Spacetime + Quantum Mechanics,
Particles and Strings,
Polytopes and Binary Geometries

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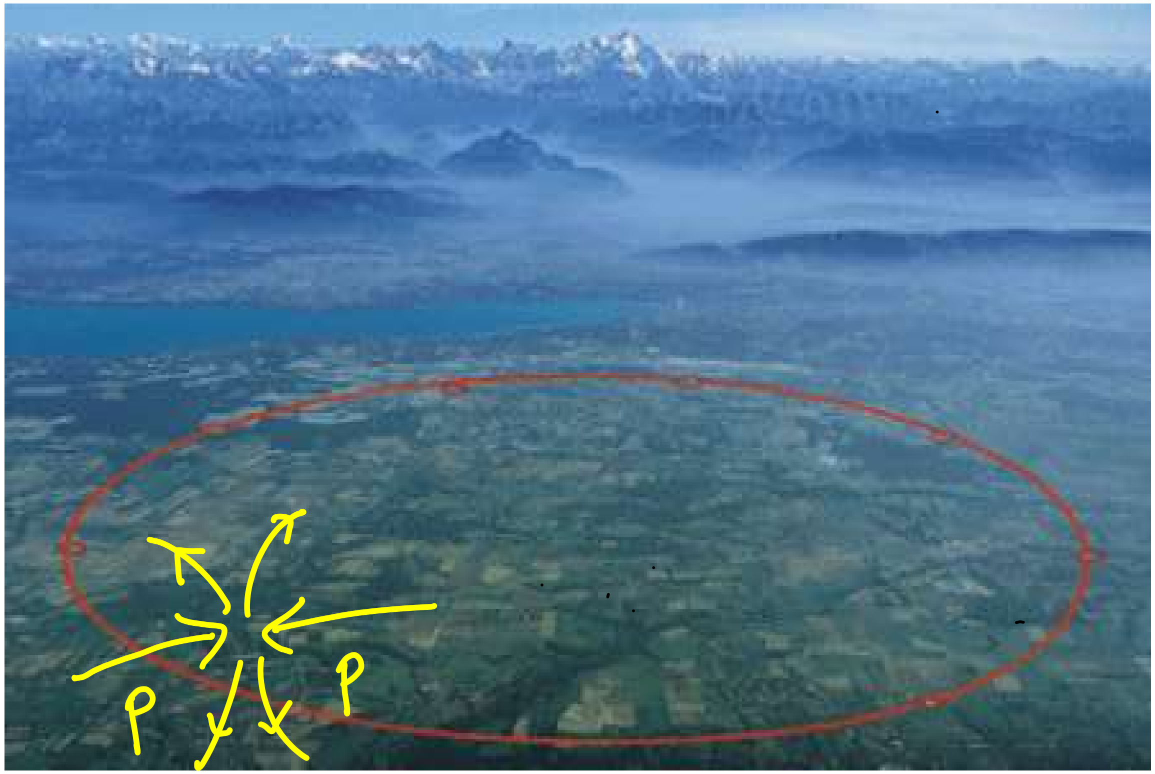
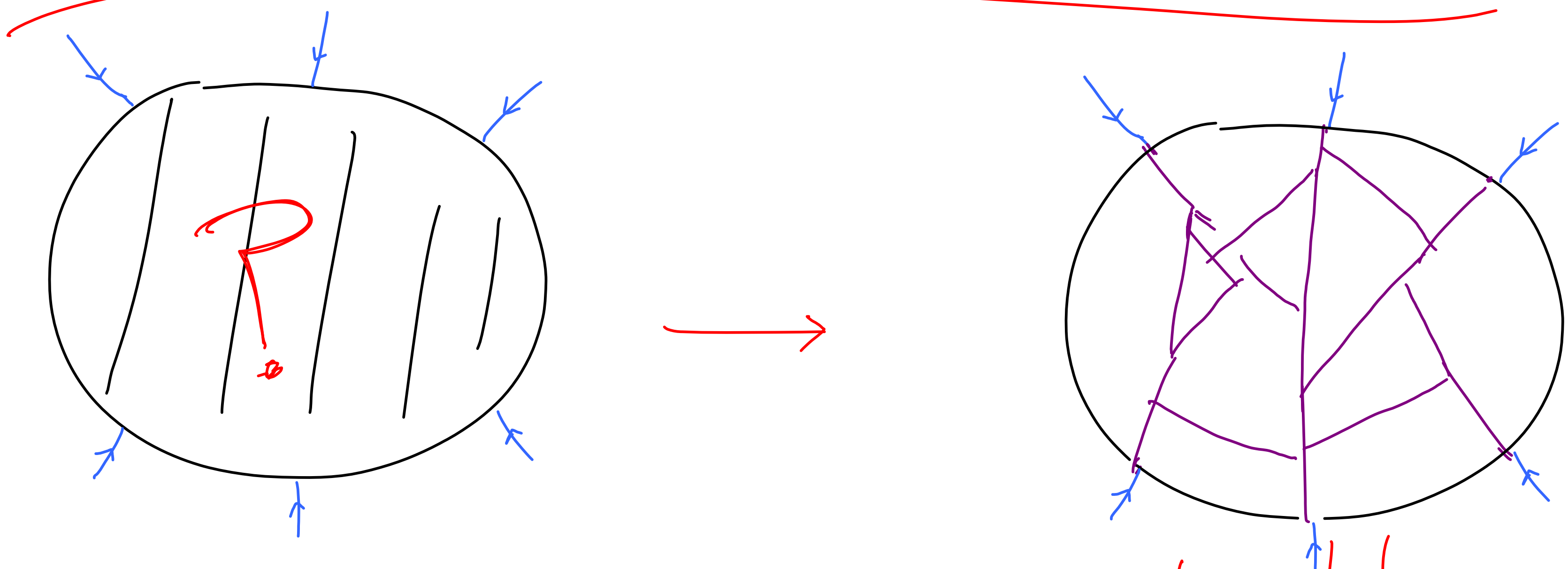
Scattering Amplitudes



Most Basic Observable in Fundamental Physics

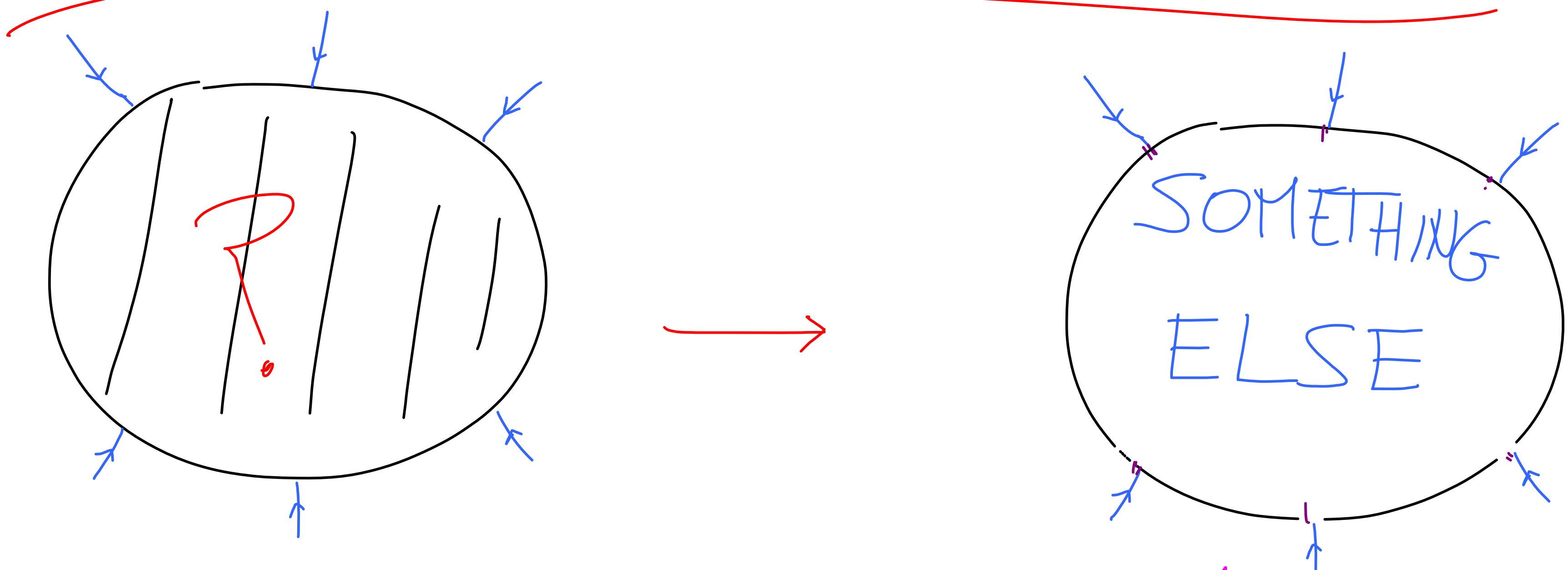
Directly Controlled by Principles of Spacetime + Quantum Mechanics

What is the Q to which A is the Answer?



Local evolution in Spacetime, Unitary evolution in Hilbert Space

What is the Q to which A is the Answer?

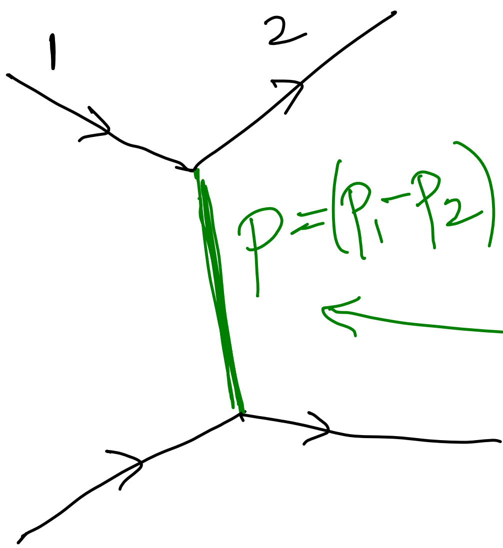


+ see how "something else" gives rise to the properties of Amplitudes we ordinarily ascribe to ST+QM!

"On-shell vs. Off-shell"

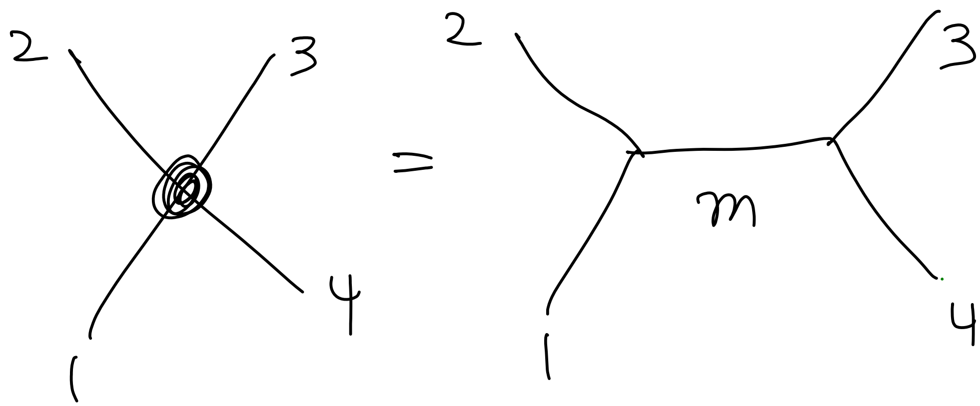
$p^\mu = (E, \vec{p})$

$p^2 = E^2 - \vec{p}^{\,2} = m^2$ "on-shell"



: Propagator

$1/(p^2 - M^2)$, p ~~off~~ shell!



