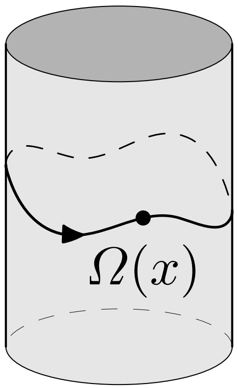
Integrability in AdS/CFT, Part I: Classical Strings and Spin Chains



Niklas Beisert

Princeton University

Mathematical Structures
in String Theory

KITP, Santa Barbara
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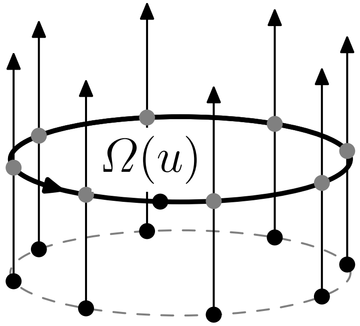


Based on work in collaboration with:
V. Kazakov, K. Sakai, M. Staudacher, A. Tseytlin, K. Zarembo
`hep-th/0310252,0410253,0502226.0503200.0509084.`

Introduction

AdS/CFT Conjecture

- $\mathcal{N} = 4$ gauge theory (exactly) dual to IIB superstrings on $AdS_5 \times S^5$.
- Spectrum should agree. Would like to test.
- Strong/weak duality: Cannot use perturbation theory on both sides.
- Hope: Both models appear integrable.

Outline

- Perturbative comparison of spectra
- Integrability of the classical superstring sigma model
- From a classical solution to an algebraic curve

Assume: Classical, non-interacting strings.

AdS/CFT as a Strong/Weak Duality

AdS/CFT predics spectra of strings and gauge theory to match. [Maldacena hep-th/9711200]

String theory expansion of energies E at large λ [Gubser Klebanov Polyakov]

$$E(\lambda) = \lambda^{1/4} E_0 + \lambda^{-1/4} E_1 + \lambda^{-3/4} E_2 + \lambda^{-5/4} E_3 + \ldots .$$

Gauge theory expansion of scaling dimensions D at small λ

$$D(\lambda) = D^0 + \lambda D^1 + \lambda^2 D^2 + \lambda^3 D^3 + \ldots .$$

How to confirm $E(\lambda) = D(\lambda)$?

Large Spin on S^5

Consider states with variable spin J on S^5 (and lots of other parameters).

Effective spin $\mathcal{J}$, effective coupling $\tilde{\lambda}$: [Berenstein Maldacena Nastase] [Frolov, Tseytlin hep-th/0306130]

$$\mathcal{J} = \frac{J}{\sqrt{\lambda}}, \qquad \tilde{\lambda} = \frac{\lambda}{J^2} = \frac{1}{\mathcal{J}^2}.$$

String theory expansion of energies E at large λ, fixed $\mathcal{J}$ [Gubser Klebanov Polyakov]

$$E(\lambda, J) = \lambda^{1/2} E_0(\mathcal{J}) + \lambda^0 E_1(\mathcal{J}) + \lambda^{-1/2} E_2(\mathcal{J}) + \ldots .$$

Gauge theory expansion of scaling dimensions D at small λ, fixed J

$$D(\lambda) = D^0(J) + \lambda D^1(J) + \lambda^2 D_2(J) + \lambda^3 D_3(J) + \ldots .$$

Outsmarting AdS/CFT

Expansion for large $\mathcal{J}$ at large λ: [Frolov, Tseytlin hep-th/0204226][Frolov, Tseytlin hep-th/0306130](also [Berenstein Maldacena Nastase])

$$
\begin{aligned}
E(\lambda, J) &= \lambda^{1/2}\left(\mathcal{J}\, E_0^0 + \mathcal{J}^{-1}\, E_0^1 + \mathcal{J}^{-3}\, E_0^2 + \mathcal{J}^{-5}\, E_0^3 + \ldots\right) \\
&\quad + \lambda^0\left(\mathcal{J}^{-2}\, E_1^1 + \mathcal{J}^{-4}\, E_1^2 + \mathcal{J}^{-6}\, E_1^3 + \ldots\right) \\
&\quad + \lambda^{-1/2}\left(\mathcal{J}^{-3}\, E_2^1 + \mathcal{J}^{-5}\, E_2^2 + \mathcal{J}^{-7}\, E_2^3 + \ldots\right) \\
&\quad + \ldots \\
&= J\, E_0^0 \\
&\quad + \frac{\lambda}{J} E_0^1 + \frac{\lambda^2}{J^3} E_0^2 + \frac{\lambda^3}{J^5} E_0^3 \\
&\quad + \frac{\lambda}{J^2} E_1^1 + \frac{\lambda^2}{J^4} E_1^2 + \frac{\lambda^3}{J^6} E_1^3 \\
&\quad + \frac{\lambda}{J^3} E_2^1 + \frac{\lambda^2}{J^5} E_2^2
\end{aligned}
$$

