E_{10}

AND A "SMALL TENSION EXPANSION" OF

M-THEORY

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work with M. HENNEAUX AND H. NICOLAI + ...

DH. PRL 85, 920. (2000)
DH PRL 86, 4749 (2001)
DH Julia N PLB 509, 323 (2001)
DH Rendall Weaver AHP 3, 1049 (2002)
D de Buyl H Schomblond JHEP 0208, 030 (2002)
DHN, PRL 89 221601 (2002)
DHN, CQG 20, R145 (2003)

CG1

FROM E_7 TO E_{10}

1978 CREMMER-JULIA-SCHERK $\mathcal{L}_{11}^{SUGRA} \sim R + (dA)^2 + A \wedge dA \wedge dA + \dots$

1978-79 CREMMER-JULIA: HIDDEN E_7 SYMMETRY OF $D=4$ REDUCED EOM

+ CONJECTURED E_6 IN $D=5$ AND E_8 IN $D=3$

1981 JULIA IN $D=2$: AFFINE EXTENSION $E_8^\wedge \equiv E_9$

1982 JULIA

"FINALLY, WE CAN GO TO 1 (TIME) DIMENSION: WE ARE NOW CONSIDERING THE SO-CALLED HOMOGENEOUS SPACE-TIMES. COULD IT BE THAT E_{10} BE A SYMMETRY OF HOMOGENEOUS $N'=7$ SUPERGRAVITY?"

$E_{10} = E_8^{\wedge\wedge}$

"IT IS SOMETIMES CALLED HYPERBOLIC AND DOES NOT HAVE A SIMPLE INTERPRETATION LIKE SOME EXTENSION OF A LOOP GROUP. WE ARE IN A SITUATION WHERE PHYSICS COULD PROVIDE CONCRETE AND SIMPLE REALIZATIONS OF HIGHLY ABSTRACT MATHEMATICAL OBJECTS"

RECENTLY (TD, M. HENNEAUX, H. NICOLAI '02) SOME EVIDENCE FOR $E_{10}(\mathbb{R})$ IN A "SMALL-TENSION EXPANSION" OF $SUGRA_{11}$

NB1: HERE CONTINUOUS SYMMETRY $E_{10}(\mathbb{R})$: SOLUTION → SOLUTION'

NOT SOME DISCRETE VERSION $E_{10}(\mathbb{Z})$: STATE → PHYSICALLY EQUIVALENT STATE

NB2: OTHER STRUCTURES OF E_{10} ALREADY SHOWED UP IN STRING/M THY

- SPECTRUM OF BPS STATES: ROOT LATTICE (Harvey, Moore 96)
- U-DUALITY OF M THY: WEYL GROUP (Obers, Pioline, .. 98; Banks, Fischler, Motl 99)
- GENERIC COSMOLOGICAL BEHAVIOUR: LOWEST APPROXIMATION OF COSMOLOGICAL BILLIARD = WEYL CHAMBER (TD, Henneaux '00)

(1) DESCRIBE LEADING APPROXIMATION TO DYNAMICS OF COSMOLOGIC SOLU

2

KAC-MOODY ALGEBRAS

Lie algebra $\mathcal{G} \equiv \mathcal{G}(A)$

$r \times r$ integer-valued Cartan matrix $\begin{cases} a_{ii} = +2 \\ -a_{ij} \in \mathbb{N}\,;\ i \neq j \\ a_{ij} = 0 \Rightarrow a_{ji} = 0 \end{cases}$

rank

eg

$$A_{E_{10}} = \begin{pmatrix} 2 & -1 & & & & & & & & \\ -1 & 2 & -1 & & & & & & & \\ & -1 & 2 & -1 & & & & & & \\ & & -1 & 2 & -1 & & & & & \\ & & & -1 & 2 & -1 & & & & \\ & & & & -1 & 2 & -1 & & & \\ & & & & & -1 & 2 & -1 & & -1 \\ & & & & & & -1 & 2 & -1 & \\ & & & & & & & -1 & 2 & \\ & & & & & & -1 & & & 2 \end{pmatrix}$$

DYNKIN DIAGRAM :

Chevalley-Serre generators : $\{h_i, e_i, f_i\}_{i=1\dots r}$ $\sim (J_z, J_+, J_-)$ for $SU(2) \sim A_1$

$[h_i, h_j] = 0$ ← Cartan subalgebra

$[e_i, f_j] = \delta_{ij}\, h_j$

$[h_i, e_j] = A_{ij}\, e_j$

$[h_i, f_j] = -A_{ij}\, f_j$

+ Jacobi $[a,[b,c]] + \dots = 0$

+ Serre relations

$\mathrm{ad}_{e_i}^{1-A_{ij}}\, e_j = 0 = \mathrm{ad}_{f_i}^{1-A_{ij}}\, f_j$

3

TRIANGULAR DECOMPOSITION

$$\mathfrak{g}(A) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

$\mathfrak{n}^-$ generated by $[f_{i_1}[f_{i_2}[f_{i_3}[\dots]$ modulo Serre

$\mathfrak{h}$: Cartan SA $\{h_1 \dots h_r\}$

$\mathfrak{n}^+$ generated by $[e_{i_1},[e_{i_2}[e_{i_3}[\dots]$ modulo Serre

roughly speaking: $\mathfrak{n}^- = \begin{pmatrix} 0 & & & & 0 \\ * & 0 & & & \\ * & * & 0 & & \\ * & * & * & 0 & \\ * & * & * & * & 0 \end{pmatrix}$, $\mathfrak{h} = \begin{pmatrix} * & & 0 \\ & * & \\ 0 & & * \end{pmatrix}$, $\mathfrak{n}^+ = \begin{pmatrix} 0 & * & * \\ & 0 & * \\ 0 & & 0 \end{pmatrix}$

Classification for symmetrizable Cartan matrices A:

$A \to A^s$ symmetric : A^s positive definite ⇒ Cartan's finite dimensional Lie alg.

A^s $\begin{cases}\text{one eigenvalue} = 0 \\ \text{other ones} > 0\end{cases}$ ⇒ Affine KM alg.

A^s $\begin{cases}\text{one eigenvalue} < 0 \\ \text{other ones} > 0\end{cases}$ ⇒ Lorentzian KM alg.

deleting any node from Dynkin diagram ⇒ $\begin{cases}\text{finite}, > 0 \\ \text{or} \\ \text{affine}\end{cases}$ ⇒ Hyperbolic KM alg.

