

$$\text{and } p_2 = \omega^2 - K^2(\cos\theta \operatorname{sech}(ax) + \sin\theta \tanh(ax))^2 v_A^2 .$$

In general, g is a function of x , so are the coefficients p_0 , p_1 , and p_2 . At the critical points only, where p_0 vanishes, g is independent of x , given by (18) and (19).

B Coefficients p_0, p_1, p_2

The coefficients of the wave equation $p_0(x)\xi_x''(x) + p_1(x)\xi_x'(x) + p_2(x)\xi_x(x) = 0$ are

$$p_0(x) = \left\{ \text{sech}(ax)^2 \left[\omega^2 \cosh(ax)^2 - (k + h \sinh(ax))^2 v_A^2 \right] \right. \\ \left. \left(\omega^2 \cosh(ax)^2 v_A^2 + c_s^2 \left[\omega^2 \cosh(ax)^2 - (k + h \sinh(ax))^2 v_A^2 \right] \right) \right\} \times \\ \left\{ \omega^2 \cosh(ax)^2 \left[\omega^2 - (h^2 + k^2) v_A^2 \right] + (h^2 + k^2) c_s^2 \left[-\omega^2 \cosh(ax)^2 + (k + h \sinh(ax))^2 v_A^2 \right] \right\}^{-1},$$

$$p_1(x) = - \left\{ a \text{sech}(ax)^2 \left[-3hk + hk \cosh(2ax) - 2(h^2 - k^2) \sinh(ax) \right] \right. \\ \left. v_A^2 \left[-\omega^4 \cosh(ax)^4 v_A^2 \left(\omega^2 - (h^2 + k^2) v_A^2 \right) - \right. \right. \\ \left. \left. 2 \omega^2 \cosh(ax)^2 c_s^2 \left(\omega^2 - (h^2 + k^2) v_A^2 \right) \left(\omega^2 \cosh(ax)^2 - (k + h \sinh(ax))^2 v_A^2 \right) + \right. \right. \\ \left. \left. (h^2 + k^2) c_s^4 \left(\omega^2 \cosh(ax)^2 - (k + h \sinh(ax))^2 v_A^2 \right)^2 \right] \right\} \times \\ \left\{ \omega^2 \cosh(ax)^2 \left[\omega^2 - (h^2 + k^2) v_A^2 \right] + (h^2 + k^2) c_s^2 \left[-\omega^2 \cosh(ax)^2 + (k + h \sinh(ax))^2 v_A^2 \right] \right\}^{-2},$$

$$\text{and } p_2(x) = \omega^2 - v_A^2 (k \text{sech}(ax) + h \tanh(ax))^2.$$

Parametrizing the wavenumbers $k = K \cos \theta$ in the y-direction and $h = K \sin \theta$ in the z-direction as in Section 7, and writing x-dependences in the form of the critical point conditions $g = \cos \theta \text{sech}(ax) + \sin \theta \tanh(ax)$ wherever possible, these can be expressed more conveniently as:

$$p_0 = \left\{ \left[\omega^2 (v_A^2 + c_s^2) - K^2 c_s^2 v_A^2 (\cos \theta \text{sech}(ax) + \sin \theta \tanh(ax))^2 \right] \right. \\ \left. \left[\omega^2 - K^2 v_A^2 (\cos \theta \text{sech}(ax) + \sin \theta \tanh(ax))^2 \right] \right\} \times \\ \left\{ \omega^4 - \omega^2 K^2 (v_A^2 + c_s^2) + K^4 c_s^2 v_A^2 (\cos \theta \text{sech}(ax) + \sin \theta \tanh(ax))^2 \right\}^{-1},$$

$$p_1(x) = \frac{dp_0}{dx} = \frac{A}{2} \frac{dg(x)}{dx} = \\ -2av_A^2 K^2 \left\{ [\cos \theta \tanh(ax) - \sin \theta \text{sech}(ax)] [(\cos \theta \text{sech}(ax) + \sin \theta \tanh(ax))] \right. \\ \left. \left[-\omega^4 v_A^2 \left(\omega^2 - K^2 v_A^2 \right) - 2 \omega^2 c_s^2 \left(\omega^2 - K^2 v_A^2 \right) \left(\omega^2 - H^2 v_A^2 \right) + K^2 c_s^4 \left(\omega^2 - H^2 v_A^2 \right)^2 \right] \right\} \times \\ \left\{ \omega^4 - \omega^2 K^2 (v_A^2 + c_s^2) + K^4 c_s^2 v_A^2 (\cos \theta \text{sech}(ax) + \sin \theta \tanh(ax))^2 \right\}^{-2},$$