Three-Loop Discrepancies

Expansion for large J at small λ: [NB, Minahan, Staudacher, Zarembo][Serban, Staudacher](also [Callan, Lee, McLoughlin, Schwarz, Swanson, Wu])

$$\begin{aligned}
D(\lambda, J) = J D^0_0 \\
&+ \lambda\big(J^{-1} D^1_1 + J^{-2} D^1_2 + J^{-3} D^1_3 + \ldots\big) \\
&+ \lambda^2\big(J^{-3} D^2_1 + J^{-4} D^2_2 + J^{-5} D^2_3 + \ldots\big) \\
&+ \lambda^3\big(J^{-5} D^3_1 + J^{-6} D^3_2 + J^{-7} D^3_3 + \ldots\big) + \ldots
\end{aligned}$$

$$\begin{aligned}
= J\, D^0_0 \\
&+ \frac{\lambda}{J} D^1_0 + \frac{\lambda^2}{J^3} D^2_0 + \frac{\lambda^3}{J^5} D^3_0 \\
&+ \frac{\lambda}{J^2} D^1_1 + \frac{\lambda^2}{J^4} D^2_1 + \frac{\lambda^3}{J^6} D^3_1 \\
&+ \frac{\lambda}{J^3} D^1_2 + \frac{\lambda^2}{J^5} D^2_2
\end{aligned}$$

Outsmarted by AdS/CFT

Actual expansion for large $\mathcal{J}$ at large λ: [NB, Tseytlin hep-th/0509084][Schäfer-Nameki Zamaklar]

$$
\begin{aligned}
E(\lambda, J) = \lambda^{0.5}\,\big(&\mathcal{J}\,E_0^0 + \mathcal{J}^{-1}\,E_0^1 + \mathcal{J}^{-3}\,E_0^2 + \mathcal{J}^{-5}\,E_0^3 + \ldots\big)\\
&+ \lambda^{0}\big(\mathcal{J}^{-2}\,E_1^1 + \mathcal{J}^{-4}\,E_1^2 + \mathcal{J}^{-5}\,E_1^{2.5} + \mathcal{J}^{-6}\,E_1^3 + \ldots\big)\\
&+ \lambda^{-0.5}\big(\mathcal{J}^{-3}\,E_2^1 + \mathcal{J}^{-5}\,E_2^2 + \mathcal{J}^{-6}\,E_2^{2.5} + \mathcal{J}^{-7}\,E_2^3 + \ldots\big)\\
&+ \ldots\\
= J\,E_0^0&\\
&+ \frac{\lambda}{J}\,E_0^1 + \frac{\lambda^2}{J^3}\,E_0^2 + \frac{\lambda^3}{J^5}\,E_0^3\\
&+ \frac{\lambda}{J^2}\,E_1^1 + \frac{\lambda^2}{J^4}\,E_1^2 + \frac{\lambda^{2.5}}{J^5}\,E_1^{2.5} + \frac{\lambda^3}{J^6}\,E_1^3\\
&+ \frac{\lambda}{J^3}\,E_2^1 + \frac{\lambda^2}{J^5}\,E_2^2 + \frac{\lambda^{2.5}}{J^6}\,E_2^{2.5}
\end{aligned}
$$

AdS and CFT

Attempt to avoid strong/weak duality at large spin J [Berenstein Maldacena Nastase][Frolov, Tseytlin hep-th/0306130]

- Coefficients of expansion different. [Callan, Lee, McLoughlin Schwarz, Swanson, Wu][Serban Staudacher]
- Structure of expansion different. [NB, Tseytlin hep-th/0509084][Schäfer-Nameki Zamaklar]

What next?

- Compute AdS at large λ.
- Compute CFT at small λ.
- Notice similar structures.
- Some agreement up to $\mathcal{O}(\lambda^2)$ or $\mathcal{O}(1/J^4)$.
- Understand how to interpolate to finite λ.
- Three-loop mismatch related to new terms $E_0^3 - D_0^3 = -\frac{16}{3}E_1^{2.5}$.

Overview Classical Strings

⋆ **Cast of Characters**

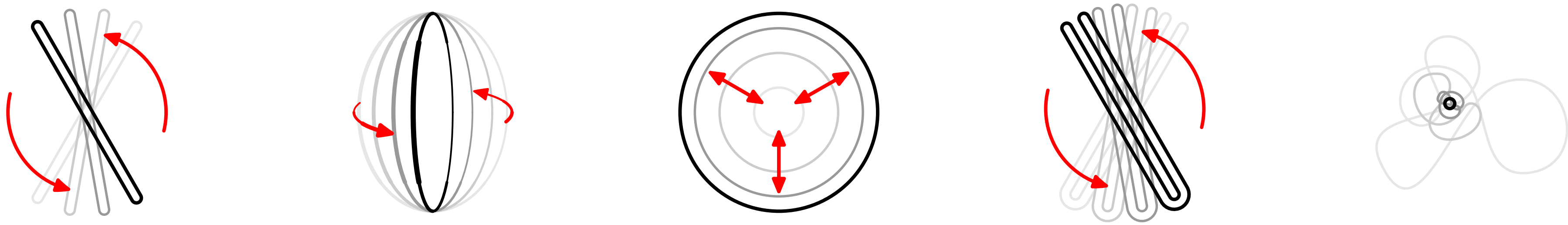
- Classical spinning string solutions
- Coset space sigma model
- Integrability, Lax connection, monodromy

