Stanley, me and Life on the Light-Cone

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In 1981 Michael Green, John Schwarz and I computed the four-point one-loop S-matrix element for *N=4* Yang-Mills and *N=8* Supergravity and found that it is given by a box-diagram with kinematical factors.

It looked to me as if there was a scalar field theory behind.

How can one describe these theories with a scalar field? (Wrong but useful idea!)

N=4 in the light-cone gauge

Make the gauge choice $A^+ = 0$

Choose $x^+ = \frac{1}{\sqrt{2}}(x^0 + x^3)$ as the time.

We can then solve for A^- since it satisfies a kinetic equation of motion and linearly combine the transverse physical degrees of freedom as

$A = 1/\sqrt{2}\,(A^1 + i\,A^2)$ and its c.c

For the fermions we choose

$$\Psi = 1/2\,\gamma_+ \gamma_- \Psi + 1/2\,\gamma_- \gamma_+ \Psi = \Psi_+ + \Psi_-$$

Similarly Ψ_- satisfies a kinetic equation of motion and can be eliminated and the two-component Ψ_+ can be written as a complex Grassmann odd field Ψ.

We can now introduce a superspace

$$x^{\pm}, \qquad x, \qquad \bar{x}, \qquad \theta^m, \qquad \bar{\theta}_n$$

and span the N=4 supersymmetry

$$\begin{aligned}
\{Q^m_+, \bar{Q}_{+n}\} &= -\sqrt{2}\delta^m_n P^+ \\
\{Q^m_-, \bar{Q}_{-n}\} &= -\sqrt{2}\delta^m_n P^- \\
\{Q^m_+, \bar{Q}_{-n}\} &= -\sqrt{2}\delta^m_n P,
\end{aligned}$$

When we act straight on a field we write *q*

The kinematical q's will be represented by

$$q_{+}^{m} = -\partial^{m} + \frac{i}{\sqrt{2}}\,\theta^{m}\,\partial^{+}\ ,\ \bar{q}_{+n} = \quad \bar{\partial}_{n} - \frac{i}{\sqrt{2}}\,\bar{\theta}_{n}\,\partial^{+}\ ,$$

and the dynamical ones as

$$q_{-}^{m} \ = \ \frac{\bar{\partial}}{\partial^{+}}\,q_{+}^{m}\ , \qquad \bar{q}_{-m} \ = \ \frac{\partial}{\partial^{+}}\,\bar{q}_{+m}\ .$$

On this space we can also represent "chiral" derivatives anticommuting with the supercharges Q.

$$d^{m} = -\partial^{m} - \frac{i}{\sqrt{2}}\,\theta^{m}\,\partial^{+},\ \ \bar{d}_{n} = \quad \bar{\partial}_{n} + \frac{i}{\sqrt{2}}\,\bar{\theta}_{n}\,\partial^{+}\ .$$

To find an irreducible representation we have to impose the the chiral constraints

$$d^m \phi = 0 \; ; \qquad \bar{d}_m \bar{\phi} = 0 \; ,$$

on a complex superfield $\phi(x^{\pm}, x, \bar{x}, \theta^m, \bar{\theta}_n)$. The solution is then that

$$\phi = \phi(x^+, y^- = x^- - \frac{i}{\sqrt{2}} \theta^m \bar{\theta}_m, x, \bar{x}, \; \theta^m).$$

It is particularly interesting to study the cases $N = 4 \times integer$. For those values one can impose a further condition on the superfield ϕ namely the "inside out" condition

$$\bar{d}_{m_1} \bar{d}_{m_2} \, ..\bar{d}_{m_{N/2-1}} \, \bar{d}_{m_{N/2}} \, \phi =$$
$$\frac{1}{2} \epsilon_{m_1 m2 \, ... m_{N-1} m_N} \, d^{m_{N/2+1}} \, d^{m_{N/2+2}} \, ... d^{m_{N-1}} d^{m_N} \, \bar{\phi}$$

$N = 4$

$$
\begin{aligned}
\phi\,(y) \;&=\; \frac{1}{\partial^+}\,A\,(y) + \frac{i}{\sqrt{2}}\,\theta^m\,\theta^n\,\overline{C}_{mn}\,(y) \\
&+\; \frac{1}{12}\,\theta^m\,\theta^n\,\theta^p\,\theta^q\,\epsilon_{mnpq}\,\partial^+\,\bar{A}\,(y) \\
&+\; \frac{i}{\partial^+}\,\theta^m\,\bar{\chi}_m(y) + \frac{\sqrt{2}}{6}\theta^m\,\theta^n\,\theta^p\,\epsilon_{mnpq}\,\chi^q(y)\;.
\end{aligned}
$$

