ORBITS!

Rounding out Young Embedded Star Clusters,
Future Structure of Dark Matter Halos,
Unambiguous Definition of Galactic Masses,
Orbital Instability in Triaxial Cusp Potentials,
and Stochastic Hill's Equations

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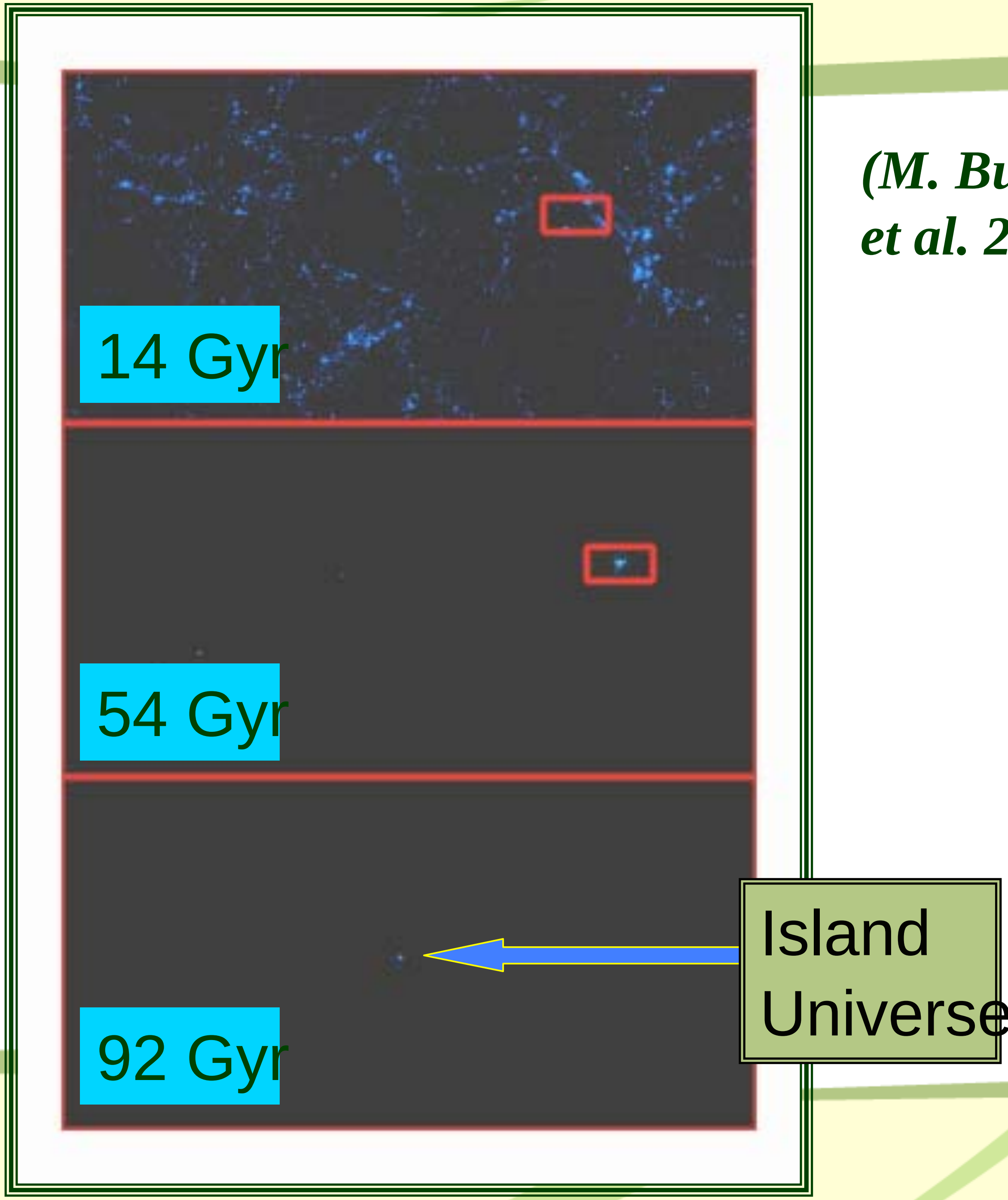
What makes young star clusters round? How do orbits affect radiation exposure?

Density profile implied by Larson's Law:

$$\frac{\partial P}{\partial \rho} = (\Delta v)^2 \propto \frac{1}{\rho} \Rightarrow \rho \propto 1/r$$

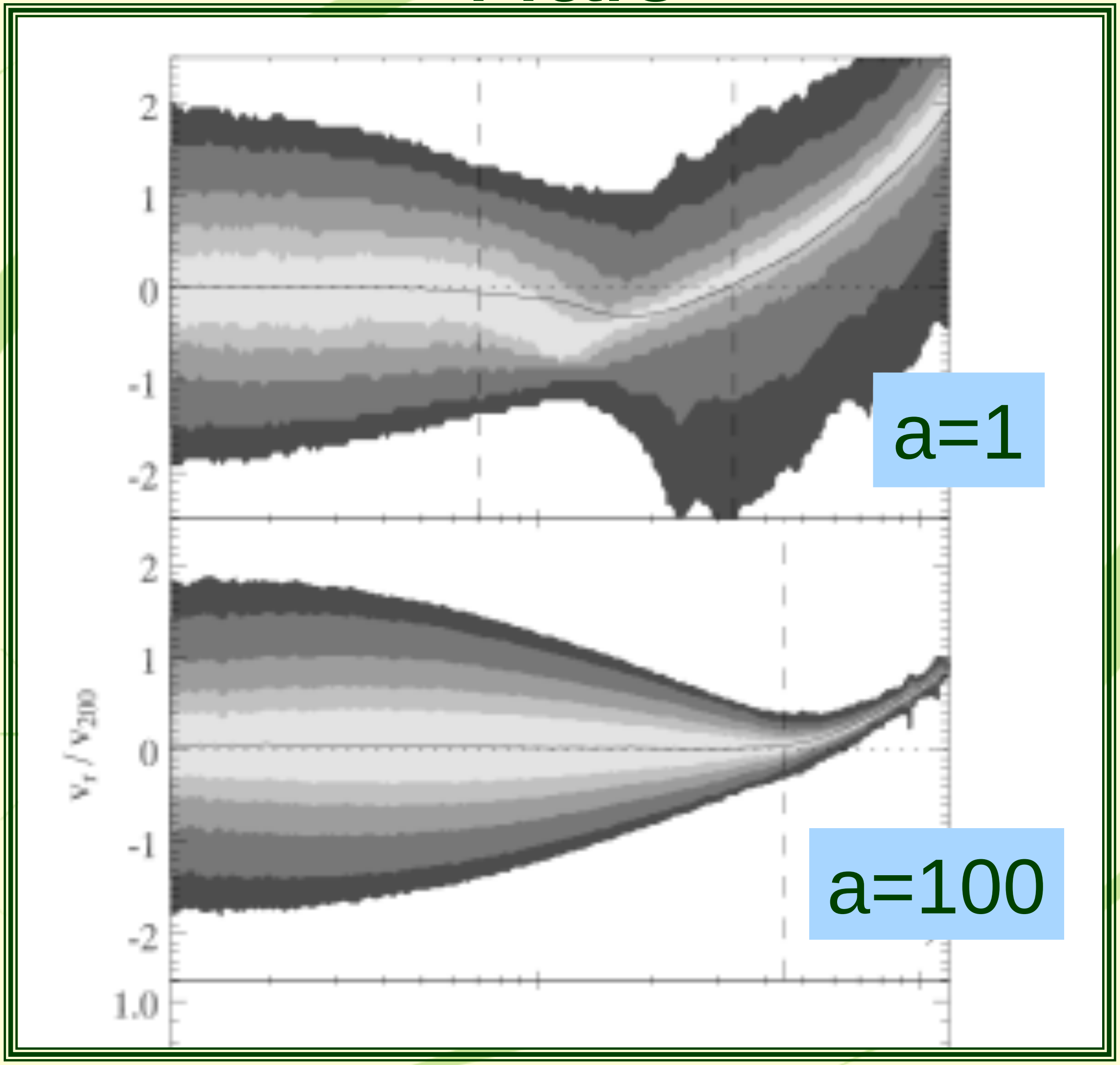
$$\rho = \frac{\rho_0}{\xi(1+\xi)^3} \quad \textit{where} \quad \xi = r/r_s$$

What is the total mass of a galaxy? Why do dark matter halos have a nearly universal form?

