Bosonization and Beyond

Remarks in honor of Stanley Mandelstam's 80th Birthday

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Return with us now to those thrilling days of yesteryear ...

Ye Grande Olde Canonical Field Theory

Review of ancient developments

- Stanley's bosonization (bosonisation?)
- Deconstruction---the simplest case
- Reconstruction---add complications

Some less ancient developments

- Operator products
- Making four-dimensional theories look two-dimensional---the tomographic transform
- A new light cone

Quantum sine-Gordon equation as the massive Thirring model*

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(Received 6 January 1975)

The sine-Gordon equation is the theory of a massless scalar field in one space and one time dimension with interaction density proportional to $\cos\beta\varphi$, where β is a real parameter. I show that if β^2 exceeds 8π, the energy density of the theory is unbounded below; if β^2 equals 4π, the theory is equivalent to the zero-charge sector of the theory of a free massive Fermi field; for other values of β, the theory is equivalent to the zero-charge sector of the massive Thirring model. The sine-Gordon soliton is identified with the fundamental fermion of the Thirring model.

$$-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\alpha}{\beta^2}\cos\beta\phi$$

$$i\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{g}{2}j^\mu j_\mu - m\bar{\psi}\psi$$

Sine-Gordon

Massive Thirring

Soliton operators for the quantized sine-Gordon equation*

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(Received 24 February 1975)

Operators for the creation and annihilation of quantum sine-Gordon solitons are constructed. The operators satisfy the anticommutation relations and field equations of the massive Thirring model. The results of Coleman are thus reestablished without the use of perturbation theory. It is hoped that the method is more generally applicable to a quantum-mechanical treatment of extended solutions of field theories.

$$\psi_1(x) = (c\mu/2\pi)^{1/2} e^{\mu/8\epsilon} \colon \exp\left[-2\pi i \beta^{-1} \int_{-\infty}^{x} d\xi\, \dot{\phi}(\xi) - \frac{1}{2} i\beta\phi(x)\right] \colon$$

$$\psi_2(x) = -i(c\mu/2\pi)^{1/2} e^{\mu/8\epsilon} \colon \exp\left[-2\pi i \beta^{-1} \int_{-\infty}^{x} d\xi\, \dot{\phi}(\xi) + \frac{1}{2} i\beta\phi(x)\right] \colon$$

Achieving Homogeneous Anticommutation

Let $A_1(\mathbf{x})$ and $A_2(\mathbf{x})$ be two expressions linear in a Bose field $\phi(\mathbf{x})$ and its canonical conjugate $\dot{\phi}(\mathbf{x})$. Then with

$$\psi_1(\mathbf{x}) = \exp[iA_1(\mathbf{x})], \quad \psi_2(\mathbf{x}') = \exp[iA_2(\mathbf{x})]$$

it follows that for $\mathbf{x} \neq \mathbf{x}'$

$$\{\psi_i(\mathbf{x}), \psi_j(\mathbf{x}')\} = \{\psi_i(\mathbf{x}), \psi_j^\dagger(\mathbf{x}')\} = \{\psi_i^\dagger(\mathbf{x}), \psi_j^\dagger(\mathbf{x}')\} = 0$$

provided

$$[A_i(\mathbf{x}), A_j(\mathbf{x}')] = in_{ij}\pi$$

for odd integer n_{ij}.

Free massless fermions in 1 space dimension

$$\partial_\pm = \partial_0 \pm \partial_1 \qquad \psi_\pm(\mathbf{x}) = \frac{1}{2}(1+\sigma_3)\psi(\mathbf{x}) \qquad \partial_\pm \psi_\pm(\mathbf{x}) = 0$$

$$\langle 0|\psi_\pm(\mathbf{x})\psi_\pm^\dagger(\mathbf{x}')|0\rangle = \frac{1}{2\pi}\frac{1}{\alpha \mp i(\mathbf{x}-\mathbf{x}')} \quad \langle 0|\psi_\pm^\dagger(\mathbf{x}')\psi_\pm(\mathbf{x})|0\rangle = \frac{1}{2\pi}\frac{1}{\alpha \pm i(\mathbf{x}-\mathbf{x}')}$$

