Studies in the Geometry of 2D Spin Glasses: From Fractals to SLE

P. Le Doussal, D. Bernard, LPTENS
A. Alan Middleton, Syracuse University

Support from NSF, ANR; KITP

"Stochastic Geometry and Field Theory", KITP
12 October, 2006

Organization

- Basic picture of 2D spin glasses (SG): model and geometry.
- Rest is a largely “experimental” talk on SG domain walls.
- Our work inspired by Amoruso, Hartmann, Hastings, Moore, cond-mat/0601711.
- Evidence for SLE
  - Multiple tests.
  - Compare with other processes (LERW, MST) to evaluate discriminatory power of the tests.

Hamiltonian with quenched disorder

Edwards-Anderson Hamiltonian for Ising spins $s_i = \pm 1$ on a lattice

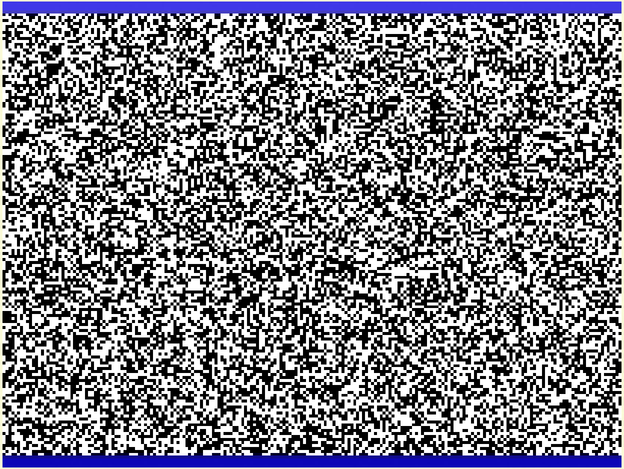
$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} s_i s_j \; ,$$

with J_{ij} Gaussian, zero mean, unit variance.

$T = 0$ phase $\Rightarrow$ replace thermodynamic average with averages over J_{ij}, selecting ground states.

Ground states

Here is a ground state (220×150), BCs periodic along x:



Call this $\alpha = \{s_i^0\}$; spin-flipped state $\beta = \{-s_i^0\}$ is also a ground state for same $\{J_{ij}\}$.

Ground States

Given spins $\{s_i\}$, let

$$F_{ij} = J_{ij} s_i s_j$$

be the "satisfaction" on the bond dual to $\langle ij \rangle$.
Flipping a block gives $F_{ij} \to -F_{ij}$ on the block boundary, with

$$\Delta E = 2 \sum_{\mathrm{blockbdy}} F_{ij}$$

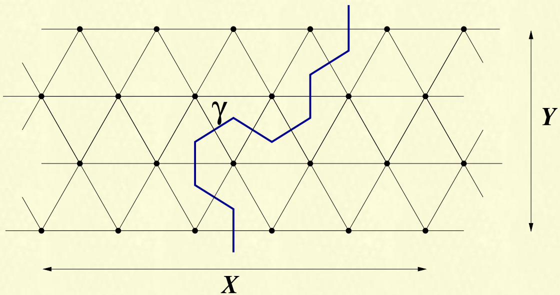
Numerically:

Use Barahona's mapping of planar ISG to a matching problem (1982).

- Maximize total satisfaction to get *exact* ground state.
- The ground state has no negative loops in F^0_{ij}. [Why do you think it is called the ground state?]
- $t_{\mathrm{solve}} \sim N^2$ in practice.

