Two-fermion approach to two-matrix integrals

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Reduced two-matrix integrals in $2N$ variables:

$$\mathbf{Z}_N^{(2)}(\mathbf{t},\tilde{\mathbf{t}}) := \prod_{i=1}^{N} \int\!\!\int_{\kappa\Gamma} d\mu(x_i,y_i)\gamma(x_i,\mathbf{t})\tilde{\gamma}(y_i,\tilde{\mathbf{t}})\Delta_N(\mathbf{x})\Delta_N(\mathbf{y})$$

where $d\mu(x,y)$ is some two-variable measure, e.g.

$$d\mu(x,y) = d\mu(x)d\tilde{\mu}(y)h(xy)\sum_{a=1}^{k}\sum_{b=1}^{l} z_{ab}\chi_a(x)\chi_b(y)$$

$$\gamma(x,\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i x^i}, \quad \tilde{\gamma}(y,\tilde{\mathbf{t}}) := e^{\sum_{i=1}^{\infty} \tilde{t}_i y^i}$$

$$\mathbf{t} = (t_1, t_2, \ldots), \quad \tilde{\mathbf{t}} = (\tilde{t}_1, \tilde{t}_2, \ldots)$$

$$\kappa = \{\kappa_{\alpha\beta}\}\Big|_{\substack{1\le\alpha\le d_1 \\ 1\le\beta\le d_2}}. \quad \Gamma = \{\Gamma_\alpha \times \tilde{\Gamma}_\beta\}$$

$$\int\!\!\int_{\kappa\Gamma} := \sum_{\alpha\beta} \kappa_{\alpha\beta} \int_{\Gamma_\alpha}\int_{\tilde{\Gamma}_\beta}$$

Examples:

• Two-matrix partition function:

$$k = l = 1, \quad z_{11} = 1, \quad \chi_1 = \chi_2 = 1$$

with, e.g.

$$f(x) = e^{V_1(x)}, \quad \tilde{f}(y) = e^{V_2(y)}, \quad h(xy) = e^{xy}$$

• Generating function for (k, l) point correlators (marginal distributions) of eigenvalues

$$\chi_a = \delta(x - X_a), \quad \chi_b = \delta(y - Y_b)$$

• Generating function for point (k, l) gap probabilities

$$\chi_a = \chi_{[\alpha_{2a-1}, \alpha_{2a}]}(x) \quad \tilde{\chi}_b = \chi_{[\beta_{2b-1}, \beta_{2b}]}(x)$$

• Generating function for point (k, l) *Janossy distributions* (Combine the above two.)

• Correlator for ratios and products of characteristic polynomials (see below)

Motivation: Two matrix models

Most **statistical properties** of the spectrum are expressible as expectation values

$$< F >= \frac{1}{\mathbf{Z}_N^{(2)}} \int F(M_1, M_2) d\Omega(M_1, M_2)$$

where the **Partition function** is

$$\mathbf{Z}_N^{(2)} := \int d\Omega(M_1, M_2)$$

For some **conjugation invariant** F's, unitarily diagonalizable matrices M_1, M_2, and certain matrix measures $d\Omega(M_1, M_2)$ this reduces to:

$$< F > \propto \prod_{i=1}^{N} \iint_{\kappa\Gamma} d\mu(x_i, y_i) \Delta_N(x) \Delta_N(y) \\ \times \tilde{F}(x_1, \ldots, x_N, y_1, \ldots y_N)$$

for a reduced $2D$ measure $d\mu(x, y)$ and suitable support.

Example: (Itzykson-Zuber (1980)

$$d\Omega(M_1, M_2) = d\mu_0(M_1) d\mu_0(M_2) e^{\mathrm{Tr}(V_1(M_1) + V_2(M_2) + M_1 M_2)}$$

In particular, **determinantal correlators**:

$$\mathbf{I}_N^{(2)} := \left\langle \prod_{a=1}^{N} \frac{\prod_{\alpha=1}^{L_1} \det(\xi_\alpha \mathbf{I} - M_1) \prod_{\beta=1}^{L_2} \det(\zeta_\beta \mathbf{I} - M_2)}{\prod_{j=1}^{M_1} \det(\eta_j \mathbf{I} - M_1) \prod_{k=1}^{M_2} \det(\mu_k \mathbf{I} - M_2)} \right\rangle$$

