KITP Santa Barbara, 1 Nov 2006

The two-dimensional Ising model and SLE

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INSTITUTE for ADVANCED STUDY

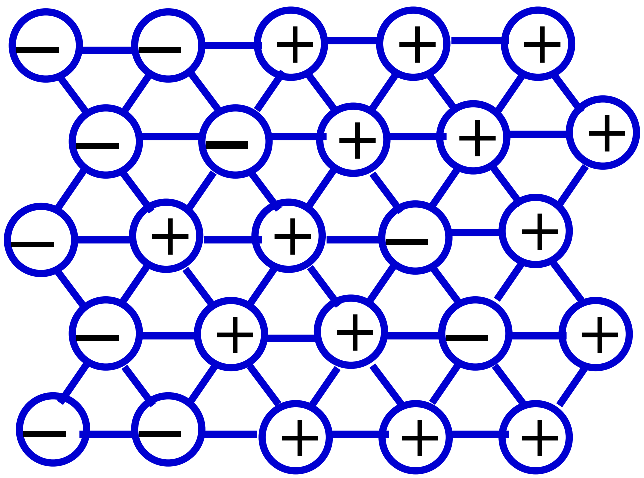
Plan

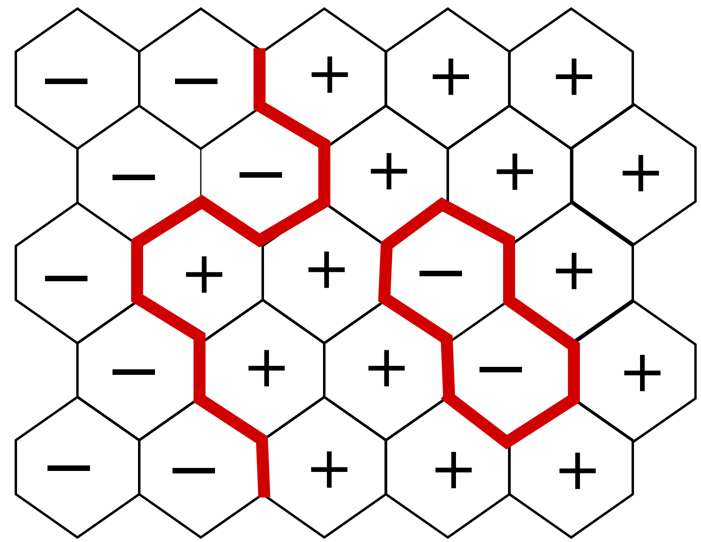
- 2D Ising model and Conformal Field Theory
- Schramm Loewner Evolution (SLE)
- FK representation and holomorphic fermion

work in collaboration with **John Cardy**

Lattice model

(Ising 1925)





$$H = -J \sum_{<ij>} \sigma_i \sigma_j - h \sum_i \sigma_i \qquad \sigma_i \in \{-1, 1\}$$

$$H = -J \sum_{<ij>} \sigma_i \sigma_j - h \sum_i \sigma_i$$

$$\begin{cases} < \mathcal{O} > = \frac{1}{Z} \sum_{\{\sigma\}} \mathcal{O}\, e^{-\beta H[\sigma]} \\ Z = \sum_{\{\sigma\}} e^{-\beta H[\sigma]} \end{cases}$$

$$\begin{cases} M = \frac{1}{N} < \sum_i \sigma_i > \\ g_{ij} = < \sigma_i \sigma_j > \end{cases}$$

Exact solution on square lattice at h=0 (Onsager 1944)

Critical point: $K \equiv \beta J = K_c \,, \quad h = 0$

$$< \sigma(r)\sigma(0) > \propto \begin{cases} \frac{e^{-r/\xi}}{r^{(d-1)/2}} & K << K_c \\ \frac{1}{r^{d-2+\eta}} & K = K_c \end{cases}$$

$\xi \to \infty$

Conformal Field Theory (CFT)

$\xi \to \infty$

universality

$\rightarrow$ Field Theory description of the scaling limit

(Wilson, Fisher 1960's : RG)

critical exponents:

Specific heat: $C \sim |T - T_c|^{-\alpha}$

Order parameter: $M \sim (T_c - T)^{\beta}$

Susceptibility: $\gamma \sim |T - T_c|^{-\gamma}$

...

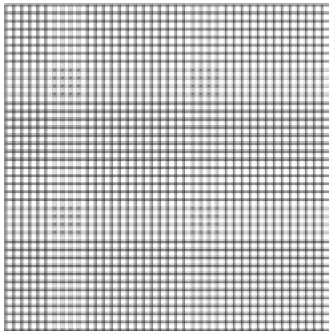
conformal invariance ⟶ Conformal Field Theory

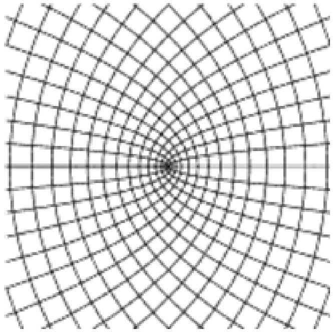
(Belavin-Polyakov-Zamolodchikov 1984)

Conformal transformations:

local translations, rotations and dilatations

preserve angles





2D: analytic functions $z \longrightarrow f(z)$

⟶ equilibrium problems exactly solvable in terms of
- central charge c
- conformal dimensions Δ_ϕ

⟶
- **critical exponents:** Δ_ϕ
- **finite size effects:** $E_0(L) = -\frac{\pi}{6L}\, c$

Transformation properties in CFT

$$\begin{cases} w = f(z) \\ \bar{w} = \bar{f}(\bar{z}) \end{cases} \longrightarrow \qquad \phi(w,\bar{w}) = \left(\frac{df}{dz}\right)^h \left(\frac{d\bar{f}}{d\bar{z}}\right)^{\bar{h}} \phi(z,\bar{z})$$

Dilatation: $\begin{cases} w = \lambda\, z \\ \bar{w} = \lambda\, \bar{z} \end{cases}$ $\qquad \phi(w,\bar{w}) = \lambda^{(h+\bar{h})}\, \phi(z,\bar{z})$

Rotation: $\begin{cases} w = e^{i\theta}\, z \\ \bar{w} = e^{-i\theta} \bar{z} \end{cases}$ $\qquad \phi(w,\bar{w}) = e^{i\,(h-\bar{h})\,\theta}\, \phi(z,\bar{z})$

$$\Delta = h + \bar{h} \qquad s = h - \bar{h}$$

Holomorphic operators: $\bar{h} = 0$

Ising model

$\mathbb{Z}_2$ symmetry $\sigma_i \to -\sigma_i$ $\Rightarrow$ CFT with $c = \frac{1}{2}$