4

Maximal Compact Subalgebra k

invariant subalgebra of $g(A)$ under Chevalley involution

$$\theta(h_i) = -h_i, \quad \theta(e_i) = -f_i, \quad \theta(f_i) = -e_i$$

$x \in k \Leftrightarrow \theta(x) = x$, i.e. k generated by multiple commutators of $e_i - f_i$

Define transposition: $E^T \equiv -\theta(E)$

i.e. $h_i^T = h_i, \quad e_i^T = f_i, \quad f_i^T = e_i$

$x \in \ \Leftrightarrow x^T = -x$, i.e. x "antisymmetric"

in other words the group $K = e^{k}$ = "orthogonal" elements $O^T = O^{-1}$

Coset space: G/K

G: KM group $e^{g(A)}$; K: maximal compact subgroup $O^T = O^{-1}$

Invariant bilinear form on $g(A)$: $\langle E_1 | E_2 \rangle$

⇒ induce a bilinear form on CSA: $\langle h_1 | h_2 \rangle$, and therefore a bilinear form on the ROOTS ≡ linear forms on h

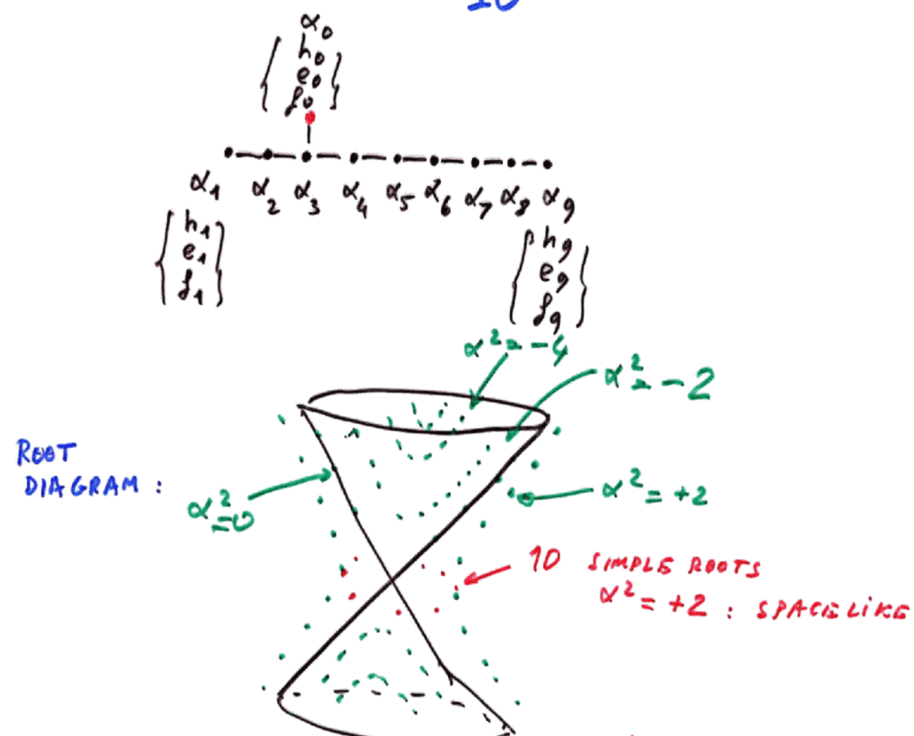
$\alpha \Rightarrow \alpha(h) \in \mathbb{R}$

then $A_{ij} = 2\dfrac{\langle \alpha_i | \alpha_j \rangle}{\langle \alpha_i | \alpha_i \rangle}$ $\quad [h, e_i] = \alpha_i(h)\, e_i$

SIMPLE ROOTS: $\alpha_i(h_j) = A_{ji}$

5

E_{10}



to each root $\alpha \rightarrow E_\alpha^{(s)}$ (growing degeneracy) : $[h, E_\alpha^{(s)}] = \alpha(h)\, E_\alpha^{(s)}$

Decomposition in simple roots: $\alpha = \sum_{i=0}^{9} m^i \alpha_i$ (m^i integers)

positive roots: $m^i \geq 0 \Leftrightarrow [e_{i_1}[e_{i_2}[e_{i_3}[\ldots]$

height of α: $ht(\alpha) = \sum_{i=0}^{9} m^i$

E4

CONSTRUCTION OF A ONE-DIM. E_{10}-INVARIANT COSET MODEL

INFINITE DIMENSIONAL COSET SPACE $E_{10}/K(E_{10})$

MAXIMAL COMPACT SUBGROUP OF THE CANONICAL REAL FORM OF E_{10}

GENERIC E_{10} GROUP ELEMENT

$$\mathcal{V} = e^{X} \quad X \in Lie(E_{10}) : X = X^i h_i + \sum_{\alpha\in\Delta^+} X^{\alpha,s} E_\alpha^{(s)} + \sum_{\alpha\in\Delta^+} Y^{\alpha,s} F_\alpha^{(s)}$$

$F_\alpha^{(s)} \equiv E_{-\alpha}^{(s)}$

IWASAWA DECOMP. OF E_{10}: $\mathcal{V} = \underset{\text{COMPACT}}{K}\ \underset{\text{ABELIAN}}{A}\ \underset{\text{"NULL"}}{N}$

CHEVALLEY INVOLUTION ω: $\omega(h_i) = -h_i$; $\omega(e_i) = -f_i$; $\omega(f_i) = -e_i$

DEFINE "TRANSPOSE": $E_{\alpha,s}^T \equiv -\omega(E_{\alpha,s}) \propto F_{\alpha,s}$

Lie $K(E_{10})$: SPANNED BY "ANTISYMMETRIC ELEMENTS": $E_{\alpha,s} - E_{\alpha,s}^T$

$E_{10}/K(E_{10})$ DEFINED BY:
- GAUGE-FIXED $\mathcal{V} = AN = e^{\underset{\text{CARTAN}}{X^i h_i}} + \sum_{\alpha\in\Delta^+} \underset{\text{ONLY RAISING}}{X^{\alpha,s} E_{\alpha,s}}$
- PROJECTED DERIVATIVE: $v_{sym} \equiv \frac{1}{2}(v + v^T)$

E_{10}-INVARIANT ONE-DIMENSIONAL COSET MODEL:

$$\boxed{S_1^{(E_{10})} = \int \frac{dt}{n(t)} \left(v_{sym} \mid v_{sym} \right) = \int \frac{dt}{n(t)} \frac{1}{4} \left(\frac{d\mathcal{V}}{dt}\mathcal{V}^{-1} + \left(\frac{d\mathcal{V}}{dt}\mathcal{V}^{-1}\right)^T \middle| \frac{d\mathcal{V}}{dt}\mathcal{V}^{-1} + \left(\frac{d\mathcal{V}}{dt}\mathcal{V}^{-1}\right)^T \right)}$$

IMPOSES THE CONSTRAINT

VELOCITY $v = \frac{d\mathcal{V}}{dt}\mathcal{V}^{-1}$

PROJECTION $\perp K$

INVARIANCE OF COSET-MODEL ACTION

$$\mathcal{V}(t) \longrightarrow k_g(t)\, \mathcal{V}(t)\, g \equiv \mathcal{V}'(t)$$

"group element of E_{10}" $\approx$ "vielbein"

compensating element of K: $k_g^T = k_g^{-1}$

arbitrary (rigid) element of $G = E_{10}$

$$v' \equiv \frac{d\mathcal{V}'}{dt}\mathcal{V}'^{-1} = \left(k \dot{\mathcal{V}} \cancel{g} + \dot{k}\mathcal{V}\cancel{g}\right) \cancel{g^{-1}}\, \mathcal{V}^{-1} k^{-1}$$

$$= k v k^{-1} + \underbrace{\dot{k} k^{-1}}$$

"antisymmetric" element: $\dot{O} O^{-1} = $ "Ω_{ij}"