Free massless bosons in 1 space dimension

$$\phi(\mathbf{x}) = \phi_+(\mathbf{x}) + \phi_-(\mathbf{x}) \qquad \partial_\pm \phi_\pm(\mathbf{x}) = 0$$

$$\phi_\pm(\mathbf{x}) = \frac{1}{2}\left[\phi(\mathbf{x}) \mp \frac{1}{\partial_1}\dot\phi(\mathbf{x})\right] = \frac{1}{2}\left[\phi(\mathbf{x}) \mp \int_{-\infty}^{\infty} d\mathbf{x}'\, \theta(\mathbf{x}-\mathbf{x}')\dot\phi(\mathbf{x}')\right]$$

$$[\phi_\pm(\mathbf{x}), \phi_\pm(\mathbf{x}')] = \pm\frac{i}{4}\epsilon(\mathbf{x}-\mathbf{x}') \qquad [\phi_+(\mathbf{x}), \phi_-(\mathbf{x}')] = \frac{i}{4}$$

Theorem

If operators A and B are each the sum of creation and destruction operators with respect to a vacuum $|0\rangle$ then

$$e^A e^B = \langle 0|e^A e^B|0\rangle{:}e^A e^B{:} = e^{\langle 0|AB+AA/2+BB/2|0\rangle} : e^A e^B{:}$$

The bosonization formula

$$\psi_+(\mathbf{x}) = \frac{1}{\sqrt{2\pi\alpha}} e^{i\sqrt{4\pi}\phi_+(\mathbf{x})} \qquad \psi_-(\mathbf{x}) = \frac{1}{\sqrt{2\pi\alpha}} e^{-i\sqrt{4\pi}\phi_-(\mathbf{x})}$$

Apply the theorem

$$\psi_+(\mathbf{x})\psi_+^\dagger(\mathbf{x}') = \frac{1}{2\pi\alpha} e^{4\pi\langle 0|\phi_+(\mathbf{x})\phi_+(\mathbf{x}') - \phi_+(\mathbf{x})\phi_+(\mathbf{x}) - \phi_+(\mathbf{x}')\phi_+(\mathbf{x}')|0\rangle} :e^{i\sqrt{4\pi}[\phi_+(\mathbf{x}) - \phi_+(\mathbf{x}')]}:$$

The Bose two-point function (to within an additive constant)

$$\langle 0|\phi_+(\mathbf{x})\phi_+(\mathbf{x}')|0\rangle = \frac{1}{4\pi} \log \frac{\alpha}{\alpha - i(\mathbf{x} - \mathbf{x}')}$$

The Fermi two-point function

$$\langle 0|\psi_+(\mathbf{x})\psi_+^\dagger(\mathbf{x}')|0\rangle = \frac{1}{2\pi\alpha} \exp\left[\log \frac{\alpha}{\alpha - i(\mathbf{x} - \mathbf{x}')}\right] = \frac{1}{2\pi} \frac{1}{\alpha - i(\mathbf{x} - \mathbf{x}')}$$

Fermion bilinears

$$\rho_+(\mathbf{x}) = -\frac{1}{2}\lim_{\mathbf{a}\to 0} :[\psi_+(\mathbf{x}+\mathbf{a}/2)\psi_+^\dagger(\mathbf{x}-\mathbf{a}/2) + \psi_+(\mathbf{x}-\mathbf{a}/2)\psi_+^\dagger(\mathbf{x}+\mathbf{a}/2)]:$$