⋆ **Results**

- Spectral curve
- Analytic properties
- String moduli (finite cut solutions)
- Integral equations

Spinning Strings

Many examples investigated: [Gubser Klebanov Polyakov][Frolov, Tseytlin hep-th/0204226][Minahan hep-th/0209047][Frolov, Tseytlin hep-th/0304255]. . .



folded | circular | pulsating | higher modes | plane waves

Ansatz, e.g. string on $\mathbb{R}_t \times S^2$: Energy $\mathcal{E} = E/\sqrt{\lambda}$, spin $\mathcal{J} = J/\sqrt{\lambda}$.

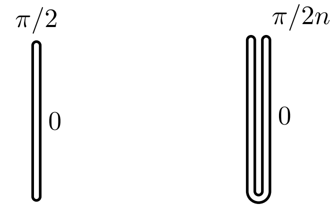
$$t(\tau,\sigma) = \mathcal{E}\,\tau, \quad \vec{X}(\tau,\sigma) = \begin{pmatrix} \sin\vartheta(\sigma)\,\cos\mathcal{J}\tau \\ \sin\vartheta(\sigma)\,\sin\mathcal{J}\tau \\ \cos\vartheta(\sigma) \end{pmatrix}.$$

Solve equations of motion and Virasoro constraint

$$\vartheta(\sigma) = \mathrm{am}(\mathcal{E}(\sigma-\sigma_0),\eta), \qquad \mathcal{J} = \eta\,\mathcal{E}.$$

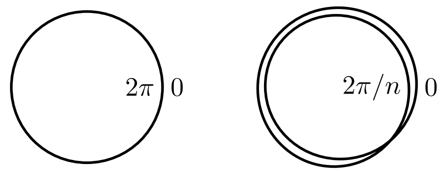
Periodicity

Folded string: $\vartheta(0) = 0$ and $\vartheta'(\pi/2n) = 0$



$$J = \sqrt{\lambda}\,\mathcal{J} = \sqrt{\lambda}\,\frac{2n}{\pi}\mathrm{K}(1/\eta), \qquad E = \sqrt{\lambda}\,\mathcal{E} = \sqrt{\lambda}\,\frac{2n}{\eta\pi}\mathrm{K}(1/\eta).$$

Circular string: $\vartheta(0) = 0$ and $\vartheta(2\pi/n) = 2\pi$



$$J = \sqrt{\lambda}\,\mathcal{J} = \sqrt{\lambda}\,\frac{2n\eta}{\pi}\mathrm{K}(\eta), \qquad E = \sqrt{\lambda}\,\mathcal{E} = \sqrt{\lambda}\,\frac{2n}{\pi}\mathrm{K}(\eta).$$

Global charges of generic solutions

$$J_k = \sqrt{\lambda}\,\mathcal{J}_k(\eta_a), \qquad S_k = \sqrt{\lambda}\,\mathcal{S}_k(\eta_a), \qquad E = \sqrt{\lambda}\,\mathcal{E}(\eta_a)$$

with algebraic, elliptic, hyperelliptic, ... functions of moduli $\{\eta_a\}$.

- Why elliptic functions? What is the meaning of moduli?

Towards a General Solution

- Too difficult to solve the equations of motion in general.
  No direct way to quantization as in flat space or plane waves.
- Near plane waves: Very difficult to expand around plane waves.
  Only expansion, but good testing ground.

Now what?

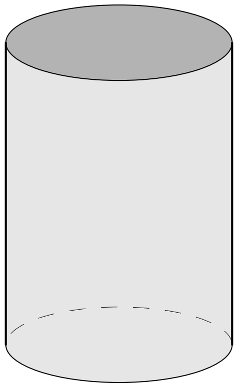
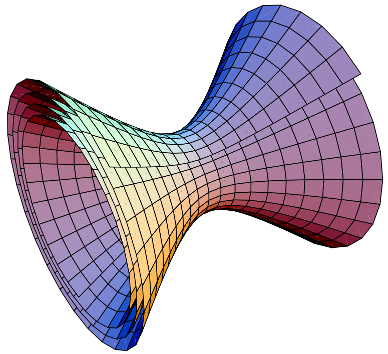
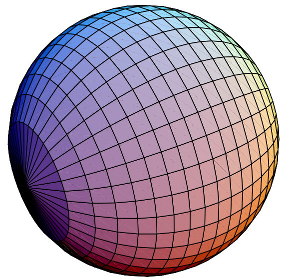
- Give up on finding exact energy spectrum.
- Classify solutions to understand structure of spectrum.
- Try to quantise that.