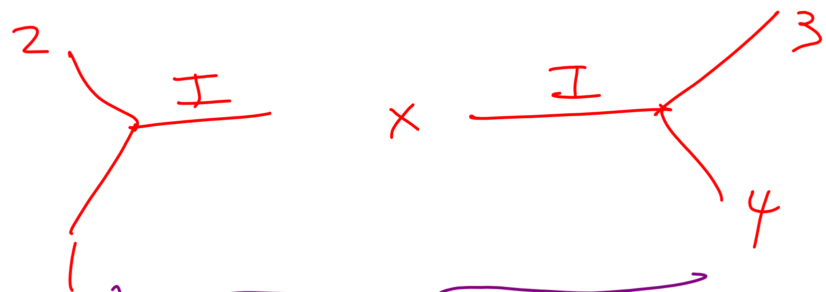
We can have resonance when $(p_1+p_2)^2 \to m^2$!

$$\text{Amp} \sim \frac{1}{(p_1+p_2)^2 - m^2} \cdot (\quad)$$

pole when "virtual particle" becomes "real"

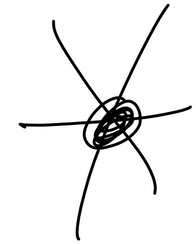
... and Quantum Mechanics tells us

$$A \sim \frac{1}{(p_1+p_2)-m^2} \cdot$$



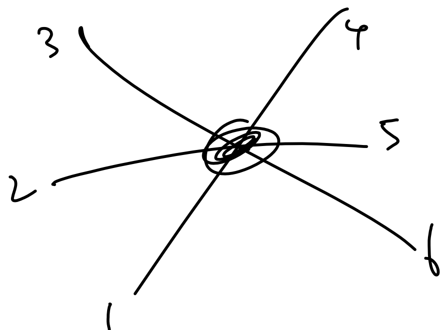
on $p_I^2 = m^2$

"Factorization"

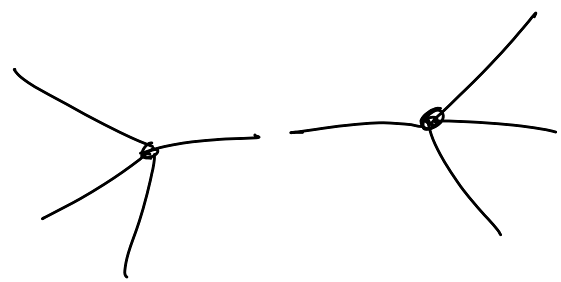
Rules of ST + QM reflected in 

* Locality: only poles when $(p_{i_1}+\cdots+p_{i_m})^2 \longrightarrow m^2$

* Unitarity: Factorization



$$\longrightarrow \frac{1}{p^2-m^2}\,\cdot$$

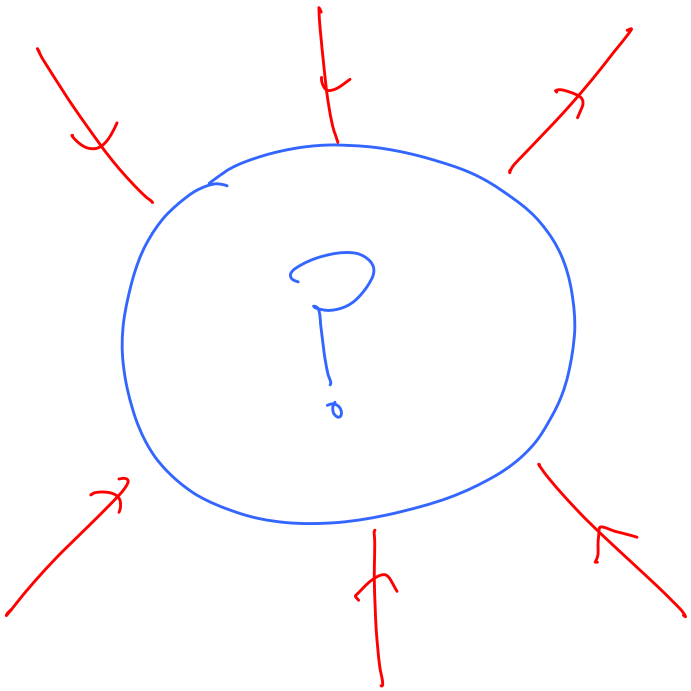


$(p_1+p_2+p_3)^2 \longrightarrow m^2$

Concrete Strategy

Find a different question — directly in kinematical space — to which $A(p_1, \ldots, p_n)$ is the answer, without referring to ST/QM.

But which nonetheless gives correct poles and correct factorization!



Emerging Picture:

Combinatorics

$\updownarrow$

Positive Geometries

$\updownarrow$

Canonical Forms

$\updownarrow$

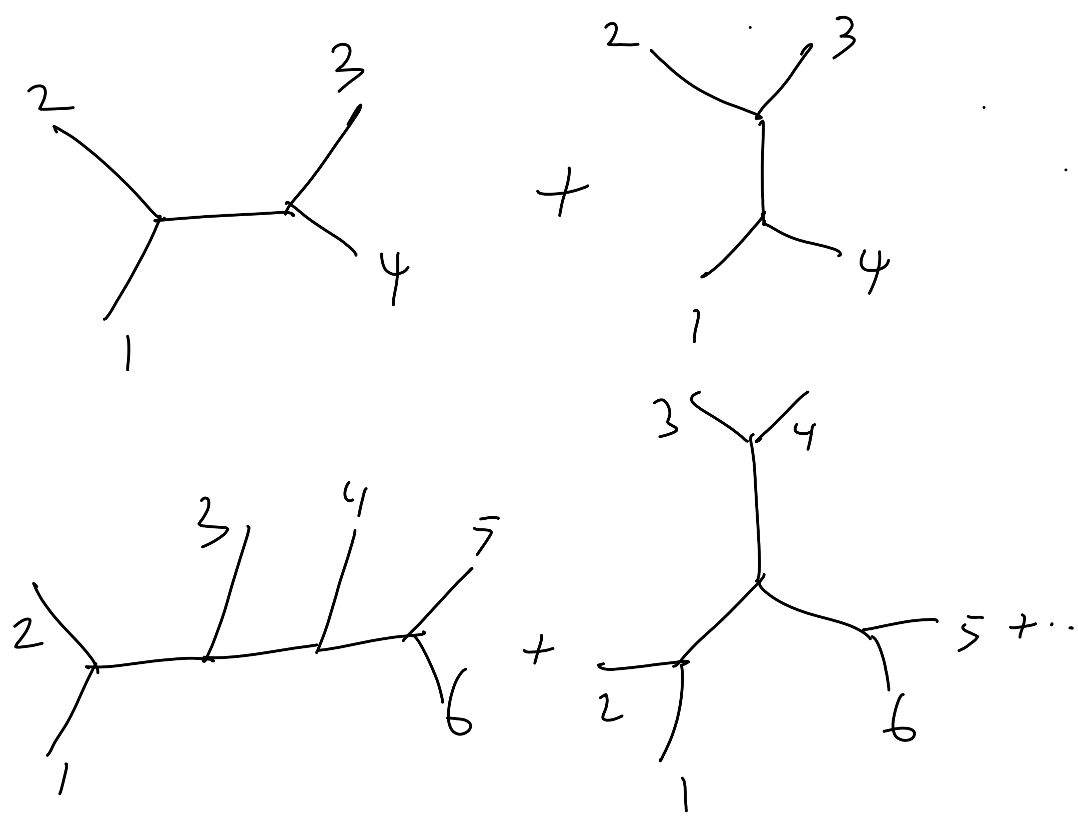
Functions / Symbols

$\updownarrow$

Non-Pert-Amplitudes

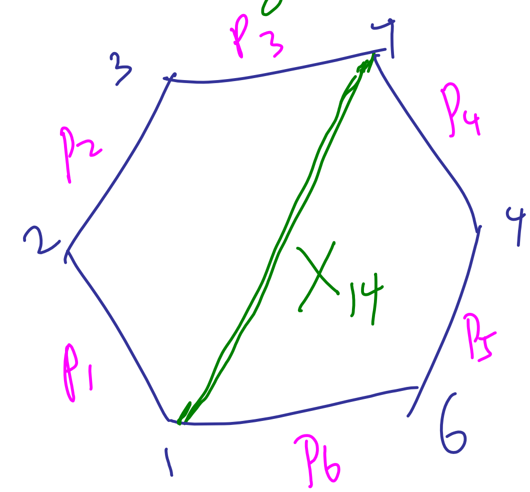
* Amplituhedra, Associahedra, Cosmological Polytopes, ...
* Describes REAL WORLD PHYSICS in reasonable approximation, exposes hidden symmetries + new mathematical structures
* Pure magic in physics right under our noses!

Simplest theory of Scalars

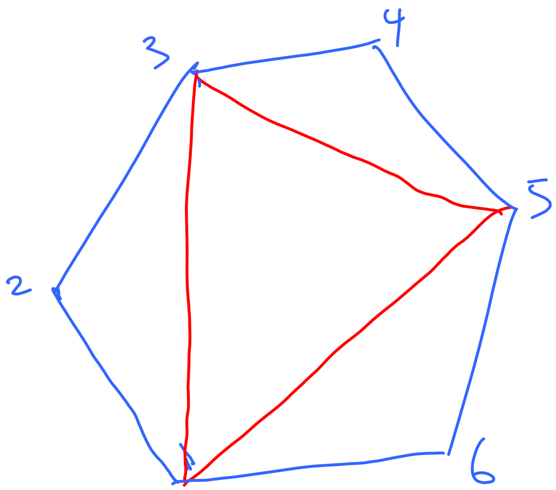


Only poles are

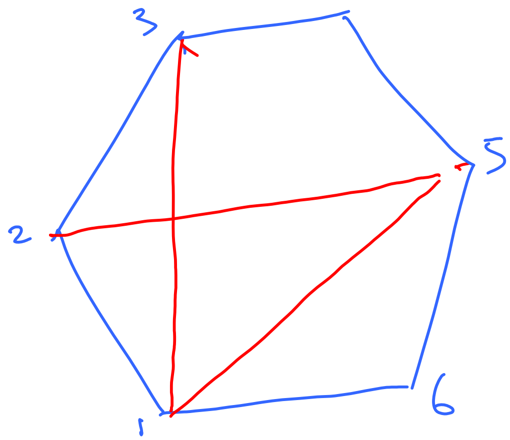
$$\underbrace{(p_i + p_{i+1} + \dots + p_{j-1})^2 - m^2}_{X_{ij}}$$



$X_{i\,i+1} = m^2 = $ "frozen"

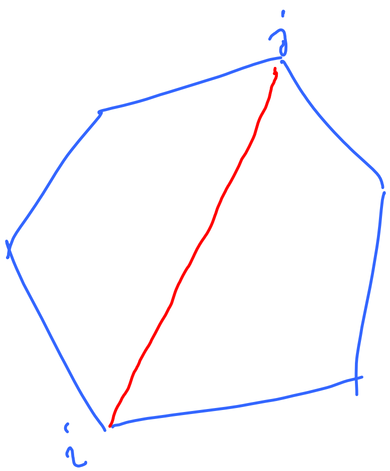


Allowed



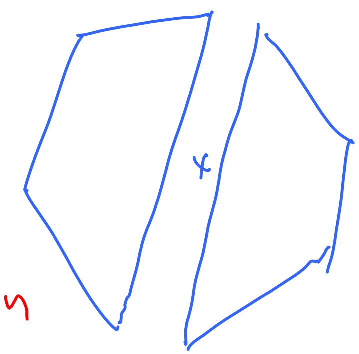
Crossing Chords not allowed.

