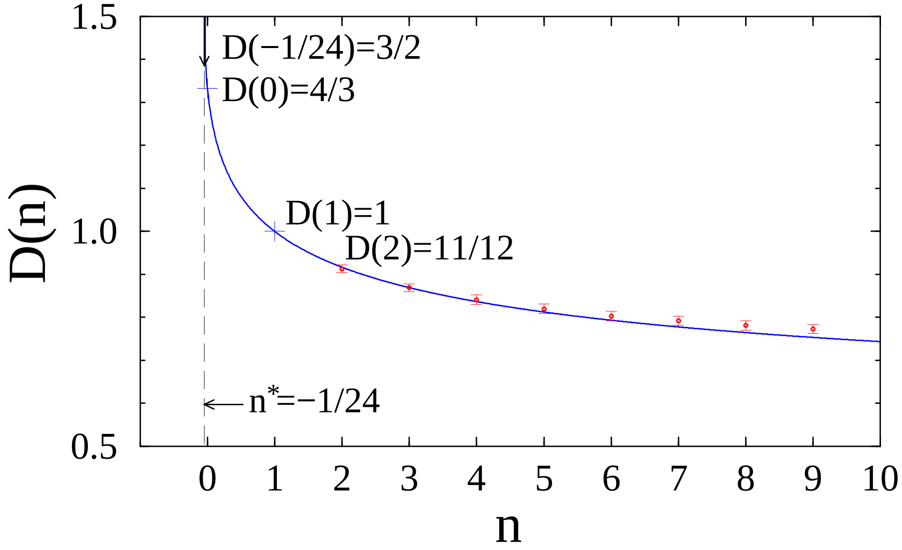
B. D., 1998 (see also Rushkin & al., 2006)

Harmonic Measure & Percolation



An accessible site (•) on the external perimeter in percolation, with *three connected crossing paths.* The entrances ⊙ of fjords close in the scaling limit. Full hull dimension: $\frac{7}{4}$. External perimeter dimension: $\frac{4}{3}$.

General Dimensions $D(n) = 2\left[x(n) - 1\right]/(n-1)$



$$D(n) = \frac{1}{2} + \frac{25}{24}\left(\sqrt{\frac{24n+1}{25}} - 1\right)^{-1}$$

for percolation, self-avoiding or random walk (D., 1999); data for percolation (red dots) (P. Meakin et al., 1988).

Multifractal Spectra

Multifractal Spectra

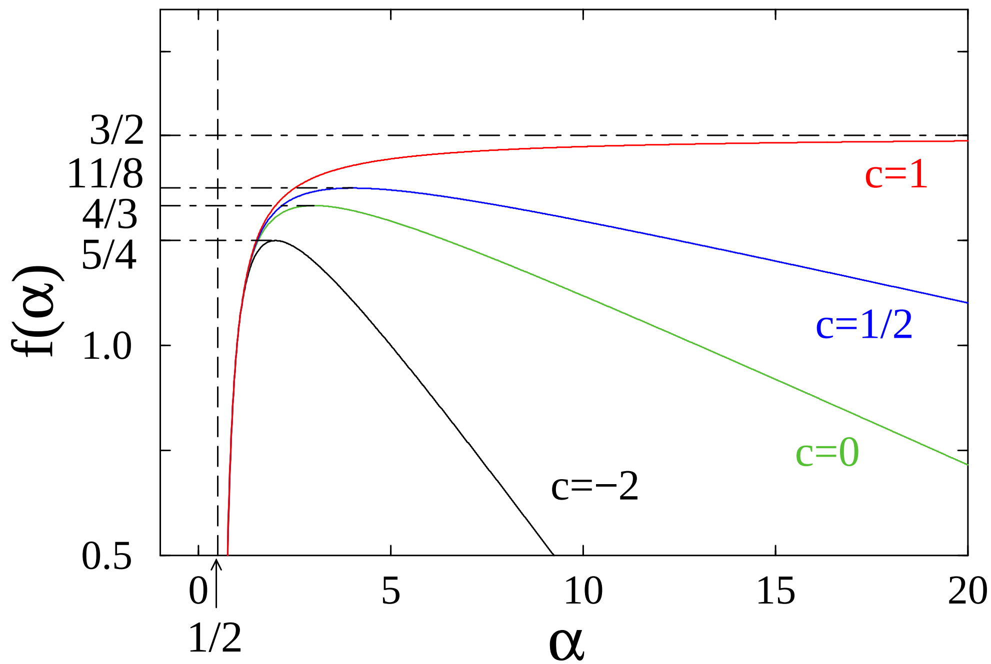
Multifractal Scaling Law

$$f(\alpha) = \alpha + b - \frac{b\alpha^2}{2\alpha - 1}, \quad b = \frac{25 - c}{12} = \frac{1}{2\kappa}\left(2 + \frac{\kappa}{2}\right)^2$$

$$\begin{aligned} f(\alpha, \lambda) &= (1+\lambda^2)\, f\left(\frac{\alpha}{1+\lambda^2}\right) - b\lambda^2 \\ &= \alpha + b - \frac{b\alpha^2}{2\alpha - 1 - \lambda^2} \end{aligned}$$