How?!

- Extract all conserved charges: Lax pair, monodromy.
- Investigate their analyticity properties.
- Reconstruct the corresponding algebraic curve.
- Discretise the curve.

Strings on $AdS_5 \times S^5$

IIB superstrings propagate on the curved superspace $AdS_5 \times S^5$

 $\longrightarrow$  $\times$  $\times$ fermi

Coset space

$$AdS_5 \times S^5 \times \text{fermi} = \frac{\mathrm{PSU}(2,2|4)}{\mathrm{Sp}(1,1) \times \mathrm{Sp}(2)} \,.$$

Decomposition of the algebra $\mathfrak{u}(2,2|4)$ to $\mathfrak{sp}(1,1) \times \mathfrak{sp}(2)$

$$j \in \mathfrak{psu}(2,2|4), \qquad j = h + q_1 + p + q_2, \qquad h \in \mathfrak{sp}(1,1) \times \mathfrak{sp}(2).$$

Algebra $j = [j_1, j_2]$ respects $\mathbb{Z}_4$-grading $h\colon 0$, $q_1: 1$, $p\colon 2$, $q_2: 3$ [Berkovits Bershadsky, Hauer Zhukov, Zwiebach]

Supersymmetric Sigma Model

Field $g(\sigma,\tau)\in \mathrm{U}(2,2|4)$ $(8\times 8$ supermatrix) with flat connection J

$$J=-g^{-1}dg=H+Q_1+P+Q_2,\qquad dJ-J\wedge J=0.$$

Coset $g\simeq gh$ with $h(\sigma,\tau)\in \mathrm{Sp}(1,1)\times\mathrm{Sp}(2)$. Action [Metsaev Tseytlin][Roiban Siegel]

$$S_\sigma=\frac{\sqrt{\lambda}}{2\pi}\int\left(\tfrac{1}{2}\operatorname{str}P\wedge *P-\tfrac{1}{2}\operatorname{str}Q_1\wedge Q_2+\mathit{\Lambda}\wedge\operatorname{str}P\right).$$

$\mathfrak{psu}(2,2|4)$ Noether current K and equation of motion

$$K=P+\tfrac{1}{2}{*Q_1}-\tfrac{1}{2}{*Q_2}-{*\mathit{\Lambda}},\qquad d{*K}-J\wedge{*K}-{*K}\wedge J=0.$$

Virasoro constraints

$$\operatorname{str}P_+^2=\operatorname{str}P_-^2=0.$$

Lax Connection

Integrability $\rightsquigarrow$ Lax pair: Family of connections [Bena Polchinski Roiban]

$$A(z) = H + \tfrac{1}{2}(z^{-2} + z^2)P + \tfrac{1}{2}(z^{-2} - z^2)(*P - \Lambda) + z^{-1}Q_1 + z\,Q_2.$$

Connection $A(z)$ flat for all values of the spectral parameter z

$$dA(z) - A(z) \wedge A(z) = 0.$$

Equivalent to flatness of J and conservation of K.

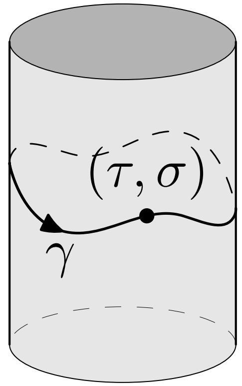
- Analytic for all $z \in \bar{\mathbb{C}}$.
- Poles at $z = 0, \infty$.
- Point $z = 1$ related to global symmetry: $A(1+\epsilon) = J - 2\epsilon *K + \ldots$.

Alternative spectral parameter x (double covering):

$$x = \frac{1+z^2}{1-z^2} \qquad z^2 = \frac{x-1}{x+1}.$$

Monodromy

Monodromy of Lax connection around closed string [Kazakov, Marshakov Minahan, Zarembo]



$$\Omega(z) = \left(\mathrm{P}\exp\oint_{-\gamma} J\right)\left(\mathrm{P}\exp\oint_{\gamma} A(z)\right).$$

Eigenvalues invariant under deformations of γ [NB, Kazakov Sakai, Zarembo]

$$\Omega(z) \simeq \mathrm{diag}\big(e^{i\hat{p}_1(z)}, \ldots, e^{i\hat{p}_4(z)} \big|\big| e^{i\tilde{p}_1(z)}, \ldots, e^{i\tilde{p}_4(z)}\big).$$

Transformation of solution $g(\sigma,\tau)$ to set of quasi-momenta $\{p_k(z)\}$.