LOCALITY



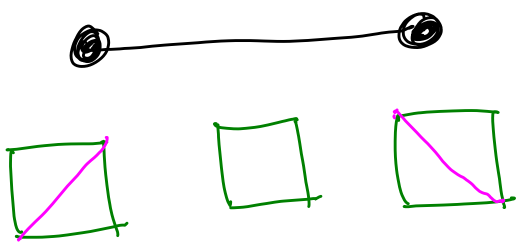
$X_{ij} \to 0$

Factorization

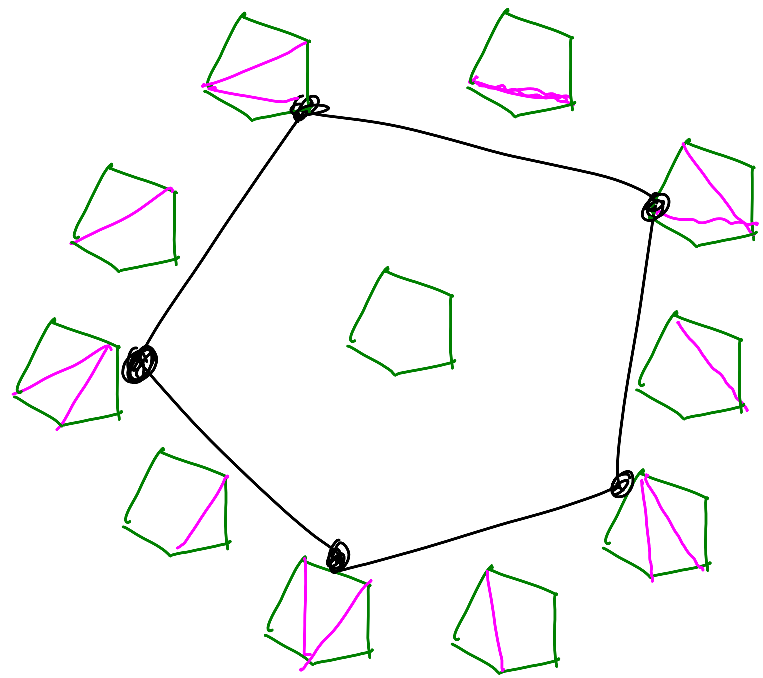


UNITARITY

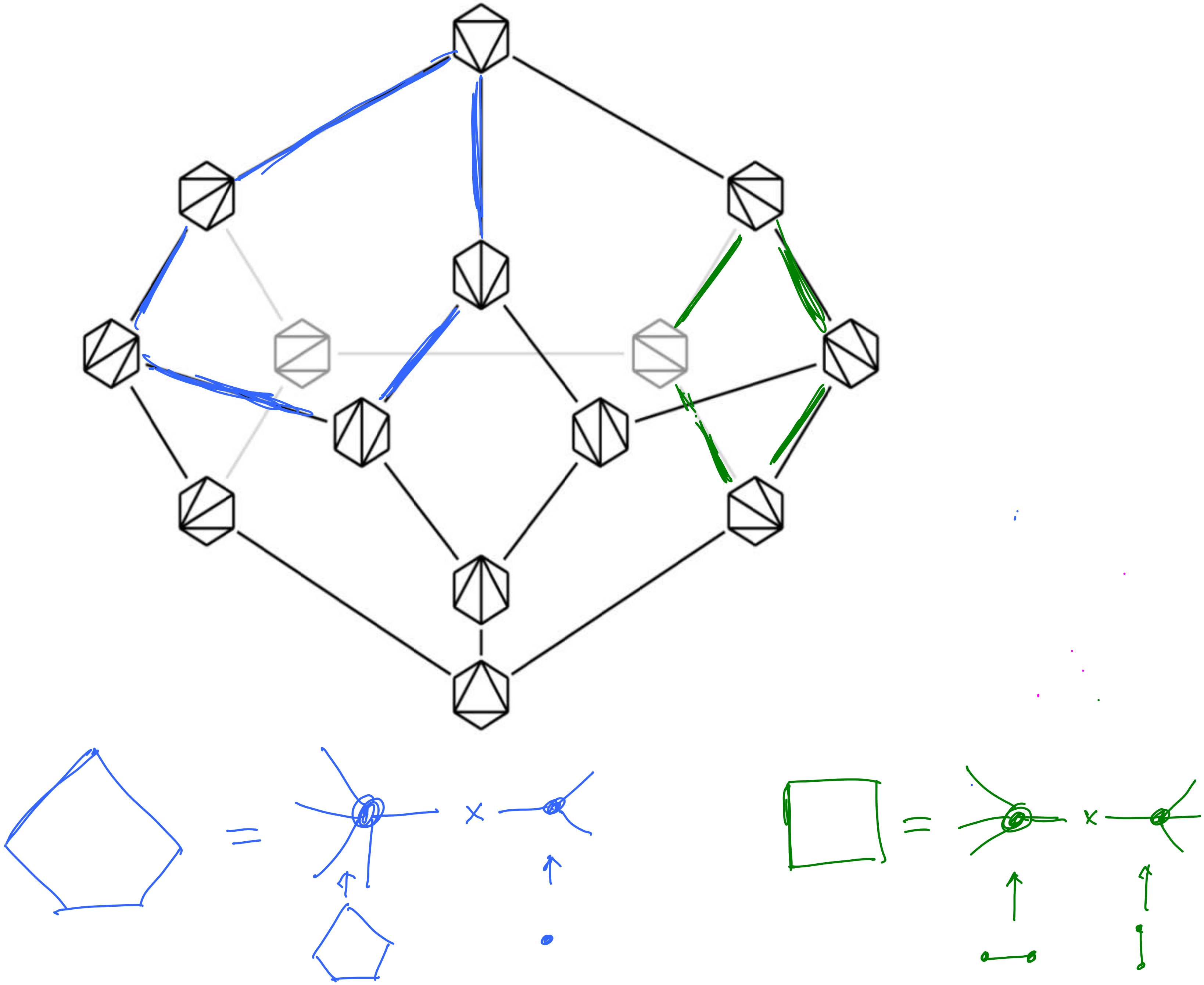
Pattern of Poles has a Shape!

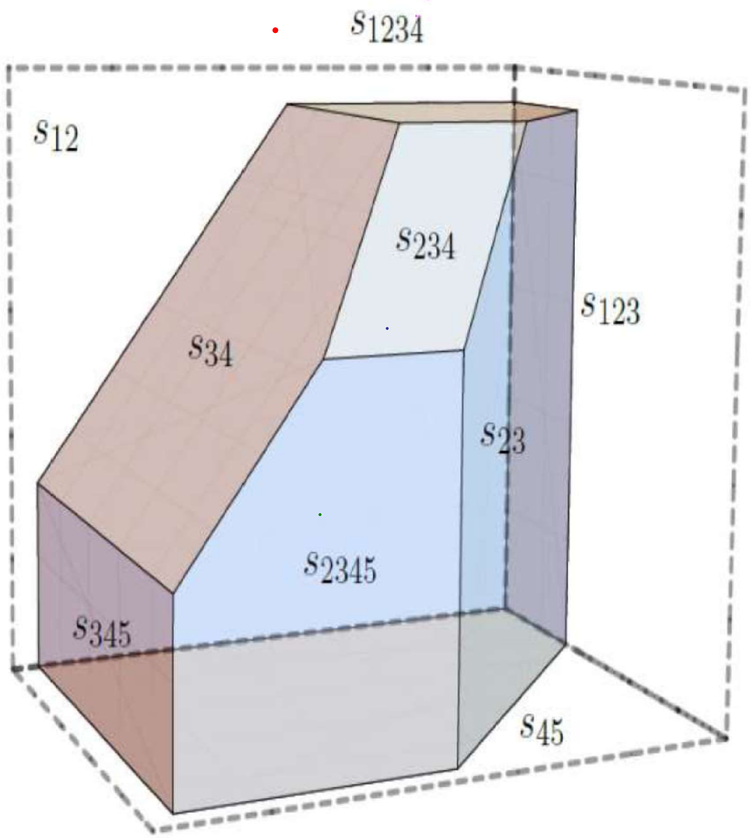


$n = 4$



$n = 5$





"ABHY"
Associahedron
in Kinematic
Space

"Type A_n Cluster Polytope"

Normal Fan = g-vector Fan

Amplitude is "the canonical form" with logarithmic $\left\{ \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \right\}$ singularities on faces of Polytope. Manifests hidden symmetry "projective invariance". One natural triangulation $\longrightarrow$ Feynman diagrams (hides proj. invariance). Other triang. possible, more efficient than Feyn. diagrams. Also other methods to get form without triangulation....

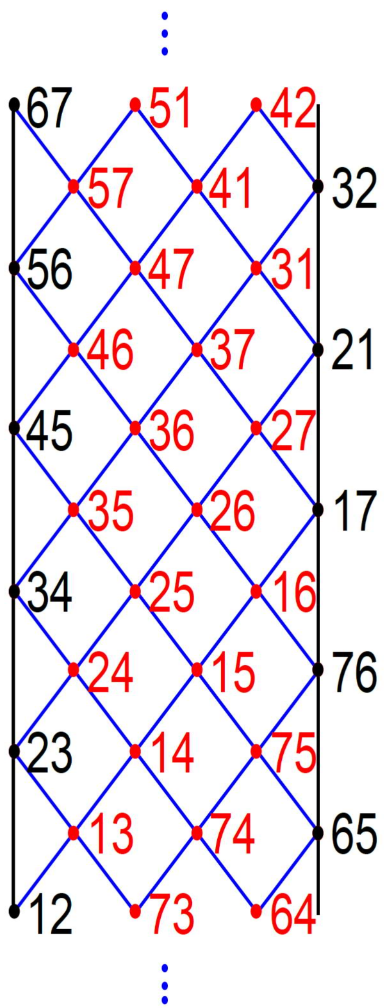
Surfacehedra $\longrightarrow$ All order Amplitudes for $\mathrm{tr}\,\phi^3$

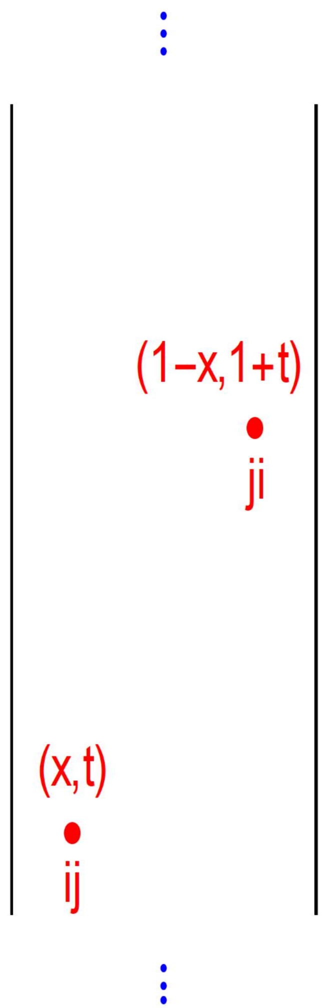
"Binary" Positive Surfacehedra $\longrightarrow$ "Baby" String Amps