For suitable measures, this reduces to **Integrals of rational symmetric functions** in $2N$ variables:

$$\mathbf{I}_N^{(2)}(\xi,\zeta,\eta,\mu) := \frac{1}{\mathbf{Z}_N^{(2)}} \prod_{i=1}^{N} \iint_{\kappa\Gamma} d\mu(x_i,y_i)\Delta_N(x)\Delta_N(y) \times \prod_{a=1}^{N} \frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha - x_a)\prod_{\beta=1}^{L_2}(\zeta_\beta - y_a)}{\prod_{j=1}^{M_1}(\eta_j - x_a)\prod_{k=1}^{M_2}(\mu_k - y_a)}$$

$$\mathbf{Z}_N^{(2)} := \prod_{i=1}^{N} \iint_{\kappa\Gamma} d\mu(x_i,y_i)\Delta_N(x)\Delta_N(y)$$

This may be expressed in terms of **Biorthogonal polynomials**

$$\iint_{\kappa\Gamma} d\mu(x,y)P_j(x)S_k(y) = \delta_{jk}, \quad \forall j,k \in \mathbf{N}$$

and their **Hilbert transforms**

$$\tilde{P}_j(\mu) := \iint_{\kappa\Gamma} d\mu(x,y)\frac{P_j(x)}{\mu - y},$$

$$\tilde{S}_j(\eta) := \iint_{\kappa\Gamma} d\mu(x,y)\frac{S_j(x)}{\eta - x}$$

Assuming generic conditions on the matrix of **bimoments**:

$$B_{jk} := \iint_{\kappa\Gamma} d\mu(x,y) x^j y^k < \infty, \quad 0, \quad \forall j,k \in \mathbf{N}$$
$$\det(B_{jk})_{0\le j,k\le N} \neq 0, \quad \forall N \in \mathbf{N}$$

implies the existence of a unique sequence of

Biorthogonal polynomials

$$\iint_{\kappa\Gamma} d\mu(x,y) P_j(x) S_k(y) = \delta_{jk},$$

normalized to have leading coefficients that are equal:

$$P_j(x) = \frac{x^j}{\sqrt{h_j}} + O(x^{j-1}), \qquad S_j(x) = \frac{y^j}{\sqrt{h_j}} + O(y^{j-1}).$$

Also, assume existence of their **Hilbert transforms**

$$\tilde{P}_j(\mu) := \iint_{\kappa\Gamma} d\mu(x,y) \frac{P_j(x)}{\mu - y},$$
$$\tilde{S}_j(\eta) := \iint_{\kappa\Gamma} d\mu(x,y) \frac{S_j(x)}{\eta - x}$$

Relation to integrable systems:

Deform the measure

$$d\Omega(M_1, M_2) \to d\Omega(M_1, M_2) e^{\mathrm{tr}(\sum_{j=0}^{\infty}(t_j M_1^j + \tilde{t}_j M_2^j)}$$
$$:= d\Omega(M_1, M_2, \mathbf{t}, \tilde{\mathbf{t}})$$

The deformed partition function

$$\mathbf{Z}_N^{(2)}(\mathbf{t}, \tilde{\mathbf{t}}) := \int d\Omega(M_1, M_2, \mathbf{t}, \tilde{\mathbf{t}})$$

is a **2-Toda τ function**.

The **biorthogonal polynomials** and the Hilbert transforms

$$\{P_j(x, \mathbf{t}, \tilde{\mathbf{t}}), \tilde{P}_j(y, \mathbf{t}, \tilde{\mathbf{t}})\}, \{\tilde{S}_j(x, \mathbf{t}, \tilde{\mathbf{t}}), S_j(y, \mathbf{t}, \tilde{\mathbf{t}})\}_{j \in \mathbf{N}}$$

are **Baker-Akhiezer** and **dual Baker-Akhiezer functions**.

Moreover, these satisfy **dual systems** of differential equations (in x and y) and recursion relations (in j) for which the deformations are **isomonodromic**

M. Bertola, B. Eynard and J. Harnad, "Duality, Biorthogonal Polynomials and Multi–Matrix Models", *Commun. Math. Phys.* **229**, 73–120 (2002).