- The $p_k(z)$ are conserved, gauge-invariant quantities.
- No (conformal/kappa) gauge fixing required.
- Analytic functions of z: Much physical information in $\{p_k(z)\}$.
- $\{p_k(z)\}$ contains all (?) action variables in Hamilton-Jacobi formalism.
- Diagonalising $\Omega(z)$ introduces (solution-dependent) singular points.

Global Charges

Expansion of Lax connection at $z = 1$:

$$A(1+\epsilon) = J - 2\epsilon \ast K + \mathcal{O}(\epsilon^2).$$

Global $\mathfrak{psu}(2,2|4)$ charges S can be read off from monodromy at $z = 1$

$$\Omega(1+\epsilon) = I - \epsilon \, \frac{4\pi \, S}{\sqrt{\lambda}} + \mathcal{O}(\epsilon^2).$$

Expansion of quasi-momenta (fix $\hat{p}_k(1) = \tilde{p}_k(1) = 0$)

$$\hat{p}_k(1+\epsilon) \sim \epsilon \, \frac{4\pi(E, S_1, S_2)}{\sqrt{\lambda}} + \ldots, \qquad \tilde{p}_k(1+\epsilon) \sim \epsilon \, \frac{4\pi(J_1, J_2, J_3)}{\sqrt{\lambda}} + \ldots$$

Conjugation Symmetry

$\mathbb{Z}_4$ property of supertranspose: $X^{\mathsf{ST},\mathsf{ST}} = \eta\, X\, \eta$, $X^{\mathsf{ST},\mathsf{ST},\mathsf{ST},\mathsf{ST}} = X$.
Conjugation of connection $J = H + Q_1 + P + Q_2$

$$C\,(H, Q_1, P, Q_2)^{\mathsf{ST}}\, C^{-1} = (-H, -iQ_1, +P, +iQ_2).$$

Map $z \mapsto iz$ conjugates Lax connection and monodromy

$$A(iz) = -C\, A^{\mathsf{ST}}(z)\, C^{-1}, \qquad \Omega(iz) = C\, \Omega^{-\mathsf{ST}}(z)\, C^{-1}.$$

Transformation of quasi-momenta with $k' = (2, 1, 4, 3)$, $\varepsilon_k = (+, +, -, -)$

$$\hat{p}_k(iz) = -\hat{p}_{k'}(z), \qquad \tilde{p}_k(iz) = 2\pi m \varepsilon_k - \tilde{p}_{k'}(z).$$

$z \mapsto -z$ is a trivial symmetry of quasi-momenta. Okay to use x

$$x = \frac{1 + z^2}{1 - z^2} \qquad z^2 = \frac{x - 1}{x + 1}.$$

Analyticity

Monodromy $\Omega(z)$ is analytic in z except at $z = 0, \infty$. Consider $z = 0$: Diagonalise Lax connection perturbatively with regular $T(z)$

$$\partial_\sigma - \bar{A}_\sigma(z) = T(z)\big(\partial_\sigma - A_\sigma(z)\big)T^{-1}(z).$$

Derivative $\partial_\sigma = \mathcal{O}(z^0)$ subleading w.r.t. $A_\sigma(z) = \mathcal{O}(1/z^2)$:

$$\begin{aligned}\bar{A}(z) &= \tfrac{1}{2}T(P_+ + \Lambda_\sigma)T^{-1}/z^2 + \mathcal{O}(1/z)\\ &= \operatorname{diag}(\alpha,\alpha,\beta,\beta||\,\alpha,\alpha,\beta,\beta)/z^2 + \mathcal{O}(1/z)\end{aligned}$$

Degeneracies due to conjugation $CP^{\mathsf{ST}}C^{-1} = P$, tracelessness $\operatorname{str} P = 0$ and Virasoro $\operatorname{str} P_+^2 = 0$.

$$\hat{p}_{1,2}(z) \sim \tilde{p}_{1,2}(z) \sim \alpha/z^2, \quad \hat{p}_{3,4}(z) \sim \tilde{p}_{3,4}(z) \sim \beta/z^2 \qquad \text{at } z = 0.$$