$\{\phi\} = \{\, \mathbb{I}\,,\, \sigma\,,\, \epsilon\,\}$

$$\langle \sigma(x)\sigma(0)\rangle = r^{-2\Delta_\sigma} \qquad \Delta_\sigma = \frac{1}{8}$$

$$\langle \epsilon(x)\epsilon(0)\rangle = r^{-2\Delta_\epsilon} \qquad \Delta_\epsilon = 1$$

$$\langle \sigma_i\, \sigma_{i+n}\rangle = \frac{1}{|n|^{d-2+\eta}} \qquad \langle \epsilon_i\, \epsilon_{i+n}\rangle = \frac{1}{|n|^{2d-2/\nu}}$$

$$\Rightarrow \begin{cases} \eta = 1/4 \\ \nu = 1 \end{cases} \quad \underset{\text{scaling laws}}{\Rightarrow} \quad \begin{cases} \alpha = 0 \\ \beta = 1/8 \\ \gamma = 7/4 \\ \delta = 15 \end{cases}$$

Fermionic Theory with c=1/2

$$S = \frac{1}{2\pi} \int d^2 z \left(\psi \partial_{\bar{z}} \psi + \bar{\psi} \partial_z \bar{\psi} + i m \bar{\psi} \psi \right) \qquad m \propto K - K_c$$

$$\partial_{\bar{z}} \psi = i \frac{m}{2} \bar{\psi}$$

CFT: $m = 0 \quad \Rightarrow \quad \psi(z, \bar{z}) = \psi(z) \qquad \Delta_\psi = s_\psi = \frac{1}{2}$

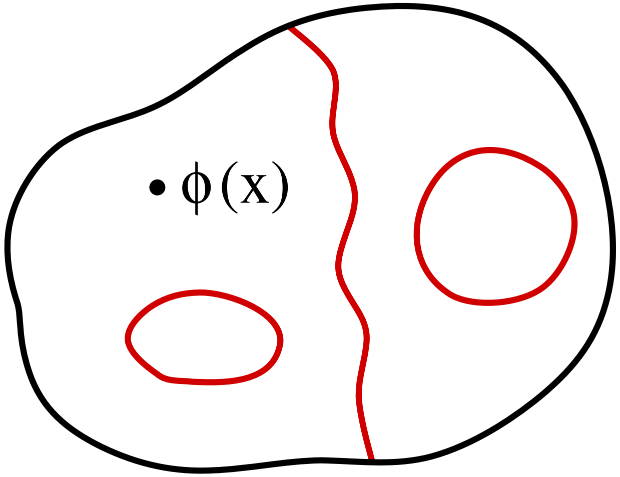
What is missing?

- rigor in relating statistical system to a given model of CFT
- full understanding of non-minimal / non-unitary theories
- satisfactory understanding of geometrical aspects

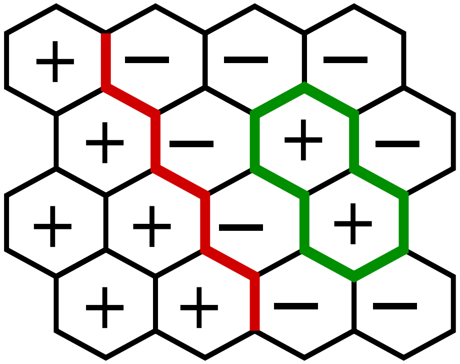
Field Theory is built on local operators

algebraic language

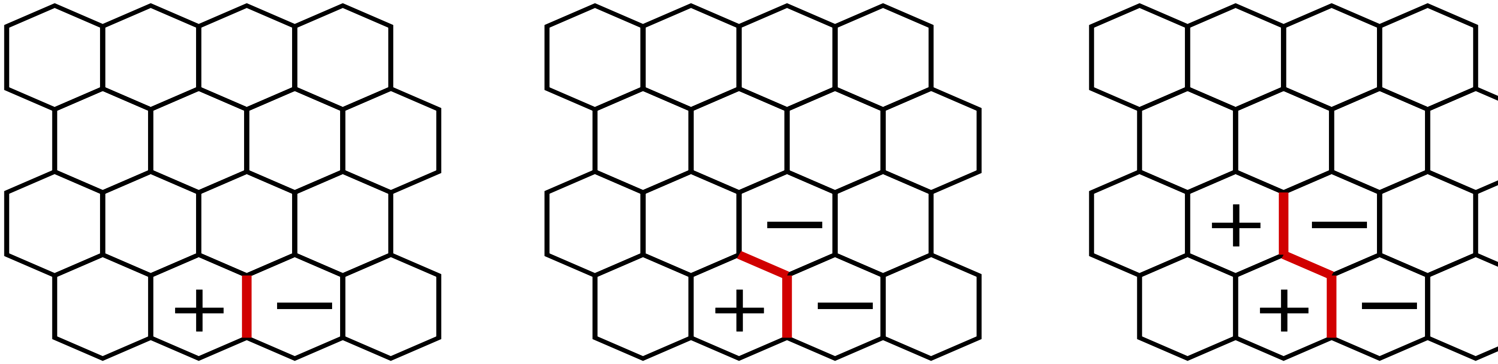
No natural language
for geometric objects like domain walls



Schramm Loewner Evolution (SLE)

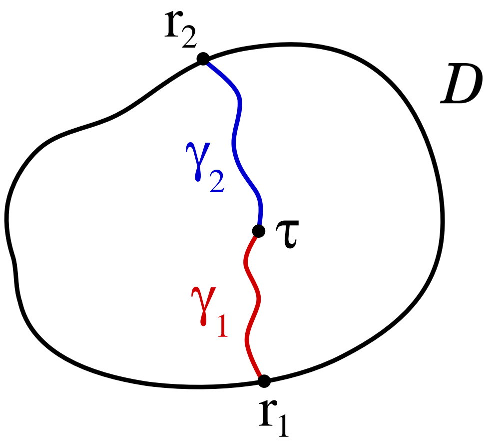


Idea: grow domain wall step by step

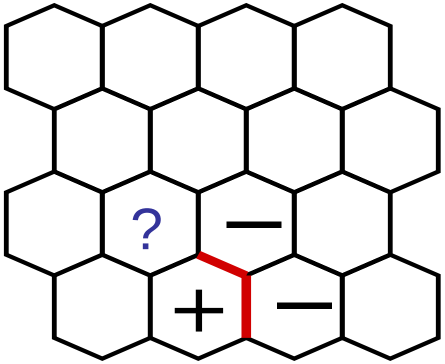


Class of curves in the continuum limit:

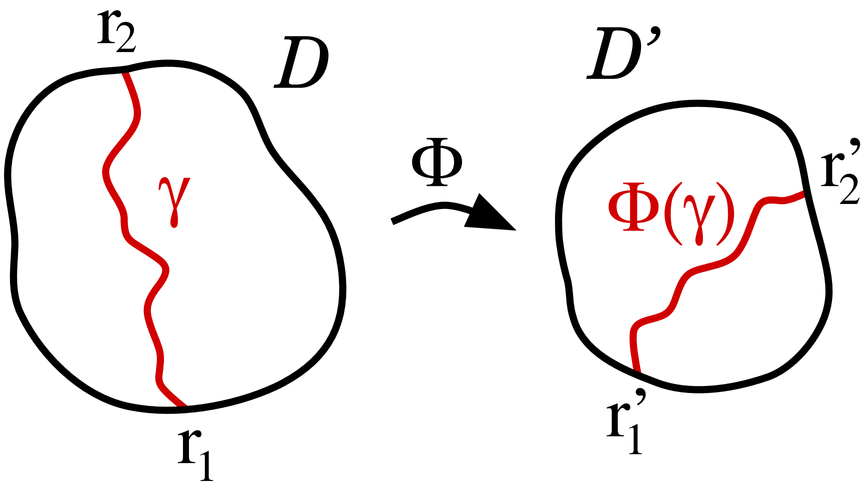
Markov property:

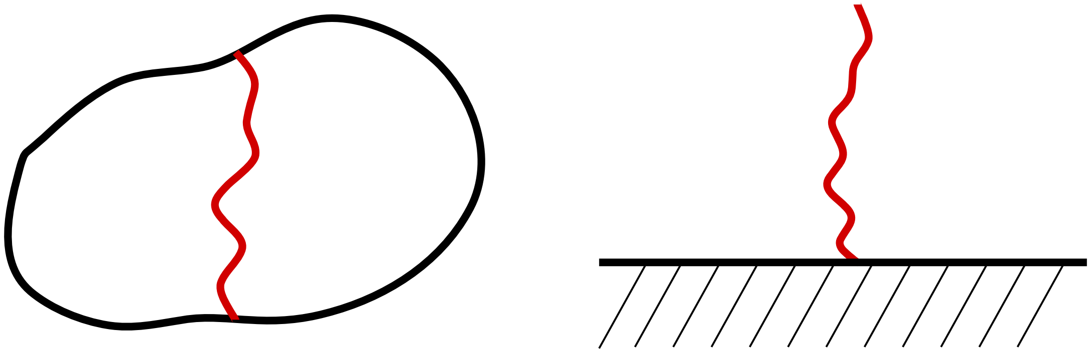


$$\mu(\gamma_2 \,|\, \gamma_1\,;\, D, r_1, r_2) = \mu(\gamma_2\,;\, D\backslash\gamma_1, \tau, r_2)$$

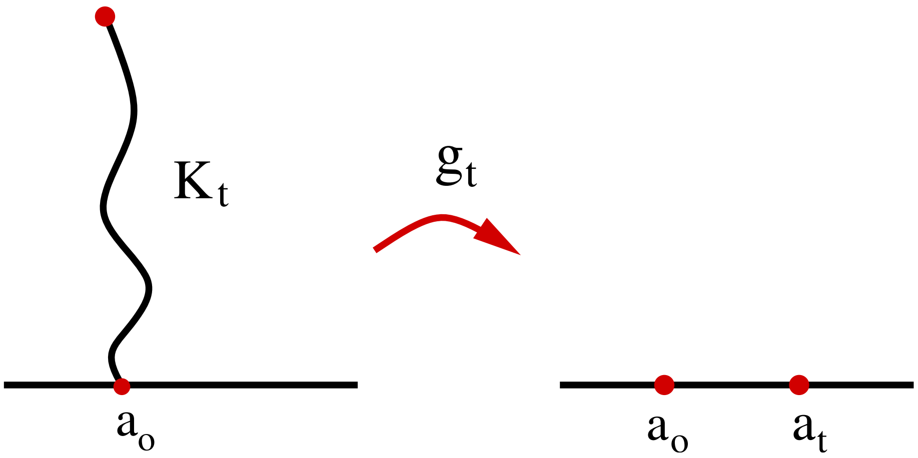


Conformal invariance: $(\Phi * \mu)(\gamma\,;\,D, r_1, r_2) = \mu(\Phi(\gamma)\,;\,D', r'_1, r'_2)$





Non self-crossing curves on the half plane from a_0 to ∞



K_t = curve grown up to time t

$$g_t \; : \; \mathbb{H} \setminus K_t \to \mathbb{H}$$

2D conformal map

Loewner equation:
(1923)

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - a_t}$$

$$a_t \in \mathbb{R}$$

Schramm:
(1999)

Markov property
conformal invariance } $\Rightarrow$

$$a_t = \sqrt{\kappa}\, B_t$$

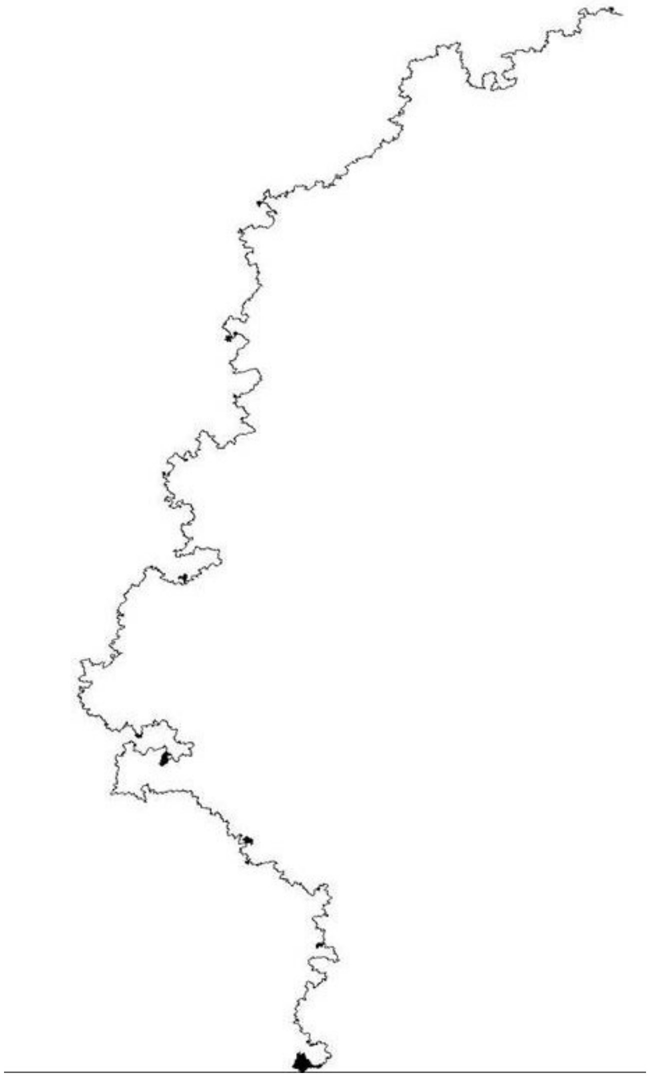
(pictures by Vincent Beffara)