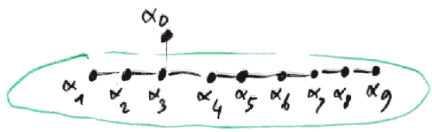
$$v'_{sym} \equiv \tfrac{1}{2}(v' + v'^T) = k\, v_{sym}\, k^{-1}$$

$$\Rightarrow \quad \langle v'_{sym} \mid v'_{sym} \rangle = \langle v_{sym} \mid v_{sym} \rangle$$

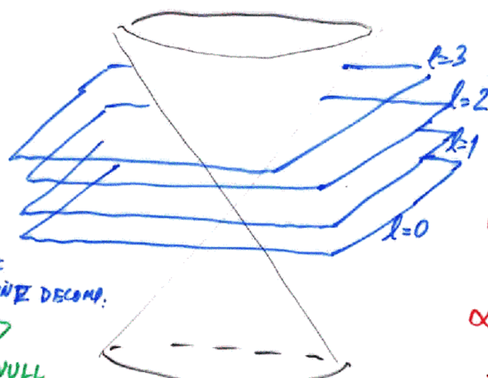
i.e. non linear realisation of $G = E_{10}$ acting on $\mathcal{V}$

E3

NEW DECOMPOSITION OF E_{10} : WRT $GL(10)$ SUBALGEBRA



$SL(10) = A_9$ SUBALGEBRA



"SPACE LIKE SLICING" OF ROOT SPACE

ROOTS

$$\alpha = \underset{\text{"LEVEL"}}{\ell}\,\alpha_0 + \sum_{j=1}^{9} m^j \alpha_j$$

$$ht(\alpha) = \ell + \sum_{j=1}^{9} m^j$$

INSTEAD OF USUAL AFFINE DECOMP.

"NULL SLICING"

$$\alpha_0 = \alpha_0^{\perp} + \alpha_0^{\parallel}$$

$\alpha_0^{\perp}$: TIME-LIKE $(\alpha_0^{\perp})^2 < 0$

$\alpha_0^{\parallel}$: IN $SL(10)$ HYPERPLANE $\{\alpha_i\}$

$\ell = 0$: $GL(10)$ SUBALGEBRA : $[K^a{}_b, K^c{}_d] = K^a{}_d \delta^c_b - K^c{}_b \delta^a_d$

$\ell = 1$: GENERATORS $E^{[a_1 a_2 a_3]}, F_{[a_1 a_2 a_3]}$: $[K, E] = E$; $[E, F] = K$

$\ell = 2$: $E^{[a_1 \cdots a_6]}, F_{[a_1 \cdots a_6]}$:

$\ell = 3$: $E^{a_0 | a_1 \cdots a_8}, F_{a_0 | a_1 \cdots a_8}$:

$\ell = 4$

$\oplus$

DETERMINATION OF COMMUTATORS : $[x, y] = z$

AND INVARIANT BILINEAR FORM : $(x | y)$

LOW LEVEL REPRESENTATIONS FOR E_{10} AND E_{11} 13

Table 1: A_9 representations in E_{10} up to level $\ell = 18$

checked

ℓ	p	m									Λ^2	dim $\mathcal{R}(\Lambda)$	mult(Λ)	μ
1	(001000000)	0	0	0	0	0	0	0	0	0	2	120	1	1
2	(000001000)	1	2	3	2	1	0	0	0	0	2	210	1	1
3	(100000010)	1	3	5	4	3	2	1	0	0	2	440	1	1
	~~(000000001)~~	~~2~~	~~4~~	~~6~~	~~5~~	~~4~~	~~3~~	~~2~~	~~1~~	~~0~~	~~0~~	~~10~~	~~8~~	~~0~~
4	(001000001)	2	4	6	5	4	3	2	1	0	2	1155	1	1
	(200000000)	1	4	7	6	5	4	3	2	1	2	55	1	1
	~~(010000000)~~	~~2~~	~~4~~	~~7~~	~~6~~	~~5~~	~~4~~	~~3~~	~~2~~	~~1~~	~~0~~	~~45~~	~~8~~	~~0~~
5	(000001001)	3	6	9	7	5	3	2	1	0	2	1848	1	1
	(100100000)	2	5	8	6	5	4	3	2	1	2	1848	1	1
	~~(000010000)~~	~~3~~	~~6~~	~~9~~	~~7~~	~~5~~	~~4~~	~~3~~	~~2~~	~~1~~	~~0~~	~~252~~	~~8~~	~~0~~
6	(100000011)	3	7	11	9	7	5	3	1	0	2	3200	1	1
	~~(000000002)~~	~~4~~	~~8~~	~~12~~	~~10~~	~~8~~	~~6~~	~~4~~	~~2~~	~~0~~	~~0~~	~~55~~	~~8~~	~~0~~
	(010001000)	3	6	10	8	6	4	3	2	1	2	8250	1	1
	(100000100)	3	7	11	9	7	5	3	2	1	0	1155	8	1
	(000000010)	4	8	12	10	8	6	4	2	1	-2	45	44	1
7	(001000002)	4	8	12	10	8	6	4	2	0	2	6160	1	1
	(000100100)	4	8	12	9	7	5	3	2	1	2	19800	1	1
	(110000010)	3	7	12	10	8	6	4	2	1	2	13860	1	1
	(001000010)	4	8	12	10	8	6	4	2	1	0	4950	8	1
	(200000001)	3	8	13	11	9	7	5	3	1	0	540	8	1
	(010000001)	4	8	13	11	9	7	5	3	1	-2	440	44	2
	(100000000)	4	9	14	12	10	8	6	4	2	-4	10	192	1
8	(000001002)	5	10	15	12	9	6	4	2	0	2	9240	1	1
	(000000200)	5	10	15	12	9	6	3	2	1	2	4950	1	1
	(100010010)	4	9	14	11	8	6	4	2	1	2	83160	1	1
	(000001010)	5	10	15	12	9	6	4	2	1	0	6930	8	1
	(011000001)	4	8	13	11	9	7	5	3	1	2	31185	1	1
	(100100001)	4	9	14	11	9	7	5	3	1	0	17280	8	2
	(000010001)	5	10	15	12	9	7	5	3	1	-2	2310	44	2
	(210000000)	3	8	14	12	10	8	6	4	2	2	1485	1	1
	(020000000)	4	8	14	12	10	8	6	4	2	0	825	8	0
	(101000000)	4	9	14	12	10	8	6	4	2	-2	990	44	2
	(000100000)	5	10	15	12	10	8	6	4	2	-4	210	192	2
9	(100000012)	5	11	17	14	11	8	5	2	0	2	14300	1	1
	(000000003)	6	12	18	15	12	9	6	3	0	0	220	8	0
	(010000110)	5	10	16	13	10	7	4	2	1	2	130680	1	1
	(100000020)	5	11	17	14	11	8	5	2	1	0	7920	8	1
	(001010001)	5	10	15	12	9	7	5	3	1	2	184800	1	1
	(200001001)	4	10	16	13	10	7	5	3	1	2	90090	1	1
	(010001001)	5	10	16	13	10	7	5	3	1	0	71280	8	2
	(100000101)	5	11	17	14	11	8	5	3	1	-2	9450	44	4
	(000000011)	6	12	18	15	12	9	6	3	1	-4	330	192	3
	(110100000)	4	9	15	12	10	8	6	4	2	2	46200	1	1
	(001100000)	5	10	15	12	10	8	6	4	2	0	13860	8	1
	(200010000)	4	10	16	13	10	8	6	4	2	0	11880	8	1
	(010010000)	5	10	16	13	10	8	6	4	2	-2	9240	44	3
	(100001000)	5	11	17	14	11	8	6	4	2	-4	1980	192	4
	(000000100)	6	12	18	15	12	9	6	4	2	-6	120	727	4
10	(001000003)	6	12	18	15	12	9	6	3	0	2	24024	1	1

⋮

$\ell = 28$ (110100011) appears 46,450,629 times!