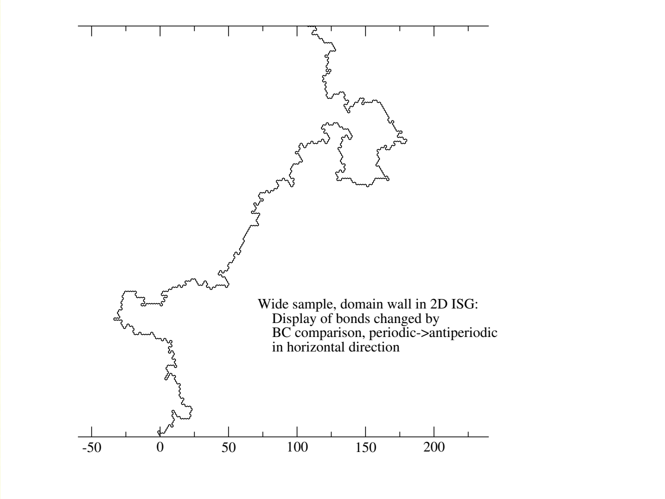
Domain Walls

⇒Changing boundary conditions, say $J_{ij} \to -J_{ij}$ on a column (P→AP), changes F_{ij} constraints.
⇒The DW γ is a shortest path in F^0_{ij} from the top to the bottom (the frustration of elementary triangles is unchanged).
⇒$E_{\mathrm{DW}} = \sum_{\gamma \text{ minimizer}} F^0_{ij}$.



L rows of W spins; $X = W$, $Y = \frac{\sqrt{3}}{2}L$.

Domain Wall



Up to 720^2 or 1024×512 with good statistics ($N > 10^4$, 1% errors).
Study “free-free” (FF) and “free-localized” (FL) BCs.

Energies and fractals

Structure is seen in the *excitations*: domain walls (Bray, Moore, McMillan) and droplets (Fisher, Huse).

The distribution of domain wall energies (centered at zero) has a width that scales with system size.

Scaling hypotheses, well confirmed by numerics:

$$|E_{\mathrm{DW}}| \sim L^{\theta}, \theta = -0.28(1).$$

$$|\gamma| \sim L^{d_f}, d_f = 1.28(1).$$

Other curves we study

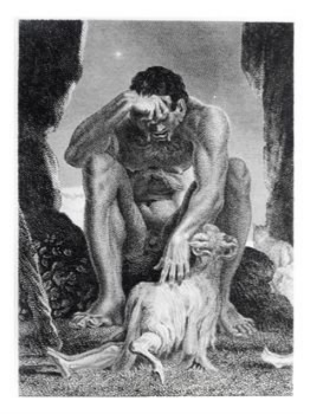
- Loop-erased random walk (LERW): execute a random walk; whenever it returns to the same point, forming a loop, clip the loop, continue with the random walk.
- [Minimal spanning tree (MST): $d_f = 1.217(3)$.]

Numerically, $d_f \approx 1.250$ for LERW, *independent* of whether the boundary conditions are reflecting or absorbing.
But SLE, predicts $d_f = \frac{5}{4}$, does not apply to reflecting BCs!

So watch out for impostors

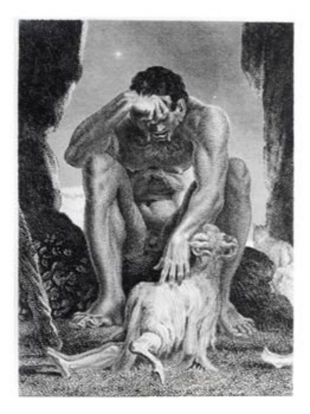
Odysseus

Sheep? Or Sheep With Odysseus?



Is it good-natured and simple, like SLE and sheep?

Sheep? Or Sheep With Odysseus?



Or is the 2DISG carrying Odysseus, like LERW with reflecting BCs with a fractal dimension [or MST - see D. Wilson papers]? Check the underside.

Geometry and boundaries in the 2D ISG

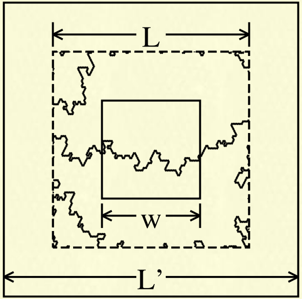
Two intertwined questions:

- Is there a unique state as $L \to \infty$?
- What is the effect of boundary conditions?

Thermodynamic limit: does it exist?

Numerical experiment:

- Solve for GS in a system of size L.
- GROW system to size L', solve for GS.
- Compare in common region, especially in box of size L.