A Functions in wave equations (11) - (13)

$$Q_{\zeta 0x}(x) = i a \operatorname{sech}(a x)^2 (h - k \sinh(a x)) c_s^2 \quad (43)$$

$$Q_{\zeta 1x}(x) = -i a \operatorname{sech}(a x)^2 (k + h \sinh(a x)) (c_s^2 + v_A^2) \quad (44)$$

$$Q_x(x) = \omega^2 - v_A^2 (k \operatorname{sech}(a x) + h \tanh(a x))^2 \quad (45)$$

$$P_{\zeta 0x}(x) = i \operatorname{sech}(a x) (k + h \sinh(a x)) c_s^2 \quad (46)$$

$$P_{\zeta 1x}(x) = i \operatorname{sech}(a x) (h - k \sinh(a x)) (c_s^2 + v_A^2) \quad (47)$$

$$d(x) = c_s^2 + v_A^2 \quad (48)$$

$$Q_{\zeta 0y}(x) = \operatorname{sech}(a x) (\omega^2 - k (k + h \sinh(a x)) c_s^2) \quad (49)$$

$$Q_{\zeta 1y}(x) = k \operatorname{sech}(a x) (-h + k \sinh(a x)) c_s^2 + \tanh(a x) (-\omega^2 + (h^2 + k^2) v_A^2) \quad (50)$$

$$P_y(x) = i (k c_s^2 + \operatorname{sech}(a x) (-h + k \sinh(a x)) v_A^2 \tanh(a x)) \quad (51)$$

$$Q_{\zeta 0z}(x) = -h \operatorname{sech}(a x) (k + h \sinh(a x)) c_s^2 + \omega^2 \tanh(a x) \quad (52)$$

$$Q_{\zeta 1z}(x) = \operatorname{sech}(a x) (\omega^2 - h (h - k \sinh(a x)) c_s^2 - (h^2 + k^2) v_A^2) \quad (53)$$

$$P_z(x) = i \operatorname{sech}(a x) [h \cosh(a x) c_s^2 + v_A^2 (h \operatorname{sech}(a x) - k \tanh(a x))] \quad (54)$$

9 Limits of ξ

Analytically solve for ξ in the limits of high and low beta, near the origin and at plus minus infinity, in terms of non-constant coefficients. Due to the complexity of the coefficients, we may need to change our method above in two ways to make the problem tractable:

- Write the wavenumbers h and k as quantities rotating with the magnetic field, as we did for the displacements. By casting the wavenumbers into the rotating basis natural to this sheared field, we can remove all explicit dependence on θ and may simplify the form of the coefficients in the wave equation and in the Frobenius expansion.
- If a general analytic solution for arbitrary displacements remains intractable, then solve the wave equation for special cases by initializing the displacement at the $-x$ boundary, e.g. for pure sound waves and pure Alfvén waves. This will permit us to track the evolution of each type of MHD wave as it propagates through the shear layer.

10 Numerical solutions

Numerically solve for ξ in high and low beta limits, for all positions x , if necessary.

where

$$p2n = (1 + s^2)^2 \omega^2 - K^2 v_A^2 (2s \cos \theta + (-1 + s^2) \sin^2 \theta)$$

Cancelling common factors, these coefficients can be rewritten without denominators to facilitate a series expansion in s :

$$p0_s = p0n p0d(1 + s^2) \quad (35)$$

$$p1_s = p1n \quad (36)$$

$$p2_s = p2n p0d^2(1 + s^2) \quad (37)$$

8.2 Frobenius method

Using the Frobenius method, we will expand about a regular singularity s_* . We assume a solution of the form

$$\xi_x = \sum_{n=0}^{\infty} b_n (s - s_*)^{\nu+n}. \quad (38)$$

Substituting this trial solution into (30), using $p0_s$, $p1_s$, and $p2_s$ above, and writing $(s - s_*) = s_1$ everywhere, we find:

$$s_1^\nu \sum_{n=0}^{\infty} b_n [t_0 s_1^n + t_1 s_1^{n+1} + t_2 s_1^{n+2} + \dots + t_{14} s_1^{n+14}] = 0 \quad (39)$$

where the coefficients t are very involved polynomial functions of s which also depend on θ .