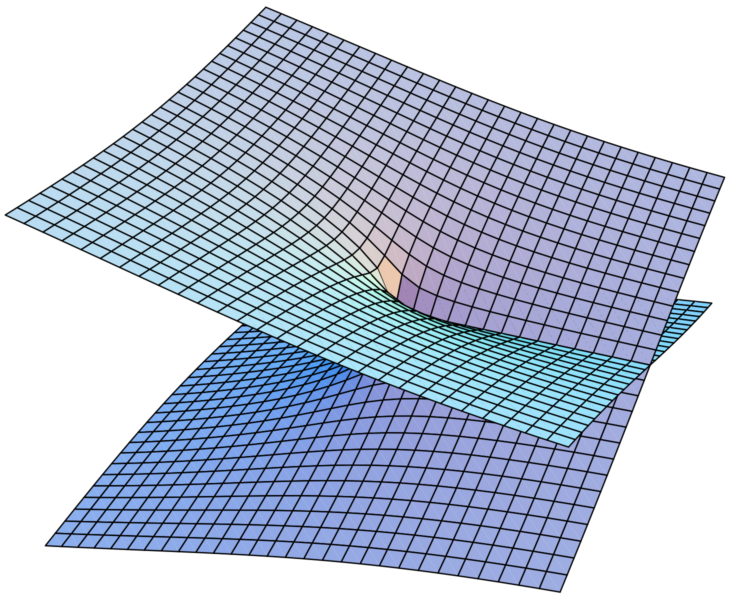
Diagonalization introduces new singularities $\{\hat{z}_a, \tilde{z}_a, z_a^*\}$ in $\hat{p}_k(z), \tilde{p}_k(z)$.

Bosonic Branch Points

Eigenvalue crossing: Consider 2×2 bosonic submatrix $\mathit{\Gamma}$ of $\mathit{\Omega}(z)$ [NB, Kazakov Sakai, Zarembo]

$$\mathit{\Gamma} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \gamma_{1,2} = \tfrac{1}{2}\left(a + d \pm \sqrt{(a-d)^2 + 4bc}\right).$$

Generic behaviour at degenerate eigenvalues $e^{ip_k(z_a)} = e^{ip_l(z_a)}$:



$$e^{ip_k(z_a)}\left(1 \pm \alpha_a \sqrt{z - z_a} + \mathcal{O}(z - z_a)\right).$$

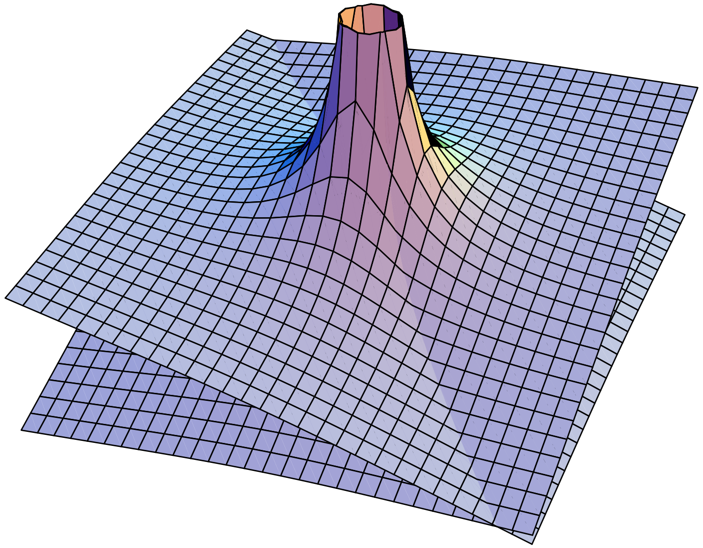
Full turn around z_a interchanges eigenvalues (labelling): Branch cut.

Fermionic Singularities

Mixed eigenvalue crossing: Consider $(1|1) \times (1|1)$ submatrix Γ of $\Omega(z)$

$$\Gamma = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right), \qquad \hat{\gamma} = \frac{bc}{d-a} + a, \quad \tilde{\gamma} = \frac{bc}{d-a} + d\,.$$