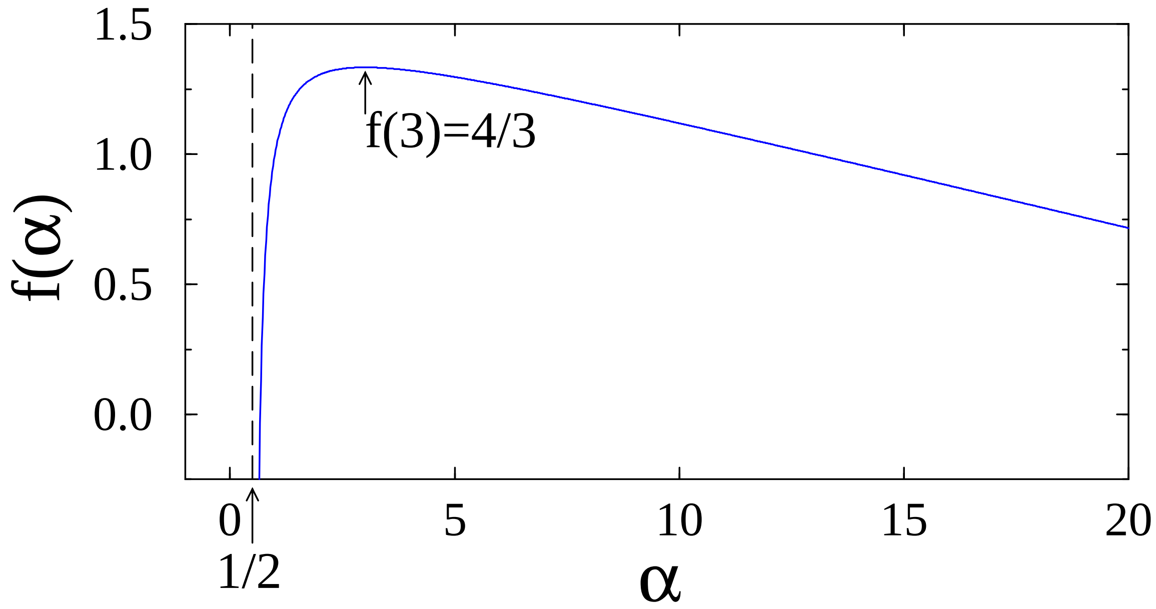
(B. D., 1998; I. Binder & B. D., 2002, 2006)

Multifractal Spectra $f(\alpha)$



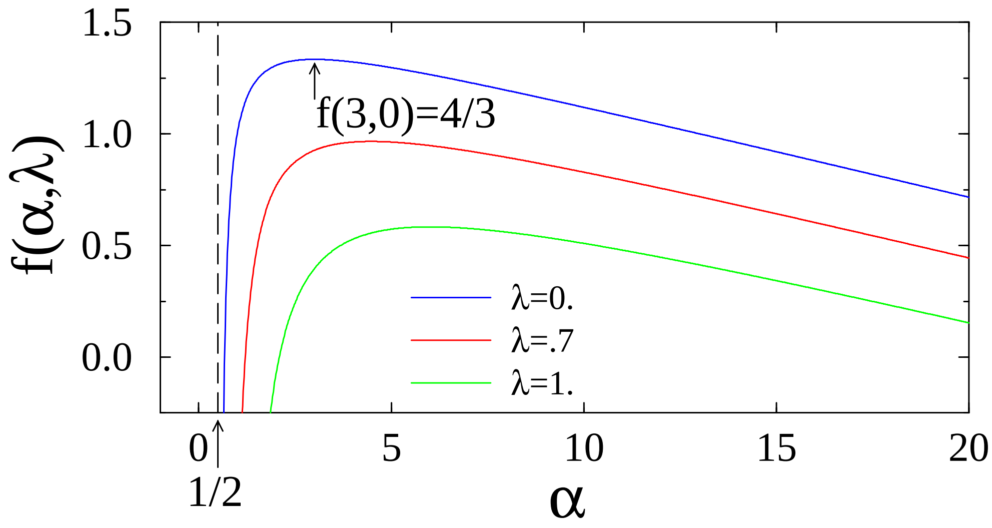
Loop-erased RW ($c = -2$, SLE_2); Brownian & percolation frontiers, and SAW's ($c = 0$, $\mathrm{SLE}_{8/3}$); Ising clusters ($c = \frac{1}{2}$, SLE_3); $Q = 4$ Potts clusters ($c = 1$, SLE_4).