E5

EXPLICIT PARAMETRIZATION OF $E_{10}/K(E_{10})$

$$\mathcal{V}(t) = e^{h^a{}_b(t) K^b{}_a}\, e^{\frac{1}{3!} A_{abc}(t) E^{abc} + \frac{1}{6!} A_{a_1 \cdots a_6}(t) E^{a_1 \cdots a_6} + \frac{1}{9!} A_{a_0 | a_1 \cdots a_8}(t) E^{a_0 | a_1 \cdots a_8} + \cdots}$$

"SL(10) field" $h^a{}_b(t)$ — SL(10) generators — $\ell = 1$ generators $A_{[abc]}(t)$ — $\ell = 2$ $A_{[a_1 \cdots a_6]}(t)$ — $\ell = 3$ $A_{a_0 | a_1 \cdots a_8}(t)$

REMINISCENT OF SIMILAR ALGEBRAIC CONSTRUCTIONS OF Cremmer, Julia, Lu, Pope 98 and West '01

EXPLICIT FORM OF E_{10}-INVARIANT ACTION $S_1 = \int dt \mathcal{L}_1^{E_{10}}$

$$\boxed{n \mathcal{L}_1^{E_{10}} = \frac{1}{4}\left(g^{ac} g^{bd} - g^{ab} g^{cd}\right) \dot{g}_{ab} \dot{g}_{cd} + \frac{1}{2}\frac{1}{3!} DA_{a_1 a_2 a_3} DA^{a_1 a_2 a_3} + \frac{1}{2}\frac{1}{6!} DA_{a_1 \cdots a_6} DA^{a_1 \cdots a_6} + \frac{1}{2}\frac{1}{9!} DA_{a_0 | a_1 \cdots a_8} DA^{a_0 | a_1 \cdots a_8} + \cdots}$$

ALL INDICES RAISED BY g^{ab}

WHERE

$$g^{ab} = e^a{}_c e^b{}_c \quad \text{WITH} \quad e^a{}_b \equiv (\exp h)^a{}_b$$

AND

$$DA_{a_1 \cdots a_3} = \dot{A}_{a_1 a_2 a_3}$$

$$DA_{a_1 \cdots a_6} = \dot{A}_{a_1 \cdots a_6} + 10\, A_{[a_1 \cdots a_3} \dot{A}_{a_4 a_5 a_6]}$$

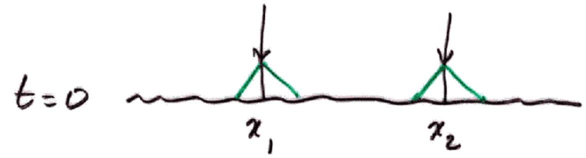
$$DA_{a_1 | a_2 \cdots a_9} = \dot{A}_{a_1 | a_2 \cdots a_9} + 42\, A_{\langle a_1 \cdots a_3} \dot{A}_{a_4 \cdots a_9 \rangle} - 42\, \dot{A}_{\langle a_1 \cdots a_3} A_{a_4 \cdots a_9 \rangle} + 280\, A_{\langle a_1 \cdots a_3} A_{a_4 \cdots a_6} \dot{A}_{a_7 a_8 a_9 \rangle}$$

NB: ALL NUMERICAL COEFFICIENTS ARE UNIQUELY FIXED (MODULO FIELD REDEFINITIONS) BY THE STRUCTURE OF E.

"SMALL TENSION EXPANSION"

NEAR SPACE-LIKE SINGULARITY

$\neq$ SPACE POINTS DECOUPLE



$\ell_{\text{HORIZON}} \sim ct$ becomes smaller than $|x_2 - x_1|$

VARIOUS WAYS OF LOOKING AT IT:

- $\boxed{\frac{\partial}{\partial x} h \ll \frac{1}{c}\frac{\partial}{\partial t} h} \rightarrow \boxed{\text{Belinskii-Khalatnikov-Lifshitz}}$ expansion: $f = f(\underset{\text{dominant}}{t}\,;\, \underset{\text{small}}{x})$
- formally as if $\boxed{c \rightarrow 0}$
- formally $S = \frac{1}{2}\int d\hat{t}\, d^{10}x \left[\rho_b (\partial_t h_{ij})^2 - T_b (\partial_x h_{ij})^2 \right]$, $\rho_b = \frac{c^2}{32\pi G}$, $T_b = \frac{c^4}{32\pi G}$
  as if $\boxed{T_{\text{bulk}} \rightarrow 0}$ "small tension", $c = \sqrt{\frac{T_b}{\rho_b}}$
- $\mathcal{H} \sim G\pi_g^2 + \frac{1}{G}(\partial_x g)^2 \sim \pi_g'^2 + \frac{1}{G^2}(\partial_x g')^2$ (rescaled)
  "strong coupling" expansion: $\boxed{G \rightarrow \infty}$

? POSSIBLE LINK WITH THE $T_s \sim M_s^2 \sim \frac{1}{\alpha'} \rightarrow 0$ LIMIT CONSIDERED BY D. GROSS '88

? INFINITE TOWER OF MASSLESS STATES

LOWEST-ORDER IN "BKL-SMALL-TENSION" EXPANSION VS FIRST-ORDER IN LEVEL EXPANSION OF E_{10}

NEGLECT ALL SPACE DEPENDENCE IN $SUGRA_{11}$

$$G_{\mu\nu}(x)\,dx^\mu dx^\nu$$

Gauss coordinates

$$\text{i.e. } ds^2 = -(\tilde N \sqrt{G})^2 (dx^0)^2 + G_{ij}(t)\, dx^i dx^j$$

rescaled lapse $\tilde N \equiv N/\sqrt{G^{10}}$

$$\mathcal{A} = \frac{1}{3!} \mathcal{A}_{\mu\nu\lambda}(x)\, dx^\mu \wedge dx^\nu \wedge dx^\lambda = \frac{1}{3!} \mathcal{A}_{ijk}(t)\, dx^i \wedge dx^j \wedge dx^k$$