M. Bertola, B. Eynard and J. Harnad, "Differential systems for biorthogonal polynomials appearing in 2-matrix models and the associated Riemann-Hilbert problem", *Commun. Math. Phys.* **243**, 193–240 (2003)

First result: Double Schur function perturbation expansion

$$\mathbf{Z}_N^{(2)}(\mathbf{t},\tilde{\mathbf{t}}) = N!\sum_{\lambda,\mu} B_{\lambda\mu} s_\lambda(\mathbf{t}) s_\mu(\tilde{\mathbf{t}})$$

where $s_\lambda(\mathbf{t})$, $s_\mu(\tilde{\mathbf{t}})$ are Schur functions corresponding to partitions $\lambda := (\lambda_1, \dots \lambda_{\ell(\lambda)})$, $\mu := (\mu_1, \dots \mu_{\ell(\mu)})$ of lengths $\ell(\lambda), \ell(\mu) \leq N$, and

$$B_{\lambda,\mu} = \det(B_{\lambda_i - i + N, \mu_j - j + N})\big|_{i,j=1,\dots,N},$$

Second result: Evaluation of symmetric rational integrals (Assume $N + L_2 - M_2 \geq N + L_1 - M_1 \geq 0$)

$$\mathbf{I}_N^{(2)} = \epsilon(L_1, L_2, M_2, M_2) \frac{\prod_{n=0}^{N+L_2-M_2-1}\sqrt{h_n}\ \prod_{n=0}^{N+L_1-M_1-1}\sqrt{h_n}}{\prod_{n=0}^{N-1} h_n}$$
$$\times \frac{\prod_{\alpha=1}^{L_1}\prod_{j=1}^{M_1}(\xi_\alpha - \eta_j)\prod_{\beta=1}^{L_2}\prod_{k=1}^{M_2}(\zeta_\beta - \mu_k)}{\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)} \ \det G,$$

where

$$\epsilon(L_1, L_2, M_2, M_2) := (-1)^{\frac{1}{2}(M_1+M_2)(M_1+M_2-1)}(-1)^{L_1 M_2}$$

and G is the $(L_2 + M_1) \times (L_2 + M_1)$ matrix

$$G = \begin{pmatrix} \overset{N+L_1-M_1}{K_{11}}(\xi_\alpha, \eta_j) & \overset{N+L_1-M_1}{K_{12}}(\xi_\alpha, \zeta_\beta) \\ \overset{N+L_1-M_1}{K_{21}}(\mu_k, \eta_j) & \overset{N+L_1-M_1}{K_{22}}(\mu_k, \zeta_\beta) \\ \tilde{S}_{N+L_1-M_1}(\eta_j) & S_{N+L_1-M_1}(\zeta_\beta) \\ \vdots & \vdots \\ \tilde{S}_{N+L_2-M_2-1}(\eta_j) & S_{N+L_2-M_2-1}(\zeta_\beta) \end{pmatrix}$$

$$\overset{J}{K}_{11}(\xi,\eta) := \sum_{n=0}^{J-1} P_n(\xi)\tilde{S}_n(\eta) + \frac{1}{\xi-\eta}$$

$$\overset{J}{K}_{22}(\mu,\zeta) := \sum_{n=0}^{J-1} \tilde{P}_n(\mu)S_n(\zeta) + \frac{1}{\zeta-\mu}$$

$$\overset{J}{K}_{21}(\mu,\eta) := \sum_{n=0}^{J-1} \tilde{P}_n(\mu)\tilde{S}_n(\eta) - H(\mu,\eta)$$

$$\overset{J}{K}_{12}(\xi,\zeta) := \sum_{n=0}^{J-1} P_n(\xi)S_n(\zeta), \quad H(\mu,\eta) := \iint_{\kappa\Gamma} \frac{d\mu(x,y)}{(\eta-x)(\mu-y)}$$

Previous work:

Schur function expansions:

• V. A. Kazakov, M. Staudacher and T. Wynter "Character Expansion Methods for Matrix Models of Dually Weighted Graphs", *Commun. Math. Phys.* **177**, 451-468 (1996)

Rational symmetric integrals; polynomial case:

G. Akemann and G. Vernizzi, "Characteristic polynomials of complex random matrix models", *Nucl. Phys.* **B 660**, 532–556 (2003).