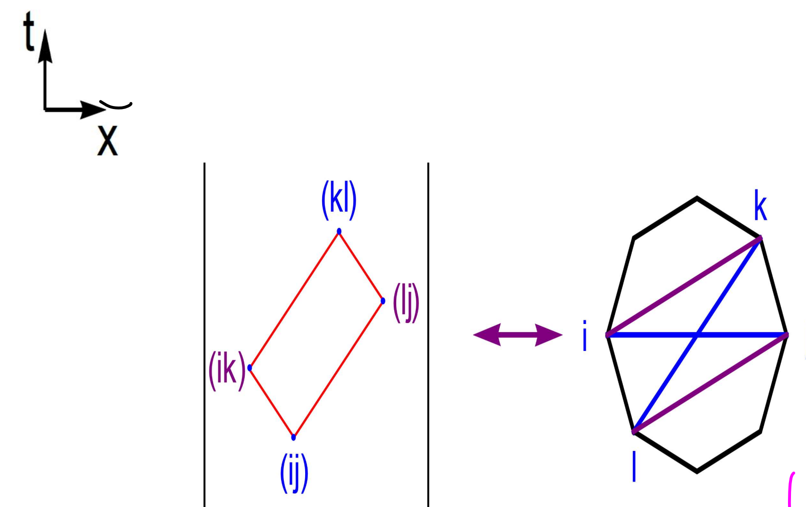
$\longrightarrow$ "Real" String Amps

Positivity, Polytopes + Particles

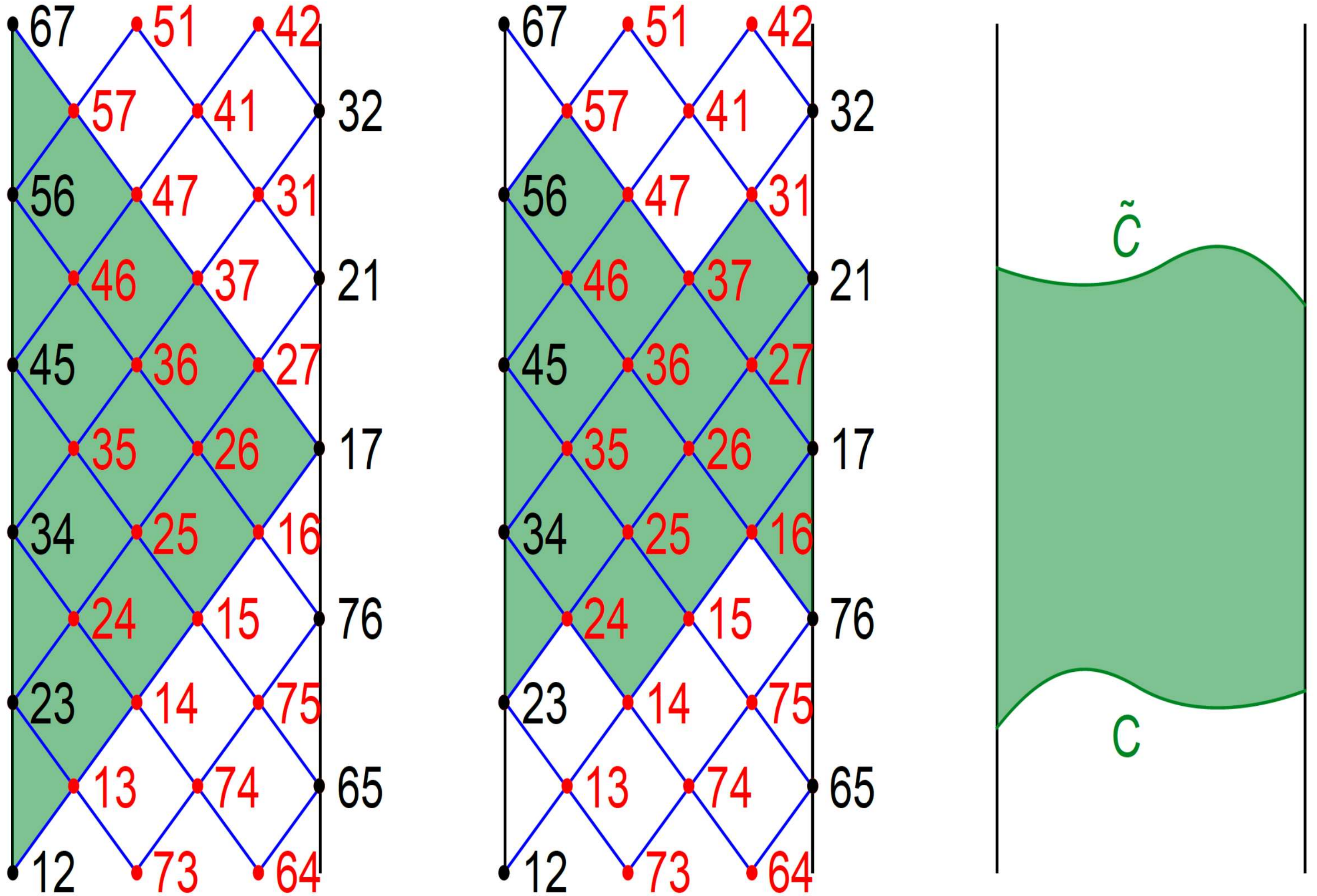
Kinematic "Spacetime"



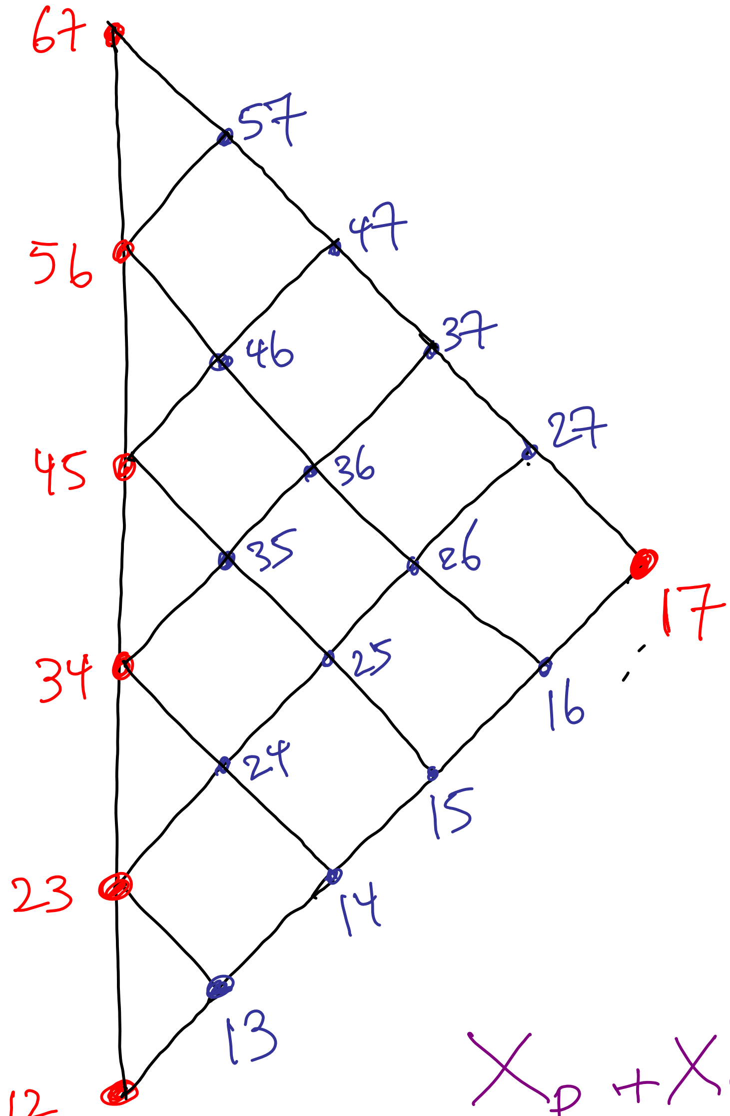




P, F corners of Causal Diamond ⟷ incompatible chords



67 51 42
57 41 32
56 47 31
46 37 21
45 36 27
35 26 17
34 25 16
24 15 76
23 14 75
13 74 65
12 73 64
67 51 42
57 41 32
56 47 31
46 37 21
45 36 27
35 26 17
34 25 16
24 15 76
23 14 75
13 74 65
12 73 64
$\tilde{C}$
C

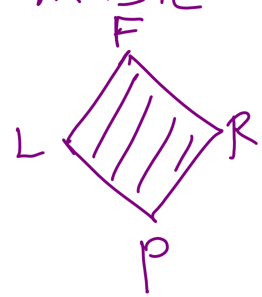


Variables X_{ij}

$X_{i\,i+1} = 0$, all

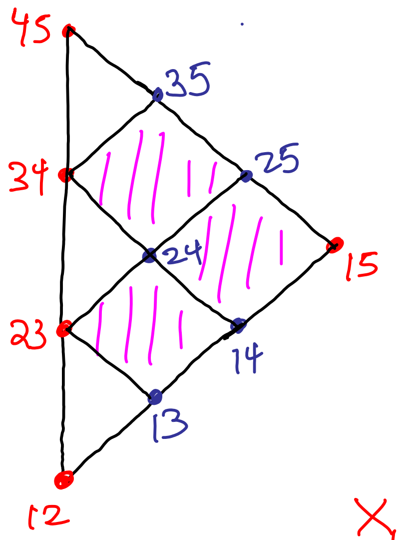
rest $X_{ij} \geq 0$

But also impose relation for every mesh:

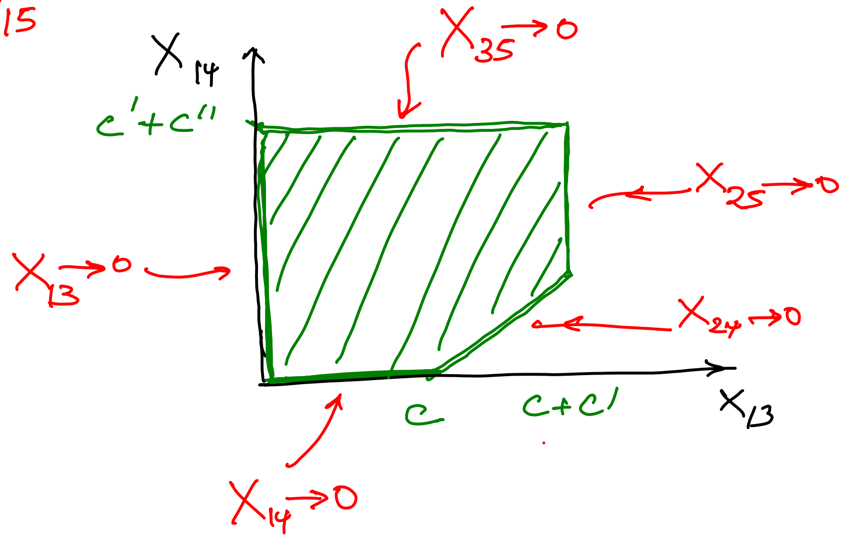


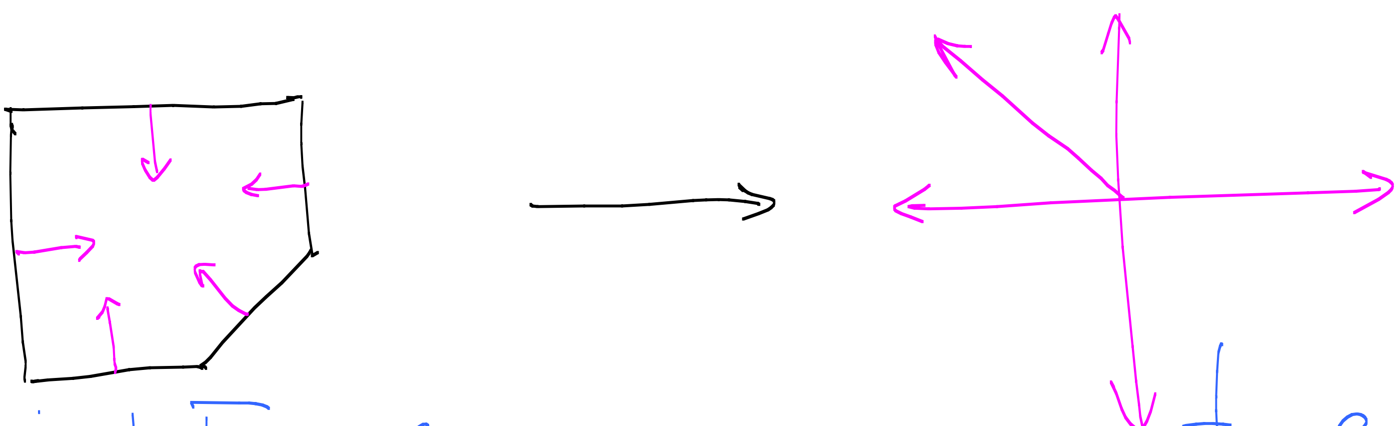
$$X_P + X_F - X_L - X_R = c_{\diamond} > 0$$