Radiation gauge ($\mathcal{A}_{0ij} = 0$)

$$S_{11}^{SUGRA} = \int d^{11}x \sqrt{G}\left[\frac{R(G)}{4} - \frac{1}{48}\mathcal{F}_{\mu\nu\rho\sigma}\mathcal{F}^{\mu\nu\rho\sigma} + \frac{2}{(12)^4}\varepsilon^{\mu_1\cdots\mu_{11}}\mathcal{F}_{\mu_1\cdots\mu_4}\mathcal{F}_{\mu_5\cdots\mu_8}\mathcal{A}_{\mu_9\cdots\mu_{11}}\right]$$

$$\downarrow 0$$

$$S_{11}^{\text{homogeneous}} = \int d^{10}x \int \frac{dx^0}{\tilde N}\left[\frac{1}{4}(G^{ik}G^{jl} - G^{ij}G^{kl})\dot G_{ij}\dot G_{kl} + \frac{1}{2}\frac{1}{3!}G^{ii'}G^{jj'}G^{kk'}\dot{\mathcal{A}}_{ijk}\dot{\mathcal{A}}_{i'j'k'}\right]$$

this coincides with the truncation of the E_{10} coset action to the levels $\ell = 0$ and $\ell = 1$ i.e

$$\mathcal{V}(t) = e^{h^a{}_b(t) K^b{}_a}\, e^{\frac{1}{3!}A_{abc}(t) E^{abc}} \quad ; \quad v \equiv \dot{\mathcal{V}}\mathcal{V}^{-1}$$

$$v_{sym} = \frac{1}{2}(v + v^T)$$

$$\Rightarrow S_1^{E_{10}/K(E_{10})} \equiv \int \frac{dt}{n(t)} \langle v_{sym} | v_{sym}\rangle$$

$$= \int \frac{dt}{n(t)}\left[\frac{1}{4}(g^{ac}g^{bd} - g^{ab}g^{cd})\dot g_{ab}\dot g_{cd} + \frac{1}{2}\frac{1}{3!}g^{aa'}g^{bb'}g^{cc'}\dot A_{abc}\dot A_{a'b'c'}\right]$$

with $g^{ab} \equiv (\exp h)^a{}_c (\exp h)^b{}_c$

NEXT-TO-NEXT-TO-LEADING ORDER IN BKL EXPANSION VS THIRD-ORDER IN LEVEL EXPANSION OF E_{10}

E_{10} COSET ELEMENT TRUNCATED TO LEVELS $\ell = 0, 1, 2, 3$

$$\mathcal{V}(t) = e^{h^a{}_b(t) K^b{}_a}\, e^{\frac{1}{3!}A_{a_1a_2a_3}(t)E^{a_1a_2a_3} + \frac{1}{6!}A_{a_1\cdots a_6}E^{a_1\cdots a_6} + \frac{1}{9!}A_{a_0|a_1\cdots a_8}E^{a_0|a_1\cdots a_8}}$$

$h^b{}_a(t)$ $\quad A_{a_1a_2a_3}$ $\quad A_{a_1\cdots a_6}$ $\quad A_{a_0|a_1\cdots a_8}$

$$g^{ab}(t) \equiv (e^h)^a{}_c (e^h)^b{}_c$$

$$S_1^{E_{10}} = \int \frac{dt}{n(t)} \langle v_{sym} | v_{sym}\rangle$$

indices raised by g^{ab}

$$= \int \frac{dt}{n(t)}\left[\frac{1}{4}(g^{ac}g^{bd} - g^{ab}g^{cd})\dot g_{ab}\dot g_{cd} + \frac{1}{2}\frac{1}{3!}\dot A_{a_1a_2a_3}\dot A^{a_1a_2a_3}\right.$$

$$\left. + \frac{1}{2}\frac{1}{6!}DA_{a_1\cdots a_6}DA^{a_1\cdots a_6} + \frac{1}{2}\frac{1}{9!}DA_{a_0|a_1\cdots a_8}DA^{a_0|a_1\cdots a_8}\right]$$

$$DA_{a_1\cdots a_6} = \dot A_{a_1\cdots a_6} + 10\, A_{[a_1\cdots a_3}\dot A_{a_4\cdots a_6]}$$

$$DA_{a_0|a_1\cdots a_8} = \dot A_{a_0|a_1\cdots a_8} + 42\, A_{\langle a_1\cdots a_3}\dot A_{a_4\cdots a_8\rangle} - 42\, \dot A_{\langle a_1\cdots a_3}A_{a_4\cdots a_8\rangle} + 280\, A_{\langle a_1\cdots a_3}A_{a_4\cdots a_6}\dot A_{a_7\cdots a_8\rangle}$$

projection on Young tableau

coefficients predicted by structure constants of E_{10}

E6

COMPARISON WITH CJS EQS OF MOTION

$$S_{11}^{(\mathrm{SUGRA})} = \int d^{11}x \sqrt{G}\left[\frac{1}{4}R(G) - \frac{1}{48}\mathcal{F}_{MNPQ}\mathcal{F}^{MNPQ} + \frac{2}{(12)^4}\varepsilon^{PQ\ldots Z}\mathcal{F}_{PQRS}\mathcal{F}_{TUVW}\mathcal{A}_{XYZ}\right]$$

$$\mathcal{F}_4 = d\mathcal{A}_3$$

EQS OF MOTION IN ZERO-SHIFT SLICING

$$\boxed{\begin{aligned} ds^2 &= -N^2(dx^0)^2 + G_{ab}\,\theta^a\theta^b \\ \mathcal{F} &= \frac{1}{3!}\mathcal{F}_{0abc}\,dx^0\wedge\theta^a\wedge\theta^b\wedge\theta^c + \frac{1}{4!}\mathcal{F}_{abcd}\,\theta^a\wedge\theta^b\wedge\theta^c\wedge\theta^d \end{aligned}}$$

WITH TIME-INDEPENDENT SPATIAL ZEHNBEIN $\boxed{\theta^a(x) = E^a{}_i(x)\,dx^i}$

CHOOSE TIME-COORDINATE x^0 SO THAT $N = \sqrt{\det G_{ab}} \equiv \sqrt{G}$

NB: PROPER TIME $T = \int N\,dx^0 \neq x^0$: $x^0 \to \infty$ AS $T \to 0$

$$\boxed{\left\{\begin{aligned} \partial_0(G^{ac}\partial_0 G_{cb}) &= \frac{1}{6}G\mathcal{F}^{a\beta\gamma\delta}\mathcal{F}_{b\beta\gamma\delta} - \frac{1}{72}G\mathcal{F}^{\alpha\beta\gamma\delta}\mathcal{F}_{\alpha\beta\gamma\delta}\delta^a_b - 2GR^a{}_b(\Gamma, C) \\ \partial_0(G\mathcal{F}^{0abc}) &= \frac{1}{144}\varepsilon^{abc\,a_1\ldots a_3\,b_1\ldots b_4}\mathcal{F}_{0a_1\ldots a_3}\mathcal{F}_{b_1\ldots b_4} + \frac{3}{2}G\mathcal{F}^{de[ab}C^{c]}{}_{de} \\ &\quad - G\,C^e{}_{de}\mathcal{F}^{dabc} - \partial_d(G\mathcal{F}^{dabc}) \\ \partial_0\mathcal{F}_{abcd} &= 6\,\mathcal{F}_{0e[ab}C^e{}_{cd]} + 4\,\partial_{[a}\mathcal{F}_{0bcd]} \end{aligned}\right.}$$

keep leading $\Gamma\Gamma \sim CC$ term in $R \sim \partial\Gamma + \Gamma\Gamma + \Gamma C$