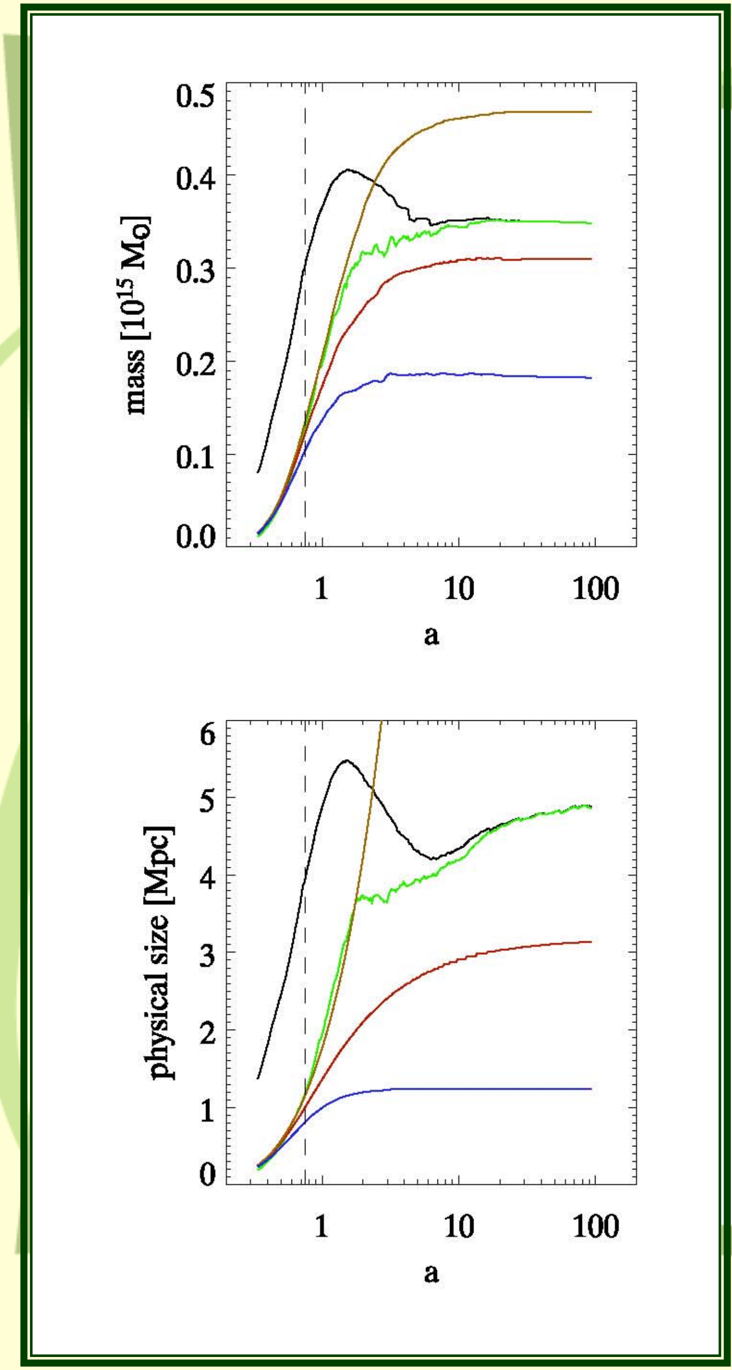


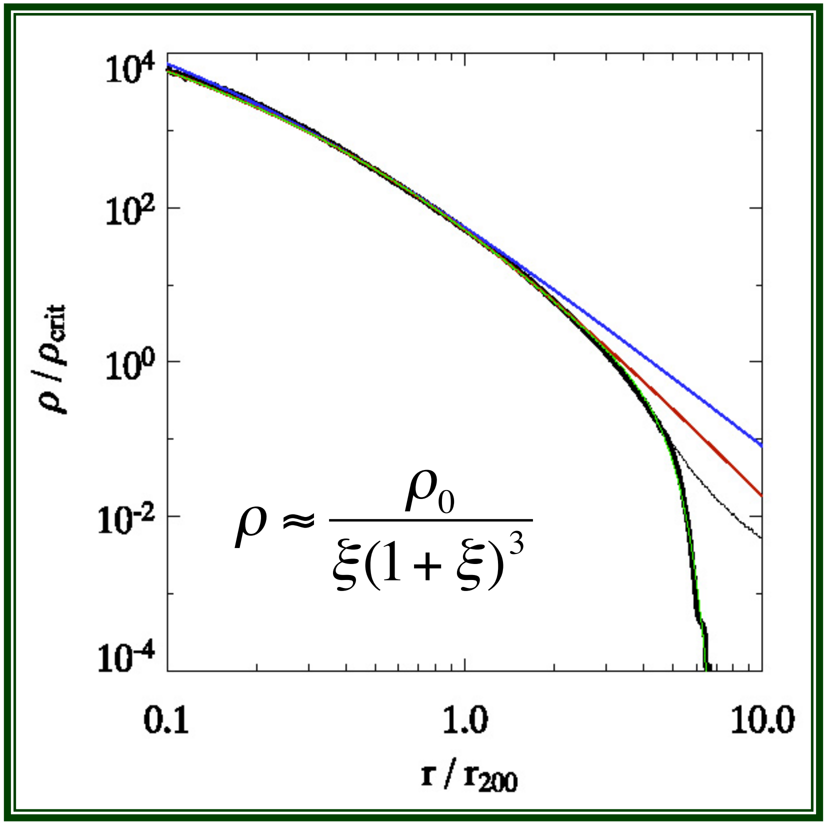
(M. Busha et al. 2003)

Phase Space of Dark Matter Halo



(M. Busha et al. 2005)



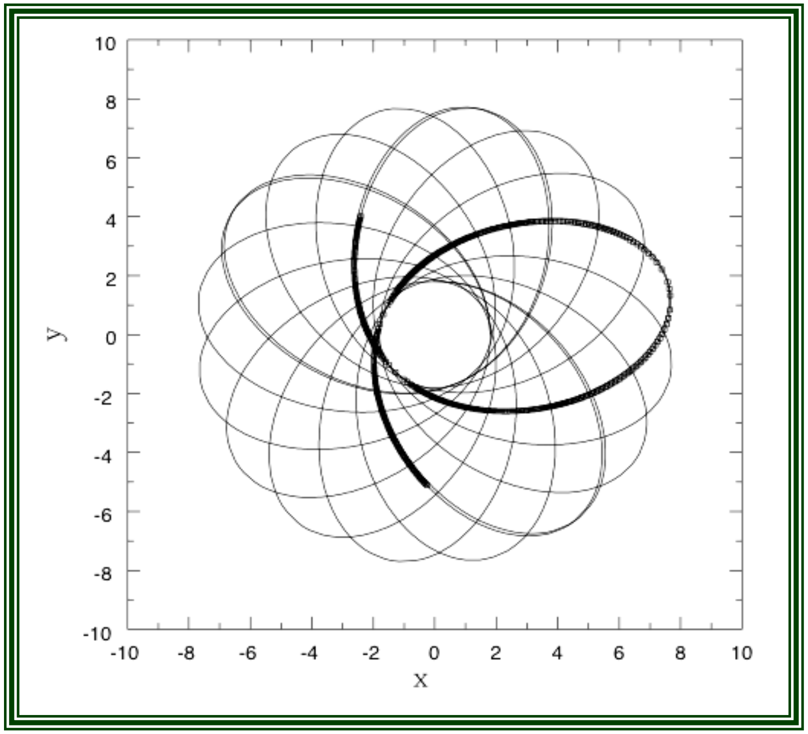


Dark matter halos approach a well-defined asymptotic form with unambiguous total mass, outer radius, density profile

WHY THESE ORBITS?

- Most of the mass is in dark matter
- Most dark matter resides in these halos
- Halos have the universal form found here for most of their lives
- Most orbital motion that will EVER occur will be THIS orbital motion $\left(\textit{factor of } 10^{74} \right)$

Spherical Limit:
Orbits look like Spirographs



Orbits in Spherical Potential

$$\rho = \frac{\rho_0}{\xi(1+\xi)^3} \Rightarrow \Psi = \frac{\Psi_0}{1+\xi}$$

$$\varepsilon \equiv |E|/\Psi_0 \quad and \quad q \equiv j^2/2\Psi_0 r_s^2$$

$$\varepsilon = \frac{\xi_1 + \xi_2 + \xi_1\xi_2}{(\xi_1 + \xi_2)(1 + \xi_1 + \xi_2 + \xi_1\xi_2)}$$

$$q = \frac{(\xi_1\xi_2)^2}{(\xi_1 + \xi_2)(1 + \xi_1 + \xi_2 + \xi_1\xi_2)}$$

$$q_{\max} = \frac{1}{8\varepsilon}\frac{(1+\sqrt{1+8\varepsilon}-4\varepsilon)^3}{(1+\sqrt{1+8\varepsilon})^2}$$