Assuming that $b_0 \neq 0$, the indicial equation $t_0 = 0$ at $n = 0$ is

$$0 = (\omega^2 - K^2 v_A^2 \sin^2 \theta) [\omega^2 (-K^2 (c_s^2 + v_A^2) + \omega^2) + K^4 c_s^2 v_A^2 \sin^2 \theta] \\ \left\{ \omega^2 [(a^2 \nu^2 - K^2)(c_s^2 + v_A^2) + \omega^2] + K^2 c_s^2 v_A^2 (K^2 - a^2 \nu^2) \sin^2 \theta \right\}$$

with roots

$$\nu = \pm \frac{K^2 \omega^2 (c_s^2 + v_A^2) - K^4 c_s^2 v_A^2 \sin^2 \theta - \omega^4}{a^2 \omega^2 (c_s^2 + v_A^2) - a^2 K^2 c_s^2 v_A^2 \sin^2 \theta}. \quad (40)$$

These can be evaluated for each critical point, in theory. While this work remains to be finished, here is the plan:

8.3 Solutions at critical points

8.3.1 Case a.

When $g = \frac{\omega}{K v_a}$ then $p_2 = 0$ and

$$s = \frac{\pm \sqrt{K^2 v_A^2 - \omega^2} - K v_A \cos \theta}{\mp \omega + K v_A \sin \theta} \quad (41)$$

8.3.2 Case b.

When $g = \frac{\omega^2 (v_A^2 + c_s^2)^2}{K c_s v_A}$ then $p_2 = \frac{-\omega^2 v_A^2}{c_s^2}$ and

$$s = \frac{\pm \sqrt{K^2 c_s^2 v_A^2 - (c_s^2 + v_A^2) \omega^2} - K c_s v_A \cos \theta}{\mp \omega \sqrt{c_s^2 + v_A^2} + K c_s v_A \sin \theta} \quad (42)$$

8 Solutions near critical points

We first transform the independent variable of the differential equation (14) from x to s . Then we apply the Frobenius method to find solutions near critical points such as those given by (26), where $x_i = 0$, x_r satisfies (18) and $s_* \equiv e^{ax_r}$. Later, we will match the solutions at critical points to solutions $\xi_x(x)$ throughout x .

8.1 Transformed differential equation

We transform equation (14) using $s \equiv e^{ax}$, $\frac{ds}{dx} = as$, $\frac{d\xi}{dx} = as\frac{d\xi}{ds}$, and $\frac{d^2\xi}{dx^2} = a^2s^2\frac{d^2\xi}{ds^2} + a^2s\frac{d\xi}{ds}$ to find

$$a^2s^2p_0\frac{d^2\xi_x}{ds^2} + (a^2s^2p_0 + as^2p_1)\frac{d\xi_x}{ds} + p_2\xi_x = 0 \quad (30)$$

and rewrite the coefficients in terms of s .

$$p0(s) = \frac{p0n}{(1+s^2)^2 p0d} \quad (31)$$

where

$$p0n = \left[(1+s^2)^2\omega^2 - K^2v_A^2(2s\cos\theta + (-1+s^2)\sin\theta)^2 \right] \\ \left\{ (1+s^2)^2v_A^2\omega^2 + c_s^2 \left[(1+s^2)^2\omega^2 - K^2v_A^2(2s\cos\theta + (-1+s^2)\sin\theta)^2 \right] \right\}$$

and

$$p0d = (1+s^2)^2\omega^2(-K^2v_A^2 + \omega^2) \\ + c_s^2K^2 \left[-(1+s^2)^2\omega^2 + K^2v_A^2(2s\cos\theta + (-1+s^2)\sin\theta)^2 \right].$$