The full action could the be found as

$$
\begin{aligned}
\mathcal{S} \;&=\; -\int d^4x \int d^4\theta\, d^4\bar{\theta} \\
&\Big\{\;\; \bar{\phi}^a\,\frac{\Box}{\partial^{+2}}\,\phi^a + \frac{4g}{3}\,f^{abc}\Big(\frac{1}{\partial^+}\,\bar{\phi}^a\,\phi^b\,\bar{\partial}\,\phi^c + \text{c.c.}\Big) \\
&-\;\; g^2\,f^{abc}\,f^{ade}\Big(\frac{1}{\partial^+}(\phi^b\partial^+\phi^c)\frac{1}{\partial^+}(\bar{\phi}^d\partial^+\bar{\phi}^e) \\
&\qquad + \frac{1}{2}\,\phi^b\bar{\phi}^c\phi^d\,\bar{\phi}^e\Big)\Big\}.
\end{aligned}
$$

With this action we (Brink, Lindgren and Nilsson 1982) proved that the perturbation expansion is finite.

We also realized that the maximal supergravity could be written in this way

$N = 8$

$$
\begin{aligned}
\phi\,(y) \;\; &= \;\; \frac{1}{\partial^{+2}}\,h\,(y) \,+\, i\,\theta^m\,\frac{1}{\partial^{+2}}\,\bar{\chi}_m\,(y) \\
\dots \;\; &+ \;\; \theta^{m\,n\,p\,r}\,\bar{C}_{m\,n\,p\,r}\,(y) \\
\dots \;\; &+ \;\; \tilde{\theta}^{(7)}_m\,\partial^+\,\chi^m(y) \,+\, \tilde{\theta}^{(8)}\,\partial^{+2}\,\bar{h}\,(y) \;\; ,
\end{aligned}
$$

How do we construct the interacting theory?

We will only consider massless theories so we solve the condition $p^2 = 0$. We then find

$$p^- = \frac{p\bar{p}}{p^+}.$$

The generator p^- is really the Hamiltonian.

We have to find representations to the super-Poincaré algebra.

Generators that involve the "time" are called dynamical (or Hamiltonians) and the others kinematical.

The dynamical ones are non-linearly realized. We have to construct all of them.

The hard ones are rotations into "time". The linear part is

$$j^- \;=\; i\,x\,\frac{\partial\bar{\partial}}{\partial^+}\;-\;i\,x^-\,\partial\;+\;i\left(\theta^\alpha\,\bar{\partial}_\alpha\;+\;\frac{i}{4\sqrt{2}\,\partial^+}\,(\,d^\alpha\,\bar{d}_\alpha-\bar{d}_\alpha\,d^\alpha\,)\right)\frac{\partial}{\partial^+}$$

The N =8 Supergravity action to first order is then

$$\int d^4x \int d^8\theta\, d^8\bar{\theta}\, \mathcal{L} \;\equiv\; \int \mathcal{L}\ ,$$

where,

$$\mathcal{L} = -\bar{\phi}\,\frac{\Box}{\partial^{+4}}\,\phi + (\,\frac{4\,\kappa}{3\,\partial^{+4}}\,\bar{\phi}\,\bar{\partial}\bar{\partial}\,\phi\,{\partial^{+}}^{2}\phi - \frac{4\,\kappa}{3\,\partial^{+4}}\,\bar{\phi}\,\bar{\partial}\,\partial^{+}\phi\,\bar{\partial}\,\partial^{+}\,\phi\,+\,c.c.\,)$$

How do we construct the four-point function?
We can do it by trial and error.
Too hard.

Instead we found a remarkable property of maximally supersymmetric theories.
(with Ananth and Ramond)

The Hamiltonian as a Quadratic Form

The usual relation is that

$$H = \frac{1}{4}\{Q^m_-, Q_{-m}\}$$

For both $N = 4$ and $N = 8$

$$H = \int \delta_{\bar{q}_{-m}} \bar{\phi} \;\; \delta_{q_-{}^m} \phi$$

Not an anticommutator, but a quadratic form.

With this form we could run a Mathematica program comparing with the four-point function of gravity.

The result was a four-point coupling with 96 terms.
(In the covariant form there are about 5000 terms.)
with Ananth, Heise and Svendsen.

