Functional RG and freezing transitions in Anderson localization

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Introduction

- Critical behavior induced by disorder:
  - Resolve the discrepancy between;
    - ★ the Gade singularity of the Density of States (DOS)[1]

$$\nu(\varepsilon) \sim \frac{1}{|\varepsilon|} \exp\left[-\# \left(|\ln|\varepsilon|| \right)^{1/2} \right],$$

    - ★ the Motrunich-Damle-Huse (MDH) singularity of the Density of States (DOS)[2]

$$\nu(\varepsilon) \sim \frac{1}{|\varepsilon|} \exp\left[-\# \left(|\ln|\varepsilon|| \right)^{2/3} \right],$$

  for a single particle in a 2*d* random environment preserving spin-rotation, time-reversal, and sublattice symmetries.

[1] Gade, NPB **398**, 499 (1993)

[2] Motrunich, Damle, and Huse, PRB **65**, 064206 (2002)

Disorder is exactly marginal I

- The key idea is the decoupling transformation under which

$$\begin{aligned}\overline{Z_{\mathrm{SUSY}}(\varepsilon - \mathrm{i}\eta)} &= \int \mathcal{D}[\psi'^{*}, \psi', \beta'^{*}, \beta'; \Phi_1, \Phi_2; \bar{\alpha}', \alpha'] \\ &\times \exp\left\{ -\int d^2\boldsymbol{r} \left[\mathcal{L}'_{\mathrm{crit}} + (\varepsilon - \mathrm{i}\eta)\mathcal{O}'_{\varepsilon-\mathrm{i}\eta} + \mathcal{L}'_{\mathrm{ghost}} \right] \right\}.\end{aligned}$$

- The action $\mathcal{L}'_{\mathrm{crit}}$ is critical:

$$\begin{aligned}
&\langle \Phi_1(z,\bar{z})\Phi_1(0)\rangle_{\mathrm{crit}} = \langle \Phi_2(z,\bar{z})\Phi_2(0)\rangle_{\mathrm{crit}} = -\frac{g_A}{2\pi}\ln|z|, \\
&\left\langle \psi'_{a-}(z)\psi'^{*}_{b-}(0)\right\rangle_{\mathrm{crit}} = +\left\langle \psi'^{*}_{b-}(z)\psi'_{a-}(0)\right\rangle_{\mathrm{crit}} = \frac{\delta_{ab}}{z}, \\
&\left\langle \psi'_{a+}(\bar{z})\psi'^{*}_{b+}(0)\right\rangle_{\mathrm{crit}} = +\left\langle \psi'^{*}_{b+}(\bar{z})\psi'_{a+}(0)\right\rangle_{\mathrm{crit}} = \frac{\delta_{ab}}{\bar{z}}, \\
&\left\langle \beta'_{a-}(z)\beta'^{*}_{b-}(0)\right\rangle_{\mathrm{crit}} = -\left\langle \beta'^{*}_{b-}(z)\beta'_{a-}(0)\right\rangle_{\mathrm{crit}} = \frac{\delta_{ab}}{z}, \\
&\left\langle \beta'_{a+}(\bar{z})\beta'^{*}_{b+}(0)\right\rangle_{\mathrm{crit}} = -\left\langle \beta'^{*}_{b+}(\bar{z})\beta'_{a+}(0)\right\rangle_{\mathrm{crit}} = \frac{\delta_{ab}}{\bar{z}}, \qquad a,b = 1,2.
\end{aligned}$$

Disorder is exactly marginal II

- The bare perturbation to criticality is

$$\begin{aligned}\mathcal{O}'_{\varepsilon-i\eta} \quad := \quad & +e^{+2\Phi_2}\left(\mathcal{B}_{12+}+\mathcal{B}_{21+}-\mathcal{F}_{12+}-\mathcal{F}_{21+}\right)\\ & +e^{-2\Phi_2}\left(\mathcal{B}_{12-}+\mathcal{B}_{21-}+\mathcal{F}_{12-}+\mathcal{F}_{21-}\right),\end{aligned}$$

where

$$\begin{aligned}\mathcal{F}_{12+} &:= \psi'^{*}_{1+}\psi'^{*}_{2-}, & \mathcal{F}_{21+} &:= \psi'^{*}_{1-}\psi'^{*}_{2+},\\ \mathcal{B}_{12+} &:= \beta'^{*}_{1+}\beta'^{*}_{2-}, & \mathcal{B}_{21+} &:= \beta'^{*}_{1-}\beta'^{*}_{2+},\\ \mathcal{F}_{12-} &:= \psi'_{2-}\psi'_{1+}, & \mathcal{F}_{21-} &:= \psi'_{2+}\psi'_{1-},\\ \mathcal{B}_{12-} &:= \beta'_{2-}\beta'_{1+}, & \mathcal{B}_{21-} &:= \beta'_{2+}\beta'_{1-}.\end{aligned}$$