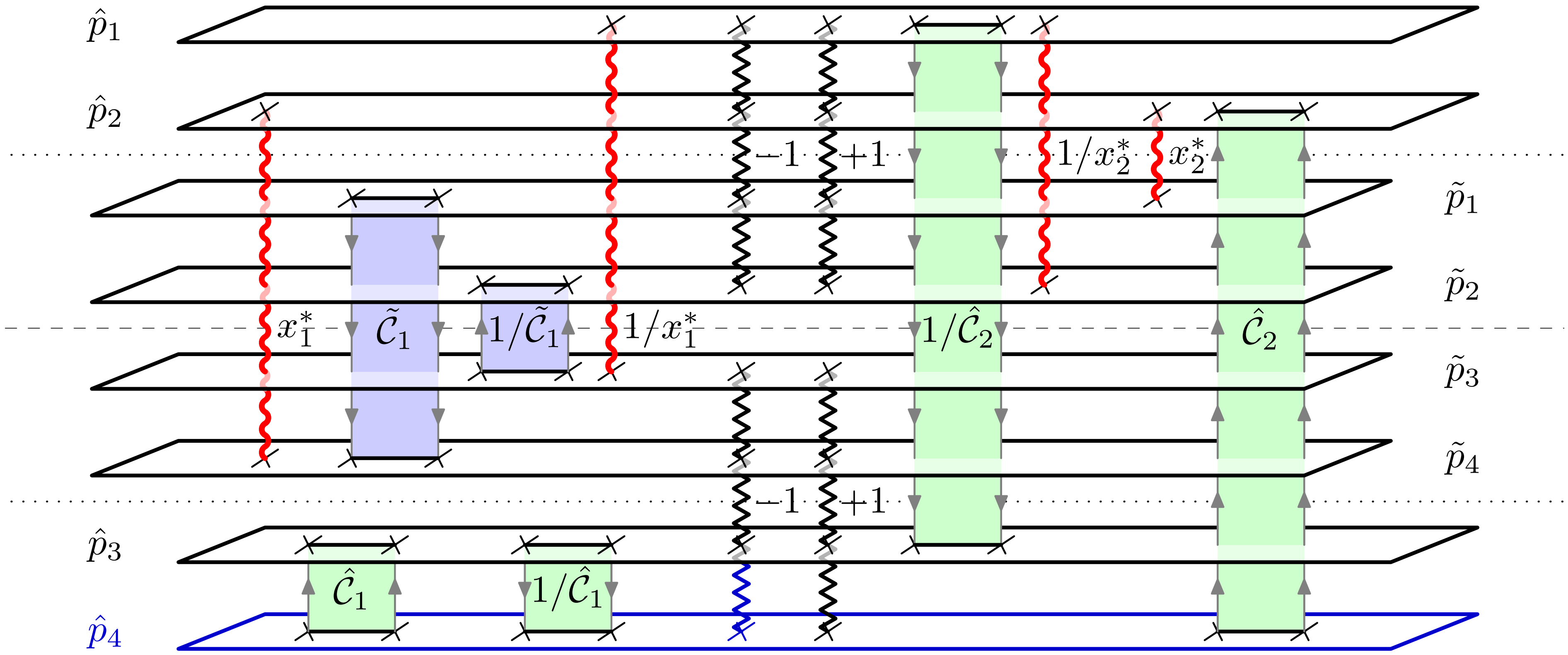
Generic behaviour at degenerate eigenvalues $e^{i\hat{p}_k(z_a^*)} = e^{i\tilde{p}_l(z_a^*)}$:



$$e^{i\hat{\tilde{p}}_k(z_a^*)} \left(\frac{\alpha_a^*}{z - z_a^*} + 1 + \mathcal{O}(z - z_a^*)\right).$$

Residue of fermionic singularity $\alpha_a^* \sim bc$ is nilpotent.

Spectral Curve

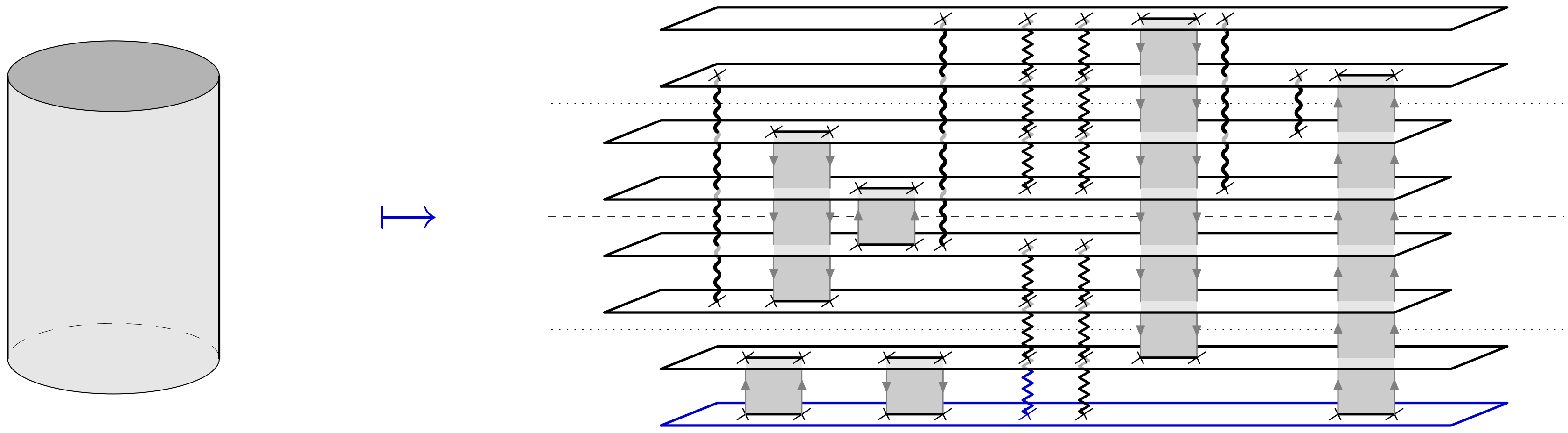


- Singularities at $x = \pm 1$; asymptotics at $x = 0, \infty$; symmetry $x \mapsto 1/x$.
- Bosonic modes: Square-roots, branch cuts (Bose condesates).
- Fermionic excitations: Poles (Pauli principle).
- Stringy spectrum of physical excitations $4 + 4|\, 8$.