If expanding system induces one domain wall, probability of change in window scales as

$$f(L'/L) \times \left(\frac{w}{L}\right)^{d-d_f},$$

with $f(x)$ approaching a constant at large x. *Confirmed for several disordered models, including 2DISG.*

BCs ⇒Hierarchy of DWs

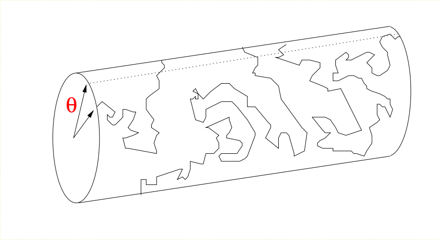
What if one fixes the boundary and moves it around?

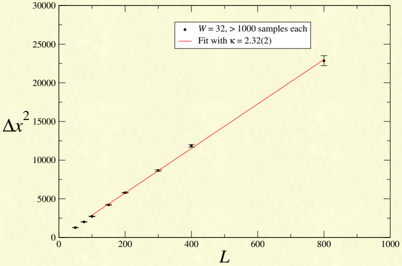
⇒Similar effect seen when overlap *all possible BCs*.
⇒Similar to that seen in problems such as non-intersecting polymer in a random medium *with positive edge weights*.

Note: fixing endpoint of 2DISG DW is awkward & expensive.

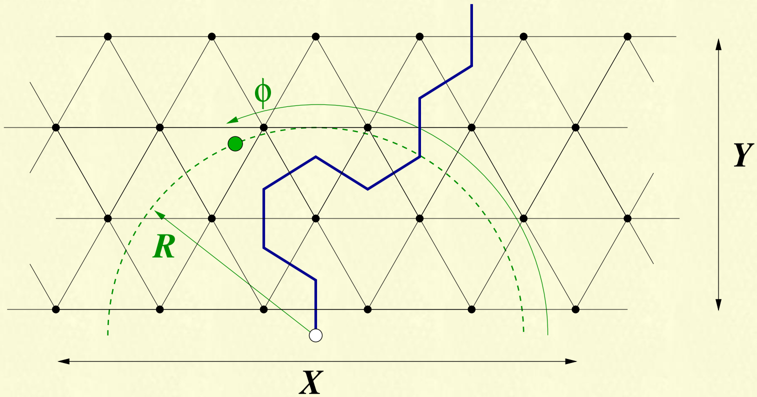
Conformal Invariance

If SLE, $d_f = 1 + \frac{\kappa}{8}$; prediction of winding number on cylinder of circumference 2π: $\langle\theta^2\rangle = \kappa L$.





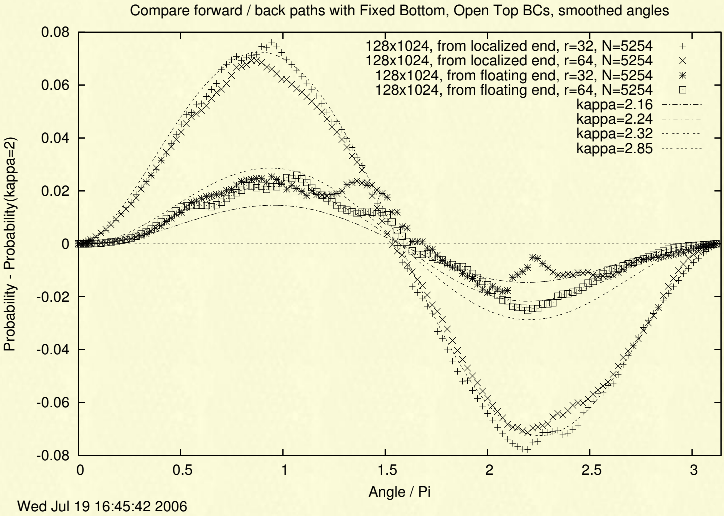
Probability of DW passing to right of a point



Schramm:

$$P_\kappa(\phi) = \frac{1}{2} + \frac{\Gamma\left(\frac{4}{\kappa}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{8-\kappa}{2\kappa}\right)} \tan(\phi) F_{12}\left(\frac{1}{2}; \frac{4}{\kappa}, \frac{3}{2}; -\tan^2(\phi)\right)$$