Complex matrix model; rational case:

M. Bergère (hep-th/0404126)

Analogous results for one-matrix models:

V.B. Uvarov, (1969) (general case), E. Brezin and S. Hikami, (2000), (polynomial integrals), Y.V. Fyodorov and E. Strahov (2003) ($N \geq M$), J. Baik, P. Deift and E. Strahov, (2003) ($N \geq M$), A. Borodin and E. Strahov (2006))

Two methods of derivation:

1. Direct method

J. Harnad and A.Yu. Orlov, "Scalar products of symmetric functions and matrix integrals", *Theor. Math. Phys.* **137**, 1676–1690 (2003).
J. Harnad and A. Yu. Orlov, "Matrix integrals as Borel sums of Schur function expansions", In: Symmetries and Perturbation theory SPT2002, eds. S. Abenda and G. Gaeta, World Scientific, Singapore, (2003).
J. Harnad and A. Yu. Orlov, "Integrals of rational symmetric functions, two-matrix models and biorthogonal polynomials", *J. Math. Phys.* **47** (in press, Nov. 2006)

2. Fermionic vacuum state expectation values

J. Harnad and A. Yu. Orlov, "Fermionic construction of partition functions for two-matrix models and double Schur function expansions", *J. Phys. A* **39**, 8783–8809 (July 2006) math-phys/0512056
J. Harnad and A.Yu. Orlov, "Fermionic approach to the evaluation of integrals of rational symmetric functions", preprint CRM-(2006)

1. Direct method:

The key tools for the Schur function expansion are:

1.1. Cauchy Littlewood identity:

$$e^{\sum_{i=1}^{\infty} i t_i \tilde{t}_i} = \sum_{\lambda} s_\lambda(\mathbf{t}) s_\lambda(\tilde{\mathbf{t}})$$

1.2. Andreief identity:

$$\prod_{a=1}^{N} \int\int d\mu(x_a, y_a) \det\phi_i(x_j) \det\psi_k(y_l)$$
$$= N! \det\left(\int\int d\mu(x,y)\phi_i(x)\psi_j(y)\right)$$
$$(1 \leq i, j, k, l \leq N)$$

For the integral of rational symmetric functions:

1.3. Multivariable partial fraction expansions:

For $N \geq M$:

$$\frac{\Delta_N(x)\Delta_M(\eta)}{\prod_{a=1}^N \prod_{j=1}^M (\eta_j - x_a)} =$$

$$(-1)^{MN} \sum_{\sigma \in S_M} \mathrm{sgn}(\sigma) \sum_{a_1 < \ldots < a_M}^{N} (-1)^{\sum_{j=1}^M a_j} \frac{\Delta_{N-M}(x[\mathbf{a}])}{\prod_{j=1}^M (\eta_{\sigma_j} - x_{a_j})}$$

For $N \leq M$:

$$\frac{\Delta_N(x)\Delta_M(\eta)}{\prod_{a=1}^N \prod_{j=1}^M (\eta_j - x_a)} =$$

$$\frac{(-1)^{\frac{1}{2}N(N-1)}}{(M-N)!} \sum_{\sigma \in S_M} \mathrm{sgn}(\sigma) \frac{\Delta_{M-N}(\eta_{\sigma_{N+1}}, \ldots, \eta_{\sigma_M})}{\prod_{a=1}^N (\eta_{\sigma_a} - x_a)}$$

1.4. Cauchy-Binet identity

If V is an oriented Euclidean vector space with volume form Ω and $(P^1, \ldots, P^L)$, $(S^1, \ldots, S^L)$ are two sets of L vectors, then the scalar product of their exterior products $(\wedge_{\alpha=1}^L P^\alpha, \wedge_{\beta=1}^L S^\beta)$ is equal to the determinant of the matrix formed from the scalar products:

$$(\wedge_{\alpha=1}^L P^\alpha, \wedge_{\beta=1}^L S^\beta) = \det G$$

$$G^{\alpha\beta} := (P^\alpha, S^\beta), \quad 1 \leq i, j \leq L$$

2) Fermionic vacuum state expectation values

Two-component fermions

$$[f_n^{(\alpha)}, f_m^{(\beta)}]_+ = [\bar{f}_n^{(\alpha)}, \bar{f}_m^{(\beta)}]_+ = 0,$$
$$[f_n^{(\alpha)}, \bar{f}_m^{(\beta)}]_+ = \delta_{\alpha,\beta}\delta_{nm}, \qquad \alpha = 1, 2$$