WHERE

$$2G_{ad}\Gamma^d{}_{bc} = C_{abc} + C_{bca} - C_{cab} + \partial_b G_{ca} + \partial_c G_{ab} - \partial_a G_{bc}$$

AND $C^a{}_{bc} = G^{ad}C_{dbc}$ ARE THE STRUCTURE COEFFICIENTS OF THE ZEHNBEIN: $\boxed{d\theta^a = \frac{1}{2}C^a{}_{bc}\,\theta^b\wedge\theta^c}$

CORRESPONDENCE $E_{10}/K(E_{10})$ coset $\longleftrightarrow$ SUGRA$_{11}$

height expansion of Hamiltonians

"Iwasawa" decomposition of $E_{10}/K(E_{10})$ element.

"triangular" gauge $\mathcal{V} = e^{h^a_b K^b_a}\, e^{A_3 E^3 + A_6 E^6 + \ldots}$

$$h^a{}_b = \begin{pmatrix} \beta^1 & \times\times \\ & \ddots \\ 0 & & \beta^{10} \end{pmatrix}$$

off-diagonal

diagonal elements

$$\Leftrightarrow \mathcal{V} = e^{\beta^\mu(t) H_\mu}\; e^{\sum_{\alpha\in\Delta_+}\sum_{s=1}^{\mu(\alpha)} \nu_{\alpha,s}(t)\,E_{\alpha,s}}$$

diagonal elements

CartanSA

positive roots

"off-diagonal (upper triangular)" elements

$\mu = 1\ldots 10$ $\quad \beta^1\ldots\beta^{10}$

$$\nu_\alpha = \left\{h^{\text{off-diagonal}}, A_{abc}, A_6, A_9, \ldots\right\}$$

gradation of roots, say by height: $ht(\alpha) = n_1 + n_2 + \ldots + n_{10}$

if $\alpha = n_1\alpha_1^{\text{simple}} + \ldots + n_{10}\alpha_{10}^{\text{simple}}$

metric in CartanSA: $-++\ldots+$

$$\mathcal{L}_{E_{10}} \simeq \frac{1}{2}G_{\mu\nu}\dot\beta^\mu\dot\beta^\nu + e^{2\alpha_1(\beta)}(\dot\nu^1)^2 + e^{2\alpha_2(\beta)}(\dot\nu^2 + \nu^1\dot\nu^1)^2 + e^{2\alpha_3(\beta)}\left[\dot\nu^3 + \nu^1\dot\nu^2 + \dot\nu^1\nu^2 + \nu^1\nu^1\dot\nu^1\right]^2 + \ldots$$

value of root on CSA element β:

$$\boxed{\alpha(\beta) \equiv \alpha_\mu\,\beta^\mu}$$

root α $\qquad$ $\beta^\mu H_\mu \in$ CSA

E_{10} Hamiltonian

define non-canonical momenta

$$\Pi_1 \equiv e^{2\alpha_1(\beta)} \dot{\nu}^1$$

$$\Pi_2 \equiv e^{2\alpha_2(\beta)} \dot{\nu}^2 + \nu^1 \dot{\nu}^1 \quad \text{etc...}$$

Poisson brackets $\{\Pi_\alpha, \Pi_\beta\} = \Omega^{\gamma}_{\alpha\beta}(\nu)\, \Pi_\gamma$

Hamiltonian

$$\mathcal{H}_{E_{10}} = \tfrac{1}{2} G^{\mu\nu} \pi_{\beta\mu} \pi_{\beta\nu} + e^{-2\alpha_1(\beta)} \Pi_1^2 + e^{-2\alpha_2(\beta)} \Pi_2^2 + e^{-2\alpha_3(\beta)} \Pi_3^2 + \dots$$

[with constraint $\mathcal{H}_{E_{10}} = 0$]

as $t^{phys} \to 0$ ($x^0 \to +\infty$) β^μ, π_β oscillate chaotically

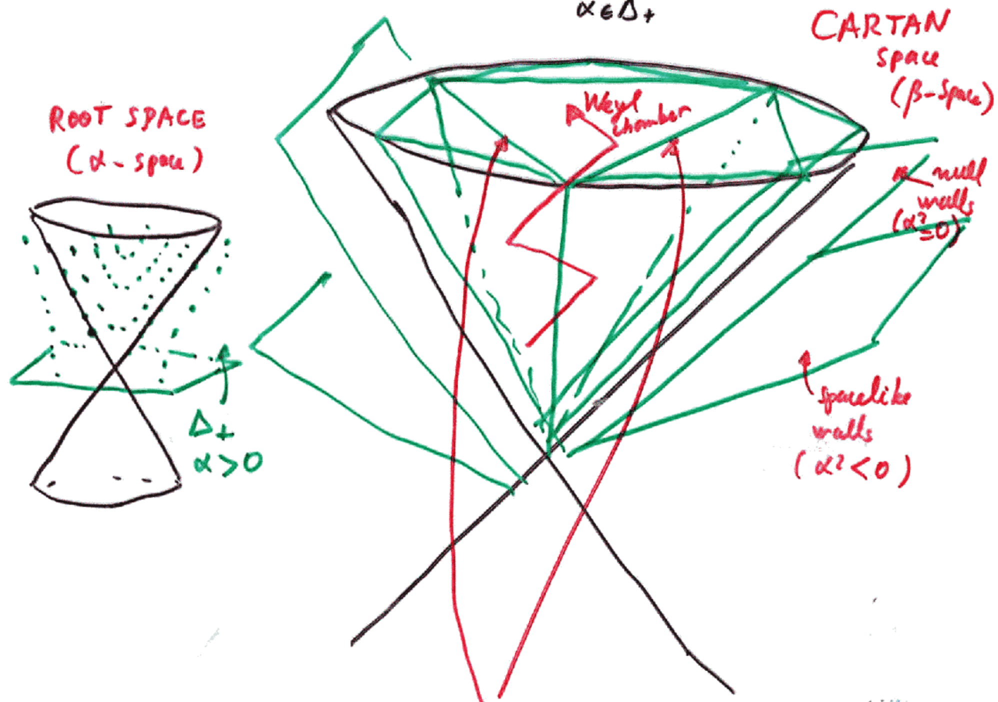
<u>but</u> ν_α, Π_α freeze $\to$ finite limits

$$\Rightarrow \quad \mathcal{H}_{E_{10}} \simeq \tfrac{1}{2} G^{\mu\nu} \pi_{\beta\mu} \pi_{\beta\nu} + \sum_{\alpha \in \Delta_+} K_\alpha e^{-2\alpha(\beta)}$$

positive roots

Hyperbolic Toda dynamics with infinite number of walls labelled by $\alpha \in \Delta_+$

$$\mathcal{H}_{E_{10}} \simeq \tfrac{1}{2} G^{\mu\nu} \pi_{\beta\mu} \pi_{\beta\nu} + \sum_{\alpha \in \Delta_+} K_\alpha e^{-2\alpha(\beta)}$$



dominant walls $\Leftrightarrow$ simple roots $\quad \sum_{i=1}^{10} K_i\, e^{-2\alpha_i^{simple}(\beta)}$

in Weyl chamber: $\alpha_i^{simple}(\beta) > 0 \Rightarrow$ simple walls are "small"

higher-height walls: $K_\alpha e^{-2\alpha(\beta)} = K_\alpha \left(e^{-\alpha_1^{simple}}\right)^{m_1} \dots \left(e^{-\alpha_{10}^{simple}}\right)^{m_{10}}$