(angular momentum of the circular orbit)

$$\xi_* = \frac{1-4\varepsilon+\sqrt{1+8\varepsilon}}{4\varepsilon}$$

(effective semi-major axis)

$$\frac{\Delta\theta}{\pi} = \frac{1}{2} + \left[(1+8\varepsilon)^{-1/4} - \frac{1}{2}\right]\left[1+\frac{\log(q/q_{\max})}{6\log 10}\right]^{3.6}$$

$$\lim_{q\to q_{\max}} \Delta\theta = \pi(1+8\varepsilon)^{-1/4}$$

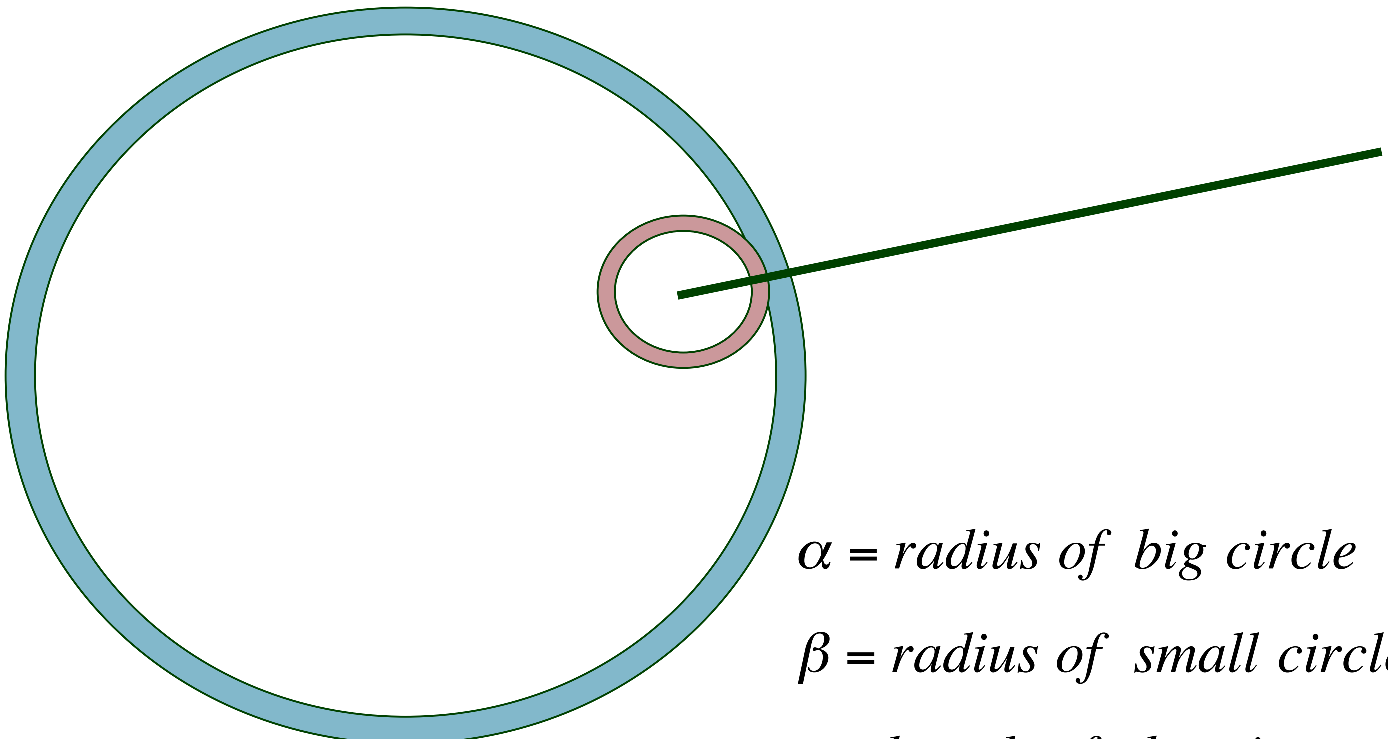
(circular orbits do not close)

These results determine the radiation exposure of a star, averaged over its orbit, as a function of energy, where the result is nearly independent of angular momentum:

$$\left\langle F_{fuv} \right\rangle \approx \frac{L_{fuv}}{8 r_s^2} \frac{A\varepsilon^{3/2}}{\cos^{-1}\sqrt{\varepsilon} + \sqrt{\varepsilon}\sqrt{1-\varepsilon}}$$

$$where \quad 1 \leq A(q) \leq \sqrt{2}$$

Spirograph Pattern (Epicycloid) given by circle turning on a circle:



α = *radius of big circle*

β = *radius of small circle*

γ = *length of drawing radius*

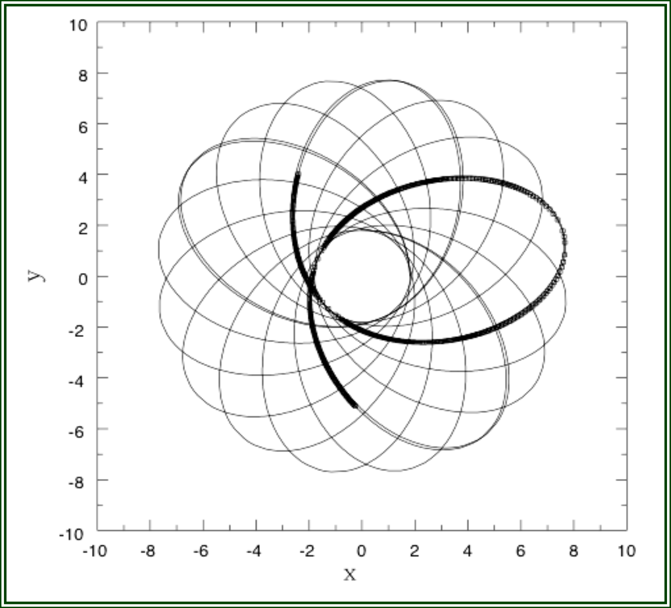
Epicycloids are NOT epicycles…

Spirographic Orbital Elements

$$(\varepsilon,\ q)$$

$$(\xi_1, \xi_2)$$

$$(\alpha, \beta, \gamma)$$



$$x(t_p) = (\alpha - \beta)\cos t_p + \gamma \cos[(\alpha - \beta)t_p / \beta]$$

$$y(t_p) = -(\alpha - \beta)\sin t_p + \gamma \sin[(\alpha - \beta)t_p / \beta]$$

(Adams & Bloch 2005)

Basic Spirographic Results

$$\varepsilon = \frac{\gamma^2 + 2\gamma - (\alpha - \beta)^2}{2\gamma[(1+\gamma)^2 - (\alpha - \beta)^2]}$$

$$q = \frac{[\gamma^2 - (\alpha - \beta)^2]^2}{2\gamma[(1+\gamma)^2 - (\alpha - \beta)^2]}$$

$$\xi_1 = \gamma - (\alpha - \beta),\;\; \xi_2 = \gamma + (\alpha - \beta),\;\; \Delta\vartheta = (\alpha - \beta)\pi / \alpha$$

$$\xi_1, \xi_2 = \gamma \pm (\alpha - \beta) \;\; so \;\; a_{sp} = \gamma,\;\; e_{sp} = \frac{\alpha - \beta}{\gamma}$$

Conservation of Energy gives transformation between physical time and parametric time:

$$\frac{dt_p}{dt} = \left(\frac{1}{1+\xi} - \varepsilon\right)\left[\frac{\gamma^2(\alpha/\beta - 1)\alpha}{\beta} + \frac{(\alpha-\beta)^2\alpha}{\beta} - \left(\frac{\alpha}{\beta} - 1\right)\xi^2\right]^{-1/2}$$

$$v_x = -\left[(\alpha-\beta)\sin t_p + \gamma\left(\frac{\alpha}{\beta} - 1\right)\sin\left(\frac{(\alpha-\beta)t_p}{\beta}\right)\right]\frac{dt_p}{dt}$$

$$v_y = \left[-(\alpha-\beta)\cos t_p + \gamma\left(\frac{\alpha}{\beta} - 1\right)\cos\left(\frac{(\alpha-\beta)t_p}{\beta}\right)\right]\frac{dt_p}{dt}$$

Triaxial Density Distributions

- Relevant density profiles include NFW and Hernquist

$$\rho_{nfw} = \frac{1}{m(1+m)^2} \qquad \rho_{Hern} = \frac{1}{m(1+m)^3}$$

- Isodensity surfaces in triaxial geometry

$$m^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \qquad a > b > c > 0$$

- In the inner limit both profiles scale as 1/r

$$m << 1 \quad \Rightarrow \quad \rho \propto \frac{1}{m}$$

Triaxial Potential

$$\Phi = \int_0^\infty du \frac{\psi(m)}{\sqrt{(u+a^2)(u+b^2)(u+c^2)}} \qquad \psi(m) = \int_\infty^{m^2} \rho(m) dm^2$$

- In the inner limit the above integral can be simplified to

$$\Phi = -I_1 + I_2$$

where I_1 is the depth of the potential well and the effective potential is given by

$$I_2 = 2\int_0^\infty du \frac{\sqrt{\xi^2 u^2 + \Lambda u + \Gamma}}{(u+a^2)(u+b^2)(u+c^2)}$$