$0 < \kappa \leq 4$

simple curve

$d_f = 1 + \frac{\kappa}{8}$

$\kappa = 2$

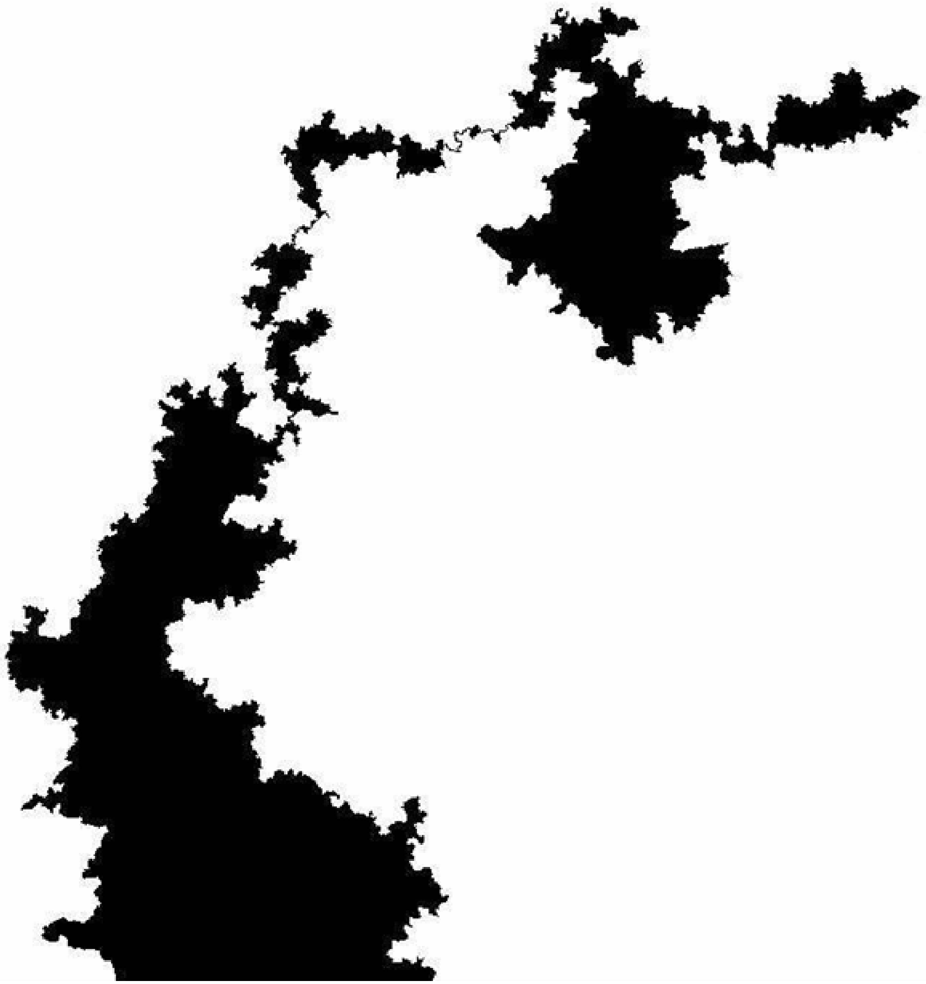


$4 < \kappa \leq 8$

self-intersecting curve

$\kappa = 6$

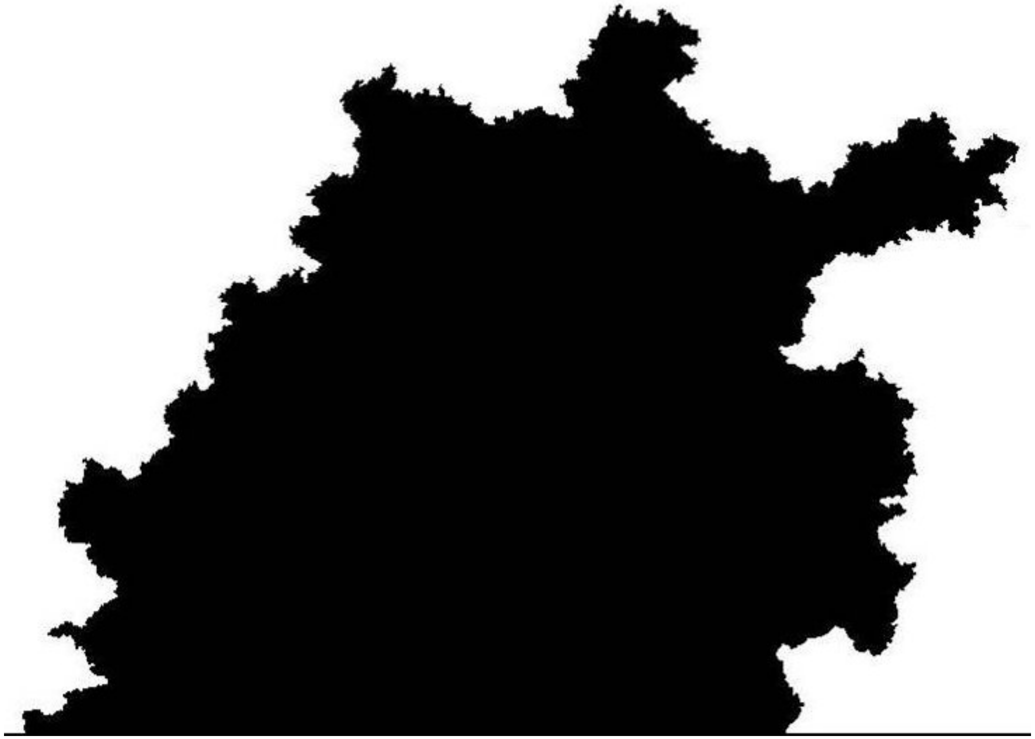
$d_f = 1 + \frac{\kappa}{8}$



$\kappa > 8$

space filling

$d_f = 2$



$\kappa = 9$

Proven results

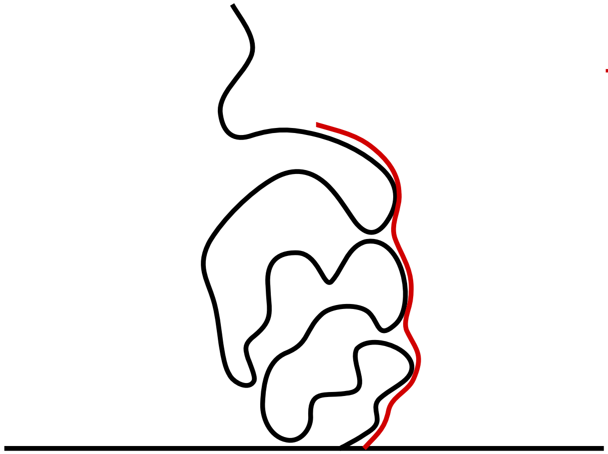
- LERW $\kappa = 2$, spanning trees $\kappa = 8$ (LSW 2000)
- percolation $\kappa = 6$ (Smirnov 2001)
- Ising (FK) $\kappa = \frac{16}{3}$ (Smirnov 2006)

Conjectures

- self-avoiding walks $\kappa = \frac{8}{3}$
- Ising $\kappa = 3$
- Q-state Potts (FK) $\sqrt{Q} = -2\cos\frac{4\pi}{\kappa}$

SLE duality: $\kappa \longleftrightarrow \frac{16}{\kappa}$

(Duplantier, Beffara)