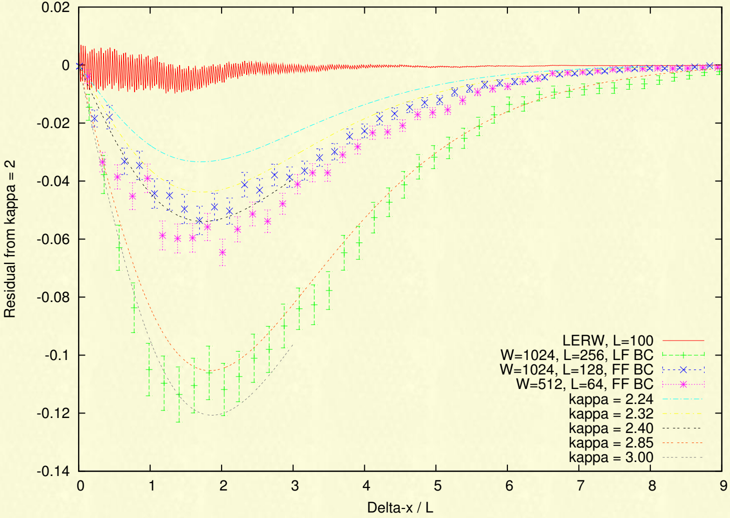
Probability of DW passing to the right of a point



Plot of residuals $P_\kappa - P_2$, where $P_2(\phi) = \frac{\phi - \sin(2\phi)/2}{\pi}$.

Endpoint distribution on a strip

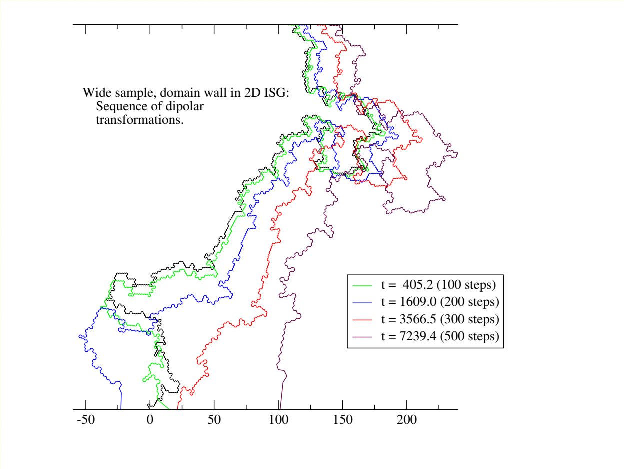
Compare with formula Bauer & Bernard, plot location of endpoint relative to start:



More detailed checks of SLE?

Are we *sure* it is just a sheep, yet?
Check the *Markov property;* gives Brownian motion for $\xi(t)$.

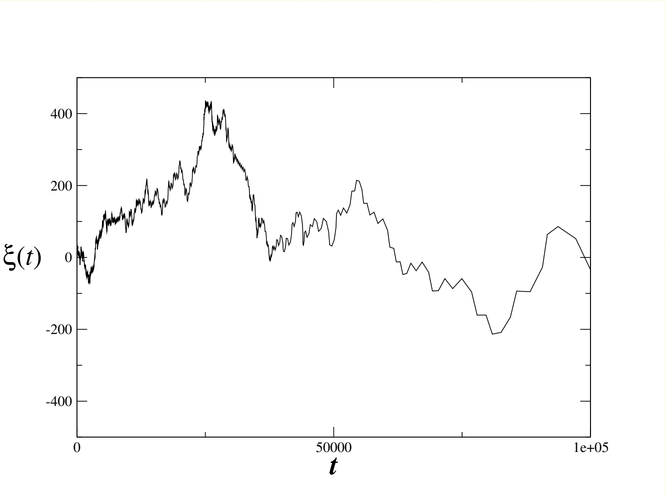
Dipolar maps



Sequence of maps $g_{t_i}(z)$ for strip height $\pi/2$ to infer $\xi(t_i)$, approximates