{Lattice Wave Equation}



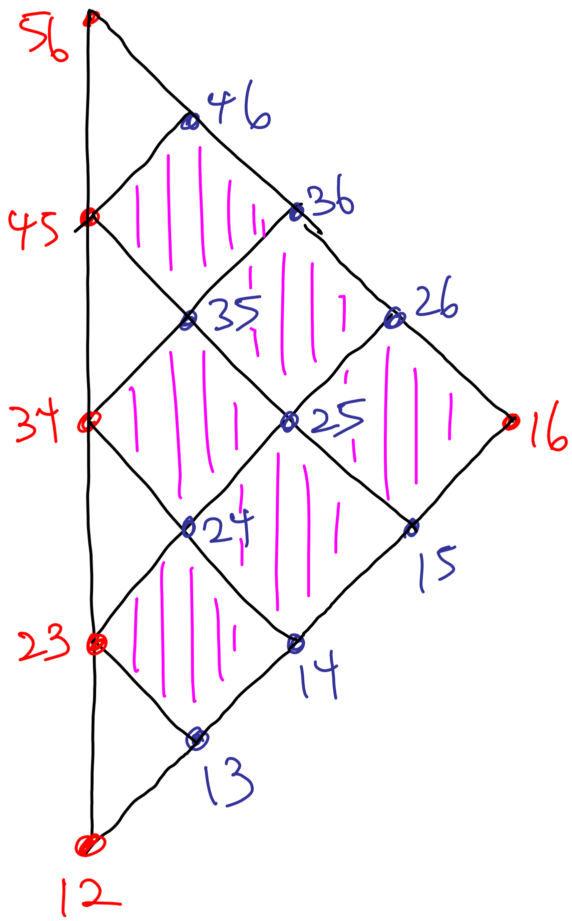
$$X_{13} + X_{24} - X_{14} - X_{23} = c$$
$$X_{14} + X_{25} - X_{24} - X_{15} = c'$$
$$X_{24} + X_{35} - X_{25} - X_{34} = c''$$





Normal Fan of Polytope

g-vector fan of A_2


56
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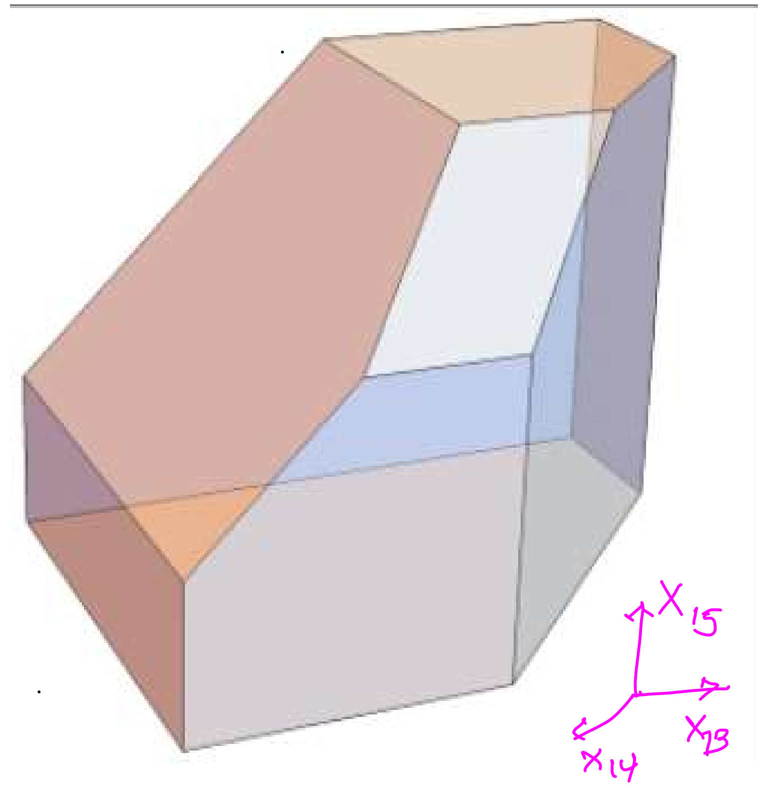
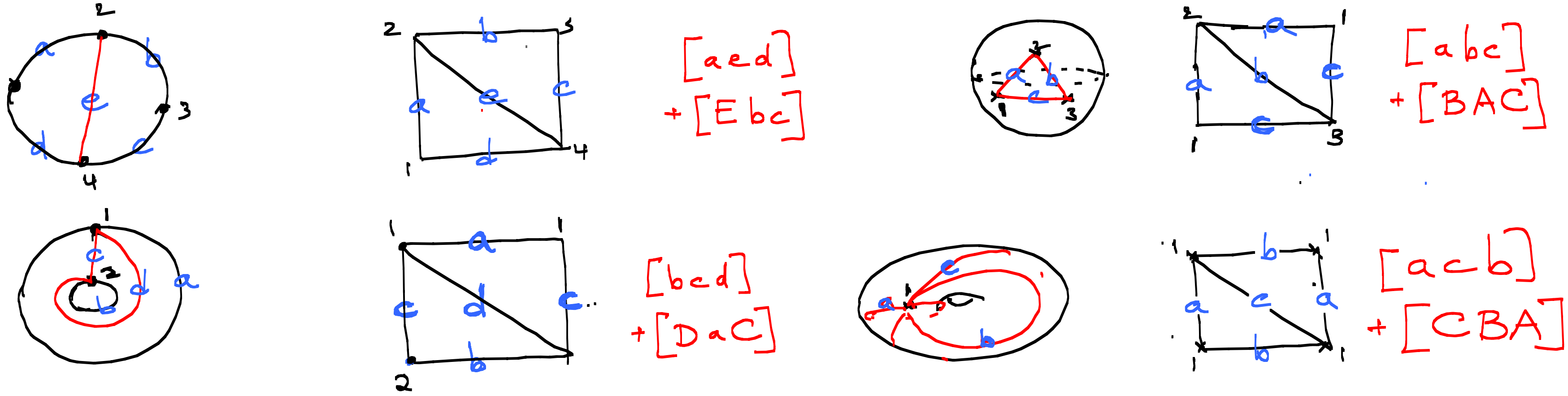
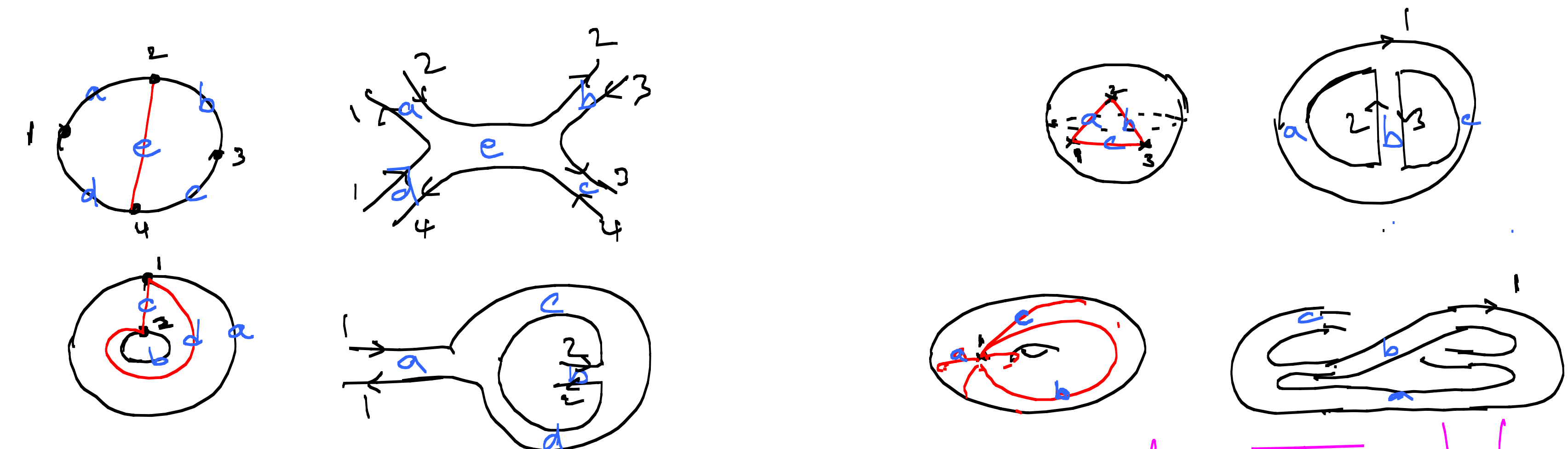

X_{15}
X_{23}
X_{14}

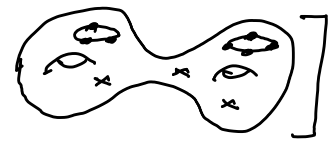




Edges of Graph ⟷ Chords of Triangulation

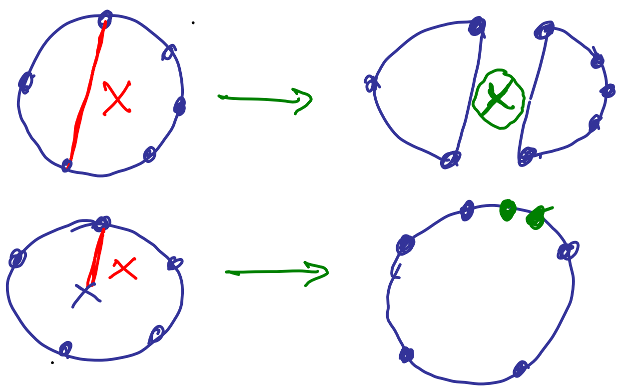
Double-line Colors ⟷ Vertices of Triang.

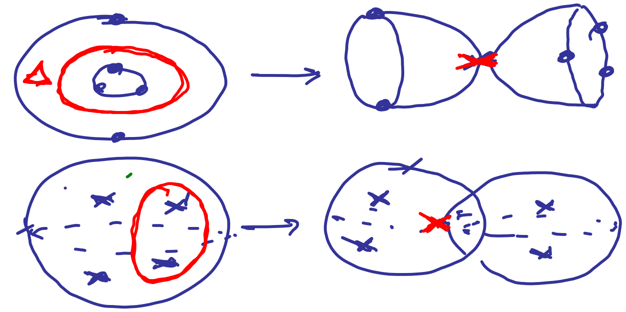
Internal Closed Color Loops ⟷ Punctures

Surfacehedra $S[$  $]$

* Facet for every (homotopy class) of Arcs
(Generally ∞, Fractal Structure)

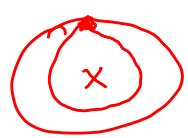
* Faces capture all "pinchings" of Surface

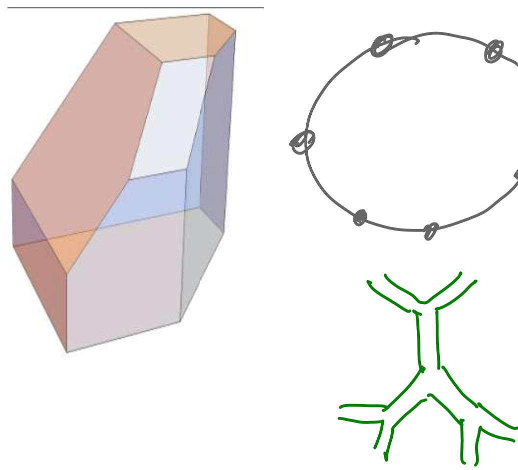


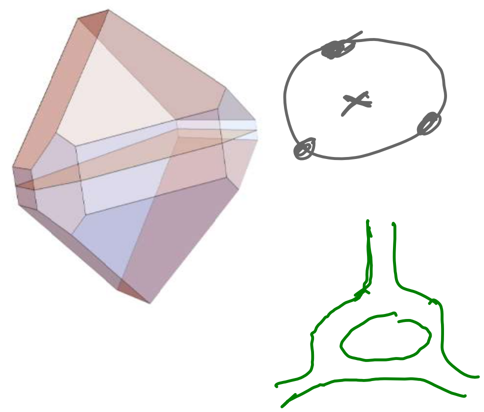


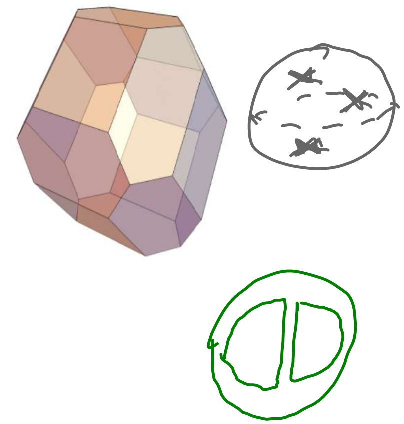
"Gravity"