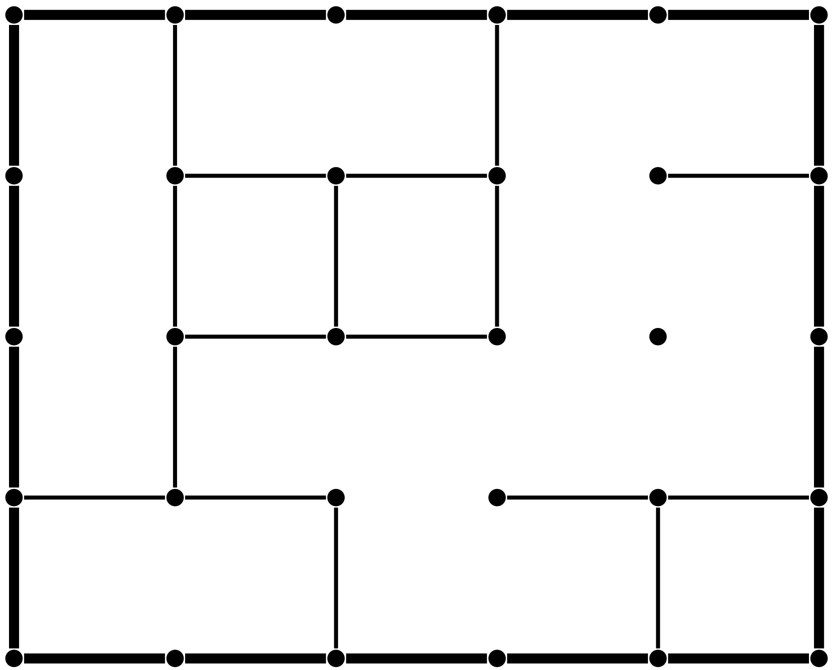
frontier of K_t

Ising: $\kappa = 3 \longleftrightarrow \kappa = \frac{16}{3}$

FK representation

(Fortuin-Kasteleyen 1972)

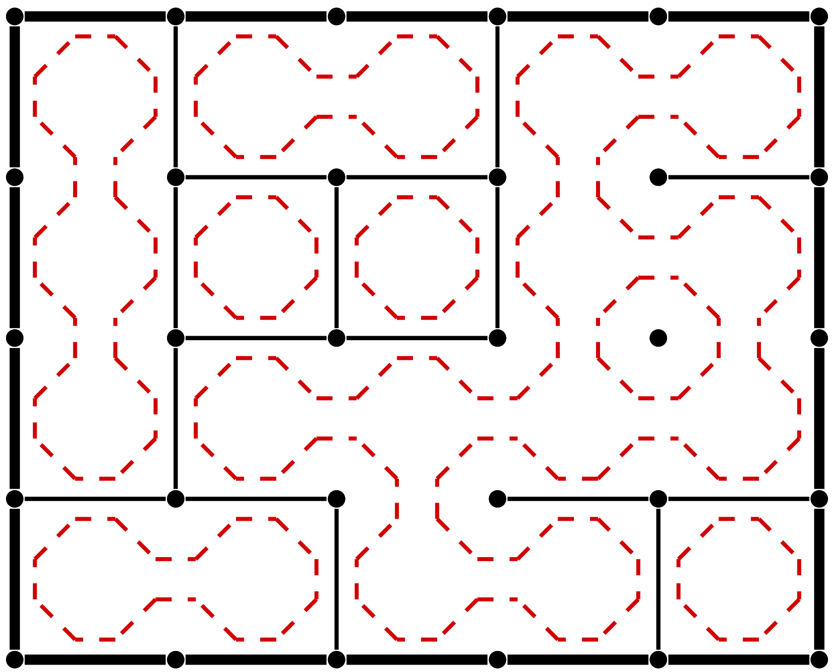
$$Z = \sum_{\{\sigma\}} e^{\sum_{<ij>} K\sigma_i\sigma_j} = \sum_{\{\sigma\}} \prod_{<ij>} \left[1 + u\,\delta(\sigma_i, \sigma_j)\right] \qquad u = e^{2K} - 1$$



$$Z = \sum_G u^b\, 2^c$$

$$Z = \sum_G u^b 2^c \quad = 2^{\frac{N}{2}} \sum_G \left(\frac{u}{\sqrt{2}}\right)^b \sqrt{2}^{\,d}$$

$$c - \ell = N - b$$

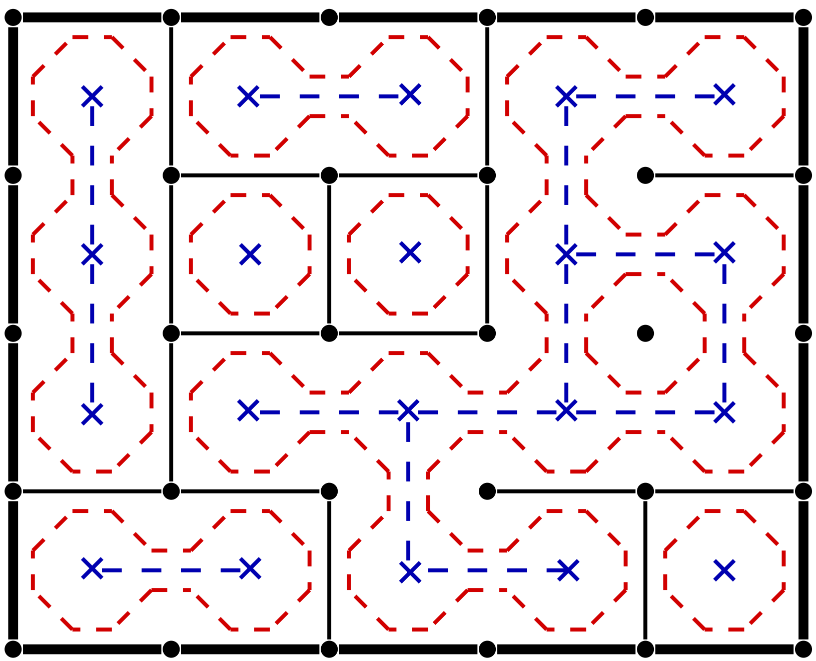


$$d = \ell + c$$

$$Z = \sum_G u^b \, 2^c \quad = 2^{\frac{N}{2}} \sum_G \left(\frac{u}{\sqrt{2}}\right)^b \sqrt{2}^{\,d}$$

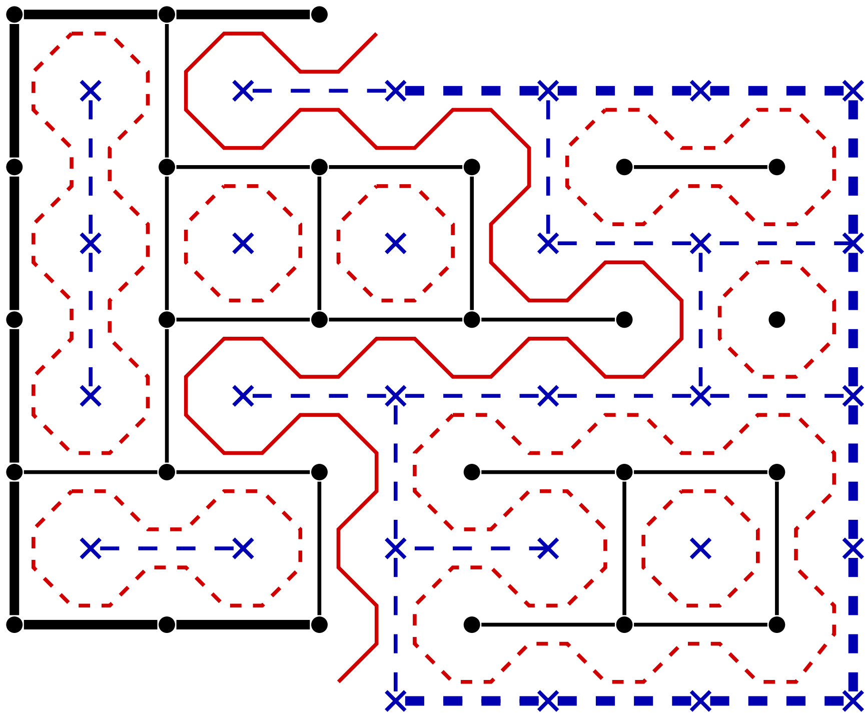
$$\propto \sum_{G^*} \left(\frac{2}{u}\right)^{b^*} 2^{\,c^*}$$