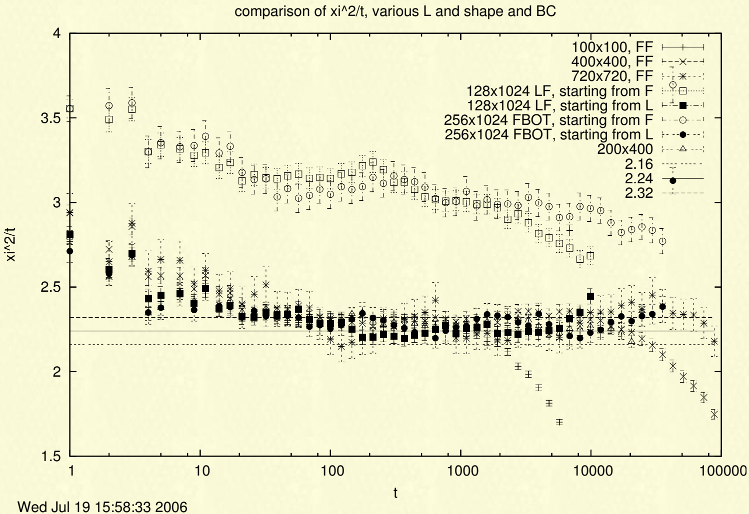
$$\frac{dg_t(z)}{dt} = \frac{2}{\tanh[g_t(z) - \xi_t]} \; ; \; g_0(z) = z$$

Dipolar maps



Dipolar maps

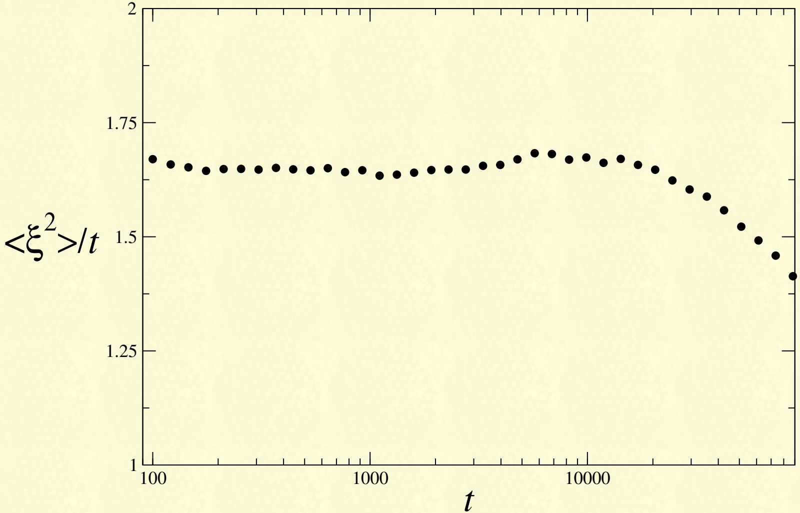
Is $\xi^2(t) = \kappa t$? Yes for FF BCs, no for FL.



Is ξ Gaussian? Yes. Is ξ correlated? Doesn't look like it.

Odysseus can hide (poorly) with $\xi(t)$

MST with $L = 256$, $N = 40000$:



(cf. expected $\kappa \approx 1.74$)

Markov Property

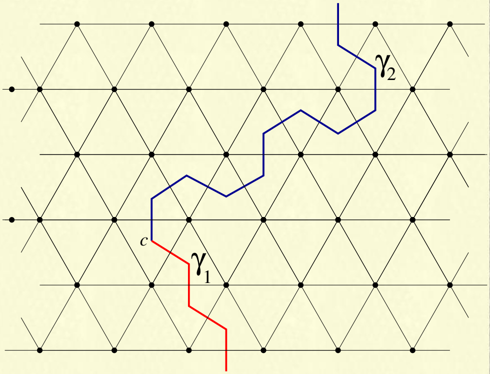
Should be given by studying $\xi(t)$, but maybe hard to see non-Markovian effects. Another approach is to directly study

$$P[\gamma_2(b,c)|\gamma_1(a,c);a,b,c,\mathbb{D}] \stackrel{?}{=} P'[\gamma_2(b,c)|c;\mathbb{D}\backslash\gamma_1,a]$$

[Note, approximate study by Hartmann, Amoruso]
"Must fail microscopically" - Hastings.