Fermionic fields

$$f^{(\alpha)}(z) := \sum_{k=-\infty}^{+\infty} z^k f_k^{(\alpha)} \ , \quad \bar{f}^{(\alpha)}(z) := \sum_{k=-\infty}^{+\infty} z^{-k-1} \bar{f}_k^{(\alpha)}$$

Right and left vacuum vectors: $|0,0\rangle$, $\langle 0,0|$

$$f_m^{(\alpha)}|0,0\rangle = 0 \qquad (m < 0), \qquad \bar{f}_m^{(\alpha)}|0,0\rangle = 0 \qquad (m \geq 0),$$
$$\langle 0,0|f_m^{(\alpha)} = 0 \qquad (m \geq 0), \qquad \langle 0,0|\bar{f}_m^{(\alpha)} = 0 \qquad (m < 0)$$

Wick's theorem implies, for linear elements of the Clifford algebra

$$\langle 0,0|w_1 \cdots w_N \bar{w}_N \cdots \bar{w}_1|0,0\rangle = \det\left(\langle 0,0|w_i \bar{w}_j|0,0\rangle\right)\,|_{i,j=1,\dots,N}$$

Remark: This is just the **Cauchy-Binet** identity!

Charged vacuum states

$$\langle n^{(1)}, n^{(2)}| := \langle 0,0|C_{n^{(1)}}C_{n^{(2)}},$$

where

$$C_{n^{(\alpha)}} := \bar{f}_0^{(\alpha)} \cdots \bar{f}_{n^{(\alpha)}-1}^{(\alpha)} \quad \text{if} \quad n^{(\alpha)} > 0$$
$$C_{n^{(\alpha)}} := f_{-1}^{(\alpha)} \cdots f_{n^{(\alpha)}}^{(\alpha)} \quad \text{if} \quad n^{(\alpha)} < 0$$
$$C_{n^{(\alpha)}} := 1 \quad \text{if} \quad n^{(\alpha)} = 0$$

and

$$|n^{(1)}, n^{(2)}\rangle := \bar{C}_{n^{(2)}}\bar{C}_{n^{(1)}}|0,0\rangle$$

where

$$\bar{C}_{n^{(\alpha)}} := f_{n^{(\alpha)}-1}^{(\alpha)} \cdots f_0^{(\alpha)} \quad \text{if} \quad n^{(\alpha)} > 0$$
$$\bar{C}_{n^{(\alpha)}} := \bar{f}_{n^{(\alpha)}}^{(\alpha)} \cdots \bar{f}_{-1}^{(\alpha)} \quad \text{if} \quad n^{(\alpha)} < 0$$
$$\bar{C}_{n^{(\alpha)}} := 1 \quad \text{if} \quad n^{(\alpha)} = 0$$

Let

$$g := e^A, \quad A := \int\int f^{(1)}(x)\bar{f}^{(2)}(y)d\mu(x,y),$$

Define two sequences of commuting operators

$$H_k^{(\alpha)} := \sum_{n=-\infty}^{+\infty} f_n^{(\alpha)}\bar{f}_{n+k}^{(\alpha)}\ , \quad k \neq 0, \quad \alpha = 1,2.$$

$$H(\mathbf{t}, \tilde{\mathbf{t}}) := \sum_{k=1}^{\infty} H_k^{(1)} t_k - \sum_{k=1}^{\infty} H_k^{(2)} \tilde{t}_k$$

$\mathbf{Z}_N^{(2)}(\mathbf{t},\tilde{\mathbf{t}})$ as a 2-Toda τ function

$$\begin{aligned}\tau_N(\mathbf{t},\tilde{\mathbf{t}}) &:= \langle N,-N|e^{H(\mathbf{t},\tilde{\mathbf{t}})}g|0,0\rangle \\ &:= \langle N,-N|e^{H(\mathbf{t},\tilde{\mathbf{t}})}e^{A}|0,0\rangle \\ &= \frac{1}{N!}(-1)^{\frac{1}{2}N(N+1)}\mathbf{Z}_N^{(2)}(\mathbf{t},\tilde{\mathbf{t}})\end{aligned}$$