ξ, Λ, Γ are polynomial functions of x, y, z, a, b, c

Triaxial Forces

$$F_x = \frac{-2\,\mathrm{sgn}(x)}{\sqrt{\left(a^2-b^2\right)\left(a^2-c^2\right)}}\ln\left(\frac{2G(a)\sqrt{\Gamma}+2\Gamma-a^2\Lambda}{2a^2\xi G(a)+\Lambda a^2-2a^4\xi^2}\right)$$

$$F_y = \frac{-2\,\mathrm{sgn}(y)}{\sqrt{\left(a^2-b^2\right)\left(b^2-c^2\right)}}\left[\sin^{-1}\left(\frac{\Lambda-2b^2\xi^2}{\sqrt{\Lambda^2-4\Gamma\xi^2}}\right)-\sin^{-1}\left(\frac{2\Gamma/b^2-\Lambda}{\sqrt{\Lambda^2-4\xi^2\Gamma}}\right)\right]$$

$$F_z = \frac{-2\,\mathrm{sgn}(z)}{\sqrt{\left(a^2-c^2\right)\left(b^2-c^2\right)}}\ln\left(\frac{2G(c)\sqrt{\Gamma}+2\Gamma-c^2\Lambda}{2c^2\xi G(c)+\Lambda c^2-2c^4\xi^2}\right)$$

$$G(u)=\xi^2u^4-\Lambda u^2+\Gamma$$

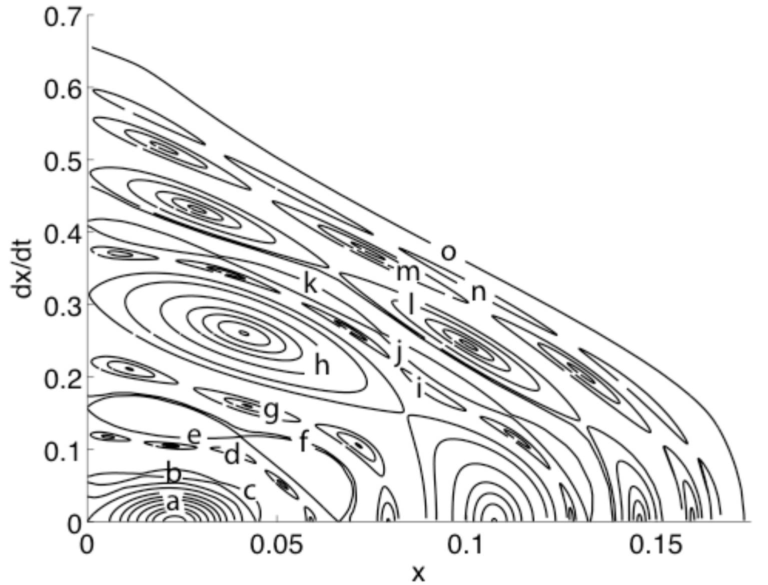
$$\xi^2=x^2+y^2+z^2$$

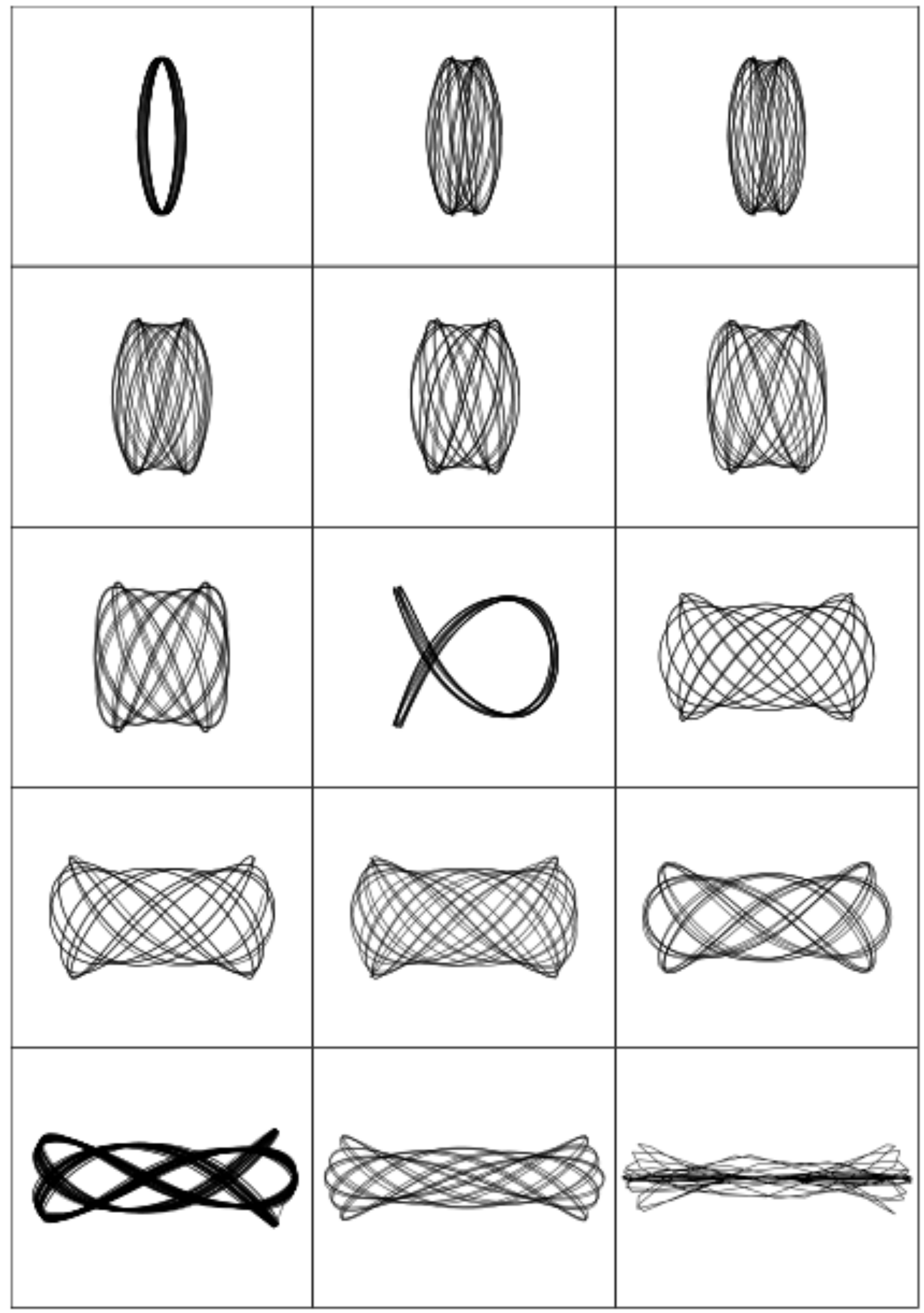
$$\Lambda=\left(b^2+c^2\right)x^2+\left(a^2+c^2\right)y^2+\left(a^2+b^2\right)z^2$$

$$\Gamma=b^2c^2x^2+a^2c^2y^2+a^2b^2z^2$$

(Adams et al. 2007)

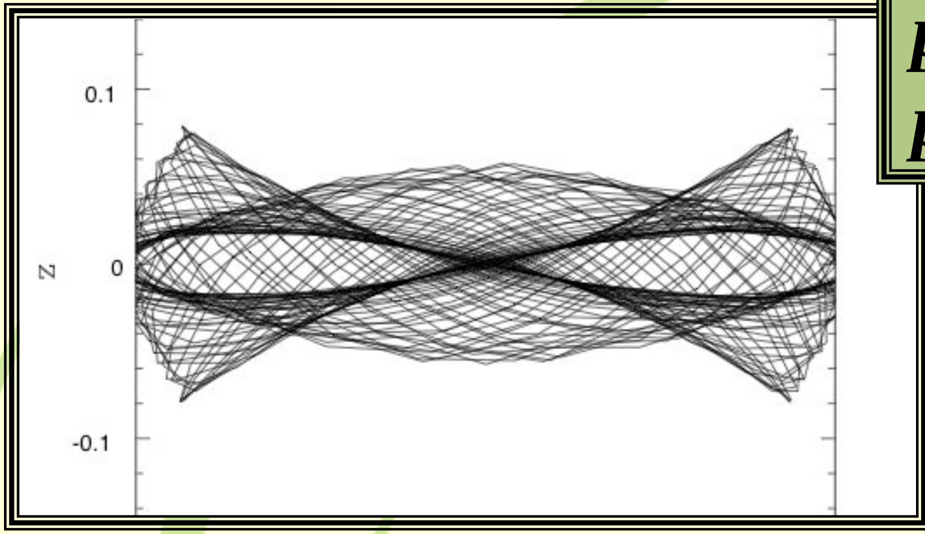
Orbit Gallery

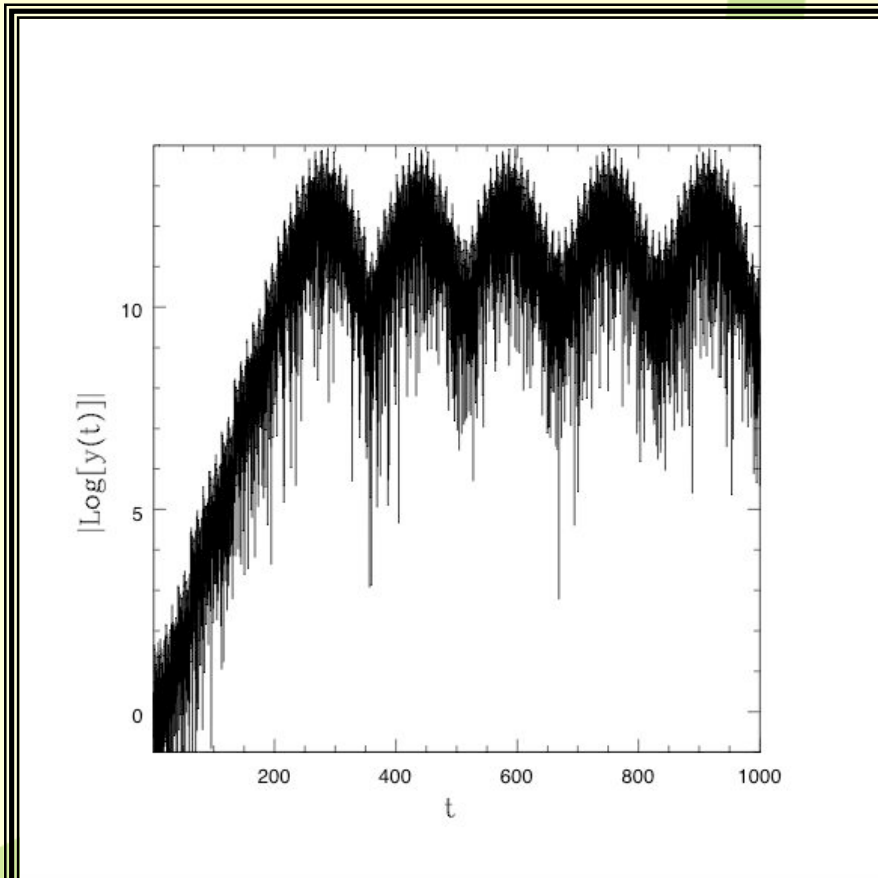




INSTABILITIES

Orbits in any of the principal planes are unstable to motion perpendicular to the plane.