- In the definition of the LDOS the operator to be averaged over is

$$\begin{aligned}\mathcal{O}_{\nu} \quad = \quad & +e^{+2\Phi_2}\left(\mathcal{B}_{12+}+\mathcal{B}_{21+}+\mathcal{F}_{12+}+\mathcal{F}_{21+}\right)\\ & +e^{-2\Phi_2}\left(\mathcal{B}_{12-}+\mathcal{B}_{21-}-\mathcal{F}_{12-}-\mathcal{F}_{21-}\right).\end{aligned}$$

Selecting the perturbations at criticality I

- $\forall \boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^4$,

$$
\begin{aligned}
\mathcal{O}_{\boldsymbol{m},\boldsymbol{n}} &= \left(\mathcal{F}_{12+}\right)^{m_{12+}} \left(\mathcal{F}_{21+}\right)^{m_{21+}} \left(\mathcal{B}_{12+}\right)^{n_{12+}} \left(\mathcal{B}_{21+}\right)^{n_{21+}} \\
&\times \left(\mathcal{F}_{12-}\right)^{m_{12-}} \left(\mathcal{F}_{21-}\right)^{m_{21-}} \left(\mathcal{B}_{12-}\right)^{n_{12-}} \left(\mathcal{B}_{21-}\right)^{n_{21-}} \\
&\times e^{2\left(m_{12+}+m_{21+}+n_{12+}+n_{21+}-m_{12-}-m_{21-}-n_{12-}-n_{21-}\right)\Phi_2},
\end{aligned}
$$

carries the scaling dimension

$$
\begin{aligned}
x_{\boldsymbol{m},\boldsymbol{n}} &= \left(m_{12+} - m_{12-}\right)^2 + \left(m_{21+} - m_{21-}\right)^2 \\
&+ \left|n_{12+} - n_{12-}\right| + \left|n_{21+} - n_{21-}\right| \\
&- \frac{g_A}{\pi} \Big[\left(m_{12+} - m_{12-}\right) + \left(m_{21+} - m_{21-}\right) \\
&\qquad + \left(n_{12+} - n_{12-}\right) + \left(n_{21+} - n_{21-}\right) \Big]^2 .
\end{aligned}
$$

Selecting the perturbations at criticality II

- For given
  $N = m_{12+} + m_{21+} + n_{12+} + n_{21+} - m_{12-} - m_{21-} - n_{12-} - n_{21-}$,
  the most relevant $\mathcal{O}_{\boldsymbol{m},\boldsymbol{n}}$ are given by

$$
\begin{aligned}
&\left(\mathcal{B}_{12\pm}\right)^{n}\left(\mathcal{B}_{21\pm}\right)^{N-n} e^{\pm 2N\Phi_2}, && N \geq n \geq 0,\\
&\left(\mathcal{F}_{12\pm}\right)\left(\mathcal{B}_{12\pm}\right)^{n}\left(\mathcal{B}_{21\pm}\right)^{N-1-n} e^{\pm 2N\Phi_2}, && N-1 \geq n \geq 0,\\
&\left(\mathcal{F}_{21\pm}\right)\left(\mathcal{B}_{12\pm}\right)^{n}\left(\mathcal{B}_{21\pm}\right)^{N-1-n} e^{\pm 2N\Phi_2}, && N-1 \geq n \geq 0,\\
&\left(\mathcal{F}_{12\pm}\right)\left(\mathcal{F}_{21\pm}\right)\left(\mathcal{B}_{12\pm}\right)^{n}\left(\mathcal{B}_{21\pm}\right)^{N-2-n} e^{\pm 2N\Phi_2}, && N-2 \geq n \geq 0,
\end{aligned}
$$

  with the $(2 \times 4N)$-fold degenerate scaling dimension

$$
x_N = N - \frac{g_A}{\pi} N^2 .
$$

Selecting the perturbations at criticality III

- Truncate the family $\{\lambda_{\boldsymbol{m},\boldsymbol{n}}, \mathcal{O}_{\boldsymbol{m},\boldsymbol{n}}\}$ to the subset $\{Y_{\pm;N}, \mathcal{A}_{\pm;N}\}$

$$\begin{aligned}\mathcal{A}_{\pm;N} \quad := \quad & \frac{e^{+2N\Phi_2}}{N!}\Big[(\mathcal{B}_{12\pm}+\mathcal{B}_{21\pm})^N \\ & \mp N(\mathcal{F}_{12\pm}+\mathcal{F}_{21\pm})(\mathcal{B}_{12\pm}+\mathcal{B}_{21\pm})^{N-1} \\ & +N(N-1)\mathcal{F}_{12\pm}\mathcal{F}_{21\pm}(\mathcal{B}_{12\pm}+\mathcal{B}_{21\pm})^{N-2}\Big].\end{aligned}$$