To prove this, we use:

$$\langle N,-N|\prod_{i=1}^{N} f^{(1)}(x_i)\bar{f}^{(2)}(y_i)|0,0\rangle = (-1)^{\frac{1}{2}N(N+1)}\Delta_N(x)\Delta_N(y)$$
$$\langle N,-N|\prod_{i=1}^{k} f^{(1)}(x_i)\bar{f}^{(2)}(y_i)|0,0\rangle = 0 \quad \text{if } k\neq N$$

and

$$\langle N,-N|A^k|0,0\rangle = 0 \qquad \text{if} \quad k\neq N$$

and

$$\begin{aligned}&\langle N,-N|e^{H(\mathbf{t},\tilde{\mathbf{t}})}f^{(1)}_{h_1}\bar{f}^{(2)}_{-h'_1-1}\cdots f^{(1)}_{h_N}\bar{f}^{(2)}_{-h'_N-1}|0,0\rangle \\ &\quad = (-1)^{\frac{1}{2}N(N+1)}s_\lambda(\mathbf{t})\mathbf{s}_\mu(\tilde{\mathbf{t}})\end{aligned}$$
$$h_i := \lambda_i - i + N, \quad h'_j := \mu_j - j + N$$

For the rational integrals, define:

$$F^{(1)}(\xi) := f^{(1)}(\xi_{L_1}) \cdots f^{(1)}(\xi_1)$$

$$F^{(2)}(\mu) := f^{(2)}(\mu_{M_2}) \cdots f^{(2)}(\mu_1)$$

$$\bar{F}^{(1)}(\eta) := \bar{f}^{(1)}(\eta_{M_1}) \cdots \bar{f}^{(1)}(\eta_1)$$

$$\bar{F}^{(2)}(\zeta) := \bar{f}^{(2)}(\zeta_{L_2}) \cdots \bar{f}^{(2)}(\zeta_1)$$

Then

$$\mathbf{I}_N^{(2)} = c_N \langle N_1, -N_2 | F^{(2)}(\mu) F^{(1)}(\xi)\; g\; \bar{F}^{(1)}(\eta) \bar{F}^{(2)}(\zeta) | 0, 0 \rangle$$

where $N_1 := N + L_1 - M_1, \qquad N_2 := N + L_2 - M_2$

$$c_N(\xi, \zeta, \eta, \mu) := \frac{\tilde{\epsilon}(L_1, L_2, M_1, M_2)}{\prod_{n=0}^{N-1} h_n} \frac{\prod_{\alpha=1}^{L_1} \prod_{j=1}^{M_1} (\xi_\alpha - \eta_j)}{\prod_{\beta=1}^{L_2} \prod_{k=1}^{M_2} (\zeta_\beta - \mu_k)} \times \Delta_{L_1}(\xi) \Delta_{L_2}(\zeta) \Delta_{M_1}(\eta) \Delta_{M_2}(\mu),$$

$$\tilde{\epsilon}(L_1, L_2, M_1, M_2)) := (-1)^{\frac{1}{2}N(N+1) + L_2(L_1+M_1) + N(L_1+M_1) + M_2 L_2}$$

Introduce the **"Dressed" fermionic field operators:**

$$d^{(1)}(x) := \sum_{n \geq m \geq 0}^{+\infty} f_n^{(1)} K_{nm} x^m + \sum_{n=-\infty}^{-1} f_n^{(1)} x^n$$

$$\bar{d}^{(1)}(x) := \sum_{m \geq n \geq 0}^{+\infty} x^{-m-1} (K^{-1})_{mn} \bar{f}_n^{(1)} + \sum_{n=-\infty}^{-1} \bar{f}_n^{(1)} x^{-n-1}$$

$$d^{(2)}(y) := \sum_{m \geq n \geq 0}^{+\infty} f_{-n-1}^{(2)} \tilde{K}_{nm} y^{-m-1} + \sum_{n=-\infty}^{-1} \bar{f}_{-n-1}^{(2)} y^{-n-1}$$

$$\bar{d}^{(2)}(y) := \sum_{n \geq m \geq 0}^{+\infty} y^m (\tilde{K}^{-1})_{mn} \bar{f}_{-n-1}^{(2)} + \sum_{n=-\infty}^{-1} \bar{f}_{-n-1}^{(2)} y^n$$