Brownian Multifractal Spectrum



$$f(\alpha) = \alpha + b - \frac{b\,\alpha^2}{2\alpha - 1}, \quad b = \frac{25}{12}$$

Brownian Mixed Spectrum $f(\alpha, \lambda)$



$f(\alpha, \lambda)$ for the Brownian frontier, percolation and SAW, and various spiralling rates λ.

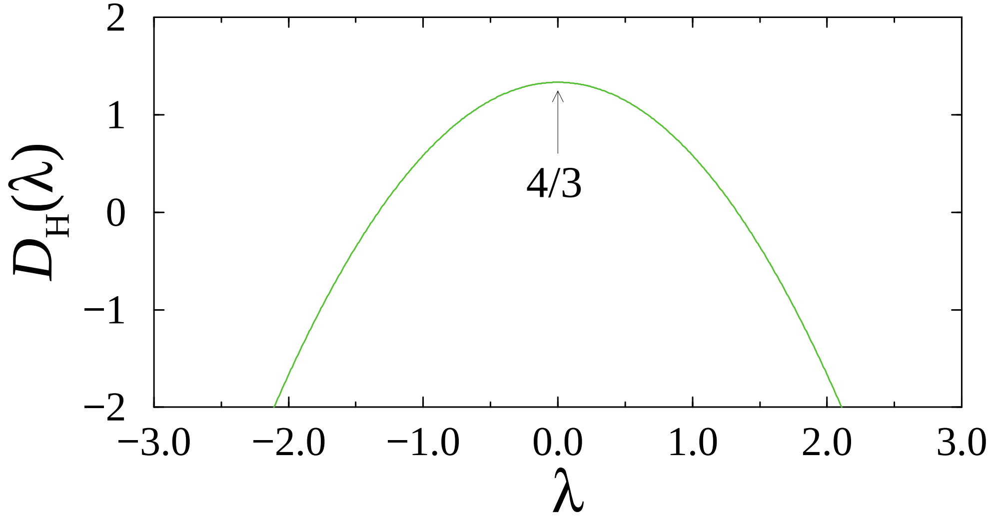
The maximum $f(\alpha = 3, \lambda = 0) = 4/3$ is the Hausdorff dimension of the frontier.

Rotation Dimensions

$$D(\lambda) = \sup_{\alpha} f(\alpha, \lambda)$$
$$= D_{\mathrm{EP}} - (b - D_{\mathrm{EP}})\,\lambda^2$$

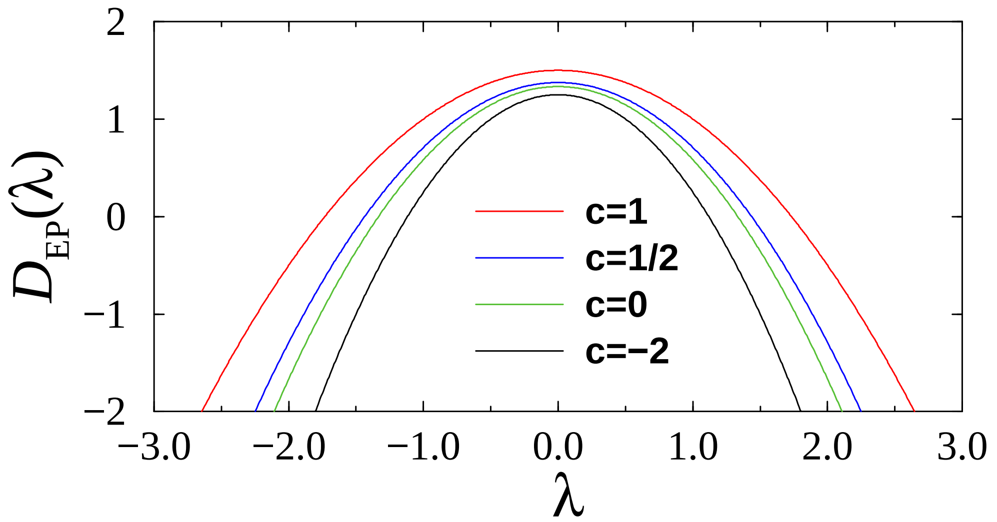
$$b \geqslant 2, \quad D_{\mathrm{EP}} \leqslant \frac{3}{2}$$