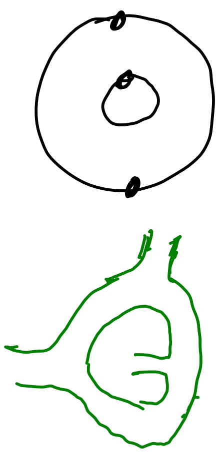
Entangled Product $S_L \boxtimes S_R$

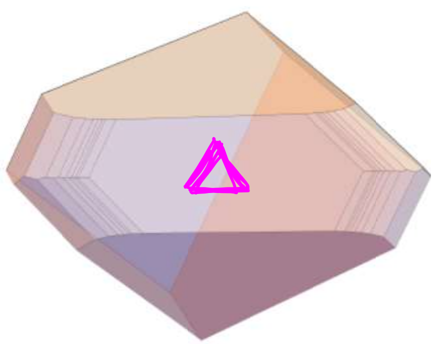
{* Normal fan = g-vector fan, but with facets for
Δ's = limiting rays; also slightly bigger than cluster
fan, we allow  ; all tagged/untagged incompatible}

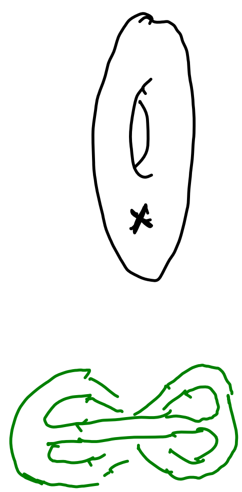


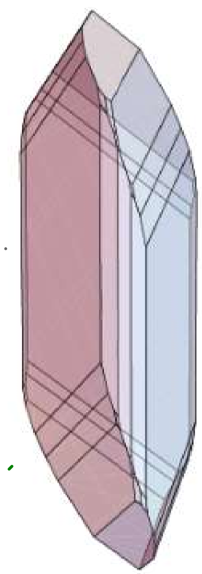


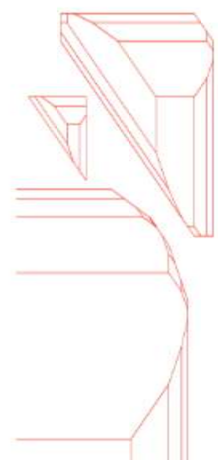


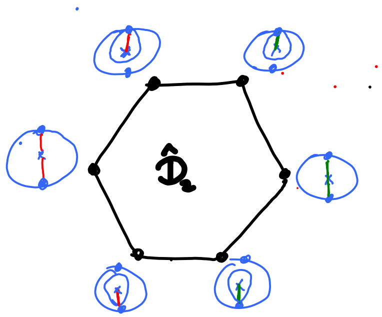


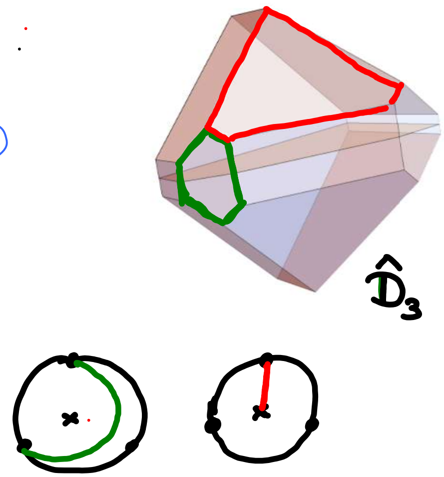


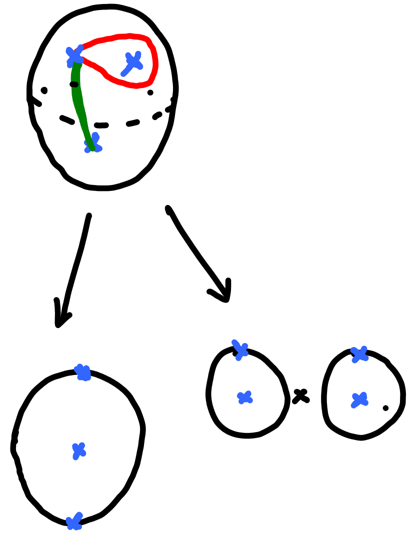


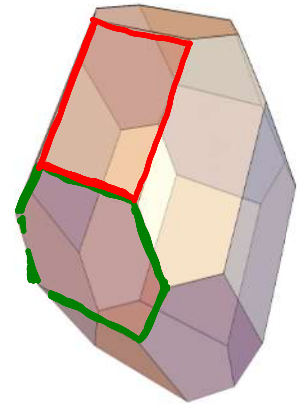




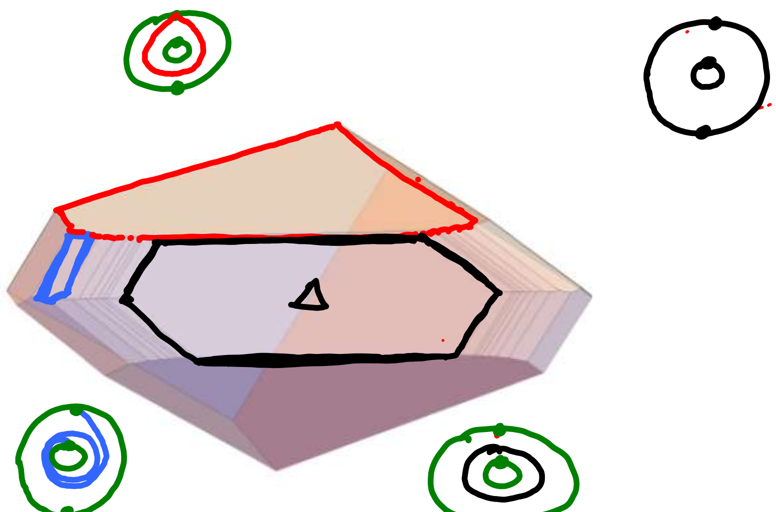




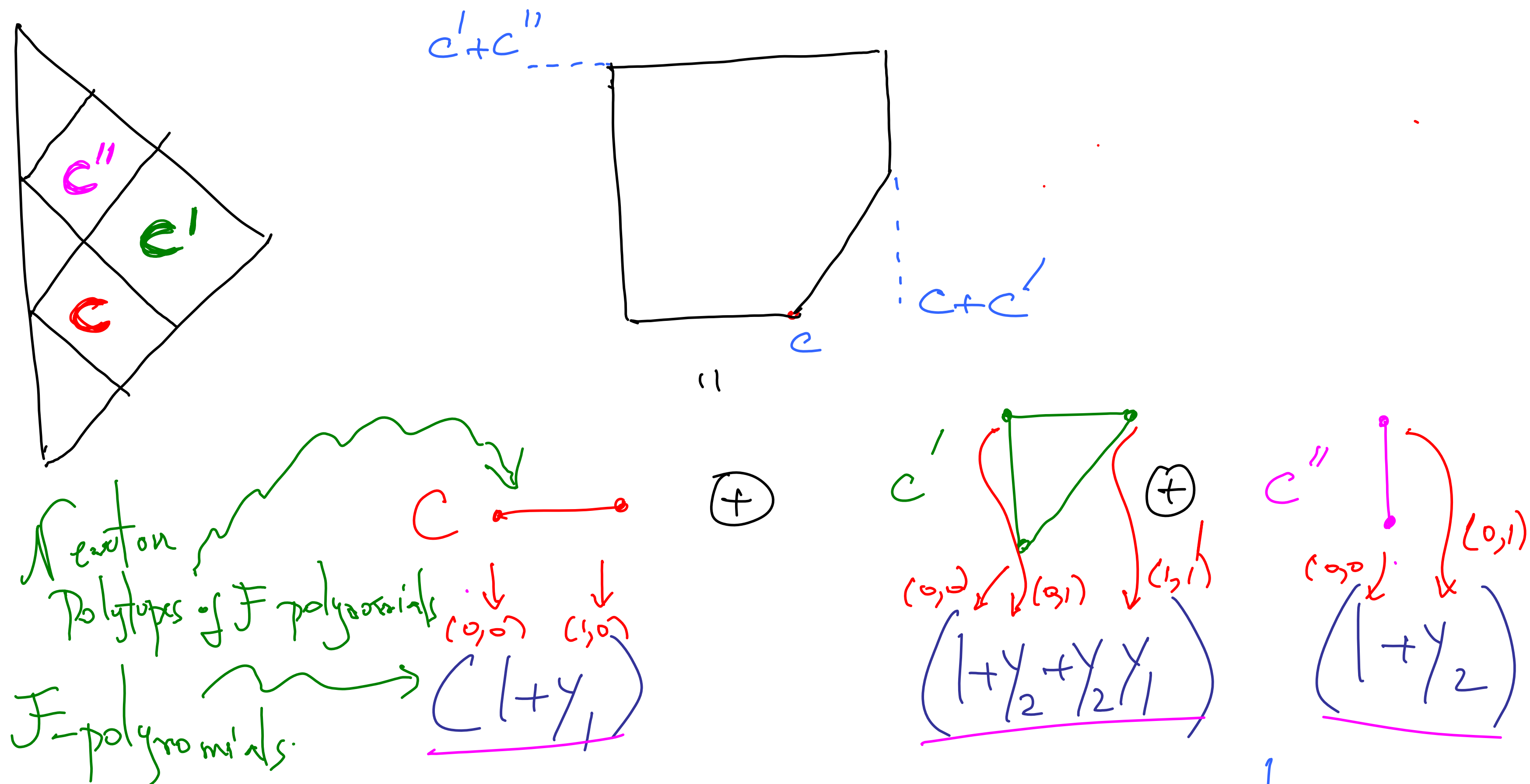




"Chamfered Cube"

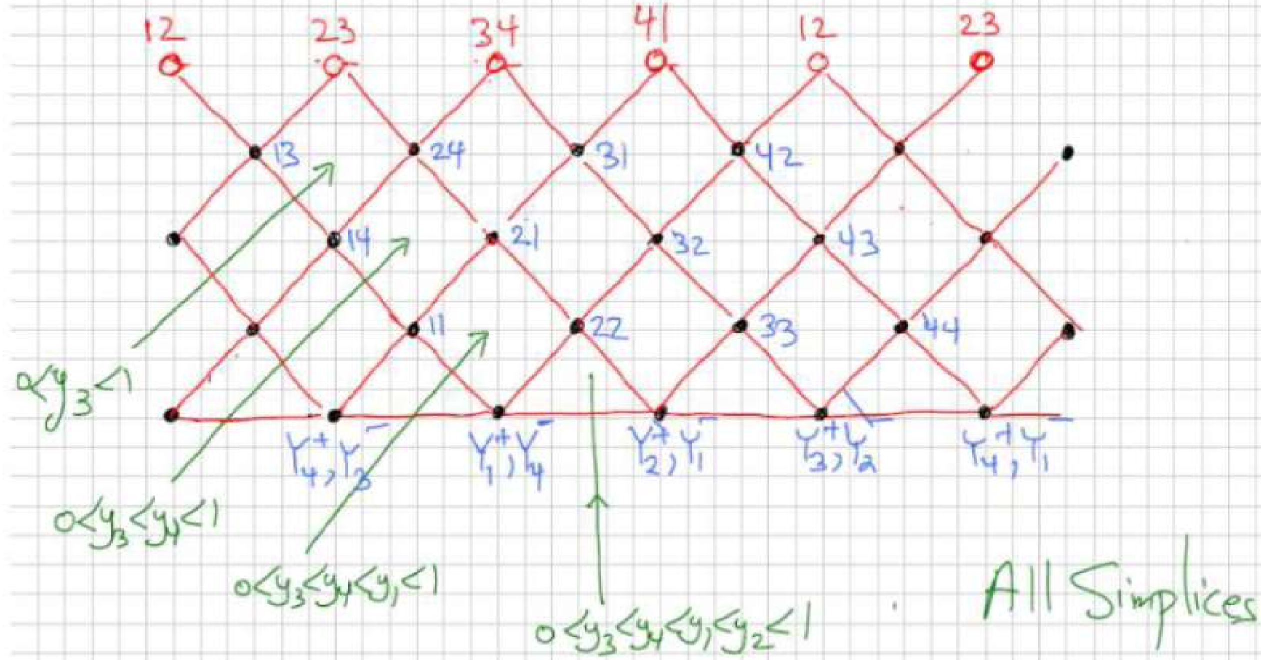


Polytopes are "Sums" of Simple Pieces



Polytopes are naturally "Minkowski Sums" of simple pieces directly associated with Binary Geometry