- The dominant OPE's within the subset $\{\mathcal{A}_{\pm;N}\}$ correspond to charge fusion such as

$$\begin{aligned}&\mathcal{A}_{+;N}(z,\bar{z})\,\mathcal{A}_{+;N'}(0,0) = \\ &|z|^{x_{N+N'}-x_N-x_{N'}}\begin{pmatrix} N+N' \\ N \end{pmatrix}\mathcal{A}_{+;N+N'}(0,0)+\cdots, \\ &\qquad\qquad 0<N,N'\in\mathbb{N},\end{aligned}$$

  whereby

$$x_N = N - \frac{g_A}{\pi}N^2 \Rightarrow x_{N+N'} - x_N - x_{N'} < 0.$$

Selecting the perturbations at criticality IV

- The OPE's that correspond to charge annihilation such as

$$\mathcal{A}_{+;N_+}(z,\bar{z})\,\mathcal{A}_{-;N_-}(0,0) = |z|^{x_{N_+-N_-}-x_{N_+}-x_{N_-}} \begin{pmatrix} N_+ - 1 \\ N_- \end{pmatrix} \mathcal{A}_{+;N_+-N_-}(0,0) + \cdots\,, \qquad 0 < N_- < N_+ \in \mathbb{N},$$

can be neglected since

$$x_N = N - \frac{g_A}{\pi} N^2,\ N_+ > \frac{\pi}{g_A} \Rightarrow x_{N_+-N_-} - x_{N_+} - x_{N_-} > 0.$$

Selecting the perturbations at criticality V

- ***Neglecting all annihilation processes gives***

$$0 = \left(\sum_{N=1}^{\infty} \beta_{Y_N} \frac{\partial}{\partial Y_N} - x_1 \right) \nu_{\mathrm{av}}(\varepsilon; \boldsymbol{x}) + \cdots,$$

$$\beta_{Y_N} = (2 - x_N) Y_N + \pi \sum_{N'=1}^{N-1} \binom{N}{N'} Y_{N'} Y_{N-N'} + \cdots,$$

$$x_N = N - \frac{g_A}{\pi} N^2,$$

$$Y_N(l = 0) = \frac{\varepsilon - i\eta}{\mathfrak{a}^{x_1 - 2}} \delta_{1,N}.$$

Case $g_M > 0$ I

Integration over disorder in the average SUSY partition function induces SUSY current-current interactions exclusively. Therein lies the nearly conformal nature[3] of the theory:

Both ε and g_A are relevant whereas g_M is exactly marginal:[4]

$$\beta_\varepsilon = (2 - x_\varepsilon)\varepsilon, \qquad \beta_{g_A} = \frac{(g_M \zeta)^2}{2\pi^2}, \qquad \beta_{g_M} = 0,$$

$$\text{with } x_\varepsilon = \zeta\,[1 + \mathcal{O}(g_M)] - \frac{g_A}{\pi}\zeta^2 \text{ and } \zeta \; := \frac{1}{1 + g_M/\pi}.$$

Computation of the anomalous scaling dimensions of $\mathcal{O}_{\boldsymbol{m},\boldsymbol{n}}$

$$x_{\boldsymbol{m},\boldsymbol{n}} =$$
$$\Big[(m_{12+} - m_{12-})^2 + (m_{21+} - m_{21-})^2 - (n_{12+} - n_{12-})^2 - (n_{21+} - n_{21-})^2\Big]\,\zeta\,[1 + \mathcal{O}($$
$$+ (n_{12+} - n_{12-})^2 + (n_{21+} - n_{21-})^2 + |n_{12+} - n_{12-}| + |n_{21+} - n_{21-}| + \mathcal{O}(g_M$$
$$- \big[(m_{12+} - m_{12-}) + (m_{21+} - m_{21-}) + (n_{12+} - n_{12-}) + (n_{21+} - n_{21-})\big]^2 \frac{g_A}{\pi}\zeta$$

Case $g_M > 0$ II

The average LDOS is conjectured to obey

$$0 = \left[z_A \varepsilon \frac{\partial}{\partial \varepsilon} + \beta_{g_A} \frac{\partial}{\partial g_A} - (2 - z_A) \right] \nu_{\mathrm{av}}(\varepsilon; \boldsymbol{x})$$

whereby

$$z_A \quad := \quad \begin{cases} 2 - \zeta + \frac{g_A}{\pi}\zeta^2 + \mathcal{O}(g_M), & \text{for } g_A \zeta^2 < 2\pi, \\ 4\sqrt{\frac{g_A}{2\pi}}\zeta - \zeta - \mathcal{O}(g_M), & \text{for } g_A \zeta^2 \geq 2\pi. \end{cases}$$

random phase XY model

The random phase XY model is defined by

$$H_{XY}[\phi, A] := \sum_{\langle ij \rangle} J_{ij} \left[1 - \cos\left(\phi_i - \phi_j - A_{ij} \right) \right] .$$