Brownian Rotation Dimensions



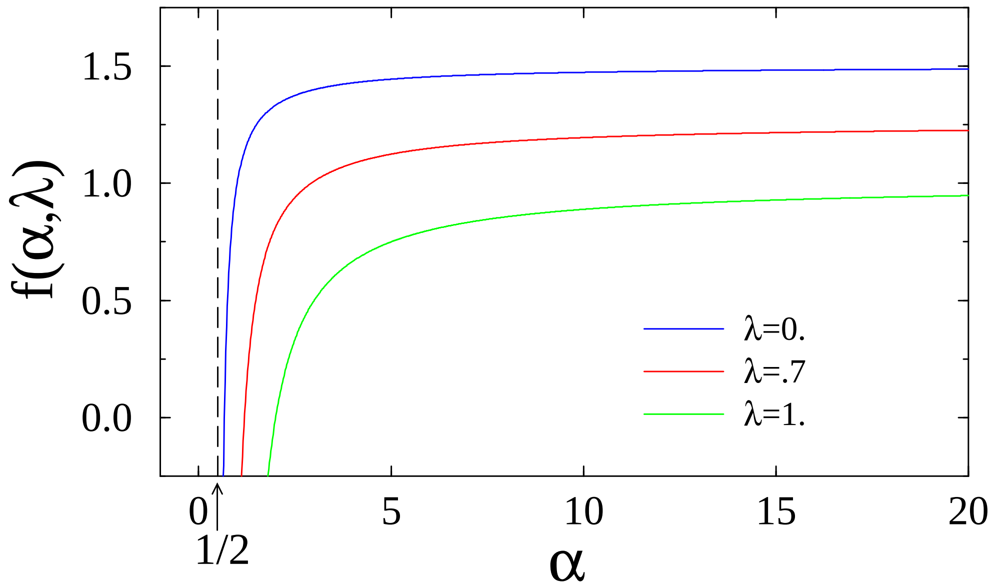
$$D(\lambda) = \sup_\alpha f(\alpha, \lambda) = \frac{4}{3} - \frac{3}{4}\lambda^2$$

Dimension $D(\lambda)$ of λ -spiral points on the Brownian frontier.



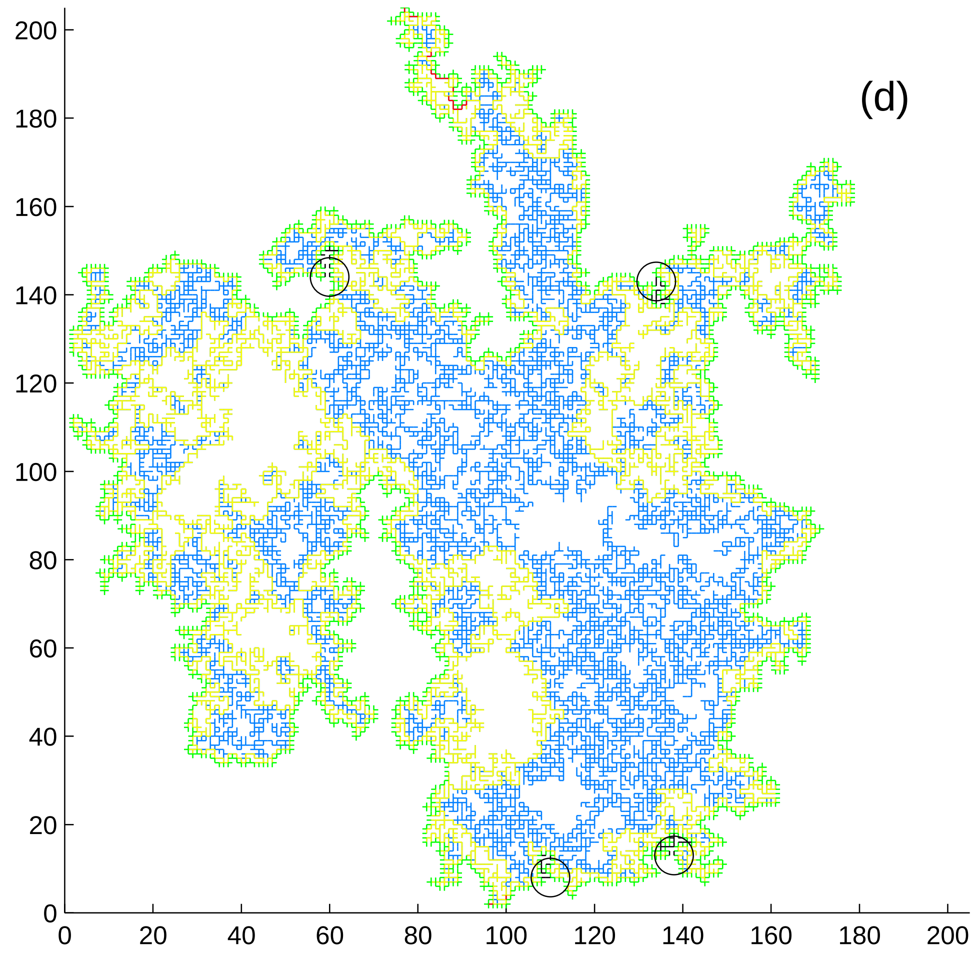
Dimensions $D(\lambda)$ of the external frontier as a function of rotation rate λ: loop-erased RW ($c=-2; \mathrm{SLE}_2$); Brownian & percolation frontiers, and SAW's ($c=0; \mathrm{SLE}_{8/3}$); Ising clusters ($c=\frac{1}{2}; \mathrm{SLE}_3$); $Q=4$ Potts clusters ($c=1; \mathrm{SLE}_4$) (or "Ultimate Norway").