The second coefficient of the differential equation becomes:

$$p1(s) = as\frac{dp0(s)}{ds} = \frac{p1n}{(1+s^2)^3 p1d} \quad (32)$$

where

$$p1n = 4asK^2v_A^2(\cos\theta - s^2\cos\theta + 2s\sin\theta)(2s\cos\theta + (-1+s^2)\sin\theta) \\ \left\{ (1+s^2)^4v_A^2\omega^4(K^2v_A^2 - \omega^2) - 2c_s^2(1+s^2)^2\omega^2(K^2v_A^2 - \omega^2) \right. \\ \left. \left[-(1+s^2)^2\omega^2 + 4s^2K^2v_A^2\cos^2\theta + 4s(-1+s^2)K^2v_A^2\cos\theta\sin\theta + (-1+s^2)^2K^2v_A^2\sin^2\theta \right] \right. \\ \left. + c_s^4K^2 \left[-(1+s^2)^2\omega^2 + 4s^2K^2v_A^2\cos^2\theta + 4s(-1+s^2)K^2v_A^2\cos\theta\sin\theta + (-1+s^2)^2K^2v_A^2\sin^2\theta \right]^2 \right\}$$

and

$$p1d = p0d^2 \quad (33)$$

and the third coefficient of the differential equation becomes:

$$p2(s) = \frac{p2n}{(1+s^2)^2} \quad (34)$$

The real and imaginary components of this equation are, respectively:

$$g = \frac{2 \cosh(ax_r) [\cos \theta \cos(ax_i) + \sin \theta \sinh(ax_r)]}{\cos(2ax_i) + \cosh(2ax_r)} \quad (21)$$

$$0 = \frac{2 \sin(ax_i) [\sin \theta \cos(ax_i) + \cos \theta \sinh(ax_r)]}{\cos(2ax_i) + \cosh(2ax_r)} \quad (22)$$

We solve these two equations simultaneously for the two unknowns x_r and x_i . The imaginary equation has solutions where $x_i = 0$ and where $\cos(ax_i) \tan \theta = -\sinh(ax_r)$. For nonzero x_i we find one set of solutions:

$$\tanh(ax_r) = \pm \frac{b(\theta)}{g} \text{ where } b(\theta) = \sin \theta (\sin^2 \theta - \cos^2 \theta) \quad (23)$$

$$\cos^2(ax_i) = c^2(\theta) = \frac{\cos^2 \theta}{\sin^2 \theta} \left[\frac{b^2(\theta)}{g^2 - b^2(\theta)} \right] \quad (24)$$

Substituting $x_i = 0$ into (22), we find a second solution for x_r , which simply recovers the definition

$$g = \cos \theta \operatorname{sech}(ax_r) + \sin \theta \tanh(ax_r). \quad (25)$$

Graphical analysis (Figs.3,4) shows that any angle θ satisfies these critical point relations if $g \geq 1$, but the angle is restricted at lower frequencies or shorter wavelengths.

Fig.3: Since $|\tanh(ax_r)| \leq 1$, there are critical points x_r at all θ as long as $g \geq 1$. Caption: $\sin \theta (\sin^2 \theta - \cos^2 \theta)$ vs θ

Fig.4: Since $|\cos(ax_i)| \leq 1$, there are critical points x_i at all θ as long as $g \geq 1$. Caption: $\frac{\cos \theta}{\sin \theta} \sqrt{\left[\frac{b^2}{g^2 - b^2} \right]}$ vs θ

7.1 Critical points in terms of transformed variable

We seek solutions to the displacement $\xi_x(x)$ near the critical points where $x_i = 0$ and g is a constant given by (18). To this end we rewrite (25) using $s \equiv e^{ax}$, $\tanh x = \frac{s^2 - 1}{s^2 + 1}$, and $\operatorname{sech} x = \frac{2s}{s^2 + 1}$, which yields (cf Ia.74)

$$s_r = \frac{-\cos \theta \pm \sqrt{1 - g^2}}{\sin \theta \mp g}. \quad (26)$$

For reference, another set of critical points comprises $ax_r = \ln s_r$ and $ax_i = \ln s_i$ where, from (23) and (24),

$$s_r^2 = \frac{\pm g + b(\theta)}{\pm g - b(\theta)} \quad (27)$$

and

$$s_i^2 = c(\theta) \pm \sqrt{c^2(\theta) + 1}. \quad (28)$$

We may also expand $\xi_x(x)$ about this s_r when $x_i = 0$, or $\cos(ax_i) = c(\theta) = 1$. In that case, $b^2(\theta) = g^2 \sin^2 \theta$. Substituting this condition into (28) yields

$$s_r^2 = \frac{\pm g(1 + \sin \theta)}{\pm g(1 - \sin \theta)}. \quad (29)$$

Whether we expand about an s_r given by (26) or (29), we need to first write the coefficients of the wave equation (14) in terms of $s \equiv e^{ax}$.