Apply the theorem

$$\begin{aligned}\rho_+(\mathbf{x}) &= -\frac{i}{2\pi\mathbf{a}} :e^{i\sqrt{4\pi}[\phi_+(\mathbf{x}+\mathbf{a}/2)-\phi_+(\mathbf{x}-\mathbf{a}/2)]}: \\ &= \frac{1}{\sqrt{\pi}}\partial_1\phi_+(\mathbf{x})\end{aligned}$$

$$\rho_-(\mathbf{x}) = \frac{1}{\sqrt{\pi}}\partial_1\phi_-(\mathbf{x})$$

$$j^0(x) = \rho_+(\mathbf{x}) + \rho_-(\mathbf{x}) = \frac{1}{\sqrt{\pi}}\partial_1\phi(\mathbf{x})$$

Impose continuity condition and conclude

$$j^1(\mathbf{x}) = -\frac{1}{\sqrt{\pi}}\partial_0\phi(\mathbf{x}) = \rho_+(\mathbf{x}) - \rho_-(\mathbf{x})$$

Reconstruction

This bosonization gives free massless Fermi $\leftrightarrow$ free massless Boson.

Adding a mass term

$$-m{:}[\psi_+^\dagger(\underline{\mathrm{x}})\psi_-(\underline{\mathrm{x}}) + [\psi_-^\dagger(\underline{\mathrm{x}})\psi_+(\underline{\mathrm{x}})]{:} = -\frac{m}{\pi\alpha}\cos[\sqrt{4\pi}\phi(\underline{\mathrm{x}})]$$

gives the Bose field a sine-Gordon interaction and leaves the Fermi field free but massive.

Linear combinations of the exponents A_1 and A_2

$$A_1' = \cosh\gamma A_1 - \sinh\gamma A_2 \qquad A_2' = \cosh\gamma A_2 - \sinh\gamma A_1$$

supply a Thirring interation to the fermions, Any mass term becomes

$$-\frac{m}{\pi\alpha}\cos[\sqrt{4\pi}\cosh\gamma\,\phi(\underline{\mathrm{x}})]$$

What comes next?

Massless scalar propagator in D dimensions:

$$\langle 0|T[\phi(x)\phi(0)]|0\rangle = -iG_{\mathrm{B}}(x) = \frac{1}{4\pi^{D/2}}\frac{\Gamma(D/2-1)}{(x^2)^{D/2-1}}$$

Massless spinor propagator in D dimensions:

$$\langle 0|T[\psi_\alpha(x)\psi^\dagger_\beta(0)]|0\rangle = -i[G_{\mathrm{F}}(x,0)\gamma^0]_{\alpha\beta} = i\frac{\Gamma(D/2)}{2\pi^{D/2}}\frac{(\vec{\alpha}\cdot\mathbf{r}+t)_{\alpha\beta}}{(x^2)^{D/2}}$$

Spatial-derivative equation:

$$\begin{aligned}
&\langle 0|T[j_{\gamma\delta}(y)\psi_\alpha(x)\psi^\dagger_\beta(x')]|0\rangle \\
&\quad = \langle 0|T[\psi_\alpha(x)\psi^\dagger_\gamma(y)]|0\rangle\,\langle 0|T[\psi_\delta(y)\psi^\dagger_\beta(x')]|0\rangle \\
&\quad = i\frac{\Gamma(D/2)}{2\pi^{D/2}}\frac{(\vec{\alpha}\cdot\{\mathbf{x}-\mathbf{y}\}+x^0-y^0)_{\alpha\gamma}}{(\{x-y\}^2)^{D/2}}\langle 0|T[\psi_\delta(y)\psi^\dagger_\beta(x')]|0\rangle
\end{aligned}$$

Spatial-derivative equation:

$$\begin{aligned}
&\langle 0|T[j_{\gamma\delta}(y)\psi_\alpha(x)\psi^\dagger_\beta(x')]|0\rangle \\
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&\quad = i\frac{\Gamma(D/2)}{2\pi^{D/2}}\frac{(\vec{\alpha}\cdot\{\mathbf{x}-\mathbf{y}\}+x^0-y^0)_{\alpha\gamma}}{(\{x-y\}^2)^{D/2}}\langle 0|T[\psi_\delta(y)\psi^\dagger_\beta(x')]|0\rangle
\end{aligned}$$