Sample, sample, sample, compute P

E.g., 4×10^7samples, 6×6 spins.
Pick a γ_1: filter data for DWs on original strip $\mathbb{D}$ by γ_1, compute probability P of γ_2.
Compare with Probability P' of γ_2 in $\mathbb{D} \backslash \gamma_1$, given space of $J_{ij}(\mathbb{D} - \gamma_1)$, where DW starts at c.



Example: 6×6**, 2DISG,** $\gamma_1 = (\mathrm{R}, \mathrm{U}, \mathrm{R}, \mathrm{U})$

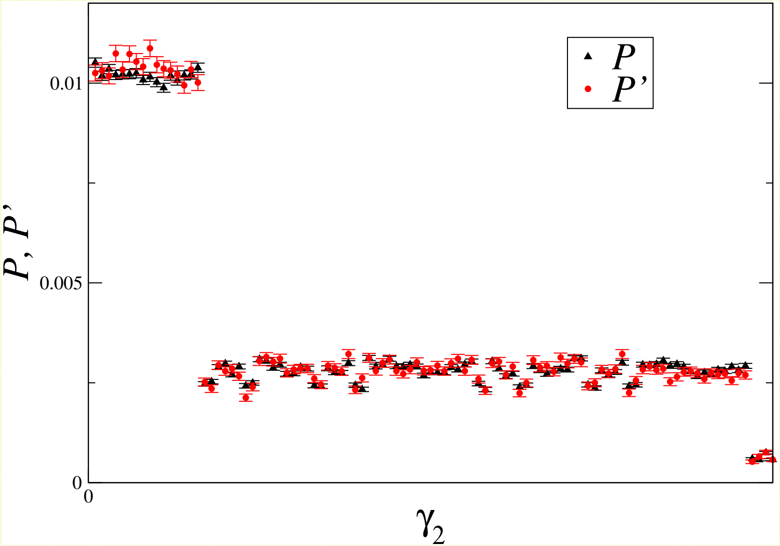
Sort the possible $\gamma_2$lexicographically, compare P and P'.
E.g., $\gamma_2 =$ (34583 distinct paths *seen*)

γ_2	$N'(\gamma_2)$
VRVRVRVRV	8157
VRVRVRVLV	7900
$\vdots$	$\vdots$
VLVLVRVRV	7808
$\vdots$	$\vdots$
VRVRVLRLVLV	2255
$\vdots$	$\vdots$
VLVRVRLRLVLRLV	456
$\vdots$	$\vdots$
LVLRLRVRLVLRVLVRVRVLVLRVLVRVRLVLRLVRVLRLV	1

Errors like $\sqrt{N(\gamma_2)}$.

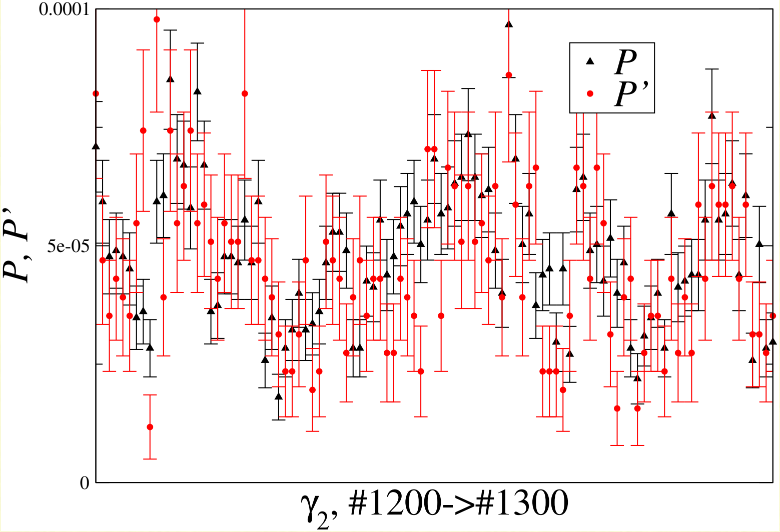
Compare P and P', ISG

2DISG, FF BCs; $\gamma_1 = UVUV$, $L = 6$, $P = P(\gamma_2|\gamma_1;\mathbb{D})$, $P'(\gamma_2|c;\mathbb{D}-\gamma_1)$.