$$p_1(x) = \frac{dp_0(x)}{dx}$$

$$p_2(x) = \omega^2 - K^2 g(x)^2 v_A^2 \text{ and}$$

$$g(x) \equiv \cos \theta \operatorname{sech}(ax) + \sin \theta \tanh(ax).$$

We have parametrized the wavenumbers in terms of an amplitude K and the angle θ between k and h . In the y -direction, $k = K \cos \theta$, and in the z -direction, $h = K \sin \theta$. The differential equation is self-adjoint:

$$\left[\frac{d}{dx} \left\{ p_0(x) \frac{d}{dx} \right\} + p_2(x) \right] \xi_x(x) = 0. \quad (15)$$

Notice that (14) is singular where either $p_0(x)$ or the denominator of $p_1(x)$ vanishes.

Once $\xi_x(x)$ is determined, it can then be used to find the amplitudes of the two rotating displacement vectors $\vec{\zeta}_{\parallel}(x)$ and $\vec{\zeta}_{\perp}(x)$.

$$\zeta_1(x) = \xi_x'(x) \left(\frac{P_y(x) Q_{\zeta_{0z}}(x) - P_z(x) Q_{\zeta_{0y}}(x)}{Q_{\zeta_{0y}}(x) Q_{\zeta_{1z}}(x) - Q_{\zeta_{0z}}(x) Q_{\zeta_{1y}}(x)} \right) \text{ and} \quad (16)$$

$$\zeta_0(x) = - \left(\frac{Q_{\zeta_{1z}}(x) \zeta_1(x) + P_z(x) (\xi_x)'(x)}{Q_{\zeta_{0z}}(x)} \right). \quad (17)$$

This completely specifies the displacement vector. Most of the rest of this work is devoted to finding $\xi_x(x)$.

7 Critical points

We find three sets of critical points x for the differential equation (14). Each critical point condition determines a value of the function $g(x) \equiv \cos \theta \operatorname{sech}(ax) + \sin \theta \tanh(ax)$. $p_0(x)$ vanishes where

$$g^2 = \left(\frac{\omega}{v_A K} \right)^2 \text{ or } g^2 = \left(\frac{\omega}{v_A K} \right)^2 \left(\frac{v_A^2 + c_s^2}{c_s^2} \right) \quad (18)$$

and the denominator of $p_1(x)$ vanishes where

$$g(x)^2 = \left(\frac{\omega}{v_A K} \right)^2 \left[\frac{K^2 (c_s^2 + v_A^2) - \omega^2}{K^2 c_s^2} \right]. \quad (19)$$

This second value of g is not of interest, since the denominator of $p_0(x)$ vanishes at the same point. The critical points may be real or imaginary, so $g(x_r + ix_i) = \cos \theta \operatorname{sech}(ax_r + iax_i) + \sin \theta \tanh(ax_r + iax_i)$. Using the identities

$$\begin{aligned} \operatorname{sech}(x + iy) &= \frac{\cos(y) \cosh(x) + i \sin(y) \sinh(x)}{\cosh(x)^2 - \sin(y)^2} \text{ and} \\ \tanh(x + iy) &= \frac{\sinh(2x) + i \sin(2y)}{\cosh(2x) + \cos(2y)} \end{aligned}$$

the critical point conditions become

$$g = \cos \theta \left[\frac{\cos(ax_i) \cosh(ax_r) + i \sin(ax_i) \sinh(ax_r)}{\cosh(ax_r)^2 - \sin(ax_i)^2} \right] + \sin \theta \left[\frac{\sinh(2ax_r) + i \sin(2ax_i)}{\cosh(2ax_r) + \cos(2ax_i)} \right]. \quad (20)$$

$$\vec{\zeta}_\perp(x) = \zeta_1(x) [0, -\tanh(ax), \operatorname{sech}(ax)]. \quad (7)$$

Both $\vec{\zeta}_\parallel(x)$ and $\vec{\zeta}_\perp(x)$ rotate with the same handedness as the field. We seek solutions $\vec{\zeta}(x, y, z)$ which satisfy the wave equation below.