Here K and $\tilde{K}$ are semi-infinite lower triangular matrices defined by

$$P_n(x) = \frac{1}{\sqrt{h_n}}(x^n + \sum_{m=0}^{n-1} K_{nm} x^m),$$

$$S_n(y) = \frac{1}{\sqrt{h_n}}(y^n + \sum_{m=0}^{n-1} \tilde{K}^{-1}_{mn} y^m), \quad n \geq 0$$

This can be achieved by a **dressing transformation**

$$\begin{aligned} d^{(1)}(x) &:= \Omega^{(1)} f^{(1)}(x) {\Omega^{(1)}}^{-1} \\ \bar{d}^{(1)}(x) &:= \Omega^{(1)} \bar{f}^{(1)}(x) {\Omega^{(1)}}^{-1} \\ d^{(2)}(y) &:= \Omega^{(2)} f^{(1)}(y) {\Omega^{(2)}}^{-1} \\ \bar{d}^{(2)}(y) &:= \Omega^{(2)} \bar{f}^{(2)}(y) {\Omega^{(1)}}^{-1} \end{aligned}$$

where

$$\Omega^{(1)} = e^{\sum_{n>m\geq 0} \omega^{(1)}_{nm} f^{(1)}_n \bar{f}^{(1)}_m}$$

$$\Omega^{(2)} = e^{\sum_{n>m\geq 0} \omega^{(2)}_{nm} f^{(2)}_{-n-1} \bar{f}^{(2)}_{-m-1}}$$

$$e^{\omega^{(1)}} = K, \quad e^{\omega^{(2)}} = \bar{K}$$

Notice that, for any N_1 and N_2,

$$\langle N_1, -N_2 | \hat{\Omega}^{(\alpha)} = \langle N_1, -N_2 | \ , \quad \hat{\Omega}^{(\alpha)} |0,0\rangle = |0,0\rangle \ , \quad \alpha = 1,2$$

It follows that

$$\begin{aligned}&\langle N_1, -N_2|F^{(2)}(\mu)F^{(1)}(\xi)\; g\; \bar{F}^{(1)}(\eta)\bar{F}^{(2)}(\zeta)|0,0\rangle\\ &= \langle N_1, -N_2|D^{(2)}(\mu)D^{(1)}(\xi)\; \bar{D}^{(1)}(\eta)\bar{D}^{(2)}(\zeta)Q|0,0\rangle\end{aligned}$$

where

$$\begin{aligned}D^{(1)}(\xi) &:= d^{(1)}(\xi_{L_1})\cdots d^{(1)}(\xi_1)\\ D^{(2)}(\mu) &:= d^{(2)}(\mu_{M_2})\cdots d^{(2)}(\mu_1)\\ \bar{D}^{(1)}(\eta) &:= \bar{d}^{(1)}(\eta_{M_1})\cdots \bar{d}^{(1)}(\eta_1)\\ \bar{D}^{(2)}(\zeta) &:= \bar{d}^{(2)}(\zeta_{L_2})\cdots \bar{d}^{(2)}(\zeta_1)\\ Q &:= \prod_{n=0}^{\infty} e^{-h_n f_n^{(1)} \bar{f}_{-n-1}^{(2)}}\end{aligned}$$

Now define

$$\begin{aligned}a_1(\xi) &:= EQd^{(1)}(\xi)Q^{-1}E^{-1}\\ \bar{a}_2(\zeta) &:= EQ\bar{d}^{(2)}(\zeta)Q^{-1}E^{-1}\\ \bar{a}_1(\eta) &:= EQ\bar{d}^{(1)}(\eta)Q^{-1}E^{-1}\\ a_2(\mu) &:= EQd^{(2)}(\mu)Q^{-1}E^{-1}\end{aligned}$$