Higher Symmetries for $N = 4$ Yang-Mills Theory

We know that the $d = 4$ theory is conformally invariant, i.e. under $PSU(2,2\,|\,4)$ even for the quantum case. We can in fact construct the whole theory by closing the conformal algebra by guessing the correct dynamical supersymmetry generator Q_{-}.

(With Kim and Ramond)

Higher Symmetries for $N = 8$ Supergravity Theory

$N = 8$ Supergravity, unlike $N = 4$ Yang-Mills, is not superconformal invariant; however, it does have the non-linear Cremmer-Julia $E_{7(7)}$ symmetry.

How do we implement the $E_{7(7)}$ symmetry?

Go back to covariant component form (Cremmer, Julia and Freedman, de Wit)

$$\mathcal{L} \;=\; \mathcal{L}_S + \mathcal{L}_V + \mathcal{L}_{\text{others}}$$

$\mathcal{L}_S$ is a Coleman-Wess-Zumino non-linear Lagrangian. The $E_{7(7)}$ is clear.

$\mathcal{L}_V$ can be written as

$$\mathcal{L}_V \;=\; -\tfrac{1}{8} F^{\mu\nu ij} G^{ij}_{\mu\nu} \;,$$

The Lagrangian is quadratic in the field strengths.
Introduce the self-dual *complex* field strengths

$$\mathbb{F}^{\mu\nu\, ij} \;=\; \tfrac{1}{2}\left(F^{\mu\nu\, ij} + i\widetilde{F}^{\mu\nu\, ij}\right)$$

and

$$\mathbb{G}^{\mu\nu\, ij} \;=\; \tfrac{1}{2}\left(G^{\mu\nu\, ij} + i\widetilde{G}^{\mu\nu\, ij}\right)$$

The equations of motion are given by

$$\partial_\mu G^{\mu\nu ij} \;=\; \partial_\mu \left(\mathbb{G}^{\mu\nu\, ij} + \overline{\mathbb{G}}^{\mu\nu\, ij}\right) \;=\; 0 \;,$$

while the Bianchi identities read

$$\partial_\mu \widetilde{F}^{\mu\nu ij} \;=\; \partial_\mu \left(\mathbb{F}^{\mu\nu\, ij} - \overline{\mathbb{F}}^{\mu\nu\, ij}\right) \;=\; 0 \;.$$

Assemble in one column vector with 56 complex entries

$$Z^{\mu\nu} = \begin{pmatrix} \mathbb{G}^{\mu\nu\,ij} + \mathbb{F}^{\mu\nu\,ij} \\ \mathbb{G}^{\mu\nu\,ij} - \mathbb{F}^{\mu\nu\,ij} \end{pmatrix} \equiv \begin{pmatrix} X^{\mu\nu\,ab} \\ Y^{\mu\nu}{}_{ab} \end{pmatrix} ,$$

where a, b are $SU(8)$ indices, with upper(lower) antisymmetric indices for $\mathbf{28}(\overline{\mathbf{28}})$.

This is a 56 under $E_{7(7)}$.

The 70 transformations are

$$\begin{aligned} \delta X^{\mu\nu\,ab} &= \Xi^{abcd}\, Y^{\mu\nu}{}_{cd} \\ \delta Y^{\mu\nu}{}_{ab} &= \overline{\Xi}_{abcd}\, X^{\mu\nu\,cd}, \end{aligned}$$

We now specialize to the light-cone gauge. We choose $A^+ = 0$ and solve for A^-. We then make non-linear field redefinitions, $A^{ij} \rightarrow B^{ij}$ and $C^{ijkl} \rightarrow D^{ijkl}$ to get rid of "time" derivatives" in the interaction terms.

This will mix up the fields and the Hamiltonian is no longer quadratic in B^{ij}.

We can now read off the $E_{7(7)}/SU(8)$ transformations in the vector and scalar fields.

However, the other fields now take part in the transformations!

The $\frac{E_{7(7)}}{SU(8)}$ quotient symmetry must commute with the other symmetries in particular with the supersymmetry. $[\delta_{70}, \delta_S]\varphi = 0$.

(There is no $E_{7(7)}$ supergroup.)

By using that we get the transformations for all fields in the multiplet.