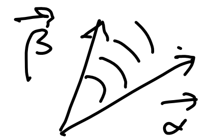
Minkowski Summands: 1-loop



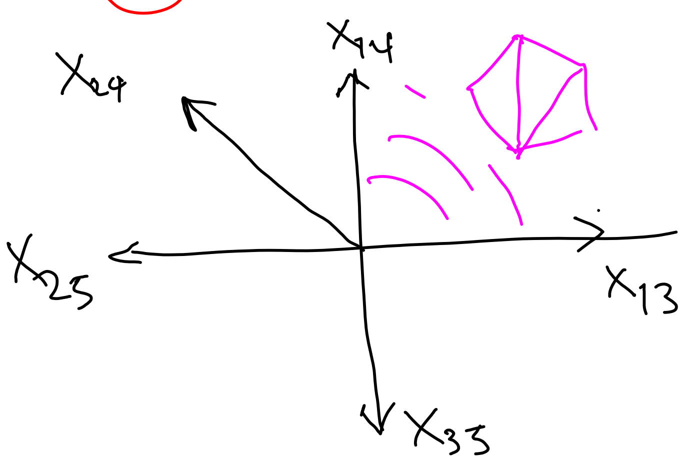
simplex $0<x_i<x_{i+1}<\cdots<x_j<1 \Longleftrightarrow [ij] = 1+y_j+y_jy_{j-1}+\cdots+y_jy_{j-1}\cdots y_i$

Schwinger Param $\longleftrightarrow$ Cones

$$\frac{1}{XY} = \int e^{-t_1 X} e^{-t_2 Y} = \int e^{-(t_1 t_2)(\vec{\alpha}\ \vec{\beta})^{-1}\binom{X}{Y}}$$



Diagrams $\longleftrightarrow$ Complete Fan



$$A = \int dt_1\, dt_2 \prod_X u_X^{X}$$

$$u_X(t_1,t_2) = e^{-\alpha_X(t_1,t_2)}$$

Piecewise Linear

Remarkably: $\alpha(t)$ has "global" description!

$$-\alpha_{13} = -t_1 - \max(0, -t_1)$$

$$-\alpha_{14} = -t_2 + \max(0, -t_1) - \max(0, -t_2, -t_1)$$

$$-\alpha_{24} = \max(0, -t_2, -t_2 - t_1) - \max(0, -t_1) - \max(0, -t_2)$$

$$\vdots$$

"Tropical" expressions: Handed to Us By "Minkowski Summands!"

$$\boxed{\mathcal{A} = \int \frac{\langle \hat{t}\, d^{n-1} \hat{t} \rangle}{\mathcal{U}(\alpha)^{D/2-\gamma}\, \mathcal{F}(\alpha)^{\gamma}}}$$

Familiar "Symanzik" Form ... but α_{ij}, α_i are all present, piecewise linear/tropical, combined into a single compact integral over S^{N-1}:

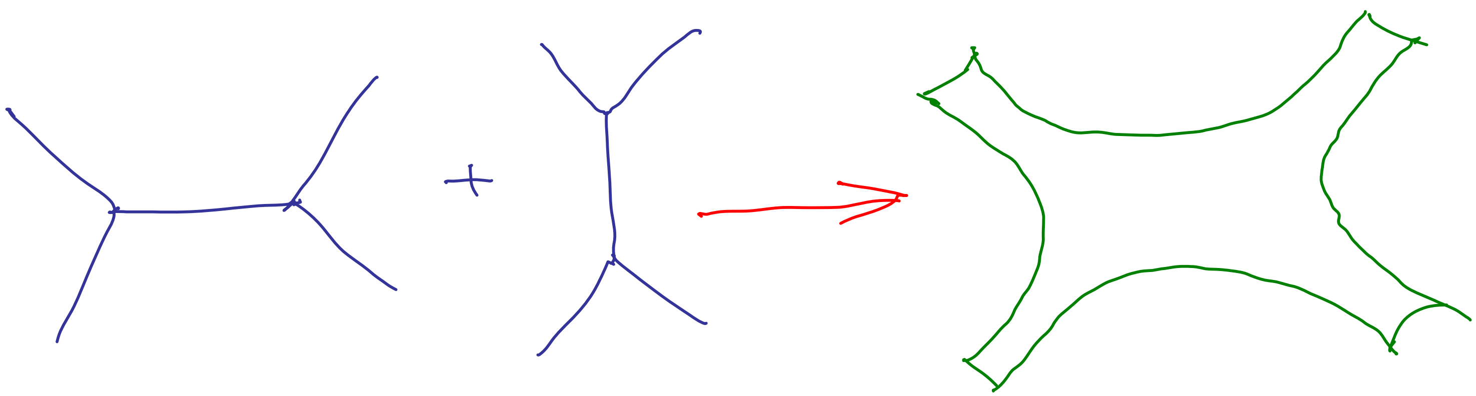
$(n-4)$-sphere for tree level, $(n-1)$ sphere 1-loop etc.

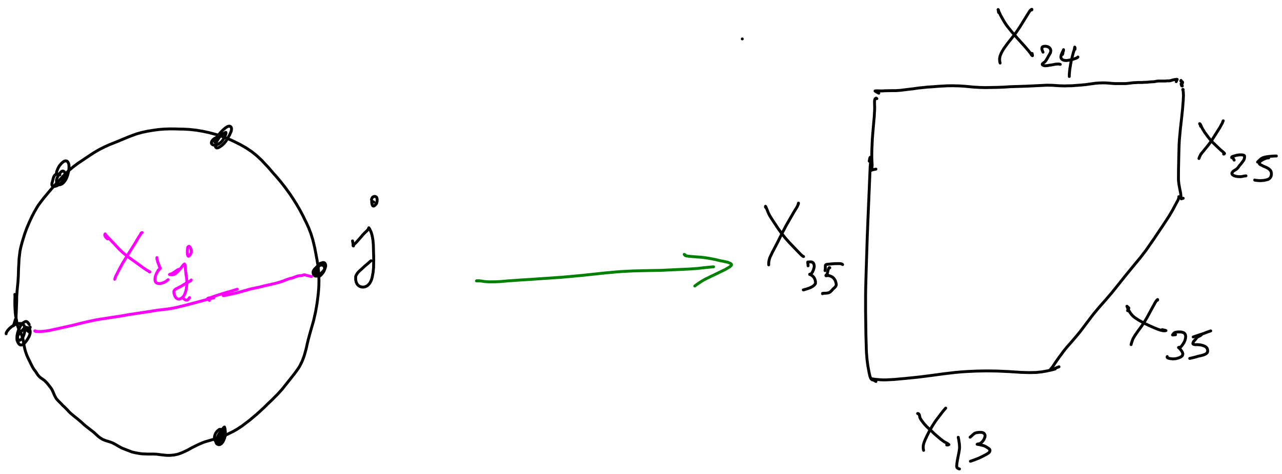
Surfacehedra $\Leftarrow$ " Minkowski Sum $\longleftrightarrow$ Unified Schwinger Rep. of Amplitude

"All-Loop Dual Surfacehedra"

"Binary" Positive Geometry:

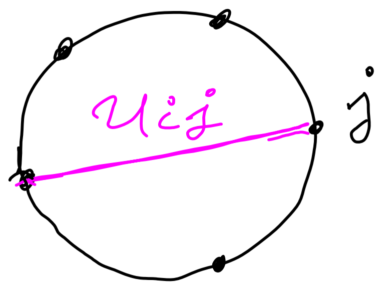
Particles $\longrightarrow$ Strings





Linear Inequalities Capture

Combinatorics + Compatibility + Factorization

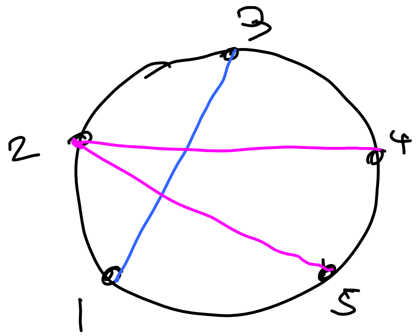


$$u_{ij} + \prod_{\substack{kl \\ \text{crossing} \\ ij}} u_{kl} = 1.$$

- Remarkably give a variety of correct dim!
- Positivity: $u_{ij} \geq 0 \longrightarrow u_{ij} \leq 1$

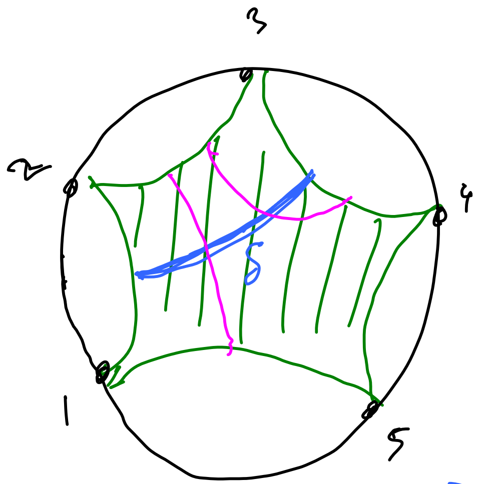
If $u_{ij} \to 0$, all u's for crossing chords $\longrightarrow 1$!

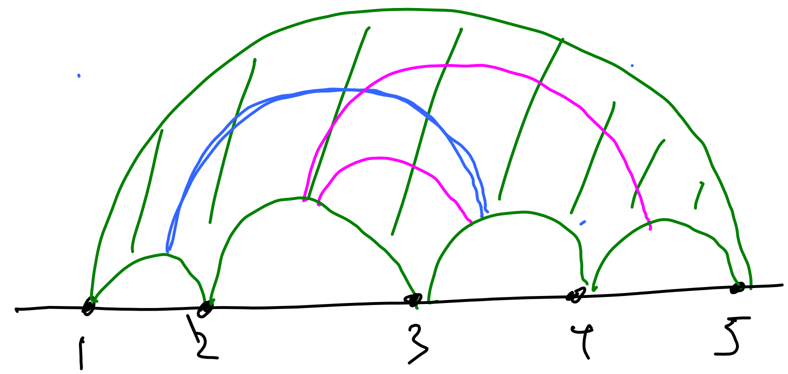
"Binary" geometry of compatibility combinatorics



$u_{13} + u_{24}u_{25} = 1$ + cyclic.