5 The wave equation

Momentum conservation and Maxwell's equations yield the wave equation

$$\rho_0 \frac{d^2 \vec{\zeta}}{dt^2} = \vec{\nabla} \left(\gamma P \vec{\nabla} \cdot \vec{\zeta} \right) + \vec{J} \times \vec{B}. \quad (8)$$

Assuming perturbations of the form $e^{i(ky + hz - \omega t)}$, the linearized wave equation is

$$\omega^2 \vec{\zeta} + c_s^2 \vec{\nabla} \left(\vec{\nabla} \cdot \vec{\zeta} \right) + \frac{v_A^2}{B_0^2} \left[\left(\vec{\nabla} \times \vec{B} \right) \times \vec{b}_1 + \left(\vec{\nabla} \times \vec{b}_1 \right) \times \vec{B} \right] = 0 \quad (9)$$

where the Alfvén speed $v_A = \frac{B_0}{\sqrt{4\pi\rho}}$, the sound speed $c_s = \sqrt{\frac{\gamma P}{\rho}}$, the mean magnetic field $\vec{B}(x)$ is given by (1), and the perturbations to the mean field are

$$\vec{b}_1(x, y, z) = \vec{\nabla} \times \left(\vec{\zeta}(x, y, z) \times \vec{B}(x) \right). \quad (10)$$

The x, y, and z components of the wave equation are, respectively:

$$Q_{\zeta 0x}(x) \zeta_0(x) + Q_{\zeta 1x}(x) \zeta_1(x) + Q_x(x) \xi_x(x) + P_{\zeta 0x}(x) \zeta_0'(x) + P_{\zeta 1x}(x) \zeta_1'(x) + d(x) \xi_x''(x) = 0 \quad (11)$$

$$Q_{\zeta 0y}(x) \zeta_0(x) + Q_{\zeta 1y}(x) \zeta_1(x) + P_y(x) \xi_x'(x) = 0 \quad (12)$$

$$Q_{\zeta 0z}(x) \zeta_0(x) + Q_{\zeta 1z}(x) \zeta_1(x) + P_z(x) \xi_x'(x) = 0. \quad (13)$$

The wave equation thus depends on three displacements and their derivatives, and on functions of x listed in Appendix A. It is not tractable in this form.

6 Differential equation

By combining the three components of the wave equation, we are able to find a single ordinary second-order differential equation for the displacement along the magnetic axis,

$$p_0(x) \xi_x''(x) + p_1(x) \xi_x'(x) + p_2(x) \xi_x(x) = 0, \quad (14)$$

where

$$p_0(x) = \left\{ \left[\omega^2 (v_A^2 + c_s^2) - K^2 c_s^2 v_A^2 g(x)^2 \right] \left[\omega^2 - K^2 v_A^2 g(x)^2 \right] \right\} \times \left\{ \omega^4 - \omega^2 K^2 (v_A^2 + c_s^2) + K^4 c_s^2 v_A^2 g(x)^2 \right\}^{-1},$$

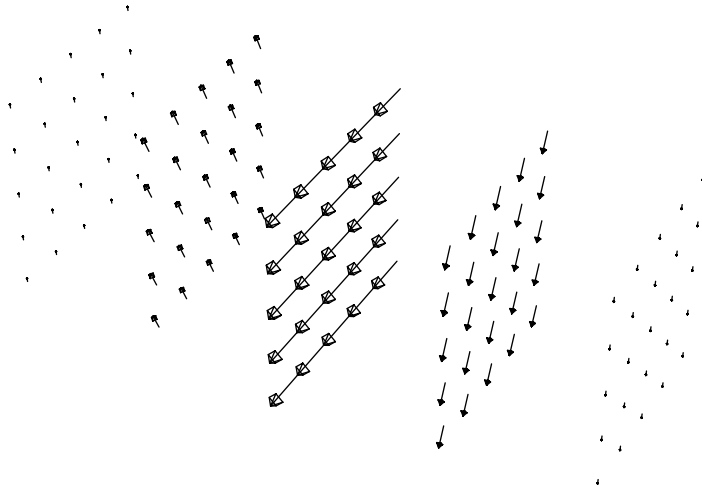


Figure 2: Sheared current sheet of width $1/a$: the x axis is horizontal, and the current is purely in the negative y direction in the middle, at $x=0$.