The Pivotal $c = 1$ Case



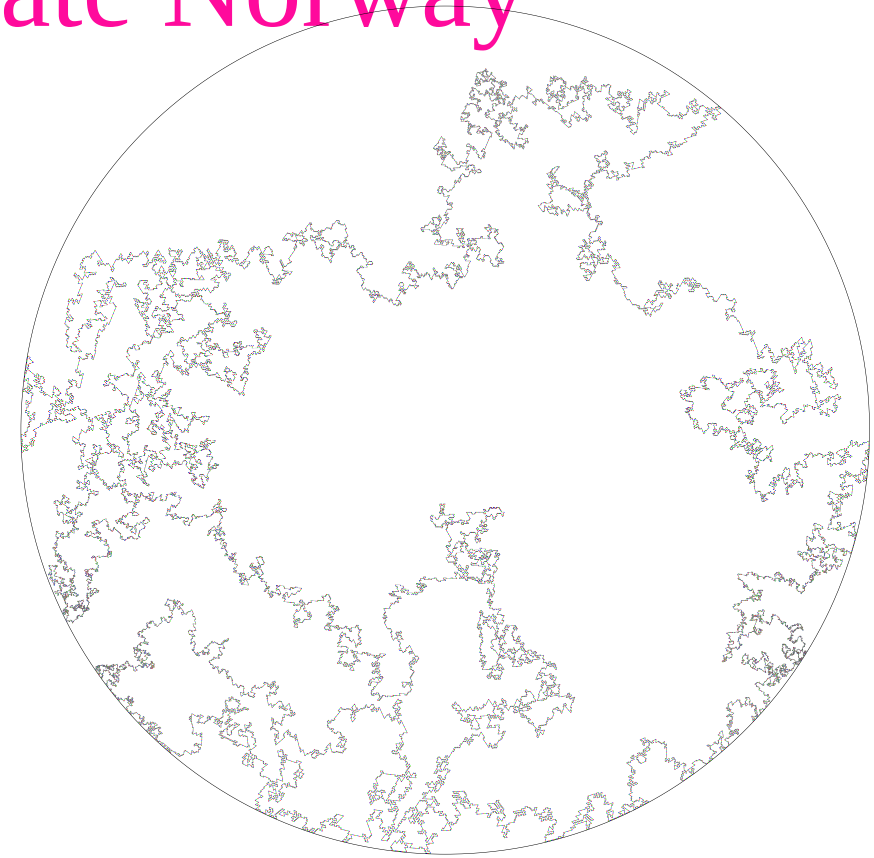
Left-sided mixed spectra $f(\alpha, \lambda)$ *for the "Ultimate Norway" ($c = 1$).*

Potts FK Cluster $(Q = 4)$



Cluster; *Hull*; *External Perimeter*.

Ultimate Norway



The "Ultimate Norway", i.e. the frontier of a $Q=4$ *Potts cluster or* $\mathrm{SLE}_{\kappa=4}$, *the self-dual conformally invariant random curve* $(c=1)$ *with maximal Hausdorff dimension* $D=3/2$ *(courtesy of D. Wilson).*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