$\alpha = m_1 \alpha_1^{simple} + \dots + m_r \alpha_r^{simple}$

$\lll$ simple walls

SUGRA$_{10}$ Hamiltonian

$$\mathcal{L}_{11} = \sqrt{G}\,R(G) - \frac{1}{2}\frac{1}{(p+1)!} F_{\mu_1\cdots\mu_{p+1}} F^{\mu_1\cdots\mu_{p+1}} + CS \text{ terms}$$

$$\mathcal{H}_{11} = \underbrace{\pi_g^{ij}\pi_{ij} - \frac{1}{d-1}\pi_{g\,i}^{\;i}\pi_{g\,j}^{\;j}}_{\text{kinetic terms of } G_{ij}} \; - \underbrace{g\,R^{(10)}}_{\text{potential terms of gravity}} + \underbrace{\frac{1}{2}\frac{1}{p!}\pi_A^{d_1\cdots d_p}\pi_{A\,d_1\cdots d_p}}_{\text{electric energy}} + \underbrace{\frac{1}{2(p+1)} F_{d_1\cdots d_{p+1}} F^{d_1\cdots d_{p+1}}}_{\text{magnetic energy}} + \cdots$$

Iwasawa decomposition: $G_{ij} = \sum_a e^{-2\beta_a} \mathcal{N}^a_i \mathcal{N}^a_j$

diagonal metric elements β_a, $a = 1\dots 10$; off-diagonal

$$\boxed{\theta^a \equiv \mathcal{N}^a_i\, dx^i}$$

$$\boxed{d\theta^a = -\frac{1}{2} C^a_{bc}\, \theta^b \wedge \theta^c}$$

$$\mathcal{H}_{SUGRA_{11}} = \underbrace{\frac{1}{4}\left(\sum(\pi_{\beta a})^2 - \frac{1}{d-1}\left(\sum \pi_{\beta a}\right)^2\right)}_{\text{Lorentzian metric on } \pi_\beta} + V^{sym} + V^{grav} + V^{elect} + V^{magnetic}$$

$$\left\{\begin{aligned}
V^{sym} &= \frac{1}{2}\sum_{a<b} e^{-2(\beta^b-\beta^a)} \left(P_a^{\;j}\mathcal{N}^b_j\right)^2 \\
V^{grav} &= -gR = \frac{1}{4}\sum{}' e^{-2\alpha_{abc}(\beta)} \left(C^a_{\;bc}\right)^2 + \sum e^{-2\mu_a(\beta)} F(\partial\beta, \partial\beta, \partial C) \\
V^{elect} &= \frac{1}{2p!}\sum e^{-2e_{a_1\cdots a_p}(\beta)} \left(\mathcal{E}^{a_1\cdots a_p}\right)^2 \\
V^{magnetic} &= \frac{1}{2(p+1)!}\sum e^{-2m_{a_1\cdots a_{p+1}}(\beta)} \left(\mathcal{F}_{a_1\cdots a_{p+1}}\right)^2
\end{aligned}\right.$$

Iwasawa-θ^a basis components of $\pi^{d_1\cdots d_p}$, $\mathcal{F}_{d_1\cdots d_{p+1}}$

$$\alpha_{abc} = 2\beta^a + \sum_{e\neq abc}\beta^e \qquad \mu_a(\beta) = \sum_{c\neq a}\beta^c$$

$$e_{a_1\cdots a_p} = \beta^{a_1} + \cdots + \beta^{a_p}$$

$$m_{a_1\cdots a_{p+1}} = \sum_{b\neq a_1\cdots a_{p+1}} \beta^b$$

CORRESPONDENCE

$\mathcal{L}^{(D=1)}_{E_{10}/K(E_{10})}$ | $\mathcal{L}^{(D=11)}_{SUGRA}$

truncate $\mathcal{V}(t) = e^{h_b^a K_a^{\;b}}\, e^{A_3 E^3 + A_6 E^6 + A_9 E^9}$

$h_a^{\;b}(t)$, $A_{abc}(t)$, $A_{a_1\cdots a_6}(t)$, $A_{a_0|a_1\cdots a_8}(t)$

- look at E.O.M. $\leftrightarrow$ Hamiltonian
- truncate in EOM the terms that would correspond to height ≥ 0 in H

i.e. deriving from $\delta H = e^{-m_1\alpha_1^{simple}(\beta) - \cdots - m_{10}\alpha_{10}^{simple}(\beta)}$ with $m_1 + \cdots + m_{10} \geq 30$

$G_{\mu\nu}(x^0, \vec{x})$ $\quad \mathcal{A}_{\mu\nu\lambda}(x^0; \vec{x})$

$$ds^2 = -(N\sqrt{G})^2 (dx^0)^2 + G_{ab}\,\theta^a_{(x)}\theta^b_{(x)}$$

$$\mathcal{F} = d\mathcal{A} = \frac{1}{3!}\mathcal{F}_{0abc}\, dx^0 \wedge \theta^a \wedge \theta^b \wedge \theta^c + \frac{\mathcal{F}_{abcd}}{4!}\theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d$$

$$\theta^a(x) = e^a_{\;i}(x)\, dx^i$$

$$d\theta^a = \frac{1}{2} C^a_{\;bc}\, \theta^b \wedge \theta^c$$

- look at EOM
- truncate away terms corresponding to $\delta H = e^{-\sum m_i \alpha_i^{simple}}$ with $\sum m_i \geq 30$

The Result is identical, at each point $\vec{x}$, under the map

$$(e^h)^a_{\;c}\,(e^h)^b_{\;c}(t) \;\;\rightleftharpoons\;\; G^{ab}(t, \vec{x})$$