How can $\frac{E_{7(7)}}{SU(8)}$ commute when $SU(8)$ does not, and

$$[\delta_{70}, \delta_{70}] = \delta_{SU(8)}?$$

Consider the Jacobi identity

$$([[\delta_{70}, \delta_{70}]\, , \delta_S] + [[\delta_S, \delta_{70}]\, , \delta_{70}] + [[\delta_{70}, \delta_S]\, , \delta_{70}])\varphi = 0$$

Since $[\delta_S, \delta_{70}]\, \delta_{70}\varphi \neq 0$, it works! $\delta_{70}\varphi$ non-linear! We only claim that $[\delta_S, \delta_{70}]\, \varphi = 0$. All fields including the graviton transform under $\frac{E_{7(7)}}{SU(8)}$ and into each other.

Some of the transformations

Vectors:

$$\boldsymbol{\delta}\,\overline{B}_{ij} \;=\; -\,\kappa\,\Xi^{klmn}\left(\frac{1}{4}\overline{D}_{ijkl}\,\overline{B}_{mn}+\frac{1}{4!}\frac{1}{\partial^+}\overline{D}_{klmn}\partial^+\overline{B}_{ij}-\frac{1}{4!}\epsilon_{ijklmnrs}\frac{1}{\partial^+}B^{rs}\partial^+h\right.$$
$$\left.+\frac{i}{3!}\frac{1}{\partial^+}\overline{\chi}_{klm}\,\overline{\chi}_{ijn}-\frac{i}{3!}\,\epsilon_{ijklmrst}\frac{1}{\partial^+}\chi^{rst}\overline{\psi}_n\right)$$
$$+\,\kappa\,\overline{\Xi}_{ijkl}\,\frac{1}{\partial^+}\,\left(\frac{1}{4}\,D^{klmn}\,\partial^+\overline{B}_{mn}-\frac{1}{\partial^+}\,B^{kl}\,\partial^{+2}\,h\right.$$
$$\left.+\frac{i}{4(3!)^2}\overline{\chi}_{mnp}\overline{\chi}_{rst}\epsilon^{klmnprst}-3\,i\,\frac{1}{\partial^+}\chi^{kln}\partial^+\overline{\psi}_n\right) \tag{1}$$

Gravitini:

$$\boldsymbol{\delta}\,\overline{\psi}_i \;=\; -\,\kappa\,\Xi^{mnpq}\left(\frac{1}{4!\cdot 3!}\epsilon_{mnpqirst}D^{rstu}\overline{\psi}_u+\frac{1}{4!}\frac{1}{\partial^+}\overline{D}_{mnpq}\partial^+\overline{\psi}_i+\frac{1}{4!}\overline{D}_{mnpq}\overline{\psi}_i\right.$$
$$\left.-\frac{1}{4!}\epsilon_{mnpqirst}\frac{1}{\partial^+}\chi^{rst}\partial^+h+\frac{1}{4}\overline{\chi}_{imn}\overline{B}_{pq}+\frac{1}{3!}\frac{1}{\partial^+}\overline{\chi}_{mnp}\partial^+\overline{B}_{iq}\right) \tag{2}$$

Gravition:

$$\boldsymbol{\delta}\,h \;=\; -\,\kappa\,\Xi^{mnpq}\left(\frac{1}{4!}\frac{1}{\partial^+}\overline{D}_{mnpq}\,\partial^+h+\frac{1}{8}\,\overline{B}_{mn}\overline{B}_{pq}+\frac{i}{\partial^+}\overline{\chi}_{mnp}\,\overline{\psi}_q\right)\;. \tag{3}$$

We then find that we can write the order κ transformation as

$$\delta\varphi = \frac{\kappa}{4!}\;\Xi^{mnpq}\;\frac{1}{\partial^{+2}}\,(\bar{d}_m\bar{d}_n\bar{d}_p\bar{d}_q\frac{1}{\partial^{+}}\varphi\,\partial^{+3}\varphi$$
$$-\quad 4\bar{d}_m\bar{d}_n\bar{d}_p\varphi\,\bar{d}_q\partial^{+2}\varphi$$
$$+\quad 3\bar{d}_m\bar{d}_n\partial^{+}\varphi\,\bar{d}_p\bar{d}_q\partial^{+}\varphi) + \cdots\;.$$

This expression is in fact unique! It can be rewritten in a very efficient form

$$\frac{\kappa}{4!}\,\Xi^{mnpq}\left(\frac{\partial}{\partial\eta}\right)_{mnpq}\frac{1}{\partial^{+2}}\left(e^{\eta\hat{\bar{d}}}\partial^{+3}\varphi\,e^{-\eta\hat{\bar{d}}}\partial^{+3}\varphi\right)\Bigg|_{\eta=0},$$

where $\hat{\bar{d}} = \frac{\bar{d}}{\partial^{+}}$.