Take $y-x=s$, $s^0=0$:

$$\psi_\delta(y)=\psi_\delta(x)+\mathbf{s}\cdot\nabla_x\psi_\delta(x)$$

As $|s|\to 0$

$$j_{\gamma\delta}(y)\psi_\alpha(x)\to -i\frac{\Gamma(D/2)}{2\pi^{D/2}}\frac{1}{s^D}(\vec{\alpha}\cdot\mathbf{s})_{\alpha\gamma}(\mathbf{s}\cdot\nabla_x)\psi_\delta(x)+{:}\psi^\dagger_\gamma(x)\psi_\delta(x)\psi_\alpha(x){:}$$

Average over $D-2$ dimensional hypersphere

$$j_{\gamma\delta}(x)*\psi_\alpha(x)=-i\frac{\Gamma(D/2)}{2\pi^{D/2}(D-1)}(\vec{\alpha}\cdot\nabla_x)_{\alpha\gamma}\psi_\delta(x)+{:}\psi^\dagger_\gamma(x)\psi_\delta(x)\psi_\alpha(x){:}$$

$$\mathcal{M}^a(y) = M^a_{\gamma\delta} j_{\gamma\delta}(y) = \mathrm{tr}\,(M^a j)$$

$$[(\vec{\alpha}\cdot\nabla_x)M^a\psi(x)]_\alpha = i\frac{2\pi^{D/2}(D-1)}{\Gamma(D/2)}\left[\mathcal{M}^a(x) * \psi_\alpha(x) - :\psi^\dagger_\gamma(x)M^a_{\gamma\delta}\psi_\delta(x)\psi_\alpha(x):\right]$$

The last term is not present for $D > 2$.

$$D=2: \quad \alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}:$$

$M^a = 1_2$:

$$\begin{aligned}\partial_1 \begin{pmatrix} \psi_1(x) \\ -\psi_2(x) \end{pmatrix} &= 2i\pi \left[j^0(x) * \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} - :\psi^\dagger_\gamma(x)\psi_\gamma(x) \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}: \right] \\ &= 2\pi i \begin{pmatrix} j^0(x) * \psi_1(x) - :\psi^\dagger_2(x)\psi_2(x)\psi_1(x): \\ j^0(x) * \psi_2(x) - :\psi^\dagger_1(x)\psi_1(x)\psi_2(x): \end{pmatrix}\end{aligned}$$

$M^a = \alpha$:

$$\begin{aligned}\partial_1 \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} &= 2i\pi \left[j^1(x) * \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} - :\{\psi^\dagger_1(x)\psi_1(x) - \psi^\dagger_2(x)\psi_2(x)\} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}: \right] \\ &= 2\pi i \begin{pmatrix} j^1(x) * \psi_1(x) + :\psi^\dagger_2(x)\psi_2(x)\psi_1(x): \\ j^1(x) * \psi_2(x) - :\psi^\dagger_1(x)\psi_1(x)\psi_2(x): \end{pmatrix}\end{aligned}$$

$$\partial_1\psi_1(x) = i\pi[j^0(x) + j^1(x)] * \psi_1(x), \quad -\partial_1\psi_2(x) = i\pi[j^0(x) - j^1(x)] * \psi_2(x)$$

$D = 4$:

$$\{\vec{\alpha}\cdot\mathbf{n}, \vec{\alpha}\cdot\nabla\}\psi(x) = 6\pi^2 i[\psi^\dagger(x)\vec{\alpha}\cdot\mathbf{n}\psi(x) * \psi(x) + \psi^\dagger(x)\psi(x) * \vec{\alpha}\cdot\mathbf{n}\psi(x)] = \mathbf{n}\cdot\nabla\psi(x)$$

$$T^{0i}(x) = 3\pi^2[\psi^\dagger(x)\alpha^i\psi(x) * \psi^\dagger(x)\psi(x)]$$

See "Fermion Avatars of the Sugawara Model"

by Coleman, Gross and Jackiw

Make four dimensions look like two

Completeness relation:

$$\delta^{(3)}(\mathbf{r}-\mathbf{r}') = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} = \frac{1}{(2\pi)^3}\int d\Omega_{\mathbf{n}} \int_0^\infty dk\, k^2\, e^{ik\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}')}$$

$$= \frac{1}{16\pi^3}\int d\Omega_{\mathbf{n}} \int_{-\infty}^\infty dk\, k^2\, e^{ik\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}')}$$

$$= \frac{1}{16\pi^3}\int d\Omega_{\mathbf{n}} \int_{-\infty}^\infty dk\, k^2 \int_{-\infty}^\infty dy \int_{-\infty}^\infty dy'\, \delta(y-\mathbf{n}\cdot\mathbf{r})\delta(y'-\mathbf{n}\cdot\mathbf{r}')e^{ik(y-y')}$$

$$= \frac{1}{16\pi^3}\int d\Omega_{\mathbf{n}} \int_{-\infty}^\infty dy \int_{-\infty}^\infty dy'\, \delta'(y-\mathbf{n}\cdot\mathbf{r})\delta'(y'-\mathbf{n}\cdot\mathbf{r}') \int_{-\infty}^\infty dk\, e^{ik(y-y')}$$

$$= \frac{1}{8\pi^2}\int d\Omega_{\mathbf{n}} \int_{-\infty}^\infty dy\, \delta'(y-\mathbf{n}\cdot\mathbf{r})\delta'(y-\mathbf{n}\cdot\mathbf{r}')$$

Define tomographic transform:

$$\tilde{\phi}(y,\mathbf{n}) = \frac{1}{2\pi}\int d^3\mathbf{r}\, \delta'(y-\mathbf{n}\cdot\mathbf{r})\, \phi(\mathbf{r})$$

Inverse transform:

$$\phi(\mathbf{r}) = \frac{1}{4\pi}\int_{-\infty}^\infty dy \int d\Omega_{\mathbf{n}}\, \delta'(y-\mathbf{n}\cdot\mathbf{r})\, \tilde{\phi}(y,\mathbf{n})$$

Fermionic completeness relation:

$$\delta^{(3)}(\mathbf{r}-\mathbf{r}')\delta_{\alpha\beta} = \frac{1}{8\pi^2}\int d\Omega_{\mathbf{n}} \int_{-\infty}^{\infty} dy\, \delta'(y-\mathbf{n}\cdot\mathbf{r})\delta'(y-\mathbf{n}\cdot\mathbf{r}')(1+\vec{\alpha}\cdot\mathbf{n})_{\alpha\beta}$$

$$= \frac{1}{4\pi^2}\int d\Omega_{\mathbf{n}} \int_{-\infty}^{\infty} dy\, \delta'(y-\mathbf{n}\cdot\mathbf{r})\delta'(y-\mathbf{n}\cdot\mathbf{r}') \sum_{a=+,-} u^a_\alpha(\mathbf{n}) u^{a\,\dagger}_\beta(\mathbf{n})$$

Spinors:

$$(\vec{\alpha}\cdot\mathbf{n})u^{\pm}(\mathbf{n}) = u^{\pm}(\mathbf{n}), \quad \gamma^5 u^{\pm}(\mathbf{n}) = \pm u^{\pm}(\mathbf{n}), \quad u^{+}(\mathbf{n}) = \begin{pmatrix} v(\mathbf{n}) \\ 0 \end{pmatrix}, \quad u^{-}(\mathbf{n}) = \begin{pmatrix} 0 \\ v(-\mathbf{n}) \end{pmatrix}$$

$$(\vec{\sigma}\cdot\mathbf{n})v(\mathbf{n}) = v(\mathbf{n}), \quad v(\mathbf{n}) = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\varphi} \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Fermionic transform:

$$\tilde{\psi}^{\pm}(y,\mathbf{n}) = \frac{1}{2\pi}\int d^3\mathbf{r}\, \delta'(y-\mathbf{n}\cdot\mathbf{r}) \sum_{\alpha=1}^{4} u^{\pm\,\dagger}_\alpha(\mathbf{n})\psi_\alpha(\mathbf{r})$$