Unstable motion shows:
(1) exponential growth,
(2) quasi-periodicity,
(3) chaotic variations, &
(4) eventual saturation.

Perpendicular Perturbations

- Force equations in limit of small x, y, or z become

$$F_x \approx -\left(\frac{4}{a\left(\sqrt{c^2y^2+b^2z^2}+a\sqrt{y^2+z^2}\right)}\right)x \qquad F_x \approx -\omega_x^2 x$$

$$F_y \approx -\left(\frac{4}{b\left(\sqrt{c^2x^2+a^2z^2}+b\sqrt{x^2+z^2}\right)}\right)y \qquad F_y \approx -\omega_y^2 y$$

$$F_z \approx -\left(\frac{4}{c\left(\sqrt{b^2x^2+a^2y^2}+c\sqrt{x^2+y^2}\right)}\right)z \qquad F_z \approx -\omega_z^2 z$$

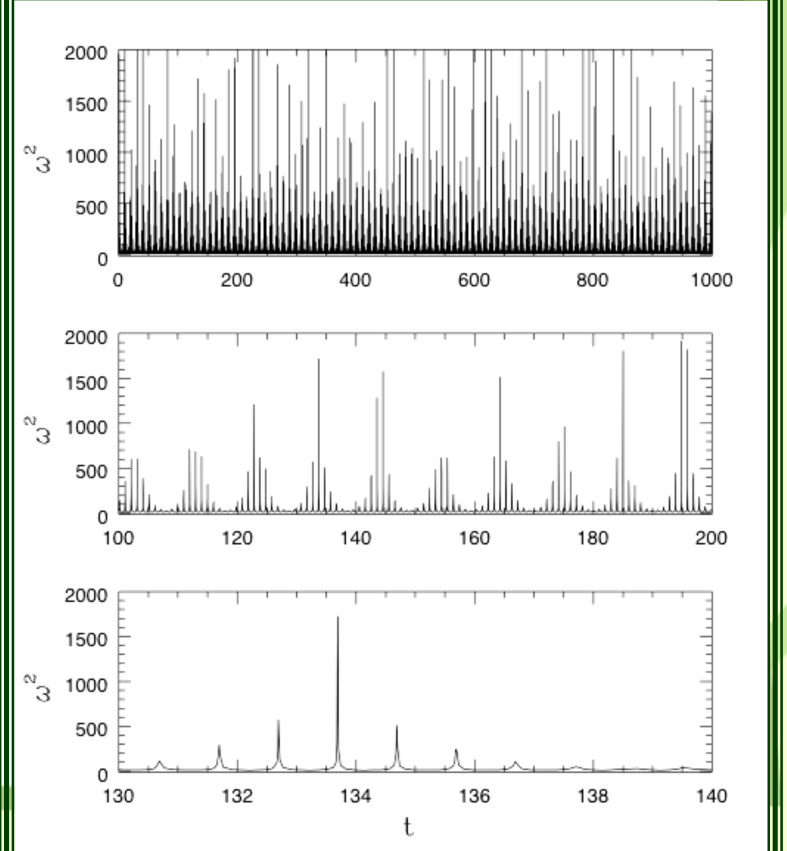
- Equations of motion perpendicular to plane have the form of Hill's equation
- Displacements perpendicular to the plane are unstable

Hill's equation

$$\frac{d^2y}{dt^2}+\frac{4/b}{\sqrt{c^2x^2+a^2z^2}+b\sqrt{y^2+z^2}}\,y=0$$

$$\frac{d^2y}{dt^2}+\left[\lambda_k+q_kQ(\mu_k t)\right]y=0$$

$$\frac{d^2y}{dt^2}+\omega^2(t)\,y=0$$



Floquet's Theorem

For standard Hill's equations (including Mathieu equation) the condition for instability is given by Floquet's Theorem (e.g., Arfken & Weber 2005; Abramowitz & Stegun 1970):

$$|\Delta| \geq 2 \quad required \quad for \quad instability$$

$$where \quad \Delta \equiv y_1(\pi) + dy_2/dt(\pi)$$

Need analogous condition(s) for the case of stochastic Hill's equation…

CONSTRUCTION OF DISCRETE MAP

To match solutions from cycle to cycle, the coefficients are mapped via the 2x2 matrix:

$$\begin{bmatrix} \alpha_b \\ \beta_b \end{bmatrix} = \begin{bmatrix} h & (h^2 - 1)/g \\ g & h \end{bmatrix} \begin{bmatrix} \alpha_a \\ \beta_a \end{bmatrix}$$

$$\text{where } h = y_1(\pi), \; g = dy_1/dt(\pi)$$

$$\text{and where } y_k(t) = \alpha_k y_{1k}(t) + \beta_k y_{2k}(t)$$

The dynamics reduced to matrix products:

$$M^{(N)} = \prod_{k=1}^{N} M_k(q_k, \lambda_k)$$

GROWTH RATES

The growth rates for the matrix products can be broken down into two separate components, the asymptotic growth rate and the anomalous rate:

$$\gamma_\infty = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \gamma(q_k, \lambda_k) \to \langle \gamma \rangle$$

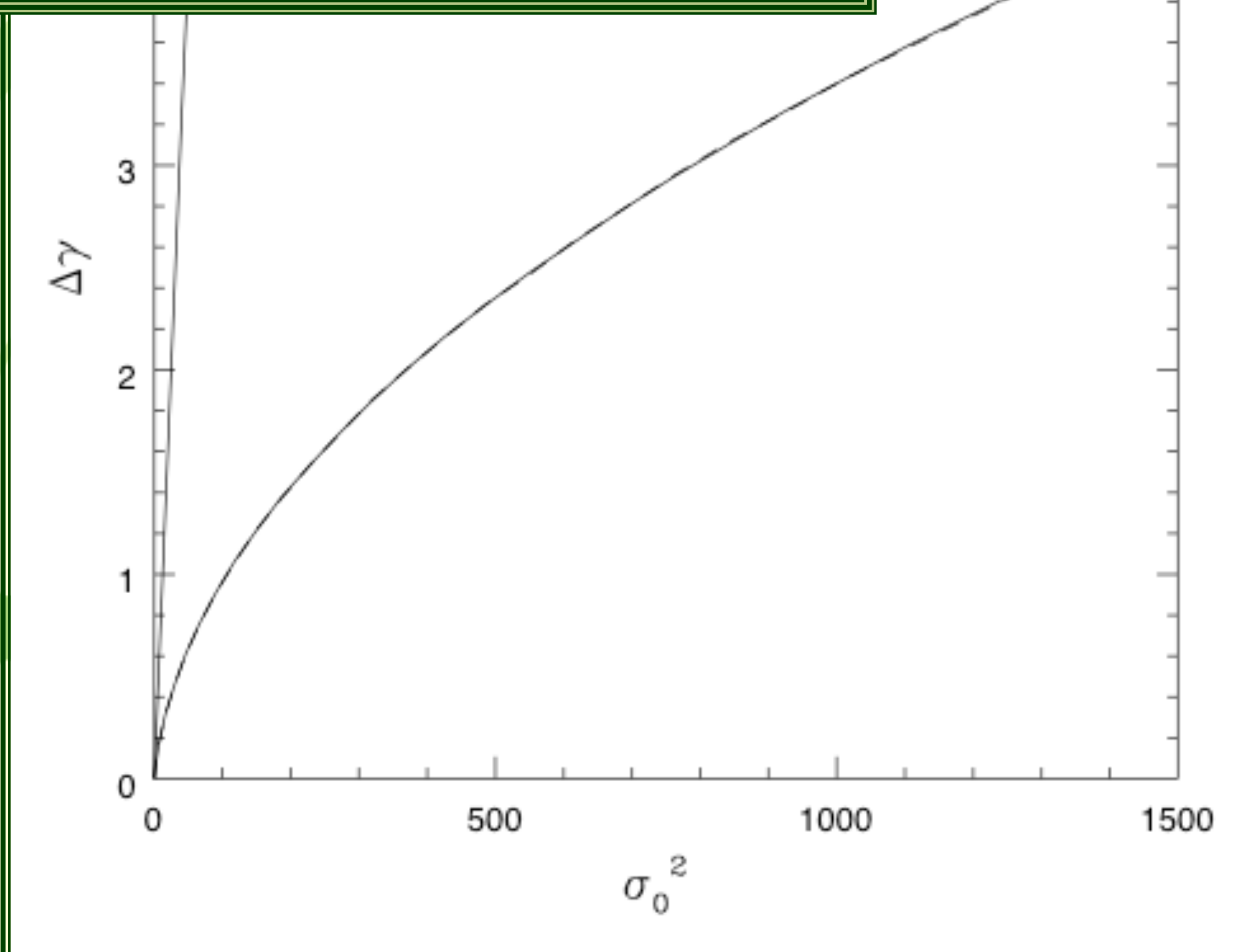
[where individual growth rates given by Floquet's Theorem]
Next: take the limit of large q, i.e., unstable limit: $h >> 1$

$$\Delta\gamma = \lim_{N \to \infty} \frac{1}{\pi N} \sum_{k=1}^{N} \ln(1 + x_{k1} / x_{k2}) - \frac{\ln 2}{\pi}$$