where

$$E := \prod_{n=0}^{N_1-1} e^{-h_n^{-1} \bar{f}_n^{(1)} f_{-n-1}^{(2)}}$$

Since

$$\langle 0,0|Q = \langle 0,0|, \qquad E|0,0\rangle = 0$$

$$\langle N_1,-N_2|Q = (-1)^{\frac{1}{2}N_1(N_1+1)}(\prod_{n=0}^{N_2-1} h_n)\langle 0,0|f^{(2)}_{-N_1-1}\cdots f^{(2)}_{-N_2}E$$

we have

$$\begin{aligned}
&\langle N_1,-N_2|F^{(2)}(\mu)F^{(1)}(\xi)\; g\; \bar{F}^{(1)}(\eta)\bar{F}^{(2)}(\zeta)|0,0\rangle \\
=&(-1)^{\frac{1}{2}N_1(N_1+1)}(\prod_{n=0}^{N_2-1} h_n) \\
&\times \langle 0,0|f^{(2)}_{-N_1-1}\cdots f^{(2)}_{-N_2}A_2(\mu)A_1(\xi)\bar{A}_1(\eta)\bar{A}_2(\zeta)|0,0\rangle
\end{aligned}$$

where

$$A_1(\xi) := a_1(\xi_{L_1})\cdots a_1(\xi_1)$$

$$A_2(\mu) := a_2(\mu_{M_2})\cdots a_2(\mu_1)$$

$$\bar{A}_1(\eta) := \bar{a}_1(\eta_{M_1})\cdots \bar{a}_1(\eta_1)$$

$$\bar{A}_2(\zeta) := \bar{a}_2(\zeta_{L_2})\cdots \bar{a}_2(\zeta_1)$$

Finally, applying **Wick's theorem** gives

$$\langle N_1, -N_2|F^{(2)}(\mu)F^{(1)}(\xi)\ g\ \bar{F}^{(1)}(\eta)\bar{F}^{(2)}(\zeta)|0,0\rangle$$
$$= \det \begin{pmatrix} \langle a_1(\xi_\alpha)\bar{a}_1(\eta_j)\rangle & \langle a_1(\xi_\alpha)\bar{a}_2(\zeta_\beta)\rangle \\ \langle a_2(\mu_k)\bar{a}_1(\eta_j)\rangle & \langle a_1(\mu_k)\bar{a}_2(\zeta_\beta)\rangle \\ \langle f^{(2)}_{N+L_1-M_1+i}\bar{a}_1(\eta_j)\rangle & \langle f^{(2)}_{N+L_1-M_1+i}\bar{a}_2(\zeta_\beta)\rangle, \end{pmatrix}$$

where $< \ .. \ >:=< 0,0| \ .. \ |0,0>$ denotes a vacuum expectation value. Evaluating the pair-wise vacuum expectation values gives

$$\langle a_1(\xi_\alpha)\bar{a}_1(\eta_j)\rangle = \frac{1}{\xi_\alpha - \eta_j} + \sum_{n=0}^{N_1-1} P_n(\xi_\alpha)\tilde{S}_n(\eta_j)$$

$$\langle a_2(\mu_k)\bar{a}_2(\zeta_\beta)\rangle = -\frac{1}{\zeta_\beta - \mu_k} - \sum_{n=0}^{N_1-1} S_n(\zeta_\beta)\tilde{P}_n(\mu_k)$$

$$\langle a_1(\xi_\alpha)\bar{a}_2(\zeta_\beta)\rangle = \sum_{n=0}^{N_1-1} P_n(\xi_\alpha)S_n(\zeta_\beta)$$

$$\langle a_2(\mu_k)\bar{a}_1(\eta_j)\rangle = -\sum_{n=0}^{N_1-1} \tilde{S}_n(\eta_j)\tilde{P}_n(\mu_k) + H(\mu_k, \eta_j)$$

$$\langle f^{(2)}_{N+L_1-M_1+i}\bar{a}_1(\eta_j)\rangle = \sqrt{h_{N+L_1-M_1+i}}\tilde{S}_{N+L_1-M_1+i}(\eta_j)$$

$$\langle f^{(2)}_{N+L_1-M_1+i}\bar{a}_2(\zeta_\beta)\rangle = \sqrt{h_{N+L_1-M_1+i}}S_{N+L_1-M_1+i}(\zeta_\beta)$$

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