$u_c = \sqrt{2}$



$d = \ell + c$

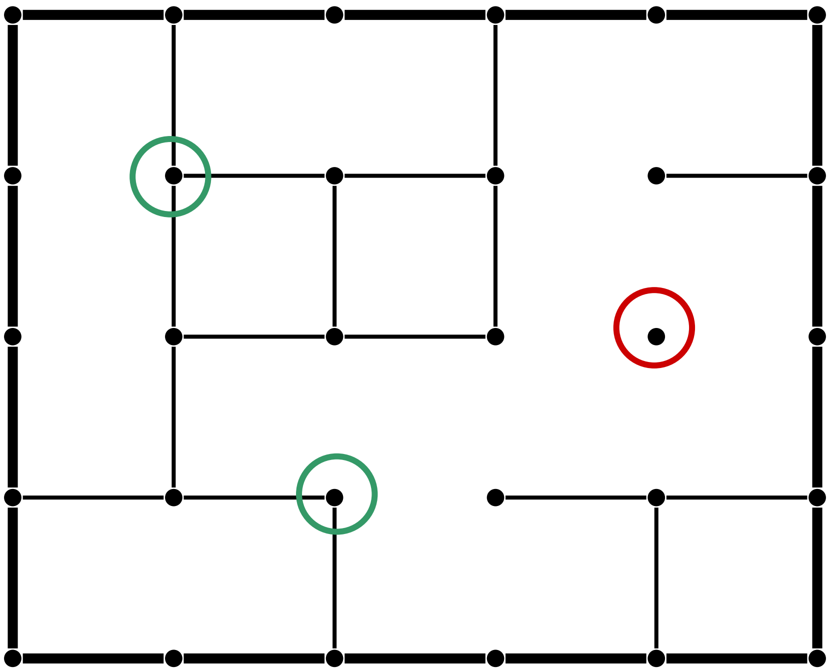
Wired / free boundary conditions → SLE with $\kappa = \frac{16}{3}$



Ising fermion

Order operator

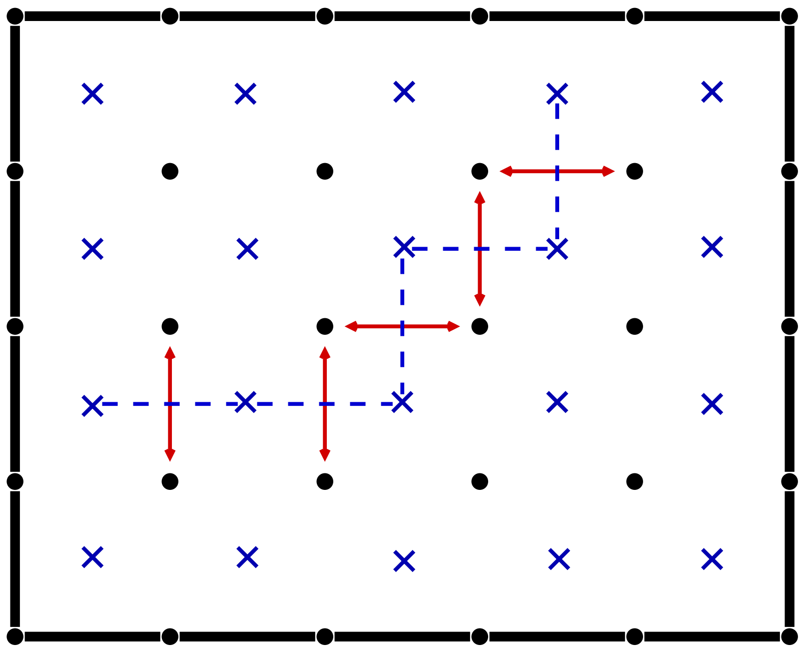
$$Z = \sum_G u^b 2^c \qquad < \sigma_i \sigma_j > = \frac{1}{Z} \sum_{G(i,j)} u^b 2^c$$



Disorder operator (Kadanoff, Ceva, Fradkin)