Can Check: solved by





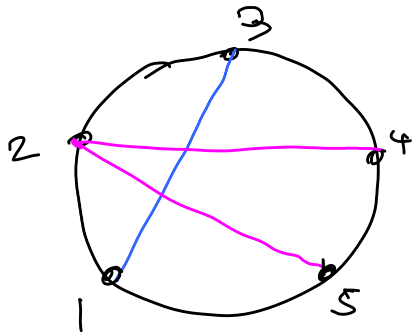
$z_1 < z_2 \cdots < z_5$

$$\left\{\cosh \delta = \frac{1+u}{1-u}\right\}$$

$$u_{ij} = \frac{(z_{i-1} - z_j)(z_i - z_{j-1})}{(z_{i-1} - z_{j-1})(z_i - z_j)}$$

Connection to Hyperbolic geometry + worldsheet

Just one representation of solutions to the "binary" equations



$$u_{13} + u_{24}u_{25} = 1 \quad + \text{cyclic.}$$

Can Check: solved by

$$u_{13} = \frac{y_1}{1+y_1}$$

$$u_{14} = \frac{(1+y_1)y_2}{1+y_2+y_2y_1}$$

$$u_{24} = \frac{1+y_2+y_2y_1}{(1+y_1)(1+y_2)}$$

$$u_{25} = \frac{1+y_2}{1+y_2+y_2y_1}$$

$$u_{35} = \frac{1}{1+y_2}$$

Note when $y_{1,2} \geq 0$

$$0 \leq u_{ij} \leq 1$$

Positive Parametrization of u_{ij}

$$u_{ij} = \frac{F_{i-1\,j}\,F_{i\,j-1}}{F_{i-1\,j-1}\,F_{ij}} \longleftarrow$$

"F-Polynomials"
Category of Quiver Repres.

Binary Geometry: Particles $\longrightarrow$ "Strings!"

"canonical dlog form" of Binary Geometry

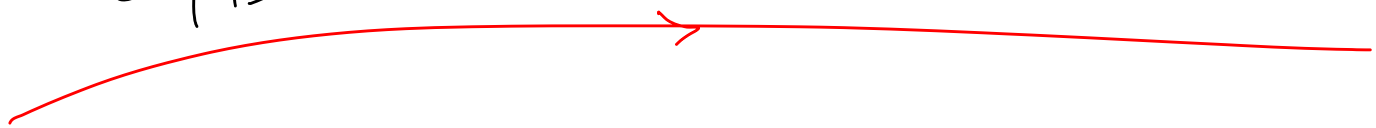
"Regulator" for every bndy $u_X \to 0$

$$A[X] = (\alpha')^N \int_0^\infty \underbrace{\frac{dy_1}{y_1} \cdots \frac{dy_n}{y_n}}_{\text{"canonical dlog form" of Binary Geometry}} \overbrace{\prod_X u_X^{\alpha' X}}^{\text{"Regulator" for every bndy } u_X \to 0}$$

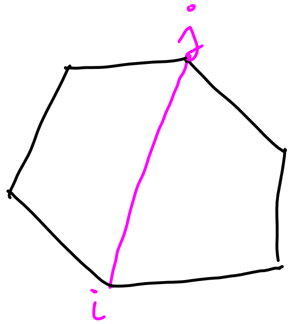
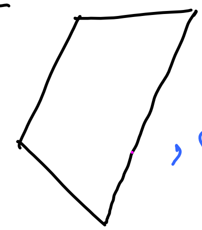
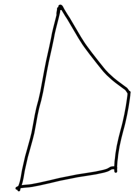
"string length", particle limit $\alpha' \to 0$

Preserves Factorization at finite α' thanks to "Binary" Property: $u_Y \to 0$, $u_{X\,\text{incomp}} \to 1$!

Also works with "Closed" Version



$$A = \int_i \pi \frac{dy_i}{y_i} \qquad \pi u_{\cancel{x}}^{2\cancel{x}} \Big|^{2}$$

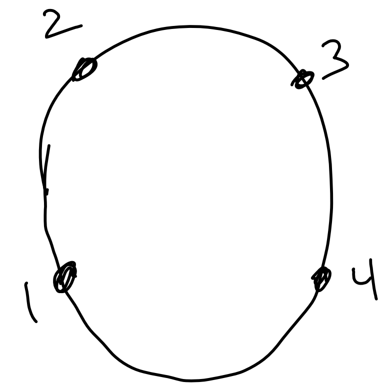
$A\Big[$  $,\alpha'\Big] \xrightarrow{X_{ij}\to 0} \dfrac{1}{X_{ij}}\, A\Big[$  $,\alpha'\Big]\, A\Big[$  $,\alpha'\Big]$

localizes to $u_{ij}\to 0$

Factorizes @ finite α' because all incomp. $u_{kl}\to 1$

“Koba-Nielsen” Tree String Amplitude.

PARTICLE $\longrightarrow$ STRING SCATTERING



$$u_{13} = \frac{y}{1+y}, \quad u_{24} = \frac{1}{(1+y)}$$

$$A\left[X_{13}, X_{24}\right] = \alpha' \int_0^\infty \frac{dy}{y} \left(\frac{y}{1+y}\right)^{\alpha' X_{13}} \left(\frac{1}{1+y}\right)^{\alpha' X_{24}}$$

$$= \frac{\Gamma[\alpha' X_{13}]\,\Gamma[\alpha' X_{24}]}{\Gamma[\alpha'(X_{13}+X_{24})]} \sim$$

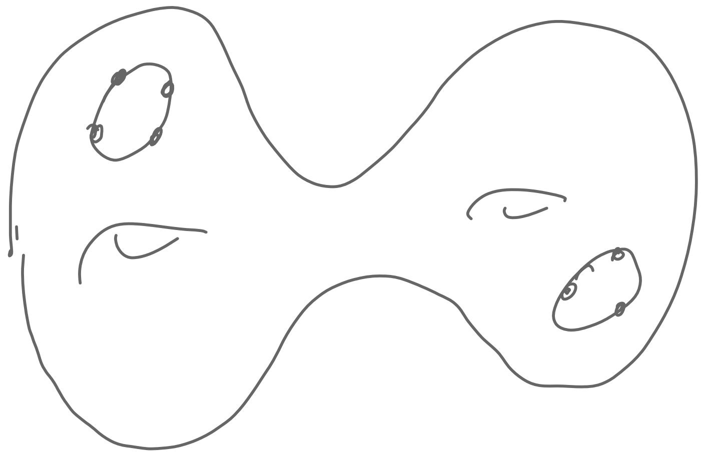
Veneziano Amplitude

$$\alpha' \to 0, \quad \to \left(\frac{1}{X_{13}} + \frac{1}{X_{24}}\right)$$

Particle Limit

INFINITELY MANY POLES → STRING EXCITATIONS

Binary Realization For All Surfaces



$$u_X + \prod_Y u_Y^{\,\mathrm{int}\#(X,Y)} = 1$$

* "Global" description of compactification of Teichmuller Space
  (c.f. "local" description of Fock–Goncharov)
* Concrete solutions of u-equations → F polynomials, Hyperbolic geom

"Minimal"
[no self-intersecting curves]

"Baby String" Amps

$$\alpha'^{n} \int_0^{\infty} \frac{dy_i}{y_i} \prod_X u_X^{\alpha' X} \Big/ \left| \int \prod \frac{dy_i}{y_i} \prod_X u_X^{\alpha' X} \right|^2$$

$\alpha' \to 0$

Field Theory Amps

"Maximal"
[include all self-intersecting curves]

"Real String" Amps!

NEW REP. OF STRING AMPS.

NOT STRINGS $\xrightarrow{\alpha' \to 0}$ PARTICLES

BUT PARTICLES $\xrightarrow[\text{Binarify}]{\alpha' \neq 0}$ STRINGS.

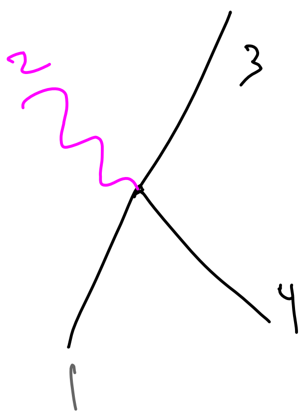
What is special about "Real Strings"?

$$\underbrace{\int_0^1 \frac{dz}{z(1-z)}\, z^{X_{13}-m^2}\,(1-z)^{X_{24}-m^2}}_{\text{Makes Sense for any } m^2 \ldots}$$

But permutation invariant for $m^2 = -1$!

$$\overbrace{\int \left| \frac{dz}{z(1-z)}\, z^{X_{13}-m^2}\,(1-z)^{X_{24}-m^2} \right|^2}^{\text{permutation invariant}}$$

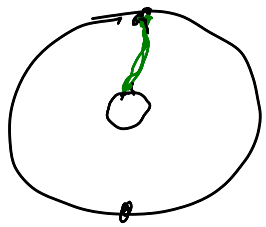
$$\underbrace{\int_0^1 \frac{dz}{z(1-z)}\, z^{X_{13}-m^2}(1-z)^{X_{24}-m^2}\left[A\,\varepsilon_2\cdot p_1(1-z) + B\,\varepsilon_2\cdot p_3\, z\right]}$$



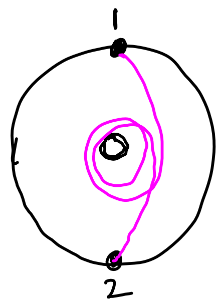
Perm. inv. in $(3,4,1)$ only if $A=-B$, $m_2^2=0$, Autom. gauge inv.

$$\overbrace{\left|\int \frac{dz}{z(1-z)}\, z^{X_{13}-m^2}(1-z)^{X_{24}-m^2}\left[A\,\varepsilon_2\cdot p_1(1-z) + B\,\varepsilon_2\cdot p_3\, z\right]\right|^2}$$

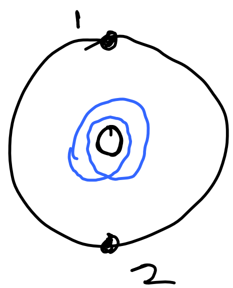
$\Longrightarrow$ Massless Gluons/Gravitons



$$A = \int d^D\ell \quad \omega_{d\log} \quad u_{Y_1}^{\frac{1}{2}\ell^2 - 1} \quad u_{Y_2}^{\frac{1}{2}(\ell+k_1)^2 - 1}$$

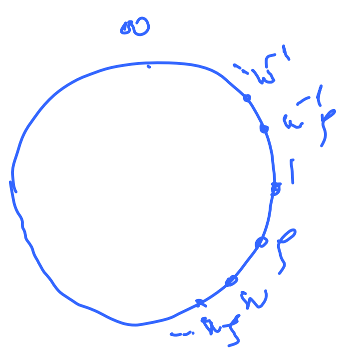


$$\prod_{n=1}^{\infty} u_{12,n}^{\frac{1}{2}k_1^2 - 1} \quad u_{21,n}^{\frac{1}{2}k_2^2 - 1} \quad \prod_{n=0}^{\infty} u_{11,n}^{\frac{1}{2}0 - 1} \quad u_{22,n}^{\frac{1}{2}0 - 1} \left(\prod_{j=1}^{\infty} u_j^{j^2} \right)^A$$



$$= \int \frac{dw\, d\rho}{w^2 (1-\rho)^2} \left(\frac{-2\pi}{\log w} \right)^{D/2} e^{-\frac{\log^2 \rho}{\log w}}$$

$$\underbrace{\left[\prod_j \frac{(1 - w^j \rho)(1 - w^j/\rho)}{(1-w^j)^2} \right]^2 \left(\prod_{j=1}^{\infty} (1 - w^j) \right)^A}_{\text{Enforce log sing. as } w \to 0}$$



$$\Longrightarrow \quad A = -24, \quad D = 26$$

Bosonic String

Outlook

Over the past decade, we have been seeing concrete examples where rules of locality + unitarity — Space-Time + QM — emerge from more elementary mathematical rubric. Common theme of "Amplituhedra" and "Surfacedra" are notions of positivity + topology in kinematic space — which breathes "physics-life" into seemingly austere / empty territory.

I fully expect to see more and more magic as we get closer and close to describing all aspects of REAL WORLD.