The Hamiltonian

We write

$$\delta_s^{dyn}\,\varphi = \delta_s^{dyn\,(0)}\,\varphi + \delta_s^{dyn\,(1)}\,\varphi + \delta_s^{dyn\,(2)}\,\varphi + \mathcal{O}(\kappa^3)$$

We can now require

$$[\,\delta_{70}\,,\,\delta_s^{dyn}\,]\,\varphi \;=\; 0$$

Here we can use the inhomogeneity of the 70 transformation

$$[\,\boldsymbol{\delta}_{70}{}^{(-1)}\,,\,\delta_s^{dyn\,(2)}\,]\,\varphi + [\,\boldsymbol{\delta}_{70}{}^{(1)}\,,\,\delta_s^{dyn\,(0)}\,]\,\varphi \;=\; 0$$

This gives the order κ^2 dynamical supersymmetry. We can the use the quadratic form to find the *Hamiltonian* to order κ^2. Much simpler than before!

Possible counterterms for N=8

Let us check first in gravity. We can write the three point coupling as

$$\begin{aligned} \delta_H^{\kappa} h &= \kappa\, \partial^{+n} \left[e^{a\hat{\bar{\partial}}} \partial^{+m} h\, e^{-a\hat{\bar{\partial}}} \partial^{+m} h \right] \Big|_{a^2} \\ &\equiv \kappa\, \partial^{+n} \left(\frac{\partial}{\partial a} \right)^2 \left[e^{a\hat{\bar{\partial}}} \partial^{+m} h\, e^{-a\hat{\bar{\partial}}} \partial^{+m} h \right] \Big|_{a=0} , \end{aligned}$$

A possible one-loop counter term is

$$\delta_H^{g_1} h \;=\; \kappa^3 \partial^{+n} \left[E \partial^{+m} h\, E^{-1} \partial^{+m} h \right] \Big|_{a^3 ,\, b} ,$$

$$E = e^{a\hat{\bar{\partial}} + b\hat{\partial}} \qquad \text{and} \qquad E^{-1} = e^{-a\hat{\bar{\partial}} - b\hat{\partial}} ,$$

Consistent with the algebra for two choices of *m* and *n*

This can in fact be generalized to all orders.

$$\delta_H^{g_l} h \;=\; \kappa^{2l+1} \partial^+ \left[E \partial^{+l} h \, E^{-1} \partial^{+l} h \right] \Big|_{a^{2+l},\, b^l} \;,$$

$$\delta_H^{g_l} h \;=\; \kappa^{2l+1} \frac{1}{\partial^{+3}} \left[E \partial^{+(l+2)} h \, E^{-1} \partial^{+(l+2)} h \right] \Big|_{a^{2+l},\, b^l} \;.$$

There is another series starting with

$$\delta_H^{g_2} h \;=\; \kappa^5 \frac{1}{\partial^{+3}} \left[E \partial^{+4} \bar{h} \, E^{-1} \partial^{+4} \bar{h} \right] \Big|_{b^6} \;.$$

We are interested in counterterms which are non-zero when we use the equation of motion.

$$\partial^- \, h \;=\; \delta_H \, h \;=\; \frac{\partial \bar{\partial}}{\partial^+} \, h \;+\; \mathcal{O}(\kappa) \;.$$

All but the third terms can be written as $\Box$ (..h...h)

There are no three-point counter terms for $N = 8$

$$\delta_H \phi = ...(..\bar{\phi}\bar{\phi})$$

since the r.h.s. is not chiral.

When we consider the four-point coupling we have to use the $E_{7(7)}$ symmetry. Remember how we obtain the four-point coupling.

$$[\, \boldsymbol{\delta}_{70}{}^{(-1)} \,,\, \delta_s^{dyn\,(2)} \,]\, \varphi + [\, \boldsymbol{\delta}_{70}{}^{(1)} \,,\, \delta_s^{dyn\,(0)} \,]\, \varphi \;=\; 0$$

The terms talk to each other pairwise. They have the same number of derivatives.

A four-point counterterm $\delta_{s,c}^{dyn\,(2)}$ must satisfy

$$[\,\boldsymbol{\delta}_{70}{}^{(-1)}\,,\;\delta_{s,c}^{dyn\,(2)}\,]\,\varphi \;=\; 0$$

Furthermore it has to satisfy all the commutations rules with the full *N = 8* superalgebra.
Well-defined problem but algebraically difficult.
We still do not have the final result.

I wish you had been part of the collaboration, Stanley!

Congratulations to the first 80!