$$< \mu_k \mu_l > \equiv \frac{Z'}{Z}$$

$$Z = \sum_{\{\sigma\}} e^{\sum_{<ij>} K\sigma_i\sigma_j}$$



$$Z \longrightarrow Z'$$

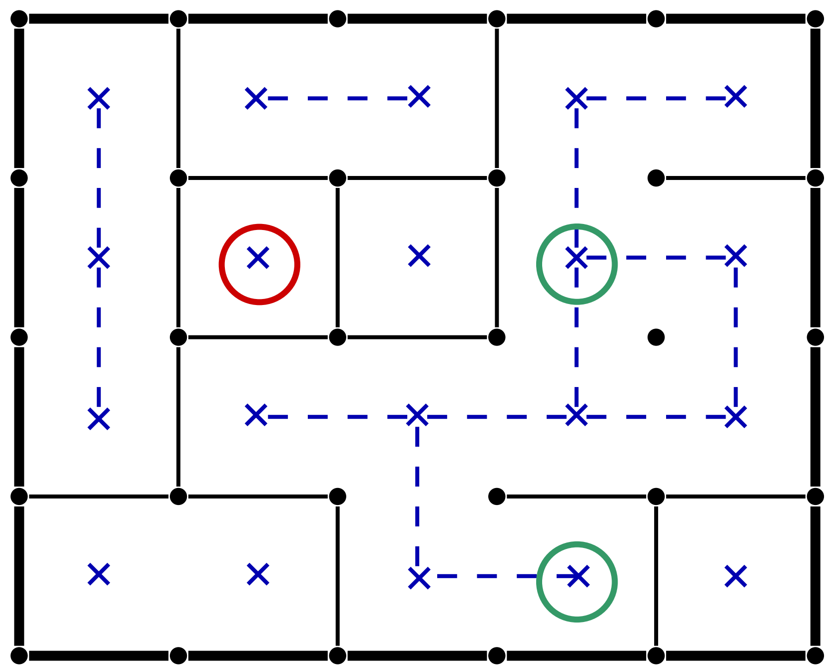
$$\uparrow$$

$$K \to -K$$

$$Z = \sum_G u^b \, 2^c$$

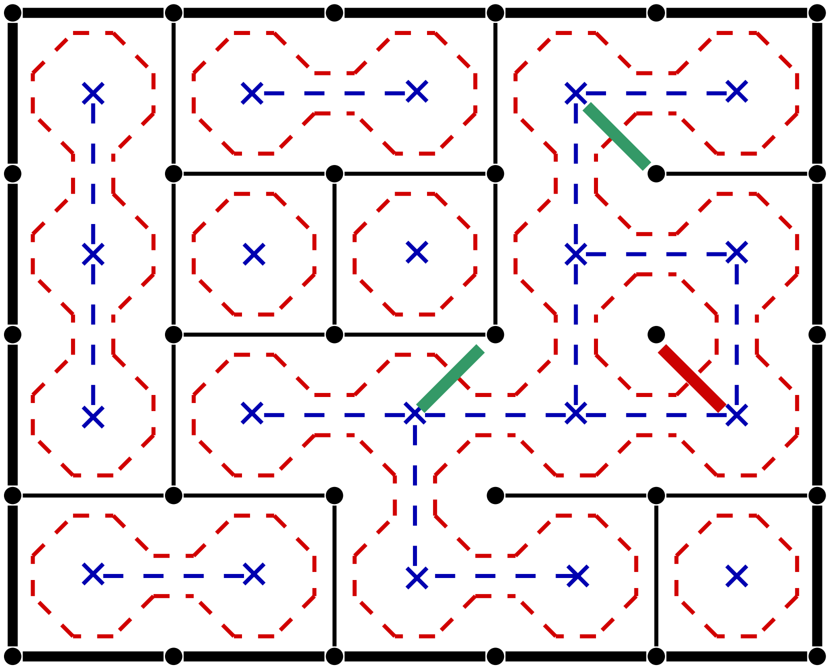
$$< \sigma_i \sigma_j > = \frac{1}{Z} \sum_{G(i,j)} u^b \, 2^c$$

$$< \mu_k \mu_l > = \frac{1}{Z} \sum_{G^*(k,l)} u^b \, 2^c$$



Fermion operator

$$\psi_p(e) = \sigma(e)\mu(e)\, e^{-i\,p\,\theta(\gamma,e)}$$



$$< \psi_p(e_1)\, \psi_p(e_2) >$$

observable of the loops

$$\mathrm{arg}[\psi_p(e)] = -p\,\theta(\gamma, e)$$

(VR, J. Cardy)

Lattice holomorphicity

$p = \frac{1}{2}$

$$\sum_{e \in C} < \psi_p(e_1)\psi_p(e) > \delta z_e = 0 \qquad \Leftrightarrow \qquad 2 \sin\left(p\frac{\pi}{2}\right) = \sqrt{2}$$

discretized version of

$$\oint_C dz < \psi_p(z_1)\,\psi_p(z) >= 0$$

$$\partial_{\bar{z}} < \psi_p(z_1)\,\psi_p(z) >= 0$$

Fermionic Theory

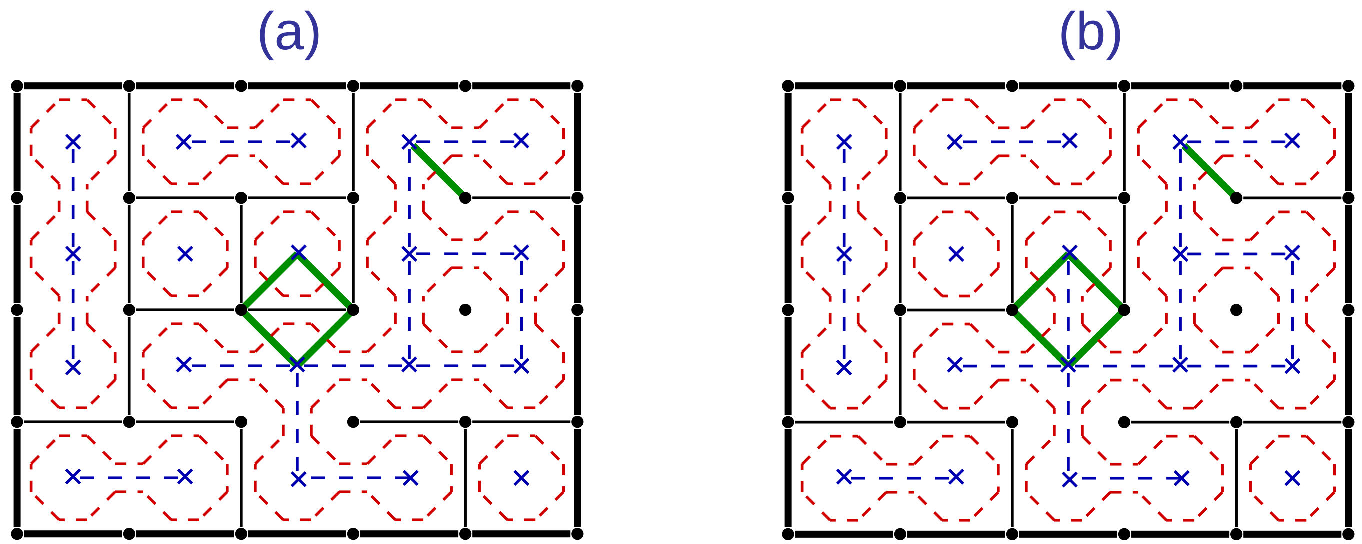
$$S = \frac{1}{2\pi} \int d^2 z \left(\psi \partial_{\bar{z}} \psi + \bar{\psi} \partial_z \bar{\psi} + i m \bar{\psi} \psi\right) \qquad m \propto K - K_c$$

$$\partial_{\bar{z}} \psi = i \frac{m}{2} \bar{\psi} \qquad \Delta_\psi = s_\psi = \frac{1}{2}$$

Proof:

$$p = \frac{1}{2}$$

$$\sum_{e \in C} < \psi_p(e_1)\psi_p(e) > \delta z_e = 0 \quad \Leftrightarrow \quad 2 \sin\left(p\frac{\pi}{2}\right) = \sqrt{2}$$



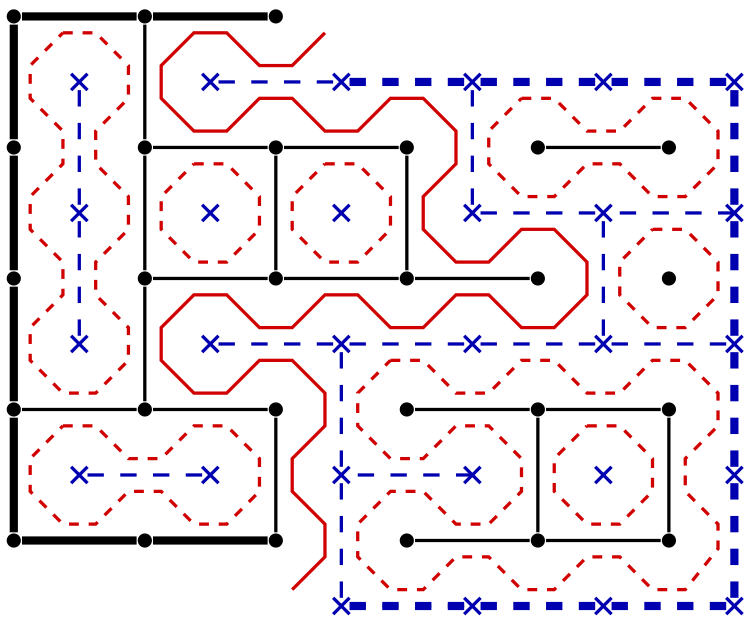
$$\left[1 + (i)e^{ip\frac{\pi}{2}}\right] \underbrace{P(a)}_{\sqrt{2}\,P(b)} + \left[1 + (i)e^{ip\frac{\pi}{2}} + (-1)e^{ip\pi} + (-i)e^{-ip\frac{\pi}{2}}\right] P(b) = 0$$