[NB, Kazakov Sakai, Zarembo]

Spectral Transformation

From embedding of world-sheet $g(\sigma, \tau)$ to spectral curve $p'(x)$.



Spectral curve encodes conserved charges of a string solution.

Algebraic Curve

Can the Riemann surface $\mathbb{M}$ be emdedded in $\mathbb{C}^2$ as an algebraic curve?

- Finite genus: Assume finitely many singularities $\{\hat{x}_a, \tilde{x}_a, x_a^*\}$. ✓
  Other solutions should be considered as limiting cases.
- Eigenvalues $e^{ip(x)}$ of $\Omega(x)$ are analytic almost everywhere. ✓
- Singularities $\{\hat{x}_a, \tilde{x}_a, x_a^*\}$ are square-root or pole singularities. ✓
- Monodromy $\Omega(x)$ has exponential singularities at $x = \pm 1$. ×
- Quasi-momentum $p(x)$ is defined modulo 2π. ×
- $p'(x)$ is unique and has only square-root and pole singularities. ✓

$p'(x)$ is the algebraic curve associated to a classical string $g(\tau, \sigma)$

$$g(\tau, \sigma) \Longrightarrow \frac{\hat{F}(x, p'(x))}{\tilde{F}(x, p'(x))} = 0, \infty$$ with $\hat{F}, \tilde{F}$ polynomial (degree 4 in p').

Simplest spinning strings have genus $0/1$: algebraic/elliptic functions.

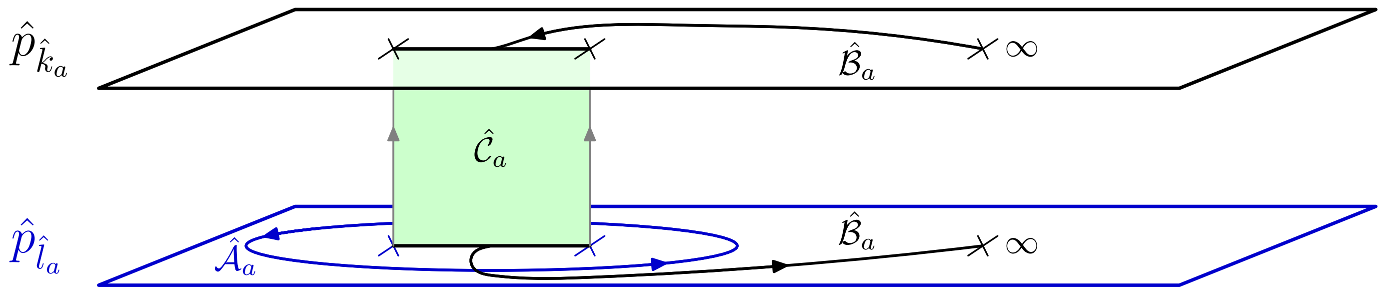
String Moduli

Single-valuedness of e^{ip}: All closed cycles must be integer

$$\oint dp \in 2\pi\mathbb{Z}.$$

Cuts/singularities: "mode number" $n_a \in \mathbb{Z}$ and "amplitude" $K_a \in \mathbb{R}$

$$\int_{\mathcal{A}_a} dp = 0, \quad n_a = \frac{1}{2\pi}\int_{\mathcal{B}_a} dp, \quad K_a = -\frac{1}{2\pi i}\oint_{\mathcal{A}_a}\left(1 - \frac{1}{x^2}\right) p(x)\, dx.$$



Solutions classified by: connection of sheets, mode numbers, amplitudes.

Integral Equations

Parametrise quasi-momenta $p(x)$ using 7 resolvents (cuts/poles)

$$G_j(x) = \int_{\mathcal{C}_{j,a}} \frac{dy\,\rho_j(y)}{1-1/y^2}\,\frac{1}{y-x} + \sum\nolimits_a \frac{\alpha_{j,a}}{1-1/x_{j,a}^2}\,\frac{1}{x_{j,a}-x}\,.$$

Integral equations with $H_j(x) = G_j(x) + G_j(1/x) - G_j(0)$

$$-2\pi n_{j,a} = \sum_{j'=1}^{7} M_{j,j'} \not{H}_{j'}(x) + F_j(x), \qquad \text{for } x \in \mathcal{C}_{j,a}, x_{j,a}.$$

$M_{j,j'}$: Cartan matrix of $\mathfrak{su}(2,2|4)$.

$F_j(x)$: Potential terms made from $G_{j'}(0)$, $G'_{j'}(0)$ and $G_{j'}(1/x)$.

Conclusions

⋆ **AdS/CFT Spectral Comparison**

- No exact perturbative comparison possible.

⋆ **IIB Strings in $AdS_5 \times S^5$**

- Classical spectral curve derived & investigated.

⋆ **Outlook**

- Integrability for gauge theory (tomorrow).
- Quantise string spectral curve.
- Find exact Bethe ansatz for AdS/CFT (if it exists).