P
P'
0.01
0.005
0
0
P, P'
γ2

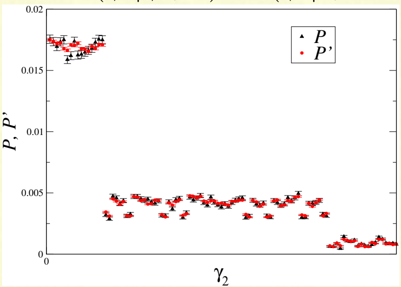
Compare P and P', ISG

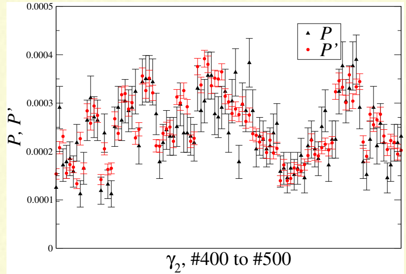
2DISG, FF BCs; $\gamma_1 = UVUV$, $L = 6$, $P = P(\gamma_2|\gamma_1;\mathbb{D})$, $P'(\gamma_2|c;\mathbb{D}-\gamma_1)$.



Markov & a sheep

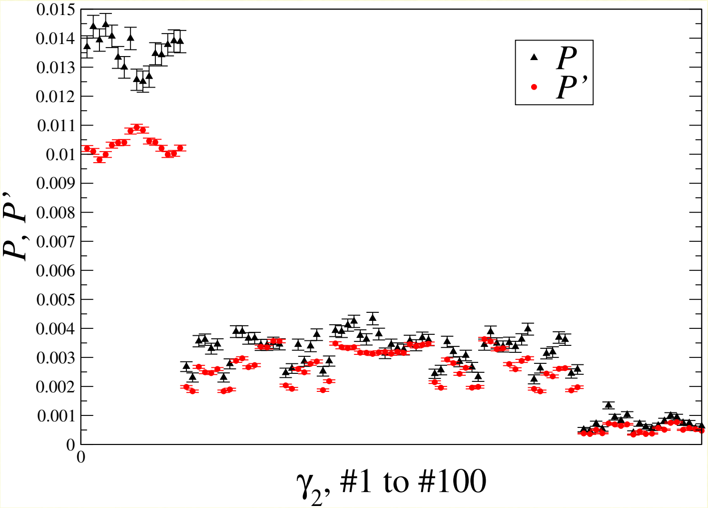
LERW with absorbing boundary conditions, $L = 6$; $\gamma_1 = UVUV$, $P = P(\gamma_2|\gamma_1;\mathbb{D})$, $P'(\gamma_2|c;\mathbb{D}-\gamma_1)$.





Markov & sheep carrying Odysseus

LERW with reflecting boundary conditions, $L = 6$; $\gamma_1 = UVUV$, $P = P(\gamma_2|\gamma_1;\mathbb{D})$, $P'(\gamma_2|c;\mathbb{D}-\gamma_1)$.

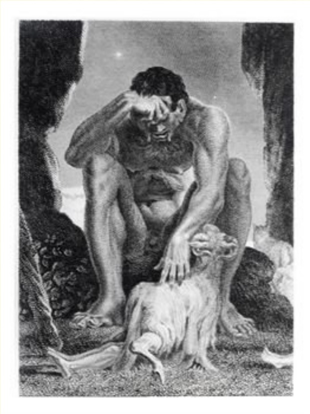


Highlights & Sequels

2DISG:

- $\kappa_{\text{eff}} = 2.30(5)$ consistent with $d_f = 1.28(1)$ for LL BCs, not LF.
- Markov property generally holds for $L \geq 4$.
- Boundary conditions

Apply same analysis path for LERW, MST, to study the utility of numerical checks.
Some MST work still in progress. Other curves? Understanding of Markov property?

Would it be so bad if Odysseus came to visit?