3 Outline of approach

- Write the displacement vector $\vec{\zeta}(x, y, z)$ in terms of displacements $\xi_x(x)$ along the x -axis and two orthogonal rotating displacement vectors $\vec{\zeta}_{\parallel}(x)$ and $\vec{\zeta}_{\perp}(x)$ in the plane of the field.
- Write the wave equation for Low's sheared force-free field, assuming constant pressure and density and negligible gravity and rotation.
- Combine the three components of the resultant wave equation to get an ordinary second-order differential equation in terms of $\xi_x(x)$ and its derivatives.
- Characterize critical points in the $\xi_x(x)$ differential equation.
- Solve for $\xi_x(x)$ and $\vec{\zeta}(x, y, z)$ near the critical points in the limits of high and low beta.
- Match solutions for $\xi_x(x)$ and $\vec{\zeta}(x, y, z)$ at the origin and $\pm\infty$ with solutions at critical points, to find displacements for all positions x .

4 The displacement vector

We write the displacement vector

$$\vec{\zeta}(x, y, z) = e^{i(ky + hz - \omega t)} \left(\xi_x(x) \hat{i} + \vec{\zeta}_{\parallel}(x) + \vec{\zeta}_{\perp}(x) \right) \quad (5)$$

in terms of two orthogonal rotating displacement vectors, one parallel to the magnetic field

$$\vec{\zeta}_{\parallel}(x) = \zeta_0(x) [0, \operatorname{sech}(ax), \tanh(ax)] \quad (6)$$

and one perpendicular to the field

1 Introduction

How do MHD waves change as they propagate into a sheared magnetic field? We attempt to find the general analytic solution for displacements in a sheared force-free magnetic field. We are able to write the MHD wave equation as an ordinary differential equation. By analyzing solutions at the critical points, we hope to be able to characterize changes to incoming waves as they travel along the sheared field, including transformations between sonic and Alfvénic waves, reflections and transmissions near critical points, and other effects.

2 The Magnetic Field

We will consider wave propagation in the presence of a constant-amplitude, sheared magnetic field. B.C. Low showed (1988) that the magnetic field in Cartesian coordinates

$$\vec{B}(x) = B_0 [0, \operatorname{sech}(ax), \tanh(ax)] \quad (1)$$

is force-free and stable. The field amplitude B_0 is fixed and its orientation in the $y - z$ plane rotates along the x axis, as in Fig.1.

The force free parameter α , defined by

$$\vec{\nabla} \times \vec{B} = \alpha \vec{B}, \quad (2)$$

is non-constant:

$$\alpha(x) = -a \operatorname{sech}(ax). \quad (3)$$

The current $\vec{J} = \vec{\nabla} \times \vec{B}$ is antiparallel and concentrated in the shear layer of thickness $1/a$, as in Fig.2.

$$\vec{J} = -a [0, \operatorname{sech}(ax)^2, \operatorname{sech}(ax) \tanh(ax)] \quad (4)$$

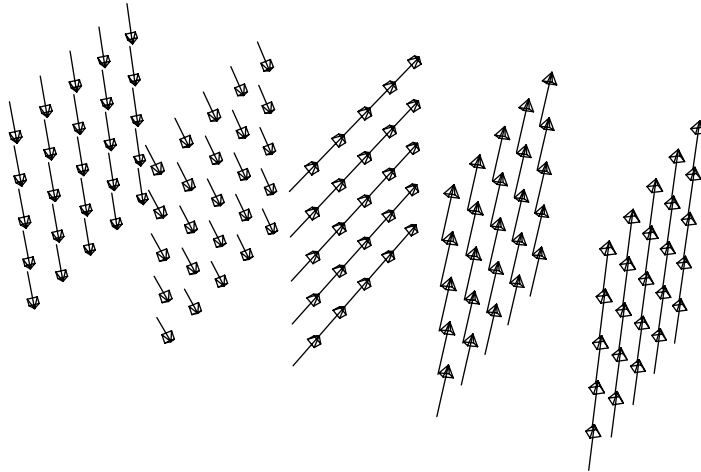


Figure 1: Sheared, force-free magnetic field: the x axis is horizontal, and the field is purely in the positive y direction in the middle, at $x=0$.