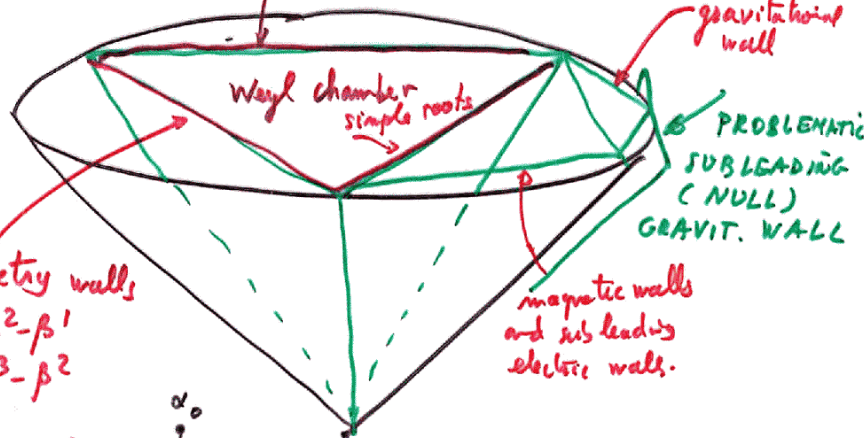
$$\dot{A}_{a_1 a_2 a_3}(t) \;\;\rightleftharpoons\;\; \mathcal{F}_{0 a_1 a_2 a_3}(t, \vec{x})$$

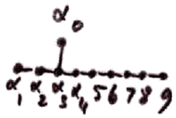
$$DA^{a_1\cdots a_6}(t) = g^{a_1 a_1'}\cdots g^{a_6 a_6'}\left[\dot{A}_{a_1'\cdots a_6'} + 10\, A_{[a_1' a_2' a_3'}\dot{A}_{a_4' a_5' a_6']}\right] = -\frac{1}{4!}\varepsilon^{a_1\cdots a_6 b_1\cdots b_4}\mathcal{F}_{b_1\cdots b_4}(t, \vec{x})$$

$$DA^{b|a_1\cdots a_8}(t) = g\cdots\left[\dot{A}_9 + 42\, A_3\dot{A}_6 + 280\, A_3 A_3 \dot{A}_3\right] = +\frac{3}{2}\varepsilon^{a_1\cdots a_8 b_1 b_2}\left[C^b_{\;b_1 b_2}(\vec{x}) + \frac{2}{9}\delta^b_{[b_1} C^c_{\;b_2]c}(\vec{x})\right]$$

only trace-free part of $C^a_{\;bc}$

FIRST PROBLEMATIC TERMS





$$\text{in } \mathcal{H}_{SUGRA} \ni -gR = \sum e^{-2\alpha_{abc}(\beta)} \left(C^a_{\ bc}\right)^2 + \sum e^{-\mu_a(\beta)} F(\partial^2\beta, \partial\beta, \partial C, C)$$

$$\alpha_{abc} = \beta^a - \beta^b - \beta^c + \sum \beta^e \qquad \mu_a(\beta) = \sum_{c \neq a} \beta^c$$

PROBLEMATIC TERMS: $(C^a_{\ ac})^2 \leftrightarrow \alpha_{aac} = \sum_{e \neq c} \beta^e \Leftrightarrow \mu_c(\beta) = \sum_{e \neq c} \beta^e$

LOWEST PROBLEMATIC TERM: $\mu_{10} = \beta^1 + \beta^2 + \dots + \beta^9$

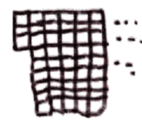
$$\mu_{10} = 3\alpha_0 + 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$$

$$\text{height}[\mu_{10}] = 3+2+4+6+5+4+3+2+1 = \underline{\underline{30}}$$

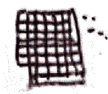
WHAT HAPPENS FOR height ≥ 30 ?

- TERMS: $\begin{cases} \partial_d (G \mathcal{F}^{dabc}) \\ \partial_a \mathcal{F}_{0bcd}] \\ \partial_a G_{bc} \to \text{in } \Gamma^{a'}_{bc} = \frac{1}{2} G^{aa'} [C_{abc} + C_{bca} - C_{cab} + \partial_b G_{ca} + \partial_c G_{ab} - \partial_a G_{bc}] \end{cases}$
  are of height > 30 and were neglected in EOM
- there is enough room in E_{10} for ALL SPATIAL GRADIENTS

$\exists$ 3 infinite towers of generators



$$E^{(b_1 b_2 b_3)}_{(a_1 \dots a_n)} \overset{?}{\longleftrightarrow} \partial^{a_1 \dots a_n} A_{b_1 b_2 b_3}$$



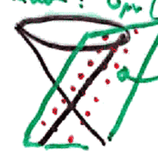
$$E^{[b_1 \dots b_6]}_{(a_1 \dots a_n)} \overset{?}{\longleftrightarrow} \partial^{a_1 \dots a_n} A_{b_1 \dots b_6}$$

$$E^{b_0 | b_1 \dots b_8}_{(a_1 \dots a_n)} \overset{?}{\longleftrightarrow} \partial^{a_1 \dots a_n} A_{b_0 | b_1 \dots b_8} \ \underset{\sim}{\text{i.e.}}\ \partial^{a_1 \dots a_n} \varepsilon_{b_1 \dots b_{10}} C^a_{\ bc}$$

- the fact that the first problematic terms $\mu_a(\beta)$ formally correspond to null roots suggest the possibility

Kac-Moody $\longrightarrow$ BORCHERDS algebras ?

- When looking at 2D-truncations: $G_{\mu\nu}(x^0, x^1)$, ... we know that E_9 is a symmetry: NON LOCAL Geroch-type



E_9 is realized on potentials $\Rightarrow$ need NON-LOCAL MAP?

CONCLUSIONS

- TANTALIZING HINTS OF INFINITE-DIMENSIONAL HIDDEN SYMMETRY $E_{10}(\mathbb{R})$ OF $SUGRA_{11}$ EOM:
  TRANSFORMING SOLUTION $\longrightarrow$ SOLUTION'
  (SIMILAR TO GEROCH GROUP $A_1^{\wedge}$ FOR $GR^{3+1}_{\downarrow\, 1+1}$ OR E_9 FOR $SUGRA^{11}_{\downarrow\, 2}$)
  [PROBABLY BROKEN DOWN TO $E_{10}(\mathbb{Z})$ FOR M-THY]
  $\uparrow$
  STATE $\to$ EQUIVALENT STATE
- BKL-like EXPANSION $\frac{1}{c}\frac{\partial h}{\partial t} \gg \frac{\partial h}{\partial x}$ OR $\leftrightsquigarrow$ height expansion $\sum_A C_A\, e^{-\alpha_A(\beta)}$
  would be a way to reveal this symmetry
- Formally a "small (bulk) tension" limit $T_b = \frac{c^4}{32\pi G} \to 0$
  ? any link with D. Gross $T_s = \frac{1}{2\pi\alpha'} \to 0$ limit ?
  though D. Gross' limit is very stringy, and we are very fieldy
- the correspondence maps: $SOL^{ns}\ SUGRA_{11} \longrightarrow$ NULL GEODESIC ON COSET SPACE $E_{10}/K(E_{10})$
  $\exists$ formally ∞ # conserved quantities, but it is not clear whether all geodesics can be ${}^{E_{10}}$mapped to a "Cartan" geodesic
- our construction extends to many other systems.
  $\{I, HET\} \leftrightarrow BE_{10}$ $\quad GR^{(d+1)} \leftrightarrow AE_d$ $\quad BOSONIC^{(26)} \leftrightarrow DE_{26}$