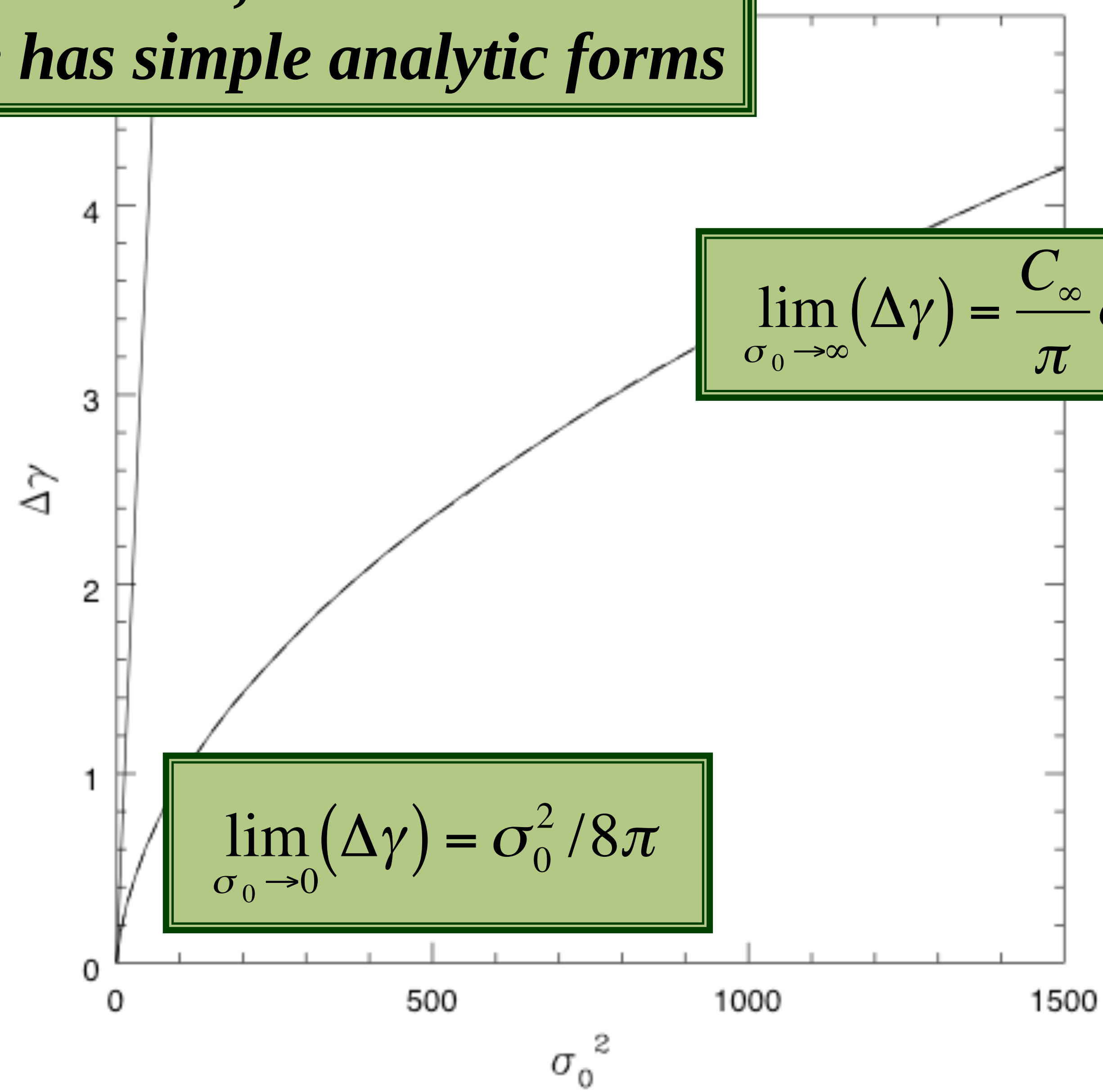
$$where \ \ x_k \equiv h_k / g_k$$

Anomalous Growth Rate as function of the variance of the composite variable

$$\xi \equiv \log[x_{k1} / x_{k2}]$$



For asymptotic limits, the Anomalous Growth Rate has simple analytic forms



$$\lim_{\sigma_0 \to \infty} (\Delta\gamma) = \frac{C_\infty}{\pi} \sigma_0$$

$$\lim_{\sigma_0 \to 0} (\Delta\gamma) = \sigma_0^2 / 8\pi$$

Basic Theorems

- **Theorem 1: Generalized Hill's equation that is non-periodic can be transformed to the periodic case with rescaling of the parameters:** $t \rightarrow \mu_k t,\ \lambda_k \rightarrow \lambda_k / \mu_k^2,\ q_k \rightarrow q_k / \mu_k^2$

- **Theorem 2: Gives anomalous growth rate for unstable limit:**

$$\Delta\gamma = \lim_{N\rightarrow\infty}(1/\pi N)\sum_{j=1}^{N}\ln\left[1 + x_{j1}/x_{j2}\right] - \ln 2/\pi$$

- **Theorem 3: Anomalous growth rate bounded by:** $\Delta\gamma \leq \dfrac{\sigma_0^2}{4\pi}$

- **Theorem 4: Gives anomalous growth rate for unstable limit for forcing function having both positive and negative signs:**

$$\Delta\gamma + \frac{\ln 2}{\pi} = \lim_{N\rightarrow\infty}\frac{1}{\pi N}\left\{ f_+ \sum_{j=1}^{N} \ln(1 + |x_{j1}/x_{j2}|) + f_- \sum_{j=1}^{N} \ln\left|1 - |x_{j1}/x_{j2}|\right| \right\}$$

(Adams & Bloch 2007)

Astrophysical Applications

- ***Dark Matter Halos***: Radial orbits are unstable to perpendicular perturbations and will develop more isotropic velocity distributions.
- ***Tidal Streams***: Instability will act to disperse streams; alternately, long-lived tidal streams place limits on the triaxiality of the galactic mass distribution.
- ***Galactic Bulges***: Instability will affect orbits in the central regions and affect stellar interactions with the central black hole.
- ***Young Stellar Clusters***: Systems are born irregular and become rounder: Instability dominates over stellar scattering as mechanism to reshape cluster.
- ***Galactic Warps***: Orbits of stars and gas can become distorted out of the galactic plane via the instability.