Inverse fermionic transform:

$$\psi_\alpha(\mathbf{r}) = \frac{1}{2\pi}\int dy \int d\Omega_{\mathbf{n}}\, \delta'(y-\mathbf{n}\cdot\mathbf{r}) \sum_a u^a_\alpha(\mathbf{n})\tilde{\psi}^a(y,\mathbf{n})$$

Bosonic lagrangian:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\int d^3\mathbf{r}\,\phi(\mathbf{r})(\nabla^2 - \partial_0^2 - m^2)\phi(\mathbf{r}) \\
&= \frac{1}{8\pi}\int d^3\mathbf{r}\int dy\int d\Omega_{\mathbf{n}}\,\phi(\mathbf{r})(\nabla^2 - \partial_0^2 - m^2)\delta'(y - \mathbf{n}\cdot\mathbf{r})\tilde{\phi}(y,\mathbf{n}) \\
&= \frac{1}{8\pi}\int d^3\mathbf{r}\int dy\int d\Omega_{\mathbf{n}}\,\phi(\mathbf{r})\delta'(y - \mathbf{n}\cdot\mathbf{r})(\partial_y^2 - \partial_0^2 - m^2))\tilde{\phi}(y,\mathbf{n}) \\
&= \frac{1}{4}\int dy\int d\Omega_{\mathbf{n}}\,\tilde{\phi}(y,\mathbf{n})(\partial_y^2 - \partial_0^2 - m^2))\tilde{\phi}(y,\mathbf{n})
\end{aligned}$$

Fermionic lagrangian:

$$\begin{aligned}
\mathcal{L} &= \int d^3\mathbf{r}\, \psi^\dagger(\mathbf{r})(i\vec{\alpha}\cdot\nabla + i\partial_0 - m\gamma^0)\psi(\mathbf{r}) \\
&= \frac{1}{2\pi}\int d^3\mathbf{r}\int dy\int d\Omega_{\mathbf{n}} \sum_{a=+,-} \psi^\dagger(\mathbf{r})(i\vec{\alpha}\cdot\nabla + i\partial_0 - m\gamma^0)\delta'(y-\mathbf{n}\cdot\mathbf{r})u^a(\mathbf{n})\tilde{\psi}^a(y,\mathbf{n}) \\
&= \frac{1}{2\pi}\int d^3\mathbf{r}\int dy\int d\Omega_{\mathbf{n}} \sum_{a=+,-} \psi^\dagger(\mathbf{r})\delta'(y-\mathbf{n}\cdot\mathbf{r})(i\partial_y + i\partial_0 - m\gamma^0)u^a(\mathbf{n})\tilde{\psi}^a(y,\mathbf{n}) \\
&= \frac{1}{2\pi}\int d^3\mathbf{r}\int dy\int d\Omega_{\mathbf{n}} \sum_{a=+,-} \psi^\dagger(\mathbf{r})\delta'(y-\mathbf{n}\cdot\mathbf{r})[u^a(\mathbf{n})(i\partial_y + i\partial_0) - mu^{-a}(-\mathbf{n})]\tilde{\psi}^a(y,\mathbf{n})) \\
&= \int dy\int d\Omega_{\mathbf{n}} \sum_{a=+,-} [\tilde{\psi}^{a\dagger}(y,\mathbf{n})(i\partial_y + i\partial_0)\psi^a(y,\mathbf{n}) - m\tilde{\psi}^{-a\dagger}(y,-\mathbf{n})\tilde{\psi}^a(y,\mathbf{n})]
\end{aligned}$$

A New Light Cone

$$\frac{t+\mathbf{n}\cdot\mathbf{r}}{\sqrt{2}}, \quad \frac{t-\mathbf{n}\cdot\mathbf{r}}{\sqrt{2}}, \quad \mathbf{r}^{\mathrm{T}}$$

Happy Birthday, Stanley!!!