Connection with SLE

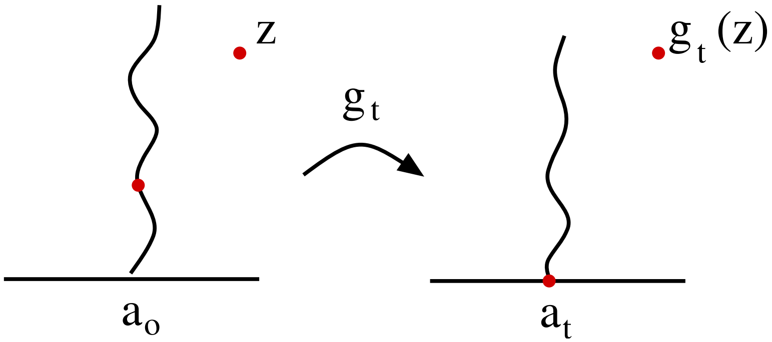
$< \psi_p(e) >$ is an observable of the domain wall

SLE has holomorphic observable of spin 1/2 at $\kappa = \frac{8}{p+1} = \frac{16}{3}$

(Smirnov)



$P(a_0, z, \bar{z})$ = probability that the curve passes to the left / right of $z \in \mathbb{H}$

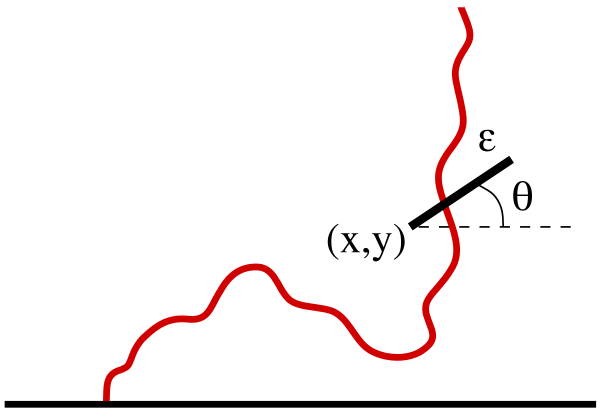


$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - a_t}$$

$$P(a_0, z, \bar{z}) = \left\langle P\left(a_0 + \sqrt{\kappa}\, dB_t\,,\ z + \frac{2dt}{z - a_0}\,,\ \bar{z} + \frac{2dt}{\bar{z} - a_0}\right)\right\rangle$$

$$\Rightarrow \quad \left(\frac{\kappa}{2}\frac{\partial^2}{\partial a_0^2} + \frac{2}{z - a_0}\frac{\partial}{\partial z} + \frac{2}{\bar{z} - a_0}\frac{\partial}{\partial \bar{z}}\right) P(a_0, z, \bar{z}) = 0$$

(B. Doyon, V.R., J. Cardy)

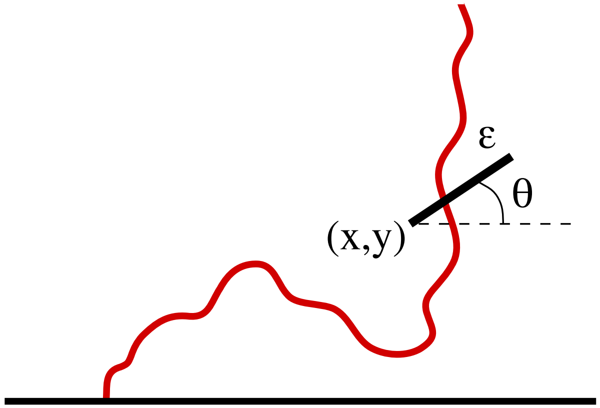


$$P(x, y, \epsilon, \theta)$$

probability that the SLE
curve passes between 2 points

$$\left\{\frac{\kappa}{2}(\partial_w + \partial_{\bar{w}})^2 + \frac{2}{w}\partial_w + \frac{2}{\bar{w}}\partial_{\bar{w}} - \left(\frac{1}{w^2} + \frac{1}{\bar{w}^2}\right)\epsilon\partial_\epsilon + \left(\frac{1}{w^2} - \frac{1}{\bar{w}^2}\right) i\partial_\theta\right\} P(w, \bar{w}, \epsilon, \theta) = 0$$

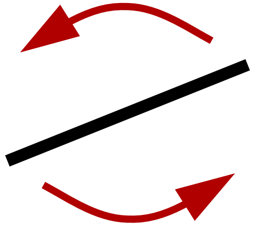
(B. Doyon, V.R., J. Cardy)



$$P(x, y, \epsilon, \theta)$$

probability that the SLE curve passes between 2 points

$$Q_p(x, y, \epsilon) = \int d\theta\, e^{-ip\theta}\, P(x, y, \epsilon, \theta)$$



p has physical meaning of spin

expansion in small ϵ : $Q_p \sim \epsilon^{\Delta_p}$

$$\left\{\frac{\kappa}{2}(\partial_w+\partial_{\bar{w}})^2+\frac{2}{w}\partial_w+\frac{2}{\bar{w}}\partial_{\bar{w}}-\left(\frac{1}{w^2}+\frac{1}{\bar{w}^2}\right)\epsilon\partial_\epsilon+\left(\frac{1}{w^2}-\frac{1}{\bar{w}^2}\right)i\partial_\theta\right\}P(w,\bar{w},\epsilon,\theta)=0$$

$$\left\{\frac{\kappa}{2}(\partial_w+\partial_{\bar{w}})^2+\frac{2}{w}\partial_w+\frac{2}{\bar{w}}\partial_{\bar{w}}-\left(\frac{1}{w^2}+\frac{1}{\bar{w}^2}\right)\epsilon\partial_\epsilon-p\left(\frac{1}{w^2}-\frac{1}{\bar{w}^2}\right)\right\}Q_p(w,\bar{w},\epsilon)=0$$

suppose $Q_p \sim \epsilon^p$

$Q_p(w,\bar{w},\epsilon)=\text{const}\times\left(\frac{\epsilon}{w}\right)^p$ is solution if $\kappa=\frac{8}{p+1}$

Conclusions

- Other models (Q-state Potts,…): parafermions
- Off-critical $\partial_{\bar{z}}\,\psi = i\,\frac{m}{2}\,\bar{\psi}$
- Loops $\longrightarrow$ CLE (Sheffield, Werner)