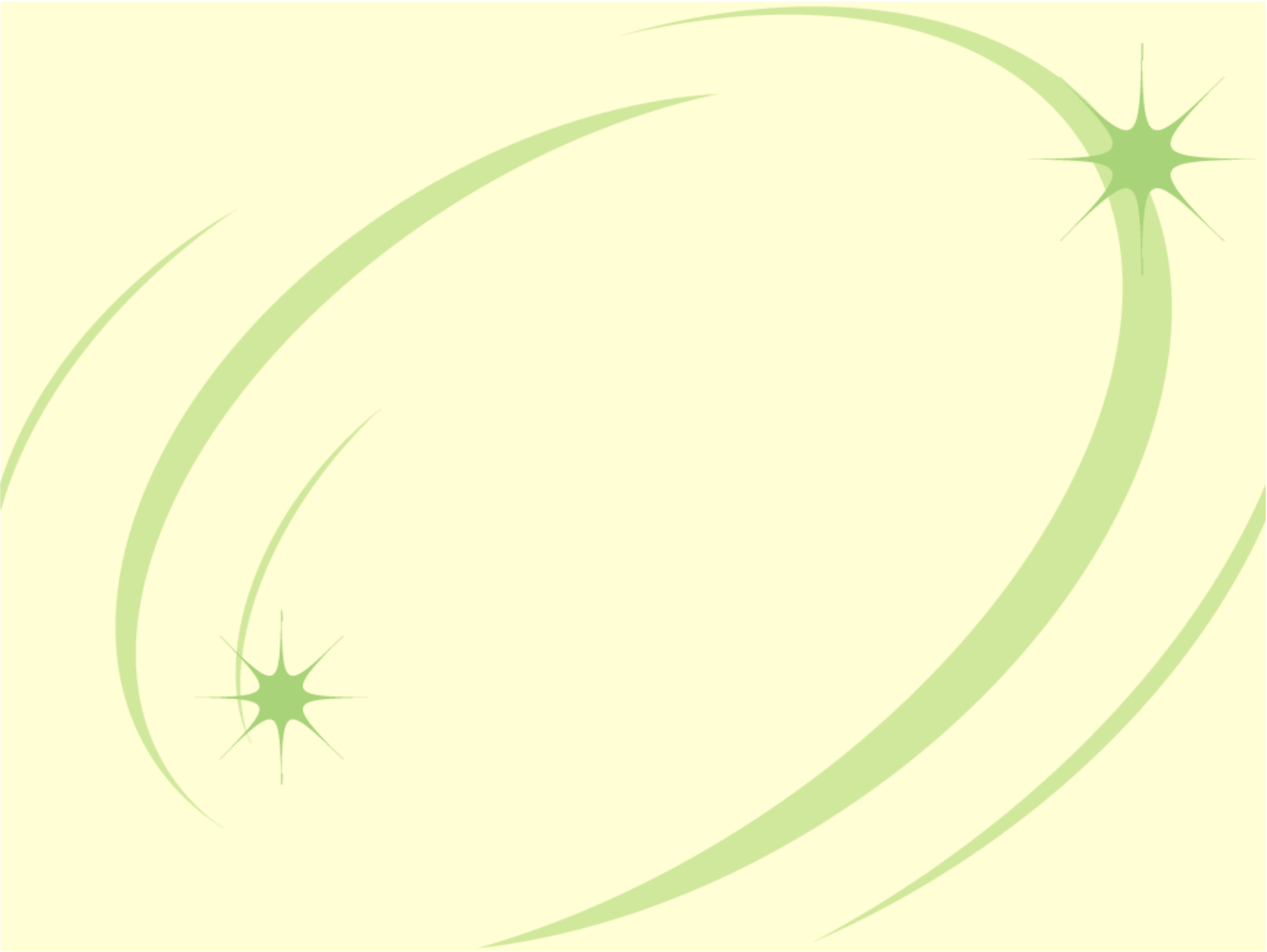
CONCLUSIONS

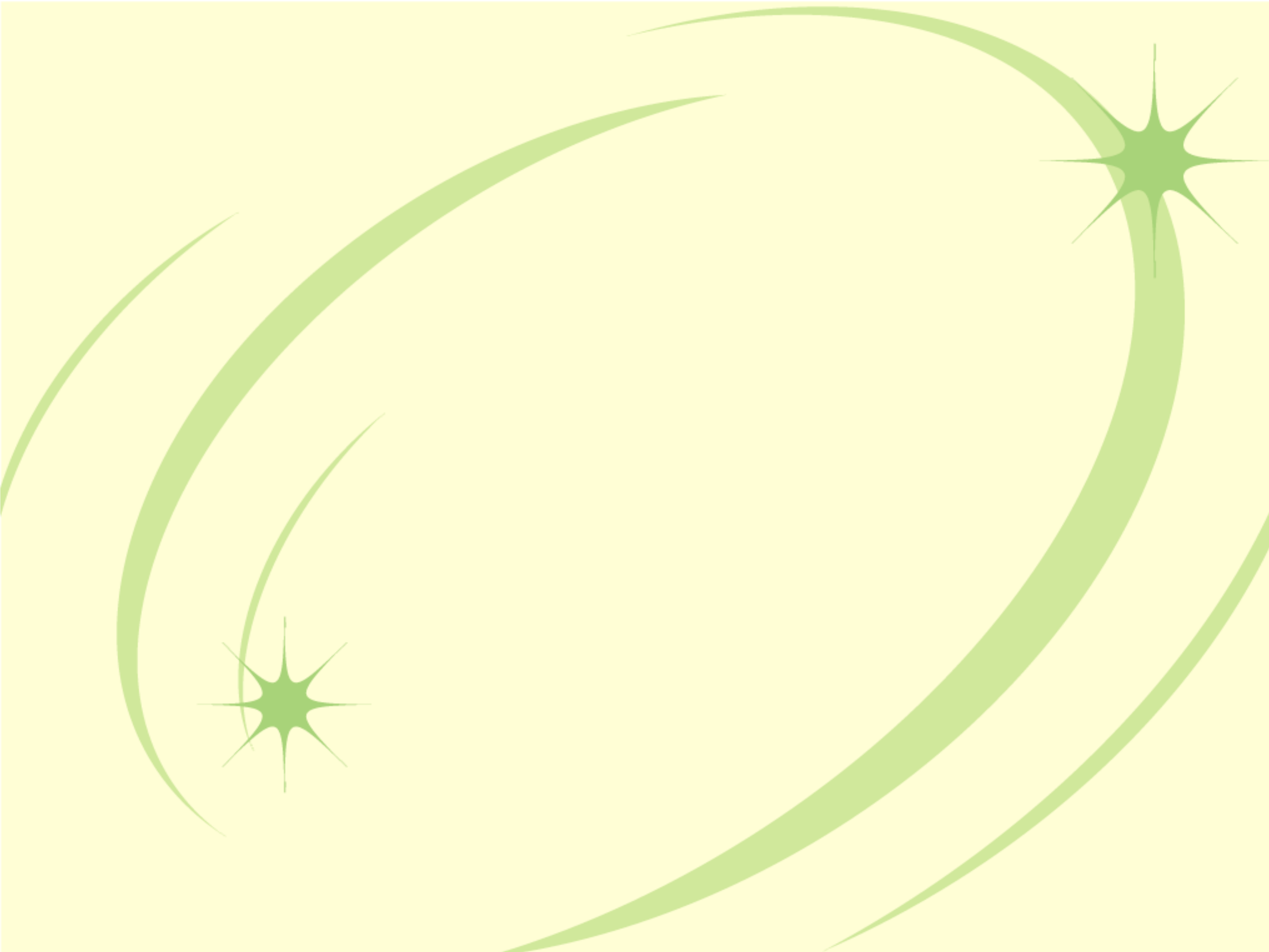
- ***Density distribution = truncated Hernquist profile for both dark matter halos and young star clusters; Analytic results for orbits in spherical limit***
- ***Analytic forms for the gravitational potential and forces in the inner limit -- Triaxial generalization***
- ***Orbits around the principal axes are Unstable***
- ***Instability mechanism described mathematically by a STOCHATIC HILL'S EQUATION***
- ***Growth rates of Stochastic Hill's Equation have Asymptotic and Anomalous parts (found analytically);***

Proof of relevant Theorems that define behavior

BIBLIOGRAPHY

- ***Asymptotic Form of Cosmic Structure, 2007 M.Busha, A.Evrard, & F.Adams, ApJ, 665, 1***
- ***Hill's Equation w. Random Forcing Terms, F. Adams & A. Bloch, 2007, submitted to SIAM J. Ap. Math.***
- ***Orbital Instability in Triaxial Cusp Potential, F. Adams et al. 2007, ApJ, in press***
- ***Orbits in Extended Mass Distributions, F. Adams & A. Bloch, 2005, ApJ, 629, 204***