It can be reinterpreted as

the random Cb gas

$$S_{\mathrm{CB}}[\Theta, \theta] := E \sum_{k} (m_k - n_k)^2 - \pi K \sum_{k \neq l} (m_k - n_k)(m_l - n_l) \ln \left| \frac{\boldsymbol{x}_k - \boldsymbol{x}_l}{l_0} \right|$$

where $K := \beta J$, $m_k \in \mathbb{Z}$ are the integer-valued charges of thermal vortices while $n_k \in \mathbb{R}$ are the real-valued charges of quenched vortices, the random Sine-Gordon model

$$\mathcal{L}_{\mathrm{SG}}[\chi, \theta] := \frac{1}{2t}(\partial_\mu \chi)^2 - \frac{h_1}{t}\cos\chi + \frac{\mathrm{i}}{\pi}\chi(\partial_\mu^2 \theta),$$

where

$$K = \frac{t}{4\pi^2}, \quad Y_1 \sim \frac{h_1}{2t}, \quad \theta(\boldsymbol{x}) = \sum_{l=1}^{N} n_l \ln\left|\frac{\boldsymbol{x} - \boldsymbol{y}_l}{l_0}\right|.$$

or the random U(1) Thirring model

$$\mathcal{L}_{\mathrm{Th}}[\bar{\psi},\psi,\theta] := \bar{\psi}(\mathrm{i}\gamma_\mu\partial_\mu + \mathrm{i}m_1)\psi - \frac{g}{2}(\bar{\psi}\gamma_\mu\psi)^2 + (\bar{\psi}\gamma_\mu\psi)(\tilde{\partial}_\mu\theta)$$

where

$$g := \frac{1}{K} - \pi, \quad \mathrm{i}m_1 \propto \frac{h_1}{4\pi^2 K}.$$

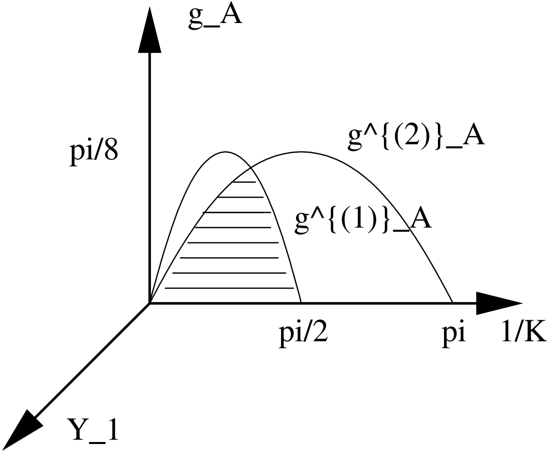

Figure: Boundaries $g_A^{(1)}(1/K)$ and $g_A^{(2)}(1/K)$ in the plane of vanishing fugacity. The shaded area represents the regime of analyticity of the fugacity expansion to fourth order in the fugacity.

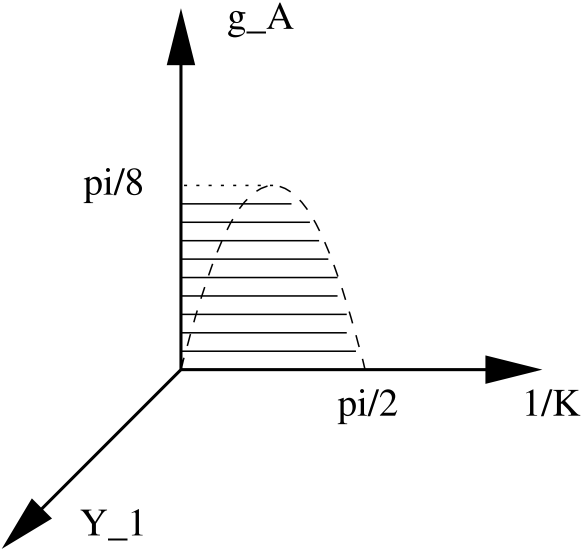

Figure: Proposed phase diagram for CB gas with quenched randomly fractionally charged vortices. $1/K$ is the reduced temperature, g_A the variance of the Gaussian disorder $\tilde{\partial}_\mu \theta$, Y_1 the charge one fugacity for thermal vortices.