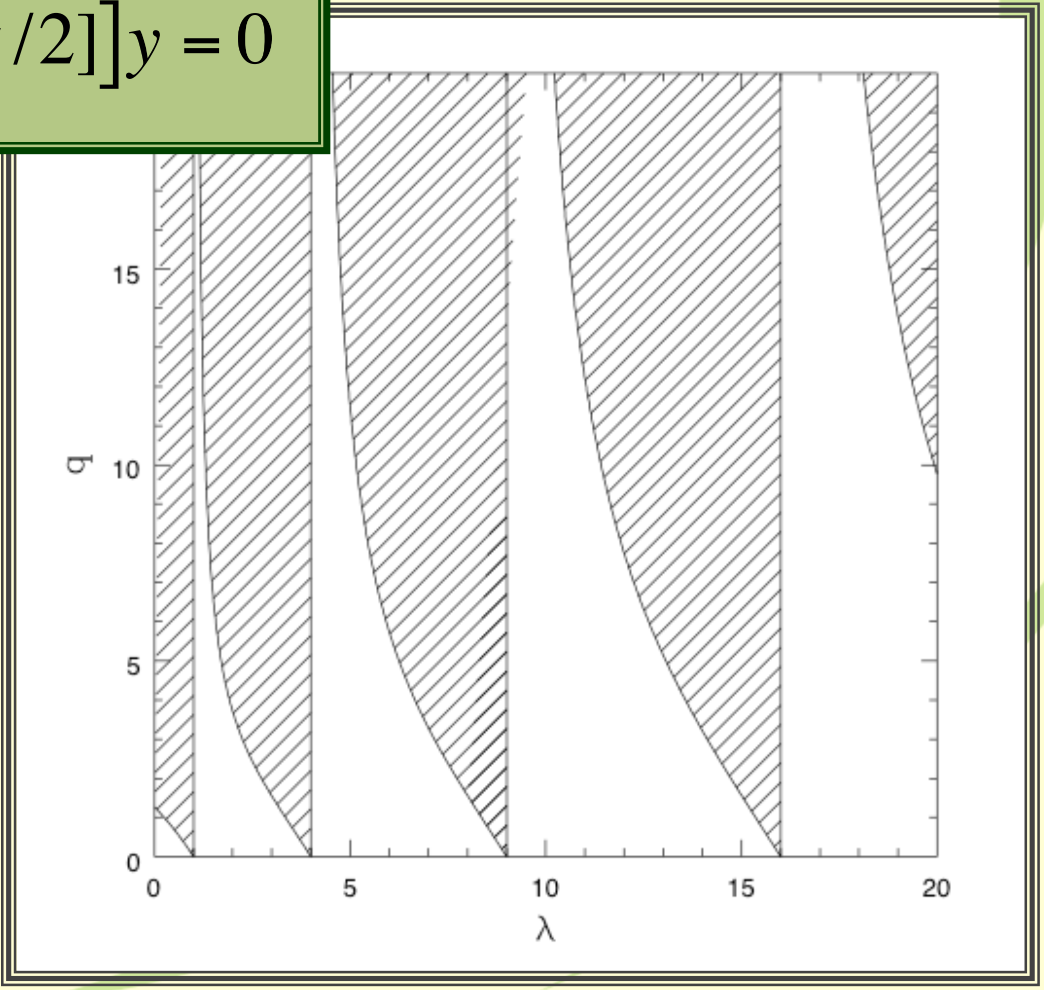




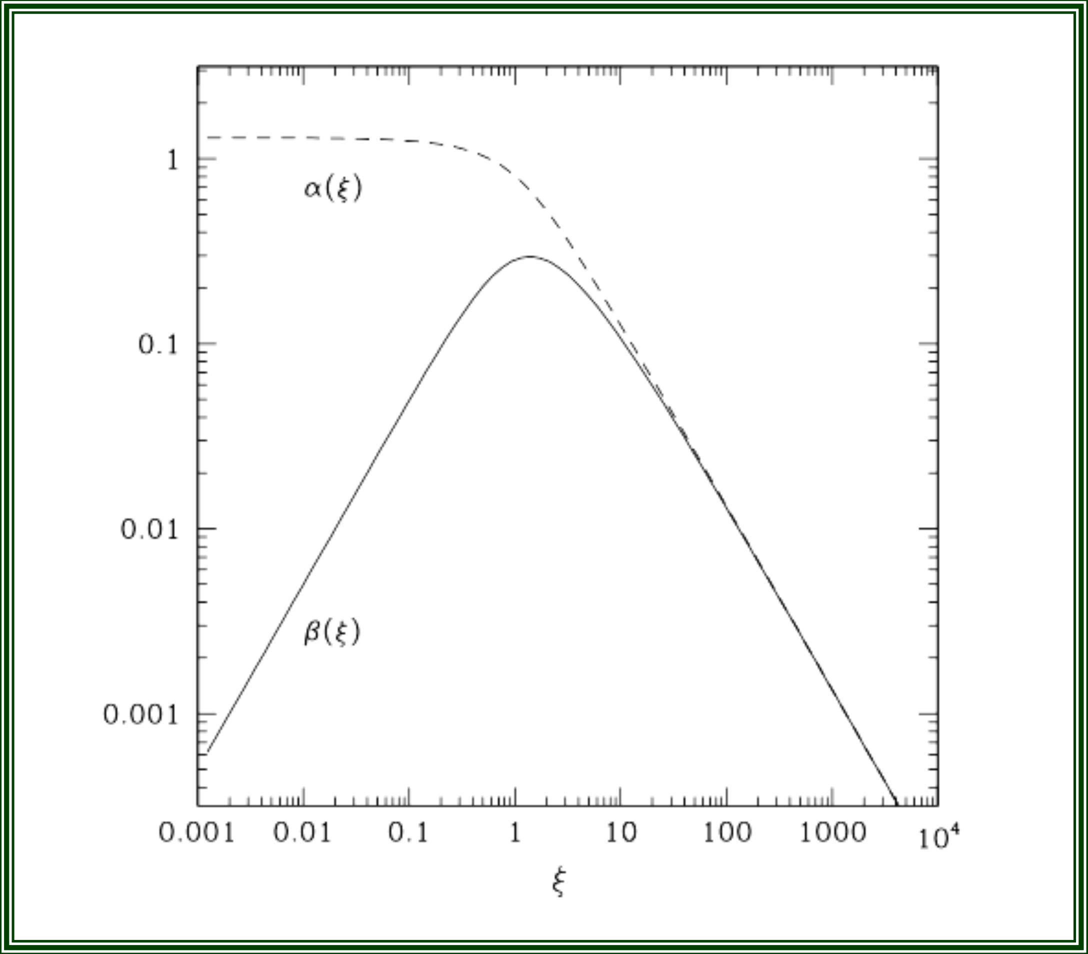
Instability Strips for Hill's Equation in Delta Function Limit

$$\frac{d^2 y}{dt^2} + \left[\lambda + q\delta[t - \pi/2]\right]y = 0$$

***q* given by distance of closest approach, λ by the crossing time**

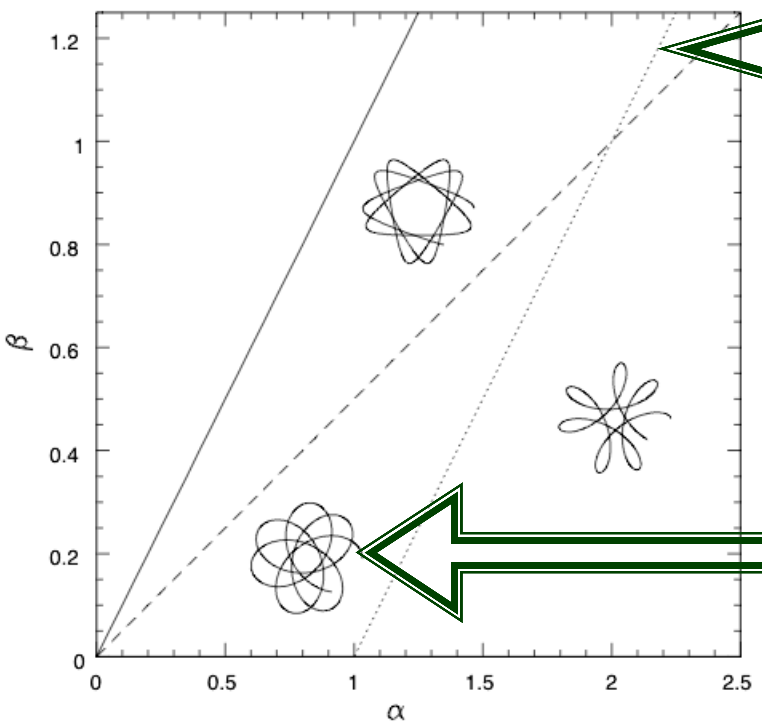


Spacetime Metric Attains Universal Fo



$$ds^2 = -[1 - A(r) - \chi^2 r^2]dt^2 + \frac{dr^2}{[1 - B(r) - \chi^2 r^2]} + r^2 d\Omega^2$$

Physical Portion of the Possible Spirographic Parameter Space



Dual Region

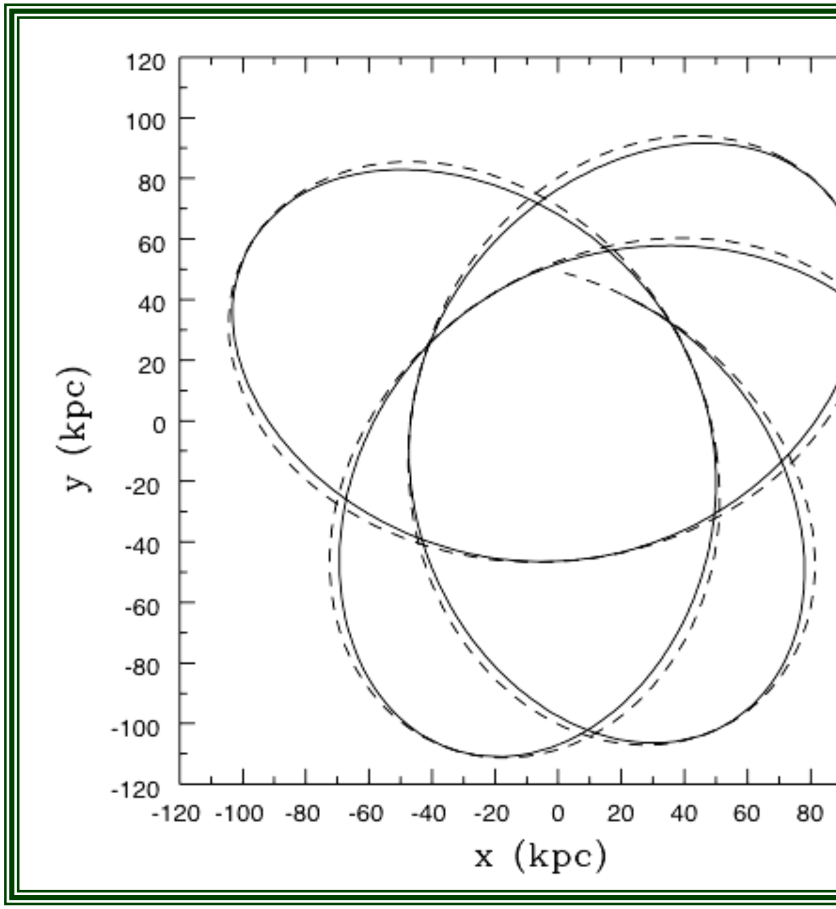
$$\gamma' = \alpha - \beta$$

$$\alpha' - \beta' = \gamma$$

$$\alpha' / \beta' = (1 - \beta / \alpha)^{-1}$$

Physical Region

Application to LMC Orbit



Spirographic approximation reproduces the orbital shape with 7 percent accuracy & conserves angular momentum with 1 percent accuracy. Compare with observational uncertainties of 10-20 percent.